Research Article

Fuzzy Generalized Conformable Fractional Derivative

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We give a new definition of fuzzy fractional derivative called fuzzy conformable fractional derivative. Using this definition, we prove some results and we introduce new definition of generalized fuzzy conformable fractional derivative.

1. Introduction

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models. Their main directions of development have been diversified, and its applications have been varied [1–4].

The derivative for fuzzy valued mappings was developed by Puri and Ralescu [5], which generalized and extended the concept of Hukuhara differentiability for set-valued mappings to the class of fuzzy mappings. Subsequently, using the H-derivative, Kaleva [6] started to develop a theory for FDE. In [7], a new well-behaved simple fractional derivative called “the conformable fractional derivative” depending just on the basic limit definition of the derivative, namely, for a function $f(0, \infty) \rightarrow \mathbb{R}$ the (conformable) fractional derivative of order $0 < q \leq 1$ of f at $t > 0$ was defined by

$$T_q f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-q}) - f(t)}{\varepsilon}, \quad (1)$$

and is defined the fractional derivative at 0 as $(T_q f)(0) = \lim_{t \rightarrow 0^+} (T_q f)(t)$. The aim of this paper is to study and generalize the fuzzy conformable fractional derivative.

2. Preliminaries

Let us denote by $\mathbb{R}_{\mathcal{F}} = \{u : \mathbb{R} \rightarrow [0, 1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties:

- (i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.
- (ii) u is fuzzy convex, i.e., for $x, y \in \mathbb{R}$ and $0 < \lambda \leq 1$:

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)]. \quad (2)$$

- (iii) u is upper semicontinuous.
- (iv) $[u]^0 = \text{cl}\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact.

Then, $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. For $0 < \alpha \leq 1$ denotes $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$, then from (i) to (iv) it follows that the α -level sets $[u]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$ is a closed bounded interval which is denoted by $[u]^\alpha = [u_1^\alpha, u_2^\alpha]$. By $P_K(\mathbb{R})$ we denote the family of all non-empty compact convex subsets of $\mathbb{R}$ and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual.

Theorem 1 (see [8]). If $u \in \mathbb{R}_{\mathcal{F}}$, then

- (i) $[u]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$.
- (ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.
- (iii) $\{\alpha_k\} \subset [0, 1]$ is a nondecreasing sequence which converges to α , then

$$[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}. \quad (3)$$

Conversely, if $A_\alpha = \{[u_1^\alpha, u_2^\alpha]; \alpha \in (0, 1)\}$ is a family of closed real intervals verifying (i) and (ii), then $\{A_\alpha\}$ defined a

fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ such that $[u]^\alpha = A_\alpha$ for $0 < \alpha \leq 1$ and $[u]^0 = \bigcup_{0 < \alpha \leq 1} A_\alpha \subset A_0$.

Lemma 1. Let $u, v : X \longrightarrow [0, 1]$ be the fuzzy sets.

Then, $u = v$ if and only if $[u]^\alpha = [v]^\alpha$ for all $\alpha \in [0, 1]$ see ([9]).

The following arithmetic operations on fuzzy numbers are well known and frequently used below. If $u, v \in \mathbb{R}_{\mathcal{F}}$, then

$$\begin{aligned} [u + v]^\alpha &= [u_1^\alpha + v_1^\alpha, u_2^\alpha + v_2^\alpha], \\ [\lambda u]^\alpha &= \lambda [u]^\alpha = \begin{cases} [\lambda u_1^\alpha, \lambda u_2^\alpha], & \lambda \geq 0, \\ [\lambda u_2^\alpha, \lambda u_1^\alpha], & \lambda < 0. \end{cases} \end{aligned} \quad (4)$$

Definition 1. Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, then w is called the H -difference of u, v and it is denoted $u \ominus v$.

