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Research Article

A Direct Method to Compare Bipolar LR Fuzzy Numbers

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We propose a new method for ordering bipolar fuzzy numbers. In this method, for comparison of bipolar LR fuzzy numbers, we use an extension of Kerre's method being used in ordering of unipolar fuzzy numbers. We give a direct formula to compare two bipolar triangular fuzzy numbers in $O(1)$ operations, making the process useful for many optimization algorithms. Also, we present an application of bipolar fuzzy number in a real life problem.

1. Introduction

Fuzzy sets are useful mathematical structures to represent a collection of objects whose boundary is vague. There is a bipolar judgmental thinking on a negative side as well as a positive side in a human decision making (see [1]). This domain has recently invoked many interesting research topics such as algebra [2, 3], psychology [4], image processing [5], and human reasoning [6].

Zhang [7] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. He defined bipolar fuzzy sets as an extension of fuzzy sets whose membership degree range is $[-1, 1]$. Also, Zhang [8] proposed a family of bipolar models. Zhou and Li [1] presented the concepts of bipolar fuzzy h-ideals and normal bipolar fuzzy h-ideals. Then, they investigated characterizations of bipolar fuzzy h-ideals by means of positive t-cut, negative s-cut, homomorphism, and equivalence relation.

Akram [9, 10] used the concept of bipolar fuzzy sets in graph theory. Talebi and Rashmanlou [11] presented some properties of the self-complement and self-weak complement bipolar fuzzy graphs. Tahmasbpour and Borzooei [12] introduced two different approaches corresponding to chromatic number of a bipolar fuzzy graph. They computed total chromatic number based on α^P-cut and α^N-cut of a bipolar fuzzy graph with the edges and vertices both being bipolar fuzzy sets.

Comparison of two fuzzy numbers is a major computational task in various algorithms. Kerre's method [13] for comparison of two unipolar fuzzy numbers is a well-known method in ordering unipolar fuzzy numbers. In this method, first, using the extension principle or α-cut computations, the fuzzy maximum of two fuzzy numbers is computed and then, using the Hamming distance, the comparison is carried out. Inspired by Kerre's method for comparison of two unipolar fuzzy numbers, we develop a method for comparison of two bipolar fuzzy numbers.

Here, we review the fundamental notions of bipolar fuzzy sets and Kerre's method for unipolar numbers in Section 2. In Section 3, we propose an extension of Kerre's method for bipolar fuzzy numbers and give a direct formula to compare two bipolar triangular fuzzy numbers. In Section 4, we present an application of bipolar fuzzy numbers in a real life problem. We conclude in Section 5.

2. Preliminaries

Here, we give some necessary definitions and new results on bipolar fuzzy set theory.

Definition 1 (see [7]). Let X be a nonempty set. A bipolar fuzzy set $\widetilde{A}$ in X is an object having the form

$$\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}}^{P}(x), \mu_{\widetilde{A}}^{N}(x) \right) \mid x \in X \right\}, \quad (1)$$

where $\mu_{\widetilde{A}}^{P}(x) : X \to [0, 1]$ and $\mu_{\widetilde{A}}^{N}(x) : X \to [-1, 0]$.

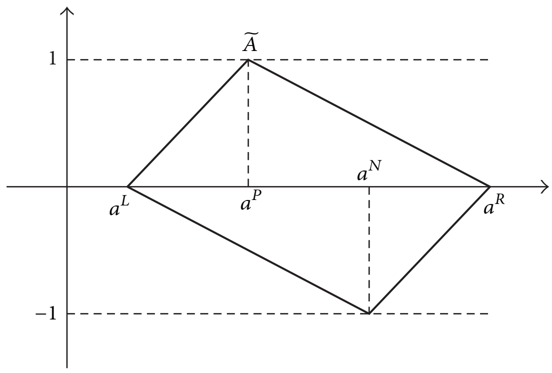

FIGURE 1: Triangular bipolar fuzzy number $a^P \neq a^N$.

Definition 2 (see [2]). Let $\widetilde{A} = (\mu^P_{\widetilde{A}}(x), \mu^N_{\widetilde{A}}(x))$ be a bipolar-valued fuzzy set and $(s,t) \in [-1,0] \times [0,1]$. The sets $A^P_t = \{x \in X \mid \mu^P_{\widetilde{A}}(x) \geq t\}$ and $A^N_s = \{x \in X \mid \mu^N_{\widetilde{A}}(x) \leq s\}$ are, receptively, called the positive t-cut of $\widetilde{A}$ and the negative s-cut of $\widetilde{A}$. For every $k \in [0,1]$, the set

$$A_k = A^P_k \cap A^N_{-k} \tag{2}$$

is called the k-cut of $\widetilde{A}$.

We now define a bipolar triangular fuzzy number.

