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Research Article

Duals and Matrix Classes Involving Cesàro Type Classes of Sequences of Fuzzy Numbers

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We first define Cesàro type classes of sequences of fuzzy numbers and equip the set with a complete metric. Then we compute the Köthe-Toeplitz dual and characterize some related matrix classes involving such classes of sequences of fuzzy numbers.

1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy sets and fuzzy set operations as an extension of the classical notion of the set theory. Later on several authors have discussed different aspects of the theory of fuzzy sets and applied it in various areas of science and engineering such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy possibility theory, fuzzy measures of fuzzy events, and fuzzy mathematical programming. Nowadays, fuzzy set theory is used as a powerful mathematical tool in solving complex real life problems which yields a notion of uncertainty and vagueness. Matloka [2] introduced bounded and convergent sequences of fuzzy numbers and studied their properties. In [3], Nanda studied sequences of fuzzy numbers and proved that every Cauchy sequence of fuzzy numbers is convergent. Since then, different classes of sequences of fuzzy numbers were introduced and studied by various authors. For the works on convergence of fuzzy sequences and series, we refer to Nuray and Savaş [4], Diamond and Kloeden [5], Matloka [2], Esi [6], Kaleva [7], Nanda [8],[3], Dubois and Prade [9], Altınok, Çolak, and Altın [10], Stojaković and Stojaković [11],[12], and Mursaleen, Srivastava, and Sharma [13]. In [14], Subrahmanyam defined the Cesàro summability of sequences of fuzzy numbers and proved some related Tauberian theorems. Some interesting results related to Cesàro summability method of sequences of fuzzy numbers and the Tauberian conditions which guarantee the convergence of summable sequences of fuzzy numbers can be found in Subrahmanyam [14], Talo and Çakan [15], Altın, Mursaleen, and Altınok [16], and Yavuz [17],[18].

Definition 1 (Goetschel and Voxman [19]). A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $v : \mathbb{R} \longrightarrow [0,1]$, which satisfies the following four conditions:

(i) v is normal; i.e., there exists an $x_0 \in \mathbb{R}$ such that $v(x_0) = 1$.
(ii) v is fuzzy convex; i.e., $v[\lambda x + (1-\lambda)y] \geq \min\{v(x), v(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.
(iii) v is upper semicontinuous.
(iv) The set $[v]_0 = \overline{\{x \in \mathbb{R} : v(x) > 0\}}$ is compact, where $\overline{\{x \in \mathbb{R} : v(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : v(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it the space of fuzzy numbers. λ-level set $[v]_\lambda$ of $v \in E^1$ is defined by $[v]_\lambda = \begin{cases} \{t\in\mathbb{R}: v(t)\geq\lambda\}, & (0<\lambda\leq 1), \\ \overline{\{t\in\mathbb{R}: v(t)>\lambda,\}}, & (\lambda=0). \end{cases}$

The set $[v]_\lambda$ is a closed, bounded, and nonempty interval for each $\lambda \in [0,1]$ which is defined by $[v]_\lambda = [v^-(\lambda), v(\lambda)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined as

$$\overline{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r. \end{cases} \tag{1}$$

Definition 2 (Talo and Başar [20]). Let $x, y, z \in E^1$ and $k \in \mathbb{R}$. Then the operations addition, scalar multiplication, and product are defined on E^1 by

$$\begin{aligned} x + y = z &\Longleftrightarrow \\ [z]_\lambda = [x]_\lambda + [y]_\lambda &\Longleftrightarrow \\ z^-(\lambda) &= x^-(\lambda) + y^-(\lambda), \\ z^+(\lambda) &= x^+(\lambda) + y^+(\lambda) \\ [kx]_\lambda &= k[x]_\lambda \quad \forall \lambda \in [0,1] \end{aligned} \tag{2}$$

and

$$\begin{aligned} xy = z &\Longleftrightarrow \\ [z]_\lambda &= [x]_\lambda [y]_\lambda \\ &\forall \lambda \in [0,1], \end{aligned} \tag{3}$$

where it is immediate that

$$\begin{aligned} z^-(\lambda) = \min\{&x^-(\lambda) y^-(\lambda), x^-(\lambda) y^+(\lambda), x^+(\lambda) \\ &\cdot y^-(\lambda), x^+(\lambda) y^+(\lambda)\} \end{aligned} \tag{4}$$

and

$$\begin{aligned} z^+(\lambda) = \max\{&x^-(\lambda) y^-(\lambda), x^-(\lambda) y^+(\lambda), x^+(\lambda) \\ &\cdot y^-(\lambda), x^+(\lambda) y^+(\lambda)\} \end{aligned} \tag{5}$$

for all $\lambda \in [0,1]$.