Theorem 2 (see [10])

(i) Let us denote

$$\bar{0} = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases} \quad (5)$$

Then, $\bar{0} \in \mathbb{R}_{\mathcal{F}}$ be a neutral element with respect to $+$, i.e., $u + \bar{0} = \bar{0} + u \quad u \in \mathbb{R}_{\mathcal{F}}$:

- (i) With respect to $\bar{0}$, none of $u \in \mathbb{R}_{\mathcal{F}}/\mathbb{R}$ has opposite in $\mathbb{R}_{\mathcal{F}}$.
- (ii) For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a + b) \cdot u = a \cdot u + b \cdot u$, for general $a, b \in \mathbb{R}$ the above property does not hold.
- (iii) For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$.
- (iv) For any $\lambda, \nu \in \mathbb{R}$ and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \cdot (\nu \cdot u) = (\lambda \cdot \nu) \cdot u$.

Define $d : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}_+ \cup \{0\}$ by the following equation:

$$d(u, v) = \sup_{\alpha \in [0, 1]} d_H([u]^\alpha, [v]^\alpha), \quad \text{for all } u, v \in \mathbb{R}_{\mathcal{F}}, \quad (6)$$

where d_H is the Hausdorff metric:

$$d_H([u]^\alpha, [v]^\alpha) = \max\{|u_1^\alpha - v_1^\alpha|, |u_2^\alpha - v_2^\alpha|\}. \quad (7)$$

It is well known that $(\mathbb{R}_{\mathcal{F}}, d)$ is a complete metric space. We list the following properties of $d(u, v)$:

$$\begin{aligned} d(u + w, v + w) &= d(u, v), \\ d(u, v) &= d(v, u), \\ d(ku, kv) &= |k|d(u, v), \\ d(u, v) &\leq d(u, w) + d(w, v), \end{aligned} \quad (8)$$

for all $u, v, w \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$.

Let (A_k) be a sequence in $P_K(\mathbb{R})$ converging to A . Then, theorem in [6] gives us an expression for the limit.

Theorem 3 (see [6]). If $d(A_k, A) \longrightarrow 0$ as $k \longrightarrow \infty$, then

$$A = \bigcap_{k \geq 1} \overline{\bigcup_{m \geq k} A_m}. \quad (9)$$

Let $I = (0, a) \subset \mathbb{R}$ be an interval. We denote by $C(I, \mathbb{R}_{\mathcal{F}})$ that the space of all continuous fuzzy functions on I is a complete metric space with respect to the metric:

$$h(u, v) = \sup_{t \in I} d(u(t), v(t)). \quad (10)$$

3. The Fuzzy Conformable Fractional Differentiability

Definition 2. Let $F : I \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy function. q th order “fuzzy conformable fractional derivative” of F is defined by

$$T_q(F)(t) = \lim_{\varepsilon \longrightarrow 0^+} \frac{F(t + \varepsilon t^{1-q}) \ominus F(t)}{\varepsilon} \quad (11)$$

$$= \lim_{\varepsilon \longrightarrow 0^+} \frac{F(t) \ominus F(t - \varepsilon t^{1-q})}{\varepsilon},$$

for all $t > 0$ and $q \in (0, 1)$. Let $F^{(q)}(t)$ stands for $T_q(F)(t)$. Hence,

$$F^{(q)}(t) = \lim_{\varepsilon \longrightarrow 0^+} \frac{F(t + \varepsilon t^{1-q}) \ominus F(t)}{\varepsilon} = \lim_{\varepsilon \longrightarrow 0^+} \frac{F(t) \ominus F(t - \varepsilon t^{1-q})}{\varepsilon}. \quad (12)$$

If F is q -differentiable in some $(0, a)$ and $\lim_{t \longrightarrow 0^+} F^{(q)}(t)$,

$$F^{(q)}(0) = \lim_{t \longrightarrow 0^+} F^{(q)}(t), \quad (13)$$

and the limits exist (in the metric d).