Definition 3. A bipolar triangular fuzzy number is defined as a quadruple $\widetilde{A} = (a^L, a^P, a^N, a^R)$ with positive and negative membership functions $\mu^P_{\widetilde{A}}(x)$ and $\mu^N_{\widetilde{A}}(x)$ as follows:

$$\mu^P_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - a^L}{a^P - a^L}, & a^L \leq x < a^P \\ \dfrac{x - a^R}{a^P - a^R}, & a^P < x \leq a^R \\ 0, & \text{otherwise}, \end{cases}$$
$$\mu^N_{\widetilde{A}}(x) = \begin{cases} \dfrac{-\left(x - a^L\right)}{a^N - a^L}, & a^L \leq x < a^N \\ \dfrac{-\left(x - a^R\right)}{a^N - a^R}, & a^N < x \leq a^R \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $\mu^P_{\widetilde{A}}$ and $\mu^N_{\widetilde{A}}$ are, respectively, the membership functions of positive and negative polars (see Figure 1).

Proposition 4. *Let $\widetilde{a} = (a^L, a^P, a^N, a^R)$ and $\widetilde{b} = (b^L, b^P, b^N, b^R)$ be two bipolar fuzzy numbers. One has the following results:*

$$x > 0, \quad x \in \mathbb{R}, \; x\widetilde{a} = \left(xa^L, xa^P, xa^N, xa^R\right),$$
$$x < 0, \quad x \in \mathbb{R}, \; x\widetilde{a} = \left(xa^R, xa^N, xa^P, xa^L\right), \tag{4}$$
$$\widetilde{a} + \widetilde{b} = \left(a^L + b^L, a^P + b^P, a^N + b^N, a^R + b^R\right).$$

Proof. The results are proved by using the extension principle. □

Note 1. We denote the set of all bipolar triangular fuzzy numbers by $F(\mathbb{R})$.

2.1. Kerre's Method to Compare Two Unipolar Fuzzy Numbers. In Kerre's method [13], first, using the extension principle or α-cut computations, the fuzzy maximum of two fuzzy numbers is computed and then, using the Hamming distance, the comparison is carried out.

Definition 5. Based on Kerre's method, one says $\widetilde{M} \leq \widetilde{N}$ if and only if

$$d\left(\widetilde{N}, \widetilde{\max}\left(\widetilde{M}, \widetilde{N}\right)\right) \leq d\left(\widetilde{M}, \widetilde{\max}\left(\widetilde{M}, \widetilde{N}\right)\right), \tag{5}$$

where $\widetilde{\max}(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are defined as in Definitions 6 and 7.

Definition 6 (see [13]). The fuzzy max between two fuzzy numbers $\widetilde{M}$ and $\widetilde{N}$ is

$$\widetilde{\max}\left(\widetilde{M}, \widetilde{N}\right)(z) = \sup\left\{\min\left(\widetilde{M}(x), \widetilde{N}(y)\right) \mid \max(x,y) = z\right\}. \tag{6}$$

Definition 7 (see [13]). The Hamming distance between two fuzzy numbers $\widetilde{M}$ and $\widetilde{N}$ is

$$d\left(\widetilde{M}, \widetilde{N}\right) = \int_{-\infty}^{\infty} \left|\widetilde{M}(x) - \widetilde{N}(x)\right| dx. \tag{7}$$

It has been shown that Kerre's "≤" is transitive [13].

3. Proposed Method for Comparison of Two Bipolar Fuzzy Numbers

Here, we at first intend to extend Kerre's method [13] for comparison of two bipolar LR fuzzy numbers. We need to find fuzzy maximum for positive and negative polars.

Proposition 8. *Let $\widetilde{M} = (m^L, m^P, m^N, m^R)_{LR}$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)_{LR}$ be two bipolar LR fuzzy numbers. Then,*

$$\widetilde{\max}^P\left(\widetilde{M}, \widetilde{N}\right)(z) = \sup\left\{\min\left(\widetilde{M}(x), \widetilde{N}(y)\right) \mid \max(x,y) = z\right\},$$
$$\widetilde{\max}^N\left(\widetilde{M}, \widetilde{N}\right)(z) = \inf\left\{\max\left(\widetilde{M}(x), \widetilde{N}(y)\right) \mid \max(x,y) = z\right\}, \tag{8}$$

where $\widetilde{\max}^P(\cdot,\cdot)$ and $\widetilde{\max}^N(\cdot,\cdot)$ are, receptively, fuzzy maximum on positive and negative polars.

Proof. These are proved by using the extension principle directly. □

Let $\widetilde{M} = (m^L, m^P, m^N, m^R)$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)$ be two arbitrary bipolar LR fuzzy numbers and let $\widetilde{O}^P = \widetilde{\max}^P(\widetilde{M}, \widetilde{N})$ and $\widetilde{O}^N = \widetilde{\max}^N(\widetilde{M}, \widetilde{N})$. If $(d(\widetilde{M}, \widetilde{O}^P) - d(\widetilde{N}, \widetilde{O}^P)) + (d(\widetilde{M}, \widetilde{O}^N) - d(\widetilde{N}, \widetilde{O}^N)) \geq 0$ then $\widetilde{M} \leq \widetilde{N}$; else $\widetilde{M} \geq \widetilde{N}$.