Definition 3 (Talo and Başar [20]). Let $\mathscr{W}$ be the set of all closed bounded intervals A of real numbers such that $A = [A_1, A_2]$. Define the relation d on $\mathscr{W}$ as follows:

$$d(A,B) = \max\{|A_1 - B_1|, |A_2, B_2|\}, \tag{6}$$

where $B = [B_1, B_2] \in \mathscr{W}$. Then $(\mathscr{W}, d)$ is a complete metric space (see Diamond and Kloeden [5], Nanda [8]). Talo and Başar [20] defined the metric D on E^1 by means of Hausdorff metric d as

$$\begin{aligned} D(x,y) &= \sup_{\lambda\in[0,1]} d([x]_\lambda, [y]_\lambda) \\ &= \sup_{\lambda\in[0,1]} \max(|x^-(\lambda) - y^-(\lambda)|, |x^+(\lambda) - y^+(\lambda)|) \end{aligned} \tag{7}$$

The partial ordering relation on E^1 is defined as follows:

$$\begin{aligned} x \preccurlyeq y &\Longleftrightarrow \\ [x]_\lambda \preccurlyeq [y]_\lambda &\Longleftrightarrow \\ x^-(\lambda) &\le y^-(\lambda) \\ \text{and } x^+(\lambda) &\le y^+(\lambda) \\ &\forall \lambda \in [0,1]. \end{aligned} \tag{8}$$

Definition 4 (Talo and Başar [20]). $x \in E^1$ is a nonnegative fuzzy number if and only if $x(x_0) = 0$ for all $x_0 < 0$. It is immediate that $x \succcurlyeq \overline{0}$ if u is a nonnegative fuzzy number.

One can see that

$$\begin{aligned} D(x, \overline{0}) &= \sup_{\lambda\in[0,1]} \max\{|x^-(\lambda)|, |x^+(\lambda)|\} \\ &= \max\{|x^-(0)|, |x^+(0)|\} \end{aligned} \tag{9}$$

Lemma 5 (Bede and Gal [21]). *Let $x, y, z \in E^1$ and $k \in \mathbb{R}$. Then*

(i) (E^1, D) *is a complete metric space.*
(ii) $D(kx, ky) = |k| D(x, y)$.
(iii) $D(x + y, z + y) = D(x, z)$.
(iv) $D(x + y, z + u) \le D(x, z) + D(y, u)$.
(v) $|D(x, \overline{0}) - D(y, \overline{0})| \le D(x, y) \le D(x, \overline{0}) + D(y, \overline{0})$.

Lemma 6 (Talo and Başar [20]). *The following statements hold:*

(i) $D(xy, \overline{0}) \le D(x, \overline{0}) D(y, \overline{0})$ *for all* $x, y \in E^1$.
(ii) *If* $x_k \longrightarrow x$, *as* $k \longrightarrow \infty$ *then* $D(x_k, \overline{0}) \longrightarrow D(x, \overline{0})$ *as* $k \longrightarrow \infty$.

By $\mathscr{W}^F$ we denote the set of all single sequences of fuzzy numbers on $\mathbb{R}$.

Matloka [2] introduced bounded and convergent sequences of fuzzy numbers and studied their properties. We now quote the following definitions given by Talo and Başar [20] which we will use in a later part of this paper.

Definition 7. A sequence of fuzzy numbers (x_k) is said to be bounded if the set of fuzzy numbers consisting of the terms of the sequence (x_k) is a bounded set. That is to say that a sequence $(x_k) \in \mathscr{W}^F$ is said to be bounded if and only if there exist two fuzzy numbers m and M such that $m \preccurlyeq x_k \preccurlyeq M$ for all $k \in N$. This means that $m^-(\lambda) \le x_k^-(\lambda) \le M^-(\lambda)$ and $m^+(\lambda) \le x_k^+(\lambda) \le M^+(\lambda)$ for all $\lambda \in [0,1]$.

The fact that the boundedness of the sequence $(x_k) \in \mathscr{W}^F$ is equivalent to the uniform boundedness of the functions $x_k^-(\lambda)$ and $x_k^+(\lambda)$ on $[0,1]$. Therefore, one can say that the boundedness of the sequence $(x_k) \in \mathscr{W}^F$ is equivalent to the fact that

$$\begin{aligned} \sup_{k\in N} D(x_k, \overline{0}) &= \sup_{k\in N} \sup_{\lambda\in[0,1]} \max\{|x_k^-(\lambda)|, |x_k^+(\lambda)|\} \\ &< \infty. \end{aligned} \tag{10}$$

Definition 8. Consider the sequence of fuzzy numbers $(x_k) \in \mathscr{W}^F$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in N$ and $l \in E^1$ such that $\mathrm{D}(x_k, l) < \varepsilon$ for all $k > n_0$, then we say that the sequence is said to be convergent to the limit l and write

$$\lim_{k\longrightarrow\infty} D(x_k, l) = 0, \tag{11}$$

and we have the sets $\ell_\infty^F, \mathscr{C}^F, \mathscr{C}_{\overline{0}}^F$ consisting of the bounded, convergent, and convergent to $\overline{0}$ sequences of fuzzy numbers (Talo and Başar [20]) as follows:

$\ell_\infty^F = \{(x_k) \in \mathscr{W}^F : \sup_k D(x_k, \overline{0}) < \infty\}$.

$\mathscr{C}^F = \{(x_k) \in \mathscr{W}^F : \exists l \in E^1$ such that $\lim_{k\longrightarrow\infty} D(x_k, l) = 0\}$.

$\mathscr{C}_0^F = \{(x_k) \in \mathscr{W}^F : \lim_{k\longrightarrow\infty} D(x_k, \overline{0}) = 0\}$.