Remark 1. From the definition, it directly follows that if F is q -differentiable then the multivalued mapping F_α is q -differentiable for all $\alpha \in [0, 1]$ and

$$T_q F_\alpha = [F^{(q)}(t)]^\alpha, \quad (14)$$

where $T_q F_\alpha$ is denoted from the conformable fractional derivative of F_α of order q . The converse result does not hold, since the existence of Hukuhara difference $[x]^\alpha \ominus [y]^\alpha$, $\alpha \in [0, 1]$, does not imply the existence of H -difference $x \ominus y$.

However, for the converse result we have the following.

Theorem 4. Let $F : I \longrightarrow \mathbb{R}_{\mathcal{F}}$ satisfy the assumptions:

- (i) For each $t \in I$ there exists a $\beta > 0$ such that the H -difference $F(t + \varepsilon t^{1-q}) \ominus F(t)$ and $F(t) \ominus F(t - \varepsilon t^{1-q})$ exists for all $0 \leq \varepsilon < \beta$.
- (ii) The set-valued mappings F_α , $\alpha \in [0, 1]$ are uniformly Hukuhara conformable fractional derivative of order q with derivative $T_q F_\alpha$, i.e., for each $t \in I$ and $h > 0$ there exists a $\delta > 0$ such that

$$d_H\left(\frac{(F_\alpha(t + \varepsilon t^{1-q}) - F_\alpha(t))}{\varepsilon, T_q F_\alpha(t)}\right) < h, \quad (15)$$

$$d_H\left(\frac{(F_\alpha(t) - F_\alpha(t - \varepsilon t^{1-q}))}{\varepsilon, T_q F_\alpha(t)}\right) < h, \quad (16)$$

for all $0 \leq \varepsilon < \delta$ and $\alpha \in [0, 1]$. Then, F is q -differentiable, and the derivative is given by (14).

Proof. Consider the family $\{T_q F_\alpha \mid \alpha \in [0, 1]\}$. By definition $T_q F_\alpha(t)$ is a compact, convex, and nonempty subset of $\mathbb{R}$.

$$\begin{aligned} &\text{If } \alpha_1 \leq \alpha_2, \text{ then by assumption (i),} \\ &F_{\alpha_1}(t + \varepsilon t^{1-q}) - F_{\alpha_1}(t) \supset F_{\alpha_2}(t + \varepsilon t^{1-q}) - F_{\alpha_2}(t), \quad \text{for } 0 \leq \varepsilon < \beta, \end{aligned} \quad (17)$$

and consequently

$$T_q F_{\alpha_1} \supset T_q F_{\alpha_2}. \quad (18)$$

Let $\alpha > 0$ and α_k be a nondecreasing sequence converging to α . For $h > 0$ choose $\varepsilon > 0$ such that equation (15) holds true. Then, the triangle inequality yields

$$\begin{aligned} d_H(T_q F_\alpha(t), T_q F_{\alpha_k}(t)) &\leq d_H\left(T_q F_\alpha(t), \frac{(F_\alpha(t + \varepsilon t^{1-q}) - F_\alpha(t))}{\varepsilon}\right) + d_H\left(\frac{(F_\alpha(t + \varepsilon t^{1-q}) - F_\alpha(t))}{\varepsilon}, \frac{(F_{\alpha_k}(t + \varepsilon t^{1-q}) - F_{\alpha_k}(t))}{\varepsilon}\right) \\ &\quad + d_H\left(\frac{(F_{\alpha_k}(t + \varepsilon t^{1-q}) - F_{\alpha_k}(t))}{\varepsilon}, T_q F_{\alpha_k}(t)\right) \leq 2h + d_H \\ &\quad \cdot \left(\frac{(F_\alpha(t + \varepsilon t^{1-q}) - F_\alpha(t))}{\varepsilon}, \frac{(F_{\alpha_k}(t + \varepsilon t^{1-q}) - F_{\alpha_k}(t))}{\varepsilon}\right). \end{aligned} \quad (19)$$

By assumption (i), the rightmost term goes to zero as $k \rightarrow \infty$ and hence

$$\lim_{k \rightarrow \infty} d_H(T_q F_\alpha(t), T_q F_{\alpha_k}(t)) = 0. \quad (20)$$