3.1. Modified Kerre's Method for Comparison of Two Bipolar Fuzzy Numbers. To compute the fuzzy maximum of two bipolar LR fuzzy numbers, we need to compute fuzzy maximum for each polar as given by (8). Here, we first give a result that leads to a direct and efficient formula to compute the fuzzy maximum of two arbitrary bipolar LR fuzzy numbers. Then, applying the direct formula for $\widetilde{\max}$, we modify Kerre's method to compare two bipolar LR fuzzy numbers. Next, using our modified Kerre's method for comparing of two bipolar LR fuzzy numbers, we establish some simple formulas for comparison of bipolar triangular fuzzy numbers.

Define

$$\begin{aligned} w_z &= \{w \mid w \le z\}, \\ E_z &= \{(x, y) \mid x, y \in w_z,\ z = \max(x, y)\}. \end{aligned} \tag{9}$$

Lemma 9. *Suppose $\widetilde{M} = (m^L, m^P, m^N, m^R)_{LR}$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)_{LR}$ are two bipolar LR fuzzy numbers. Then, for positive polar and for all $z \in \mathbb{R}$, one has*

$$\max\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) \le \max\left(\widetilde{M}(z), \widetilde{N}(z)\right), \quad \forall (x, y) \in E_z, \tag{10}$$

and, for negative polar and for all $z \in \mathbb{R}$, one has

$$\min\left(\max\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) \ge \min\left(\widetilde{M}(z), \widetilde{N}(z)\right), \quad \forall (x, y) \in E_z. \tag{11}$$

Proof. First, we establish (10). Consider $z \in \mathbb{R}$. Without loss of generality, suppose $M(z) \ge N(z)$. Then, we need to show

$$\max\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) \le \widetilde{M}(z), \quad \forall (x, y) \in E_z. \tag{12}$$

There are two cases as described below.

(1) Case $x < z$. Since $(x, y) \in E_z$, $y = z$ and so

$$\begin{aligned} &\min\left(\widetilde{M}(x), \widetilde{N}(z)\right) \le \widetilde{N}(z) \le \widetilde{M}(z) \Longrightarrow \\ &\max_{x \le z}\left(\min\left(\widetilde{M}(x), \widetilde{N}(z)\right)\right) \le \widetilde{M}(z). \end{aligned} \tag{13}$$

(2) Case $y < z$. Since $(x, y) \in E_z$, $x = z$ and so

$$\begin{aligned} &\min\left(\widetilde{M}(z), \widetilde{N}(y)\right) \le \widetilde{M}(z) \Longrightarrow \\ &\max_{y \le z}\left(\min\left(\widetilde{M}(z), \widetilde{N}(y)\right)\right) \le \widetilde{M}(z). \end{aligned} \tag{14}$$

Therefore, from (13) and (14), we have

$$\max\left(\min\left(\widetilde{M}(x)\right), \widetilde{N}(y)\right) \le \widetilde{M}(z), \quad \forall (x, y) \in E_z, \tag{15}$$

and the proof is complete. In a similar manner, we can establish (11). □

Next, in Theorem 10, we give a direct formula to compute the maximum of two arbitrary bipolar LR fuzzy numbers.

Theorem 10. *Suppose $\widetilde{M} = (m^L, m^P, m^N, m^R)_{LR}$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)_{LR}$ are two arbitrary bipolar LR fuzzy numbers and $m^P \le n^P$ and $m^N \le n^N$. For $z \in \mathbb{R}$, one has*

$$\begin{aligned} &\widetilde{\max}^P\left(\widetilde{M}, \widetilde{N}\right)(z) \\ &\quad = \begin{cases} \widetilde{N}(z), & m^P < z < n^P \\ \max\left(\widetilde{M}(z), \widetilde{N}(z)\right), & z \ge n^P \\ \min\left(\widetilde{M}(z), \widetilde{N}(z)\right), & z \le m^P, \end{cases} \\ &\widetilde{\max}^N\left(\widetilde{M}, \widetilde{N}\right)(z) \\ &\quad = \begin{cases} \widetilde{N}(z), & m^N < z < n^N \\ \min\left(\widetilde{M}(z), \widetilde{N}(z)\right), & z \ge n^N \\ \max\left(\widetilde{M}(z), \widetilde{N}(z)\right), & z \le m^N. \end{cases} \end{aligned} \tag{16}$$

Proof. First, we prove $\widetilde{\max}^P(\widetilde{M}, \widetilde{N})(z)$. Let $z \in \mathbb{R}$. According to (6) and $m^P \le n^P$, we consider three cases.