Throughout the text, the summations without limits run from 0 to ∞; for example,

$$\sum_k x_k \tag{12}$$

means that

$$\sum_{k=0}^{\infty} x_k. \tag{13}$$

Definition 9 (Talo and Başar [20]). Let $(x_k) \in \mathscr{W}^F$. Then the expression

$$\sum_k x_k \tag{14}$$

is called a series corresponding to the sequence (x_k) of fuzzy number. We denote

$$s_n = \sum_{k=1}^{n} x_k \quad \forall n \in \mathbb{N}. \tag{15}$$

If the sequence (s_n) converges to a fuzzy number x, then we say that the series

$$\sum_k x_k \tag{16}$$

converges to x and write

$$\sum_k x_k = x, \tag{17}$$

which implies as $n \longrightarrow \infty$ that

$$\begin{gathered} \sum_{k=0}^{n} x_k^-(\lambda) \longrightarrow x^-(\lambda) \\ \text{and} \sum_{k=0}^{n} x_k^+(\lambda) \longrightarrow x^+(\lambda), \end{gathered} \tag{18}$$

uniformly in $\lambda \in [0, 1]$. Conversely, if the fuzzy numbers $x_k = \{(x_k^-(\lambda), x_k^+(\lambda)) : \lambda \in [0, 1]\}$,

$$\sum_k x_k^-(\lambda) = x^-(\lambda), \tag{19}$$

and

$$\sum_k x_k^+(\lambda) = x^+(\lambda) \tag{20}$$

converge uniformly in $\lambda \in [0, 1]$, then $x = \{(x^-(\lambda), x^+(\lambda) : \lambda \in [0, 1]\}$ defines a fuzzy number such that $x = \sum_{k=0}^{\infty} x_k$.

Otherwise, we say the series of fuzzy numbers diverges. Additionally, if the sequence (s_n) is bounded then we say that the series

$$\sum_k x_k \tag{21}$$

of fuzzy numbers is bounded.

Definition 10 (Talo and Başar [20]). Let μ^F be a space of convergent sequences of fuzzy numbers. The sum of a series

$$\sum_k x_k \tag{22}$$

with respect to this rule is defined by

$$\lim_{n\longrightarrow\infty} \sum_k x_k. \tag{23}$$

Definition 11. Following Khan and Rahman [22], we define the Cesàro sequence space $Ces[p, (q_n)]^F$ as follows:

If $q = (q_n)$ is a positive sequence of real numbers, then, for $1 < p < \infty$,

$$\begin{aligned} Ces[p, (q_n)]^F = \Bigg\{ & x = (x_k) \\ & \in \mathscr{W}^F : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k D(x_k, \overline{0}) \right)^p < \infty \Bigg\} \end{aligned} \tag{24}$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots + q_{2^{r+1}-1}$ and $\sum_r$ denotes summation over the range $2^r \le k < 2^{r+1}$. If $q_n = 1$ for all $n \in \mathbb{N}$, then $Ces[p, (q_n)]^F$ reduces to $Ces(p)^F$ defined by

$$\begin{aligned} & Ces(p)^F \\ & = \left\{ x = (x_k) \in \mathscr{W}^F : \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r D(x_k, \overline{0}) \right)^p < \infty \right\}. \end{aligned} \tag{25}$$

Following Maddox [23], throughout the paper we use the following inequality.

For any $G > 0$ and $a, b \in E^1$ we have

$$\begin{aligned} D(ab, \overline{0}) & \le D(a, \overline{0}) D(b, \overline{0}) \\ & \le G\left[D(a, \overline{0})^t G^{-t} + D(b, \overline{0})^p \right], \end{aligned} \tag{26}$$

where $p > 1$ and $1/p + 1/t = 1$.

The classical analogy of $Ces(p)^F$ was introduced and studied by Lim [24].

The classical Cesàro sequence space and its algebraic dual and related matrix transformations were introduced and studied by various authors like Shiue [25], Leibowitz [26], Lim [24], Khan and Khan [27],[28], Khan and Rahman [22], Johnson and Mohapatra [29], Rahman and Karim [30], etc.

The main purpose of this paper is to define and study the Cesàro sequence space $Ces[p, (q_n)]^F$ and determine the Köthe-Toeplitz dual and give some related matrix transformations.

2. Complete Metric Structure

We equip $Ces[p, (q_n)]^F$ with a metric and show that this set is complete with respect to the metric defined in the following theorem.

Theorem 12. *$Ces[p,(q_n)]^F$ is complete with the metric d^* defined by*

$$d^*(x,y)=\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k,y_k)\right)^p\right)^{1/p}, \quad (27)$$

where $x=(x_k), y=(y_k)\in Ces[p,(q_n)]^F$.

Proof. We first show that $(Ces[p,(q_n)]^F,d^*)$ is a metric space.

It is obvious that $d^*(x,y)=0 \Longleftrightarrow x=y$ and $d^*(x,y)=d^*(y,x)$.

Now we prove the triangle inequality.

Suppose $x=(x_k), y=(y_k), z=(z_k)\in Ces[p,(q_n)]^F$.

$$\begin{aligned}d^*(x,z)&=\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k,z_k)\right)^p\right)^{1/p}\\&\le\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k\left(D(x_k,y_k)+D(y_k,z_k)\right)\right)^p\right)^{1/p}\\&\le\left(\sum_{r=0}^{\infty}\left(\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k,y_k)+\frac{1}{Q_{2^r}}\sum_r q_k D(y_k,z_k)\right)\right)^p\right)^{1/p}\end{aligned} \quad (28)$$

Then, using Minkowski's inequality,

$$\begin{aligned}d^*(x,z)&\le\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k,y_k)\right)^p\right)^{1/p}\\&\quad+\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(y_k,z_k)\right)^p\right)^{1/p}\\&=d^*(x,y)+d^*(y,z)\end{aligned} \quad (29)$$