Now by Theorem 3 and (18) we have

$$T_q F_\alpha(t) = \bigcup_{k \geq 1} T_q F_{\alpha_k}(t). \quad (21)$$

If $\alpha = 0$, then using equation (3.1) in [11], we deduce as before that

$$\lim_{k \rightarrow \infty} d_H(T_q F_0(t), T_q F_{\alpha_k}(t)) = 0, \quad (22)$$

where (α_k) is a nonincreasing sequence converging to zero and consequently

$$T_q F_0(t) = \overline{\bigcup_{k \geq 1} T_q F_{\alpha_k}}. \quad (23)$$

Then, from Theorem 1 it follows that there is an element $u \in \mathbb{R}_{\mathcal{F}}$ such that

$$[u]^\alpha = T_q F_\alpha(t), \quad \text{for } \alpha \in [0, 1]. \quad (24)$$

Furthermore, let $t \in I$, $h > 0$, and $\delta > 0$ be as in (ii). Then,

$$d\left(\frac{(F(t + \varepsilon t^{1-q}) \ominus F(t))}{\varepsilon}, u\right) = \sup_{0 < \alpha \leq 1} d_H\left(\frac{(F_\alpha(t + \varepsilon t^{1-q}) - F_\alpha(t))}{\varepsilon, T_q F_\alpha(t)}\right) < h, \quad (25)$$

for all $0 \leq \varepsilon < \delta$ and similarly for $d((F(t + \varepsilon t^{1-q}) \ominus F(t))/\varepsilon, u)$. Hence, $F^{(q)}(t) = u$ has the theorem.

$$[F^{(q)}(t)]^\alpha = \left[(f_1^\alpha)^{(q)}(t), (f_2^\alpha)^{(q)}(t) \right]. \quad (26)$$

Theorem 5. Let $F: I \rightarrow \mathbb{R}_{\mathcal{F}}$ be q -differentiable. Denote $F_\alpha(t) = [f_1^\alpha(t), f_2^\alpha(t)]$, $\alpha \in [0, 1]$. Then, $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q -differentiable and

Proof. If $\varepsilon > 0$ and $\alpha \in [0, 1]$, we have

$$[F(t + \varepsilon t^{1-q}) \ominus F(t)]^\alpha = [f_1^\alpha(t + \varepsilon t^{1-q}) - f_1^\alpha(t), f_2^\alpha(t + \varepsilon t^{1-q}) - f_2^\alpha(t)]. \quad (27)$$

Dividing by ε , we have

$$\frac{[F(t + \varepsilon t^{1-q}) \ominus F(t)]^\alpha}{\varepsilon} = \left[\frac{f_1^\alpha(t + \varepsilon t^{1-q}) - f_1^\alpha(t)}{\varepsilon}, \frac{f_2^\alpha(t + \varepsilon t^{1-q}) - f_2^\alpha(t)}{\varepsilon} \right]. \quad (28)$$

Similarly, we obtain

$$\frac{[F(t) \ominus F(t - \varepsilon t^{1-q})]^\alpha}{\varepsilon} = \left[\frac{f_1^\alpha(t) - f_1^\alpha(t - \varepsilon t^{1-q})}{\varepsilon}, \frac{f_2^\alpha(t) - f_2^\alpha(t - \varepsilon t^{1-q})}{\varepsilon} \right], \quad (29)$$

and passing to the limit gives the theorem.

Note that this definition and theorem of conformable fractional derivative are very restrictive; for instance, if $F(t) = c \cdot g(t)$, where c is a fuzzy number and $g : [a, b] \rightarrow \mathbb{R}^+$ is a function and is q -differentiable for some $q \in (0, 1]$ with $g^{(q)}(t) < 0$, then F is not q -differentiable. To avoid this difficulty, we introduce a more general definition of the conformable fractional derivative for fuzzy-number-valued function.