(1) Case $m^P < z < n^P$. Since the left side of the membership function of $\widetilde{N}$ is an increasing function, we have

$$\begin{aligned} &\forall (x, y) \in E_z \Longrightarrow \\ &y \le z \Longrightarrow \\ &\widetilde{N}(y) \le \widetilde{N}(z). \end{aligned} \tag{17}$$

Then, we conclude from $\widetilde{M}(x) \le 1 = \widetilde{M}(m^P)$ and $\widetilde{N}(y) \le \widetilde{N}(z)$ that

$$\min\left(\widetilde{M}(x), \widetilde{N}(y)\right) \le \min\left(1, \widetilde{N}(z)\right), \quad \forall (x, y) \in E_z. \tag{18}$$

Therefore,

$$\begin{aligned} \min\left(\widetilde{M}(x), \widetilde{N}(y)\right) &\le \min\left(1, \widetilde{N}(y)\right) \\ &\le \min\left(1, \widetilde{N}(z)\right), \quad \forall (x, y) \in E_z. \end{aligned} \tag{19}$$

But, $\min(1, \widetilde{N}(z)) = \widetilde{N}(z)$, and so we have

$$\sup_{(x,y) \in E_z}\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) = \widetilde{N}(z), \quad m^P < z < n^P, \tag{20}$$

to complete the proof for the case.

(2) Case $z \ge n^P$. Without loss of generality, suppose that $\widetilde{M}(z) > \widetilde{N}(z)$. Then, according to Lemma 9, it is clear that

$$\max\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) \le \widetilde{M}(z). \tag{21}$$

Now, we show

$$\max\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) \ge \widetilde{M}(z). \tag{22}$$

We know that $(z, n^P) \in E_z$. Then, we have

$$\begin{aligned} &\min\left(\widetilde{M}(z), \widetilde{N}\left(n^P\right)\right) \\ &\quad \leq \max_{(x,y)\in E_z}\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right). \end{aligned} \tag{23}$$

But

$$\min\left(\widetilde{M}(z), \widetilde{N}\left(n^P\right)\right) = \min\left(\widetilde{M}(z), 1\right) = \widetilde{M}(z), \tag{24}$$

and thus

$$\max_{(x,y)\in E_z}\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) \geq \widetilde{M}(z). \tag{25}$$

Therefore, (22) is established and from (21) and (22), and the proof of the case is complete.

(3) Case $z \leq m^P$ $(\leq n^P)$. Since $\widetilde{M}(x)$ and $\widetilde{N}(y)$ are increasing functions on $[\min(m^L, n^L), m^P]$, we have

$$\begin{gathered} (x, y) \in E_z \Longrightarrow \\ x \leq z \Longrightarrow \\ \widetilde{M}(x) \leq \widetilde{M}(z), \\ (x, y) \in E_z \Longrightarrow \\ y \leq z \Longrightarrow \\ \widetilde{N}(y) \leq \widetilde{N}(z), \\ \min\left(\widetilde{M}(x), \widetilde{N}(y)\right) \leq \min\left(\widetilde{M}(z), \widetilde{N}(z)\right), \\ \forall (x, y) \in E_z. \end{gathered} \tag{26}$$

Thus, we have

$$\begin{aligned} &\max_{(x,y)\in E_z}\left(\min\left(\widetilde{M}(x), \widetilde{N}(y)\right)\right) \\ &\quad \leq \min\left(\widetilde{M}(z), \widetilde{N}(z)\right), \end{aligned} \tag{27}$$

and the proof is complete. Similarly, we can prove $\widetilde{\max}^N(\widetilde{M}, \widetilde{N})(z)$. □

Definition 11. Suppose $\widetilde{M} = (m^L, m^P, m^N, m^R)$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)$ are two arbitrary bipolar LR fuzzy numbers and let

$$\begin{aligned} r^b\left(\widetilde{M}, \widetilde{N}\right) &= \left(d\left(\widetilde{M}, \widetilde{O}^P\right) - d\left(\widetilde{N}, \widetilde{O}^P\right)\right) \\ &\quad + \left(d\left(\widetilde{M}, \widetilde{O}^N\right) - d\left(\widetilde{N}, \widetilde{O}^N\right)\right). \end{aligned} \tag{28}$$

If $r^b(\widetilde{M}, \widetilde{N}) \geq 0$ then $\widetilde{M} \leq \widetilde{N}$; else $\widetilde{M} \geq \widetilde{N}$.

Theorem 12. *Let $\widetilde{M} = (m^L, m^P, m^N, m^R)_{LR}$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)$ be two LR fuzzy numbers. If $m^P \leq n^P$ and $m^N \leq n^N$, then*

$$\begin{aligned} r^b\left(\widetilde{M}, \widetilde{N}\right) = &\int_{\min(m^L,n^L)}^{m^P}\left(\widetilde{M}(z) - \widetilde{N}(z)\right)dz \\ &+ \int_{m^P}^{n^P}\left|\widetilde{M}(z) - \widetilde{N}(z)\right|dz \\ &+ \int_{n^P}^{\max(m^R,n^R)}\left(\widetilde{N}(z) - \widetilde{M}(z)\right)dz \\ &+ \int_{\min(m^L,n^L)}^{m^N}\left(\widetilde{N}(z) - \widetilde{M}(z)\right)dz \\ &+ \int_{m^N}^{n^N}\left|\widetilde{M}(z) - \widetilde{N}(z)\right|dz \\ &+ \int_{n^N}^{\max(m^R,n^R)}\left(\widetilde{M}(z) - \widetilde{N}(z)\right)dz, \end{aligned} \tag{29}$$

where $r^b(\widetilde{M}, \widetilde{N})$ is defined as in (28).