Next, to show that $Ces[p,(q_n)]^F$ is complete under d^*, let us consider that (x^i) is a Cauchy sequence in $Ces[p,(q_n)]^F$. Then, for given $\varepsilon>0$, there exists $k_0>0$ such that

$$\begin{aligned}&d^*(x^i,x^j)<\varepsilon\\&\forall i,j\ge k_0\Longrightarrow\\&\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k^i,x_k^j)\right)^p\right)^{1/p}<\varepsilon,\end{aligned} \quad (30)$$

which implies that $D(x_k^i,x_k^j)<\varepsilon\ \forall i,j>k_0$; that is, (x_k^i) is a Cauchy sequence in E^1. So (x_k^i) converges to a limit, say $x_k\in E^1$; i.e., $\lim_{i\longrightarrow\infty}x_k^i=x_k, \forall k\in\mathbb{N}$.

Suppose $x=(x_k)$. For given $\varepsilon>0$, there exists $k_0>0$ such that, for any $t\in\mathbb{N}$,

$$\left(\sum_{r=0}^{t}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k^i,x_k^j)\right)^p\right)^{1/p}\le d^*(x^i,x^j)<\varepsilon \quad \forall i,j\ge k_0 \quad (31)$$

Letting $j\longrightarrow\infty$, we obtain

$$\left(\sum_{r=0}^{t}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k^i,x_k)\right)^p\right)^{1/p}<\varepsilon \quad \forall i\ge k_0 \quad (32)$$

Since t is arbitrary, letting $t\longrightarrow\infty$, we obtain

$$\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k^i,x_k)\right)^p\right)^{1/p}<\varepsilon \quad \forall i\ge k_0 \quad (33)$$

which implies that $d^*(x^i,x)<\varepsilon, \forall\ i\ge k_0$.

Next we show that $x=(x_k)\in Ces[p,(q_n)]^F$.

We have that $d^*(x^i,\overline{0})$ is bounded in $Ces[p,(q_n)]^F$; i.e., there exists $K>0$ such that $d^*(x,\overline{0})\le K$. Now, for any $t\in\mathbb{N}$,

$$\left(\sum_{r=0}^{t}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k^i,\overline{0})\right)^p\right)^{1/p}\le d^*(x^i,\overline{0})\le K \quad \forall i\ge k_0 \quad (34)$$

Now,

$$\begin{aligned}&\left(\sum_{r=0}^{t}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k,\overline{0})\right)^p\right)^{1/p}\\&\le\left(\sum_{r=0}^{t}\left(\frac{1}{Q_{2^r}}\sum_r q_k\left(D(x_k,x_k^i)+D(x_k^i,\overline{0})\right)\right)^p\right)^{1/p}\\&\le\left(\sum_{r=0}^{t}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k,x_k^i)\right)^p\right)^{1/p}\\&\quad+\left(\sum_{r=0}^{t}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k^i,\overline{0})\right)^p\right)^{1/p}.\end{aligned} \quad (35)$$

Letting $t\longrightarrow\infty$, we obtain

$$\left(\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D(x_k,\overline{0})\right)^p\right)^{1/p}\le\varepsilon+K<\infty. \quad (36)$$

This implies that $x=(x_k)\in Ces[p,(q_n)]^F$.

This step completes the proof. □

3. Computation of the Köthe-Toeplitz Dual

Definition 13 (Talo and Başar [20]). The Köthe-Toeplitz dual or the α–dual of a set $\mu^F\subset\mathscr{W}^F$, denoted by $\{\mu^F\}^\alpha$, is defined as follows:

$$\{\mu^F\}^\alpha:=\left\{\{(x_k)\in\mathscr{W}^F:(x_ky_k)\in\ell_1^F\quad\forall(y_k)\in\mu^F\}\right\} \quad (37)$$

where ℓ_1^F denotes the absolutely summable sequences of fuzzy numbers defined as follows:

$$\ell_1^F=\left\{(x_k)\in\mathscr{W}^F:\sum_k D(x_k,\overline{0})<\infty\right\}. \quad (38)$$

We now give the following theorem by which the Köthe-Toeplitz dual $\{Ces[p,(q_n)]^F\}^\alpha$ of $Ces[p,(q_n)]^F$ will be determined.

Theorem 14. *If $1 < p < \infty$ and $1/p + 1/t = 1$, then*

$$\left\{Ces\left[p,(q_n)\right]^F\right\}^{\alpha} = \left\{a = (a_k) : \sum_{r=0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\right)^t G^{-t} < \infty \text{ for some integer } G > 1\right\}. \quad (39)$$

Proof. Let $1 < p < \infty$ and $1/p + 1/t = 1$.

We define

$$\mu_t^F = \left\{a = (a_k) : \sum_{r=0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\right)^t G^{-t} < \infty \text{ for some integer } G > 1\right\}. \quad (40)$$

We want to show that $\{Ces[p,(q_n)]^F\}^{\alpha} = \mu_t^F$.