4. The Generalized Fuzzy Conformable Fractional Differentiability

We consider the following definition.

Definition 3. Let $F : I \rightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy function and $q \in (0, 1]$. One says, F is $q_{(1)}$ -differentiable at point $t > 0$ if there exists an element $F^{(q)}(t) \in \mathbb{R}_{\mathcal{F}}$ such that for all $\varepsilon > 0$ sufficiently near to 0 there exist $F(t + \varepsilon t^{1-q}) \ominus F(t)$, $F(t) \ominus F(t - \varepsilon t^{1-q})$, and the limits (in the metric d):

$$\lim_{\varepsilon \rightarrow 0^+} \frac{F(t + \varepsilon t^{1-q}) \ominus F(t)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{F(t) \ominus F(t - \varepsilon t^{1-q})}{\varepsilon} = F^{(q)}(t), \quad (30)$$

where F is $q_{(2)}$ -differentiable at $t > 0$ if for all $\varepsilon < 0$ sufficiently near to 0, then there exist $F(t + \varepsilon t^{1-q}) \ominus F(t)$ and $F(t) \ominus F(t - \varepsilon t^{1-q})$:

$$\lim_{\varepsilon \rightarrow 0^-} \frac{F(t + \varepsilon t^{1-q}) \ominus F(t)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^-} \frac{F(t) \ominus F(t - \varepsilon t^{1-q})}{\varepsilon} = F^{(q)}(t). \quad (31)$$

If F is $q_{(n)}$ -differentiable at $t > 0$, we denote its q -derivatives ($q \in (0, 1]$) by $F_n^{(q)}(t)$, for $n = 1, 2$.

Example 1. Let $g : I \rightarrow \mathbb{R}^+$ and define $F : I \rightarrow \mathbb{R}_{\mathcal{F}}$ by $F(t) = c \cdot g(t)$ for all $t \in I$, where c is the fuzzy number. If g is q -differentiable at $t_0 \in I$, then F is the generalized fuzzy conformable fractional derivative at $t_0 \in I$ and we have

$F^{(q)}(t_0) = c \cdot g^{(q)}(t_0)$. For instance, if $g^{(q)}(t_0) > 0$, F is $q_{(1)}$ -differentiable. If $g^{(q)}(t_0) < 0$, then F is $q_{(2)}$ -differentiable.

Remark 2. In the previous definition, $q_{(1)}$ -differentiable corresponds to Definition 3, so this differentiability concept is a generalization of Definition 2 and obviously more general. For instance, in the previous example, for $F(t) = c \cdot g(t)$ with $g^{(q)}(t_0) < 0$, we have $F^{(q)}(t_0) = c \cdot g^{(q)}(t_0)$.

Theorem 6. Let $F : I \rightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy function, where $F_\alpha(t) = [f_1^\alpha(t), f_2^\alpha(t)]$, $\alpha \in [0, 1]$:

(i) If F is $q_{(1)}$ -differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q -differentiable and

$$\left[F^{(q_{(1)})}(t) \right]^\alpha = \left[(f_1^\alpha)^{(q)}(t), (f_2^\alpha)^{(q)}(t) \right]. \quad (32)$$

(ii) If F is $q_{(2)}$ -differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q -differentiable and

$$\left[F^{(q_{(2)})}(t) \right]^\alpha = \left[(f_2^\alpha)^{(q)}(t), (f_1^\alpha)^{(q)}(t) \right]. \quad (33)$$

Proof

(i) See demonstration of Theorem 5.