Proof. From Theorem 10 and Definition 7, we have

$$\begin{aligned} &d\left(\widetilde{M}, \widetilde{O}^P\right) \\ &\quad = \int_{\min(m^L,n^L)}^{m^P}\left(\widetilde{M}(z) - \min\left(\widetilde{M}(z), \widetilde{N}(z)\right)\right)dz \\ &\quad + \int_{m^P}^{n^P}\left|\widetilde{M}(z) - \widetilde{N}(z)\right|dz \\ &\quad + \int_{n^P}^{\max(m^R,n^R)}\left(\max\left(\widetilde{M}(z), \widetilde{N}(z)\right) - \widetilde{M}(z)\right)dz, \\ &d\left(\widetilde{N}, \widetilde{O}^P\right) \\ &\quad = \int_{\min(m^L,n^L)}^{m^P}\left(\widetilde{N}(z) - \min\left(\widetilde{M}(z), \widetilde{N}(z)\right)\right)dz \\ &\quad + \int_{m^P}^{n^P}\left|\widetilde{N}(z) - \widetilde{N}(z)\right|dz \\ &\quad + \int_{n^P}^{\max(m^R,n^R)}\left(\max\left(\widetilde{M}(z), \widetilde{N}(z)\right) - \widetilde{N}(z)\right)dz. \end{aligned} \tag{30}$$

Thus, we have

$$\begin{aligned} &d\left(\widetilde{M}, \widetilde{O}^P\right) - d\left(\widetilde{N}, \widetilde{O}^P\right) \\ &\quad = \int_{\min(m^L,n^L)}^{m^P}\left(\widetilde{M}(z) - \widetilde{N}(z)\right)dz \\ &\quad + \int_{m^P}^{n^P}\left|\widetilde{M}(z) - \widetilde{N}(z)\right|dz \\ &\quad + \int_{n^P}^{\max(m^R,n^R)}\left(\widetilde{N}(z) - \widetilde{M}(z)\right)dz. \end{aligned} \tag{31}$$

Also, for negative polar we have

$$\begin{aligned}
d\left(\widetilde{M},\widetilde{O}^N\right) &= \int_{\min(m^L,n^L)}^{m^N}\left(\max\left(\widetilde{M}(z),\widetilde{N}(z)\right)-\widetilde{M}(z)\right)dz \\
&+\int_{m^N}^{n^N}\left|\widetilde{M}(z)-\widetilde{N}(z)\right|dz \\
&+\int_{n^N}^{\max(m^R,n^R)}\left(\widetilde{M}(z)-\min\left(\widetilde{M}(z),\widetilde{N}(z)\right)\right)dz, \\
d\left(\widetilde{N},\widetilde{O}^N\right) &= \int_{\min(m^L,n^L)}^{m^N}\left(\max\left(\widetilde{M}(z),\widetilde{N}(z)\right)-\widetilde{N}(z)\right)dz \\
&+\int_{m^N}^{n^N}\left|\widetilde{N}(z)-\widetilde{N}(z)\right|dz \\
&+\int_{n^N}^{\max(m^R,n^R)}\left(\widetilde{N}(z)-\min\left(\widetilde{M}(z),\widetilde{N}(z)\right)\right)dz.
\end{aligned}\tag{32}$$

Thus, we have

$$\begin{aligned}
d\left(\widetilde{M},\widetilde{O}^N\right)-d\left(\widetilde{N},\widetilde{O}^N\right) &= \int_{\min(m^L,n^L)}^{m^N}\left(\widetilde{M}(z)-\widetilde{N}(z)\right)dz \\
&+\int_{m^N}^{n^N}\left|\widetilde{M}(z)-\widetilde{N}(z)\right|dz \\
&+\int_{n^N}^{\max(m^R,n^R)}\left(\widetilde{M}(z)-\widetilde{N}(z)\right)dz.
\end{aligned}\tag{33}$$

And therefore

$$\begin{aligned}
r^b\left(\widetilde{M},\widetilde{N}\right) &= \int_{\min(m^L,n^L)}^{m^P}\left(\widetilde{M}(z)-\widetilde{N}(z)\right)dz \\
&+\int_{m^P}^{n^P}\left|\widetilde{M}(z)-\widetilde{N}(z)\right|dz \\
&+\int_{n^P}^{\max(m^R,n^R)}\left(\widetilde{N}(z)-\widetilde{M}(z)\right)dz \\
&+\int_{\min(m^L,n^L)}^{m^N}\left(\widetilde{N}(z)-\widetilde{M}(z)\right)dz \\
&+\int_{m^N}^{n^N}\left|\widetilde{M}(z)-\widetilde{N}(z)\right|dz \\
&+\int_{n^N}^{\max(m^R,n^R)}\left(\widetilde{M}(z)-\widetilde{N}(z)\right)dz.
\end{aligned}\tag{34}$$