Let $x = (x_k) \in Ces[p,(q_n)]^F$ and $a = (a_k) \in \mu_t^F$. Then, using inequality (26) together with Lemma 6, we get

$$\begin{aligned}\sum_{k=1}^{\infty} D\left(a_k x_k,\bar{0}\right) &= \sum_{r=0}^{\infty}\sum_r D\left(a_k x_k,\bar{0}\right) \\ &\le \sum_{r=0}^{\infty}\sum_r D\left(a_k,\bar{0}\right) D\left(x_k,\bar{0}\right) \\ &= \sum_{r=0}^{\infty}\sum_r \frac{1}{q_k} D\left(a_k,\bar{0}\right) q_k D\left(x_k,\bar{0}\right) \\ &\le \sum_{r=0}^{\infty}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\sum_r q_k D\left(x_k,\bar{0}\right) = \sum_{r=0}^{\infty} Q_{2^r} \\ &\cdot \max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right) \\ &\le G\sum_{r=0}^{\infty}\left[\left(Q_{2^r}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\right)^t G^{-t} \right. \\ &\left. + \left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p\right] \\ &= G\left[\sum_{r=0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\right)^t G^{-t} \right. \\ &\left. + \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p\right] < \infty,\end{aligned} \quad (41)$$

which implies that $a = (a_k) \in \{Ces[p,(q_n)]^F\}^{\alpha}$. Thus $\mu_t^F \subseteq \{Ces[p,(q_n)]^F\}^{\alpha}$.

Conversely, suppose that

$$\sum_{k=1}^{\infty} D\left(a_k x_k,\bar{0}\right) < \infty \quad (42)$$

for all $x = (x_k) \in Ces[p,(q_n)]^F$ but $a \notin \mu_t^F$. Then, for every $G > 1$,

$$\sum_{r=0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\right)^t G^{-t} = \infty. \quad (43)$$

So, following Khan and Rahman [22], we can define a sequence $0 = n(0) < n(1) < n(2) < \ldots$ such that $\gamma = 0,1,2,\ldots$ and we have

$$M_\gamma = \sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left(Q_{2^r}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\right)^t (\gamma+2)^{-t/p} > 1. \quad (44)$$

Now we define a sequence $x = (x_k)$ as follows:

$$x_{N(r)} = Q_{2^r}^t \left(A_{N(r)}\right)^t \frac{1}{D\left(a_{N(r)},\bar{0}\right)}(\gamma+2)^{-t} M_\gamma^{-1} \text{ for } n(\gamma) \le r \le n(\gamma+1)-1, \quad \gamma = 0,1,2,\ldots, \quad (45)$$

and $x_k = \bar{0}\ \forall k \neq N(r)$, where $N(r)$ is such that

$$A_{N(r)}/D\left(a_{N(r)},\bar{0}\right) < L, \quad \text{for some } L > 0 \quad (46)$$

and

$$A_{N(r)} = \max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right), \quad (47)$$

where the maximum is taken with respect to k in $[2^r, 2^{r+1})$.

Therefore,

$$\begin{aligned}\sum_{k=2^{n(\gamma)}}^{2^{n(\gamma+1)-1}} D\left(a_k x_k,\bar{0}\right) &= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^t \left(A_{N(r)}\right)^t (\gamma+2)^{-t} \\ &\cdot M_\gamma^{-1} = (\gamma+2)^{-1} \\ &\cdot M_\gamma^{-1}\sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left(Q_{2^r}\max_r\left(\frac{D\left(a_k,\bar{0}\right)}{q_k}\right)\right)^t \\ &\cdot (\gamma+2)^{-t/p} = (\gamma+2)^{-1} M_\gamma^{-1} M_\gamma = (\gamma+2)^{-1}.\end{aligned} \quad (48)$$

It follows that

$$\sum_{k=1}^{\infty} D\left(a_k x_k, \bar{0}\right) = \sum_{\gamma=0}^{\infty} (\gamma+2)^{-1} \tag{49}$$

is divergent. Moreover

$$\begin{aligned}\sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(\frac{1}{Q_{2^r}} \sum_r q_k D\left(x_k, \bar{0}\right)\right)^p \leq \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \Bigg(& L Q_{2^r}^{(t-1)} \\ & \cdot \max_r \left(\frac{D\left(a_k, \bar{0}\right)}{q_k}\right)^{(t-1)} (\gamma+2)^{-t} M_\gamma^{-1} \Bigg)^p\end{aligned} \tag{50}$$

Now

$$\begin{aligned}&\sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^{(t-1)p} \max_r \left(\frac{D\left(a_k, \bar{0}\right)}{q_k}\right)^{(t-1)p} (\gamma+2)^{tp} M_\gamma^{-p} \\ &= (\gamma+2)^{-2} M_\gamma^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(Q_{2^r} \max_r \left(\frac{D\left(a_k, \bar{0}\right)}{q_k}\right)\right)^t \\ &\cdot (\gamma+2)^{-t/p} M_\gamma^{1-p} (\gamma+2)^{1-p} = (\gamma+2)^{-2} \\ &\cdot M_\gamma^{-1} M_\gamma M_\gamma^{1-p} (\gamma+2)^{1-p} \\ &= \frac{(\gamma+2)^{-2}}{M_\gamma^{p/t} (\gamma+2)^{p/t}} \leq (\gamma+2)^{-2}.\end{aligned} \tag{51}$$

Thus,

$$\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k D\left(x_k, \bar{0}\right)\right)^p \leq L^p \sum_{r=0}^{\infty} (\gamma+2)^{-2} < \infty. \tag{52}$$

So, $x = (x_k) \in Ces[p, (q_n)]^F$, which is a contradiction to our assumption. Hence $a \in \mu_t^F$.