(ii) If $\varepsilon < 0$, $q \in (0, 1]$, and $\alpha \in [0, 1]$, then we have

$$\begin{aligned} [F(t + \varepsilon t^{1-q}) \ominus F(t)]^\alpha &= [f_1^\alpha(t + \varepsilon t^{1-q}) - f_1^\alpha(t), f_2^\alpha(t + \varepsilon t^{1-q}) \\ &\quad - f_2^\alpha(t)]. \end{aligned} \quad (34)$$

and multiplying by $1/\varepsilon$, we have

$$\frac{[F(t + \varepsilon t^{1-q}) \ominus F(t)]^\alpha}{\varepsilon} = \left[\frac{f_2^\alpha(t + \varepsilon t^{1-q}) - f_2^\alpha(t)}{\varepsilon}, \frac{f_1^\alpha(t + \varepsilon t^{1-q}) - f_1^\alpha(t)}{\varepsilon} \right]. \quad (35)$$

Similarly, we obtain

$$\frac{[F(t) \ominus F(t - \varepsilon t^{1-q})]^\alpha}{\varepsilon} = \left[\frac{f_2^\alpha(t) - f_2^\alpha(t - \varepsilon t^{1-q})}{\varepsilon}, \frac{f_1^\alpha(t) - f_1^\alpha(t - \varepsilon t^{1-q})}{\varepsilon} \right], \quad (36)$$

and passing to the limit we have $[F^{(q_{(2)})}(t)]^\alpha = [(f_2^\alpha)^{(q)}(t), (f_1^\alpha)^{(q)}(t)]$.

Theorem 7. Let $F : I \longrightarrow \mathbb{R}_{\mathcal{F}}$ and $g : I \longrightarrow \mathbb{R}$ be two q -differentiable functions at a point $t > 0$ (F is the generalized fuzzy conformable fractional derivative Definition 3 and $q \in (0, 1]$):

(i) If $g(t) \cdot T_q(g)(t) > 0$ and F is $q_{(1)}$ -differentiable, then $g \cdot F$ is $q_{(1)}$ -differentiable and

$$T_q(g \cdot F)(t) = g(t) \cdot T_q(F)(t) + F(t) \cdot T_q(g)(t). \quad (37)$$

(ii) If $g(t) \cdot T_q(g)(t) < 0$ and F is $q_{(2)}$ -differentiable, then $g \cdot F$ is $q_{(2)}$ -differentiable and

$$T_q(g \cdot F)(t) = g(t) \cdot T_q(F)(t) + F(t) \cdot T_q(g)(t). \quad (38)$$

Proof. We present the details only for the case (i), since the other case is analogous. F is $q_{(1)}$ -differentiable at $t > 0$ the H -difference $F(t) \ominus F(t + \varepsilon t^{1-q})$ exists for $\varepsilon > 0$ sufficiently small, i.e., there exists $u_1(t, \varepsilon t^{1-q}) \in \mathbb{R}_{\mathcal{F}}$, such that

$$F(t) = F(t + \varepsilon t^{1-q}) + u_1(t, \varepsilon t^{1-q}). \quad (39)$$

Also,

$$g(t) = g(t + \varepsilon t^{1-q}) + u_2(t, \varepsilon t^{1-q}), \quad (40)$$

where $u_2(t, \varepsilon t^{1-q}) = g(t) - g(t + \varepsilon t^{1-q})$ for $\varepsilon > 0$ sufficiently small.

By Theorem 2, we obtain

$$\begin{aligned} F(t) \cdot g(t) &= F(t + \varepsilon t^{1-q}) \cdot g(t + \varepsilon t^{1-q}) + F(t + \varepsilon t^{1-q}) \\ &\quad \cdot u_2(t, \varepsilon t^{1-q}) + g(t + \varepsilon t^{1-q}) \cdot u_1(t, \varepsilon t^{1-q}) \\ &\quad + u_1(t, \varepsilon t^{1-q}) \cdot u_2(t, \varepsilon t^{1-q}), \end{aligned} \quad (41)$$

that is, the H -difference $F(t) \cdot g(t) \ominus F(t + \varepsilon t^{1-q}) \cdot g(t + \varepsilon t^{1-q})$ exists and we have

$$F(t) \cdot g(t) \ominus F(t + \varepsilon t^{1-q}) \cdot g(t + \varepsilon t^{1-q}) = F(t + \varepsilon t^{1-q}) \cdot u_2(t, \varepsilon t^{1-q}) + g(t + \varepsilon t^{1-q}) \cdot u_1(t, \varepsilon t^{1-q}) + u_1(t, \varepsilon t^{1-q}) \cdot u_2(t, \varepsilon t^{1-q}). \quad (42)$$