□

We can rewrite (29) as

$$\begin{aligned}
r^b\left(\widetilde{M},\widetilde{N}\right) &= \int_{\min(m^L,n^L)}^{m^P}\left(\widetilde{M}_L(z)-\widetilde{N}_L(z)\right)dz \\
&+\int_{m^P}^{n^P}\left|\widetilde{M}_R(z)-\widetilde{N}_L(z)\right|dz \\
&+\int_{n^P}^{\max(m^R,n^R)}\left(\widetilde{N}_R(z)-\widetilde{M}_R(z)\right)dz \\
&+\int_{\min(m^L,n^L)}^{m^N}\left(\widetilde{N}_L(z)-\widetilde{M}_L(z)\right)dz \\
&+\int_{m^N}^{n^N}\left|\widetilde{M}_R(z)-\widetilde{N}_L(z)\right|dz \\
&+\int_{n^N}^{\max(m^R,n^R)}\left(\widetilde{M}_R(z)-\widetilde{N}_R(z)\right)dz.
\end{aligned}\tag{35}$$

Note that when $\widetilde{M}=(m^L,m^P,m^N,m^R)$ and $\widetilde{N}=(n^L,n^P,n^N,n^R)$ are bipolar triangular fuzzy numbers, we can simplify (35), and then the computation of $\int_{m^P}^{n^P}|\widetilde{M}_R(z)-\widetilde{N}_L(z)|dz$ can be simplified, if we can compute the intersection of M_R and N_L. Since each polar of $\widetilde{M}$ and $\widetilde{N}$ is a triangular fuzzy number, $\overline{x}^P$ below,

$$\overline{x}^P=\frac{m^Rn^P-n^Lm^P}{n^P-n^L-m^P+m^R},\tag{36}$$

is the length of the intersection point of M_R and N_L for the positive polar, and $\overline{x}^N$ below,

$$\overline{x}^N=\frac{m^Rn^N-n^Lm^N}{n^N-n^L-m^N+m^R},\tag{37}$$

is the length of the intersection point of M_R and N_L for the negative polar. We have the following proposition giving a reformulation of (29).

Proposition 13. *Let $\widetilde{M}=(m^L,m^P,m^N,m^R)$ and $\widetilde{N}=(n^L,n^P,n^N,n^R)$ be two bipolar triangular fuzzy numbers with $m^P<n^P$, $m^N<n^N$, $\overline{x}^P\in[m^P,n^P]$, and $\overline{x}^N\in[m^N,n^N]$, where $\overline{x}^P$ and $\overline{x}^N$ are defined by (36) and (37). Then, one has*

$$\begin{aligned}
r^b\left(\widetilde{M},\widetilde{N}\right) &= \int_{\min(m^L,n^L)}^{m^P}\left(\widetilde{M}_L(z)-\widetilde{N}_L(z)\right)dz \\
&+\int_{m^P}^{\overline{x}^P}\left(\widetilde{M}_R(z)-\widetilde{N}_L(z)\right)dz \\
&+\int_{\overline{x}^P}^{n^P}\left(\widetilde{N}_L(z)-\widetilde{M}_R(z)\right)dz \\
&+\int_{n^P}^{\max(m^R,n^R)}\left(\widetilde{N}_R(z)-\widetilde{M}_R(z)\right)dz \\
&+\int_{\min(m^L,n^L)}^{m^N}\left(\widetilde{N}_L(z)-\widetilde{M}_L(z)\right)dz
\end{aligned}$$

$$+\int_{m^N}^{\bar{x}^N}\left(\widetilde{N}_L(z)-\widetilde{M}_R(z)\right)dz$$
$$+\int_{\bar{x}^N}^{n^N}\left(\widetilde{M}_R(z)-\widetilde{N}_L(z)\right)dz$$
$$+\int_{n^N}^{\max(m^R,n^R)}\left(\widetilde{M}_R(z)-\widetilde{N}_R(z)\right)dz. \quad (38)$$

Note that the sign of $r^b(\cdot,\cdot)$ is adequate to determine $\widetilde{M} \le \widetilde{N}$ or $\widetilde{M} \ge \widetilde{N}$. But, for bipolar LR fuzzy number linear programming problems, in some situations we need to compute the exact value of $r^b(\cdot,\cdot)$.