That is, $\{Ces[p, (q_n)]^F\}^\alpha \subseteq \mu_t^F$.

Then, combining the two results, we obtain $\{Ces[p, (q_n)]^F\}^\alpha = \mu_t^F$.

This step completes the proof. □

4. Characterization of Matrix Classes

An infinite matrix is one of the most general linear operators between two sequence spaces. The study of theory of matrix transformations has always been of great interest to mathematicians in the study of sequence spaces, which is motivated by special results in summability theory.

Definition 15 (Talo and Başar [20]). Let $\mu_1^F, \mu_2^F \subset \mathscr{W}^F$ and $A = (a_{nk})$ be any two-dimensional infinite matrix of fuzzy numbers. Then we say that A defines a mapping from μ_1^F into μ_2^F and denote it by $A : \mu_1^F \longrightarrow \mu_2^F$ if, for every sequence $x = (x_k) \in \mu_1^F$, the A–transform of x, $Ax = \{(Ax)_n\}$, given by

$$(Ax)_n = \sum_k a_{nk} x_k \tag{53}$$

exists for each each $n \in \mathbb{N}$ and is in μ_2^F.

$A \in (\mu_1^F : \mu_2^F)$ if and only if the series on the right hand side of (53) converges for each $n \in \mathbb{N}$ and every $x = (x_k) \in \mu_1^F$ and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}}$. A sequence of fuzzy numbers $x = (x_k)$ is said to be $A-$ *summable* to α if Ax converges to α which is called the A–limit of x. Also by $A \in (\mu_1^F : \mu_2^F; P)$ we denote that A preserves the limit; that is, A–limit of x is equal to the limit of x for all $x = (x_k) \in \mu_1^F$.

We write

$$A_r(n) = \max_r \left(\frac{D\left(a_{nk}, \bar{0}\right)}{q_k}\right) \tag{54}$$

where, for each $n \in \mathbb{N}$, the maximum is taken with respect to $k \in [2^r, 2^{r+1})$.

Theorem 16. *Let $A = (a_{nk})$ be an infinite matrix of fuzzy numbers and $1 < p < \infty$. Then $A \in (Ces[p, (q_n)]^F : \ell_\infty^F)$ if there exists an integer $G > 1$ such that $U(G) < \infty$ where*

$$U(G) = \sup_n \sum_{r=0}^{\infty} \left(Q_{2^r} A_r(n)\right)^t G^{-t} \tag{55}$$

and $1/p + 1/t = 1$.

Proof. Suppose there exists an integer $G > 1$ such that $U(G) < \infty$. Let $x = (x_k) \in Ces[p, (q_n)]^F$. Then

$$\begin{aligned}D\left(A_n(x), \bar{0}\right) &= D\left(\sum_{k=1}^{\infty} a_{nk} x_k, \bar{0}\right) \leq \sum_{k=1}^{\infty} D\left(a_{nk} x_k, \bar{0}\right) \\ &= \sum_{r=0}^{\infty} \sum_r D\left(a_{nk} x_k, \bar{0}\right) \leq \sum_{r=0}^{\infty} \sum_r D\left(a_{nk}, \bar{0}\right) D\left(x_k, \bar{0}\right) \\ &= \sum_{r=0}^{\infty} \sum_r \frac{1}{q_k} D\left(a_{nk}, \bar{0}\right) q_k D\left(x_k, \bar{0}\right) \\ &\leq \sum_{r=0}^{\infty} \max_r \left(\frac{D\left(a_{nk}, \bar{0}\right)}{q_k}\right) \sum_r q_k D\left(x_k, \bar{0}\right) \\ &= \sum_{r=0}^{\infty} Q_{2^r} \max_r \left(\frac{D\left(a_{nk}, \bar{0}\right)}{q_k}\right) \frac{1}{Q_{2^r}} \sum_r q_k D\left(x_k, \bar{0}\right)\end{aligned} \tag{56}$$

Using inequality (26), we obtain

$$\begin{aligned}
&D\left(A_n(x),\bar{0}\right)\\
&\leq G\sum_{r=0}^{\infty}\left[\left(Q_{2^r}\max_r\left(\frac{D\left(a_{nk},\bar{0}\right)}{q_k}\right)\right)^t G^{-t}\right.\\
&\left.+\left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p\right]\\
&= G\left[\sum_{r=0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(a_{nk},\bar{0}\right)}{q_k}\right)\right)^t G^{-t}\right.\\
&\left.+\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p\right]\\
&= G\left[\sum_{r=0}^{\infty}\left(Q_{2^r}A_r(n)\right)^t G^{-t}\right.\\
&\left.+\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p\right]<\infty.
\end{aligned}\tag{57}$$

Therefore, $A\in(Ces[p,(q_n)]^F:\ell_\infty^F)$.