Multiplying with $1/\varepsilon$ and passing to limit with $\varepsilon > 0$, by Proof of Lemma 3.2 [12], we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{F(t) \cdot g(t) \ominus F(t + \varepsilon t^{1-q}) \cdot g(t + \varepsilon t^{1-q})}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} F(t + \varepsilon t^{1-q}) \cdot \frac{u_2(t, \varepsilon t^{1-q})}{\varepsilon} \\ &\quad + \lim_{\varepsilon \rightarrow 0^+} g(t + \varepsilon t^{1-q}) \cdot \frac{u_1(t, \varepsilon t^{1-q})}{\varepsilon} + \lim_{\varepsilon \rightarrow 0^+} \frac{u_1(t, \varepsilon t^{1-q})}{\varepsilon} \cdot u_2(t, \varepsilon t^{1-q}), \end{aligned} \quad (43)$$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{F(t) \cdot g(t) \ominus F(t + \varepsilon t^{1-q}) \cdot g(t + \varepsilon t^{1-q})}{\varepsilon} &= F(t) \cdot T_q g(t) + g(t) \cdot T_q F(t) + T_q g(t) \cdot \lim_{\varepsilon \rightarrow 0^+} u_2(t, \varepsilon t^{1-q}) \\ &= F(t) \cdot T_q g(t) + g(t) \cdot T_q F(t). \end{aligned}$$

Analogously, we obtain

$$\lim_{\varepsilon \rightarrow 0^+} \frac{F(t) \cdot g(t) \ominus F(t - \varepsilon t^{1-q}) \cdot g(t - \varepsilon t^{1-q})}{\varepsilon} = F(t) \cdot T_q g(t) + g(t) \cdot T_q F(t), \quad (44)$$

and the conclusion in the case (i) is obtained.

Theorem 8. Let $q \in (0, 1]$:

(i) If F is (1)-differentiable and F is $q_{(1)}$ -differentiable, then

$$T_{q_{(1)}} F(t) = t^{1-q} D_1^1 F(t). \quad (45)$$

(ii) If F is (2)-differentiable and F is $q_{(2)}$ -differentiable, then

Note that the definition of (n) -differentiable or (D_n^1) for $n \in 1, 2$, see [12–14].

Proof. We present the details only for Case (i), since the other case is analogous. Let $h = \varepsilon t^{1-q}$ in Definition 7, and then $\varepsilon = t^{q-1} h$. Therefore, if $\varepsilon > 0$ and $\alpha \in [0, 1]$, we have

$$[F(t + \varepsilon t^{1-q}) \ominus F(t)]^\alpha = [f_1^\alpha(t + \varepsilon t^{1-q}) - f_1^\alpha(t), f_2^\alpha(t + \varepsilon t^{1-q}) - f_2^\alpha(t)]. \quad (47)$$

Dividing by ε , we have

$$\frac{[F(t + \varepsilon t^{1-q}) \ominus F(t)]^\alpha}{\varepsilon} = \left[\frac{f_1^\alpha(t + \varepsilon t^{1-q}) - f_1^\alpha(t)}{\varepsilon}, \frac{f_2^\alpha(t + \varepsilon t^{1-q}) - f_2^\alpha(t)}{\varepsilon} \right], \quad (48)$$