Example 14. Let $\widetilde{M} = (2,4,6,10)$ and $\widetilde{N} = (2,5,8,12)$ be two bipolar triangular fuzzy numbers. Then, $\widetilde{M}_L^P$, $\widetilde{M}_R^P$, $\widetilde{N}_L^P$, $\widetilde{N}_R^P$, $\widetilde{M}_L^N$, $\widetilde{M}_R^N$, $\widetilde{N}_L^N$, and $\widetilde{N}_R^N$ are

$$\widetilde{M}_L^P(z)=\begin{cases}0, & z<2,\\ \frac{1}{2}(x-2), & 2\le z\le 4,\end{cases}$$
$$\widetilde{M}_R^P(z)=\begin{cases}0, & z>10,\\ \frac{-1}{6}(x-10), & 4< z\le 10,\end{cases}$$
$$\widetilde{N}_L^P(z)=\begin{cases}0, & z<5,\\ \frac{1}{3}(x-2), & 2< z\le 5,\end{cases}$$
$$\widetilde{N}_R^P(z)=\begin{cases}0, & z>12,\\ \frac{-1}{7}(x-12), & 5< z\le 12,\end{cases}$$
$$\widetilde{M}_L^N(z)=\begin{cases}0, & z<2,\\ \frac{-1}{4}(x-2), & 2< z\le 6,\end{cases} \quad (39)$$
$$\widetilde{M}_R^N(z)=\begin{cases}0, & z>10,\\ \frac{1}{4}(x-10), & 6< z\le 10,\end{cases}$$
$$\widetilde{N}_L^N(z)=\begin{cases}0, & z<2,\\ \frac{-1}{6}(x-2), & 2< z\le 8,\end{cases}$$
$$\widetilde{N}_R^N(z)=\begin{cases}0, & z>12,\\ \frac{1}{4}(x-12), & 8< z\le 12.\end{cases}$$

According to (38), we have $r^b(\widetilde{M},\widetilde{N}) = 4.4889$ and this means $\widetilde{M} < \widetilde{N}$.

Next, we give some corollaries, the proofs of which are straightforward.

Corollary 15. *Let $\widetilde{M} = (m^L, m^P, m^N, m^R)$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)$ be two bipolar triangular fuzzy numbers. If $m^R \le n^L$, then*

$$r^b\left(\widetilde{M},\widetilde{N}\right)=2\left(\frac{m^R-m^L}{2}+\frac{n^R-n^L}{2}\right). \quad (40)$$

Corollary 16. *Let $\widetilde{M} = (m^L, m^P, m^N, m^R)$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)$ be two bipolar triangular fuzzy numbers. If $m^P < n^P$ and $m^N = n^N$, where $\bar{y}^P = \widetilde{M}_R^P(\bar{x}^P) = \widetilde{N}_L^P(\bar{x}^P)$, with $\bar{x}^P$ as defined by (36), then*

$$r^b\left(\widetilde{M},\widetilde{N}\right)=\frac{\left(n^R-n^L\right)}{2}+\frac{\left(m^R-m^L\right)}{2} -\bar{y}^P\left(m^R-n^L\right)+\frac{\left(n^R+n^L\right)}{2} -\frac{\left(m^R+m^L\right)}{2}. \quad (41)$$

Corollary 17. *Let $\widetilde{M} = (m^L, m^P, m^N, m^R)$ and $\widetilde{N} = (n^L, n^P, n^N, n^R)$ be two bipolar triangular fuzzy numbers. If $m^P < n^P$ and $m^N < n^N$, where $\bar{y}^P = \widetilde{M}_R^P(\bar{x}^P) = \widetilde{N}_L^P(\bar{x}^P)$, with $\bar{x}^P$ as defined by (36), and $\bar{y}^N = \widetilde{M}_R^N(\bar{x}^N) = \widetilde{N}_L^N(\bar{x}^N)$, with $\bar{x}^N$ as defined by (37), then*

$$r^b\left(\widetilde{M},\widetilde{N}\right)=\frac{\left(n^R-n^L\right)}{2}+\frac{\left(m^R-m^L\right)}{2} -\bar{y}^P\left(m^R-n^L\right)+\frac{\left(n^R-n^L\right)}{2} +\frac{\left(m^R-m^L\right)}{2}+\bar{y}^N\left(m^R-n^L\right). \quad (42)$$

A property of (42) is that for two bipolar triangular fuzzy numbers such as $\widetilde{M}$ and $\widetilde{N}$ we have $r^b(\lambda\widetilde{M},\lambda\widetilde{N}) = \lambda r^b(\widetilde{M},\widetilde{N})$, where $\lambda \ge 0$.