The necessity of the above theorem is still open; i.e., we do not know if $A\in(Ces[p,(q_n)]^F:\ell_\infty^F)$ then the condition (55) holds or not. □

Theorem 17. *Let $A=(a_{nk})$ be an infinite matrix of fuzzy numbers and $1<p<\infty$. Then $A\in(Ces[p,(q_n)]^F:\mathscr{C}^F)$ if (55) holds and, for fixed $k\in\mathbb{N}$, we have*

$$\lim_{n\longrightarrow\infty}a_{nk}=\bar{\alpha}_k \tag{58}$$

Proof. Suppose conditions (55) and (58) hold. Then

$$\sum_{r=0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(\bar{\alpha}_k,\bar{0}\right)}{q_k}\right)\right)^t G^{-t}\leq U(G)<\infty. \tag{59}$$

From (59), using similar argument as in Theorem 14, it is easy to verify that the series

$$\sum_{k=1}^{\infty}D\left(\bar{\alpha}_k x_k,\bar{0}\right)<\infty. \tag{60}$$

This implies that

$$D\left(\sum_{k=1}^{\infty}\bar{\alpha}_k x_k,\bar{0}\right)\leq\sum_{k=1}^{\infty}D\left(\bar{\alpha}_k x_k,\bar{0}\right)<\infty. \tag{61}$$

Thus,

$$\sum_{k=1}^{\infty}\bar{\alpha}_k x_k \tag{62}$$

exists.

Now, for each $x=(x_k)\in Ces[p,(q_n)]^F$ and $\varepsilon>0$, we can choose an integer $m_0\geq 1$ such that

$$d_{m_0}^*\left(x,\bar{0}\right)=\sum_{r=m_0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p<\varepsilon^P. \tag{63}$$

Then

$$\begin{aligned}
&D\left(\sum_{k=2^{m_0}}^{\infty}a_{nk}x_k,\sum_{k=2^{m_0}}^{\infty}\bar{\alpha}_k x_k\right)\leq\sum_{k=2^{m_0}}^{\infty}D\left(a_{nk}x_k,\bar{\alpha}_k x_k\right)\\
&\leq\sum_{k=2^{m_0}}^{\infty}D\left(a_{nk}x_k,\bar{0}\right)+\sum_{k=2^{m_0}}^{\infty}D\left(\bar{\alpha}_k x_k,\bar{0}\right)\\
&\leq\sum_{k=2^{m_0}}^{\infty}D\left(a_{nk},\bar{0}\right)D\left(x_k,\bar{0}\right)\\
&+\sum_{k=2^{m_0}}^{\infty}D\left(\bar{\alpha}_k,\bar{0}\right)D\left(x_k,\bar{0}\right)\\
&\leq\sum_{r=m_0}^{\infty}Q_{2^r}\max_r\left(\frac{D\left(a_{nk},\bar{0}\right)}{q_k}\right)\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\\
&+\sum_{r=m_0}^{\infty}Q_{2^r}\max_r\left(\frac{D\left(\bar{\alpha}_k,\bar{0}\right)}{q_k}\right)\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\\
&\leq\sum_{r=m_0}^{\infty}Q_{2^r}\max_r\left(\frac{D\left(a_{nk},\bar{0}\right)}{q_k}\right)G^{-t}G^t\frac{1}{Q_{2^r}}\\
&\cdot\sum_r q_k D\left(x_k,\bar{0}\right)+\sum_{r=m_0}^{\infty}Q_{2^r}\max_r\left(\frac{D\left(\bar{\alpha}_k,\bar{0}\right)}{q_k}\right)G^{-t}G^t\\
&\cdot\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\\
&=\left[G^t\left(\sum_{r=m_0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(a_{nk},\bar{0}\right)}{q_k}\right)\right)^t G^{-t}\right)\right]^{1/t}\\
&\cdot\left[\sum_{r=m_0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p\right]^{1/p}\\
&+\left[G^t\left(\sum_{r=m_0}^{\infty}\left(Q_{2^r}\max_r\left(\frac{D\left(\bar{\alpha}_k,\bar{0}\right)}{q_k}\right)\right)^t G^{-t}\right)\right]^{1/t}\\
&\cdot\left[\sum_{r=m_0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k D\left(x_k,\bar{0}\right)\right)^p\right]^{1/p}
\end{aligned}$$

$$\leq \left[\left[G^t \left(\sum_{r=m_0}^{\infty} \left(Q_{2^r} \max_r \left(\frac{D\left(a_{nk}, \overline{0}\right)}{q_k} \right) \right)^t G^{-t} \right) \right]^{1/t} + \left[G^t \left(\sum_{r=m_0}^{\infty} \left(Q_{2^r} \max_r \left(\frac{D\left(\overline{\alpha}_k, \overline{0}\right)}{q_k} \right) \right)^t G^{-t} \right) \right]^{1/t} \right] \cdot \varepsilon \leq \left[G\left(U\left(G\right)\right)^{1/t} + G\left(U\left(G\right)\right)^{1/t} \right] \varepsilon = 2G\left(U\left(G\right)\right)^{1/t} \cdot \varepsilon. \tag{64}$$

It follows that

$$D\left(\sum_{k=1}^{\infty} a_{nk} x_k, \sum_{k=1}^{\infty} \overline{\alpha}_k x_k \right) \longrightarrow 0, \tag{65}$$

as $n \longrightarrow \infty$. This shows that $A \in (Ces[p,(q_n)]^F : \mathscr{C}^F)$ which proves the theorem. □

Corollary 18. *Let $A = (a_{nk})$ be an infinite matrix of fuzzy numbers and $1 < p < \infty$. Then $A \in (Ces[p,(q_n)]^F : \mathscr{C}_0^F)$ if (55) holds and, for fixed $k \in \mathbb{N}$, we have*

$$\lim_{n \longrightarrow \infty} a_{nk} = \overline{0}. \tag{66}$$

Proof. The proof is obvious. □

Data Availability

The paper is theoretical in nature and all necessary references are included in the References with proper citation within the main text.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] H. Matloka, "Sequences of fuzzy numbers," *BUSEFAL*, vol. 28, pp. 28–37, 1986.