and passing to the limit,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{[F(t + \varepsilon t^{1-q}) \ominus F(t)]^\alpha}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} \left[\frac{f_1^\alpha(t + \varepsilon t^{1-q}) - f_1^\alpha(t)}{\varepsilon}, \frac{f_2^\alpha(t + \varepsilon t^{1-q}) - f_2^\alpha(t)}{\varepsilon} \right] \\ &= \lim_{h \rightarrow 0^+} \left[\frac{f_1^\alpha(t + h) - f_1^\alpha(t)}{t^{q-1} h}, \frac{f_2^\alpha(t + h) - f_2^\alpha(t)}{t^{q-1} h} \right] \\ &= t^{1-q} \lim_{h \rightarrow 0^+} \left[\frac{f_1^\alpha(t + h) - f_1^\alpha(t)}{h}, \frac{f_2^\alpha(t + h) - f_2^\alpha(t)}{h} \right] \\ &= t^{1-q} [(f_1^\alpha)'(t), (f_2^\alpha)'(t)]. \end{aligned} \quad (49)$$

Similarly, we obtain

$$\frac{[F(t) \ominus F(t - \varepsilon t^{1-q})]^\alpha}{\varepsilon} = \left[\frac{f_1^\alpha(t) - f_1^\alpha(t - \varepsilon t^{1-q})}{\varepsilon}, \frac{f_2^\alpha(t) - f_2^\alpha(t - \varepsilon t^{1-q})}{\varepsilon} \right], \quad (50)$$

and passing to the limit and $\varepsilon = t^{q-1} h$ gives $T_{q_{(1)}} F(t) = t^{1-q} [(f_1^\alpha)'(t), (f_2^\alpha)'(t)]$.

Example 2. Let $c \in \mathbb{R}_{\mathcal{F}}$ and $g(0, a) \rightarrow \mathbb{R}$, if g is q -differentiable on $(0, a)$, then the function $F(0, a) \rightarrow \mathbb{R}$ defined by $F(t) = c \cdot g(t), t \in (0, a)$. Then, F is q_i -differentiable, for $i = 1, 2$, and we have $F^{q(i)}(t) = c \cdot g^{q(i)}(t)$ for $i = 1, 2$.

$c = (1, 2, 3)$ and. $[c]^\alpha = [1 + \alpha, 3 - \alpha]$.

(i) Let $g(t) = e^t$, so $F(t) = (1, 2, 3)e^t$ and $T_q(e^t) = t^{1-q}e^t$, then it is easy to see that $T_{q_{(1)}} F(t) = (1, 2, 3)t^{1-q}e^t$, i.e.,

$$[F^{q_{(1)}}(t)]^\alpha = [(1 + \alpha)t^{1-q}e^t, (3 - \alpha)t^{1-q}e^t]. \quad (51)$$

- (ii) Let $g(t) = e^{-t}$, so $F(t) = (1, 2, 3)e^{-t}$ and $T_q(e^{-t}) = -t^{1-q}e^{-t}$, then it is easy to see that $T_{q(2)}F(t) = -(1, 2, 3)t^{1-q}e^{-t}$, i.e.,

$$[F^{q(2)}(t)]^\alpha = [(-3 + \alpha)t^{1-q}e^{-t}, (-1 - \alpha)t^{1-q}e^{-t}]. \quad (52)$$

5. Conclusion

We have investigated generalized fuzzy conformable fractional differentiability. The conformable q -differentiability introduced here is a very general differentiability concept, being also practically applicable, and we can calculate by the fuzzy conformable derivative of the product of two functions $(T_q(f \cdot g))$ because all fractional derivatives do not satisfy the known formula $T_q(f \cdot g) = T_q(f)g + fT_q(g)$.

The disadvantage of fuzzy generalized conformable differentiability of a function seems to be that a simple fuzzy differential equation $(y^{(q)} + y = 0, 0 < q \leq 1, y(0) = y_0 \in \mathbb{R}_{\mathcal{F}})$ has not got a unique solution, so it may have several solutions. The advantage of the existence of these solutions is that we can choose the solution that reflects better the behaviour of the modelled real-world system.

For further research we propose the study for fuzzy fractional differential equations, by using the generalized conformable differentiability concept. In addition, we propose to extend the results of the present paper and to combine them with the results in [15] for fuzzy fractional differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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