4. Application of Proposed Method in a Real Life Problem

Akram [10] studied an application of bipolar fuzzy sets in graph theory. He used bipolar fuzzy set for a social group. Here, we demonstrate an application of bipolar fuzzy number in maximum weighted matching problem; matching problem has some applications in various fields such as scheduling [14] and network [15] problems. We consider each vertex to be person and weight of each edge between two vertices be the influence of each person (vertex) to another person. In general, influence can be positive or negative. Suppose $G = (V, E)$ is an arbitrary weighted graph, where $V = \{1, \ldots, n\}$ is the vertex set of G and $E \subseteq V \times V$ is the edge set of G. The maximum weighted matching problem is

$$\begin{aligned} \max \quad & \sum_{e \in E} \widetilde{w}(e)\, x(e) \\ \text{s.t.} \quad & \sum_{e=(u,v) \in E} x(e) \le 1, \quad \forall u \in V, \\ & x(e) \in \{0,1\}, \quad \forall e \in E, \end{aligned} \tag{43}$$

where $x(e) = 1$, if two persons u and v are matched to each other, and $x(e) = 0$, otherwise, and $\widetilde{w}(e)$ is the weight of edge e (giving the influence of one person to another person), considered as a bipolar fuzzy number, since influence of a person cannot always be positive. The aim is to match every person to another person so that they have a stable relation.

5. Conclusions

We proposed a new efficient method for ordering bipolar fuzzy numbers. In this method, for comparison of bipolar LR fuzzy numbers, we used an extension of Kerre's method used in ordering of unipolar fuzzy numbers. In our proposed method, we provided a formula to compare two bipolar triangular fuzzy numbers in $O(1)$ operations, making the process useful for optimization algorithms. Also, we presented an application of bipolar fuzzy number in a real life problem.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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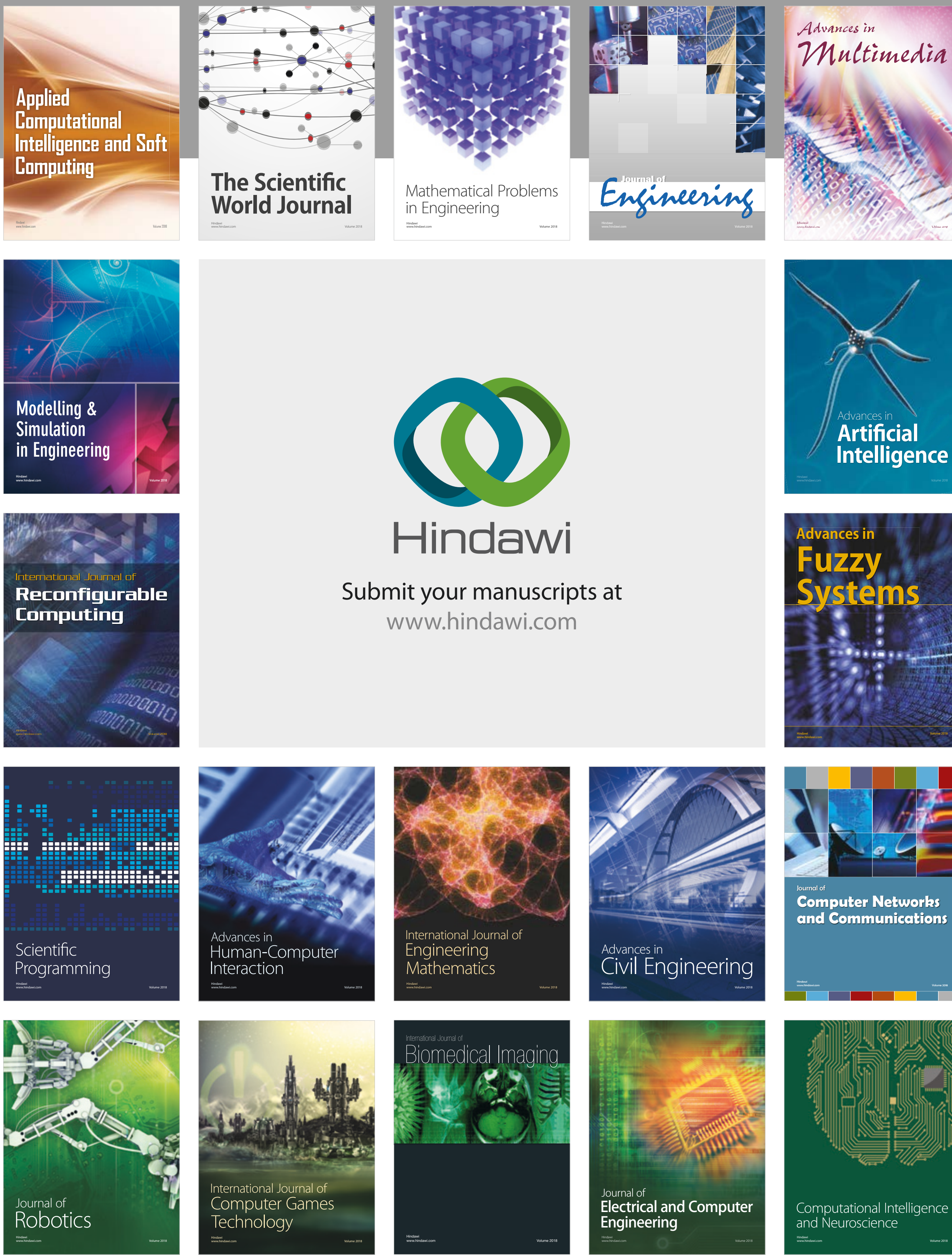
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