[3] S. Nanda, "On sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 123–126, 1989.

[4] F. Nuray and E. Savaş, "Statistical convergence of sequences of fuzzy numbers," *Mathematica Slovaca*, vol. 45, no. 3, pp. 269–273, 1995.

[5] P. Diamond and P. Kloeden, "Metric spaces of fuzzy sets," *Fuzzy Sets and Systems*, vol. 35, no. 2, pp. 241–249, 1990.

[6] A. Esi, "On some new paranormed sequence spaces of fuzzy numbers defined by Orlicz functions and statistical convergence," *Mathematical Modelling and Analysis*, vol. 11, no. 4, pp. 379–388, 2006.

[7] O. Kaleva, "On the convergence of fuzzy sets," *Fuzzy Sets and Systems*, vol. 17, no. 1, pp. 53–65, 1985.

[8] S. Nanda, *Matrix transformations and sequence spaces*, vol. 74, ICTP, Trieste, Italy, 1983.

[9] D. Dubois and H. Prade, "Operations on fuzzy numbers," *International Journal of Systems Science*, vol. 9, no. 6, pp. 613–626, 1978.

[10] H. Altinok, R. Çolak, and Y. Altin, "On the class of λ-statistically convergent difference sequences of fuzzy numbers," *Soft Computing*, vol. 16, no. 6, pp. 1029–1034, 2012.

[11] M. Stojakovic and Z. Stojakovic, "Addition and series of fuzzy sets," *Fuzzy Sets and Systems*, vol. 83, pp. 341–346, 1996.

[12] M. Stojaković and Z. Stojaković, "Series of fuzzy sets," *Fuzzy Sets and Systems*, vol. 160, no. 21, pp. 3115–3127, 2009.

[13] M. Mursaleen, H. M. Srivastava, and S. K. Sharma, "Generalized statistically convergent sequences of fuzzy numbers," *Journal of Intelligent and Fuzzy Systems*, vol. 30, no. 3, pp. 1511–1518, 2016.

[14] P. V. Subrahmanyam, "Cesàro summability of fuzzy real numbers," *Journal of Analysis*, vol. 7, pp. 159–168, 1999.

[15] Ö. Talo and C. Çakan, "On the Cesàro convergence of sequences of fuzzy numbers," *Applied Mathematics Letters*, vol. 25, pp. 676–681, 2012.

[16] Y. Altin, M. Mursaleen, and H. Altinok, "Statistical summability (C, 1) for sequences of fuzzy real numbers and a Tauberian theorem," *Journal of Intelligent and Fuzzy Systems*, vol. 21, no. 6, pp. 379–384, 2010.

[17] E. Yavuz, "Comparison theorems for summability methods of sequences of fuzzy numbers," *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, pp. 1–8, 2018.

[18] E. Yavuz, "Euler summability method of sequences of fuzzy numbers and a Tauberian theorem," *Journal of Intelligent and Fuzzy Systems*, p. 10, 2016.

[19] J. Goetschel and W. Voxman, "Elementary fuzzy calculus," *Fuzzy Sets and Systems*, vol. 18, no. 1, pp. 31–43, 1986.

[20] Ö. Talo and F. Başar, "Determination of duals of classical sets of sequences of fuzzy numbers and related matrix transformations," *Computers & Mathematics with Applications*, vol. 58, pp. 717–733, 2009.

[21] B. Bede and S. G. Gal, "Almost periodic fuzzy-number-valued functions," *Fuzzy Sets and Systems*, vol. 147, no. 3, pp. 385–403, 2004.

[22] F. M. Khan and M. F. Rahman, "Infinite matrices and Cesàro sequence spaces," *Analysis Mathematica*, vol. 23, pp. 3–11, 1997.

[23] I. J. Maddox, "Continuous and Köthe-Toeplitz dual of certain sequence spaces," *Proceedings of the Cambridge Philosophical Society*, vol. 65, pp. 431–435, 1969.

[24] K. P. Lim, "Matrix transformation in the Cesàro sequence spaces," *Kyungpook Mathematical Journal*, vol. 14, pp. 221–227, 1974.

[25] J. S. Shiue, "On the Cesàro sequence spaces," *Tamkang Journal of Mathematics*, vol. 1, pp. 19–25, 1970.

[26] G. M. Leibowitz, "A note on the Cesàro sequence spaces," *Tamkang Journal of Mathematics*, vol. 2, pp. 151–157, 1971.

[27] F. M. Khan and M. A. Khan, “The sequence space ces(p, s) and related matrix transformations,” *Review of Research Mathematics Series*, vol. 21, pp. 95–104, 1991.

[28] F. M. Khan and M. A. Khan, “Matrix transformations between Cesàro sequence spaces,” *Indian Journal of Pure and Applied Mathematics*, vol. 25, no. 6, pp. 641–645, 1994.

[29] P. D. Johnson and R. N. Mohapatra, “Density of finitely non-zero sequences in some sequence spaces,” *Mathematica Japonica*, vol. 24, no. 3, pp. 253–262, 1979.

[30] M. F. Rahman and A. B. M. Karim, “Dual spaces of generalized Cesàro sequence space and related matrix mapping,” *International Journal of Mathematics and Statistics Invention*, vol. 4, no. 4, pp. 44–50, 2016.