Research Article

μ-Fuzzy Filters in Distributive Lattices

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1. Introduction

In 1970, Mandelker [1] introduced the concept of relative annihilators as a natural generalization of relative pseudo-complement and he characterized distributive lattices with the help of these annihilators. The concept of coannihilators and μ-filters in a distributive lattice with greatest element “1” was introduced by Rao and Badawy [2] and they characterized μ-filters in terms of coannihilators. For a filter \( F \) in \( L \), \( \mu(F) = \{(x)^{+} : x \in F\} \) is an ideal in the set \( A^{+}(L) \) of all coannihilators, and conversely \( \mu(I) = \{x \in L: (x)^{+} \in I\} \) is a filter in \( L \) when \( I \) is any ideal in \( A^{+}(L) \). A filter \( F \) of \( L \) is called a μ-filter if \( \mu(F) = F \).

In 1965, Zadeh [3] mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a nonempty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate, or useful notions in describing the real-life problems, because every object encountered in this real physical world carries some degree of fuzziness. A lot of work on fuzzy sets has come into being with many applications to various fields such as computer science, artificial intelligence, expert systems, control systems, decision-making, medical diagnosis, management science, operations research, pattern recognition, neural network, and others (see [4–7]).

In 1971, Rosenfeld used the notion of a fuzzy subset of a set to introduce the concept of a fuzzy subgroup of a group [8]. Rosenfeld’s paper inspired the development of fuzzy abstract algebra. Since then, several authors have developed interesting results on fuzzy subgroups (see [9–17]), fuzzy ideals of rings (see [16, 18–21]), and fuzzy ideals of lattices (see [22–28]).

Alaba and Norahun [29] studied the concept of α-fuzzy ideals of a distributive lattice in terms of annulates. They also studied the space of prime α-fuzzy ideals of a distributive lattice. In this paper, we introduce the dual of the concept of α-fuzzy ideals which is called μ-fuzzy filters in a distributive lattice with greatest element “1.” We study the special class of fuzzy filters called μ-fuzzy filters. We prove that the set of all μ-fuzzy filters of a distributive lattice forms a complete distributive lattice isomorphic to the set of all fuzzy ideals of \( A^{+}(L) \). We also show that there is a one-to-one correspondence between the class of prime μ-fuzzy filters of \( L \) and the set of all prime ideals of \( A^{+}(L) \). We prove that every μ-fuzzy filter is the intersection of all prime μ-fuzzy filters containing it. Moreover, we study the space of all prime μ-fuzzy filters in a distributive lattice. The set of prime μ-fuzzy filters of \( L \) is denoted by \( X_{\mu} \). For a μ-fuzzy filter \( \theta \) of \( L \), open subset of \( X_{\mu} \) is of the form \( X(\theta) = \{\eta \in X_{\mu} : \theta \nsubseteq \eta\} \).
and \( V(\theta) = \{ \eta \in X^\beta : \theta \subseteq \eta \} \) is a closed set. We also show that the set of all open sets of the form \( X(x_{\beta}) = \{ \eta \in X^\beta : x_{\beta} \not\in \eta, x \in L, \beta \in [0, 1] \} \) forms a basis for the open sets of \( X^\beta \). The set of all \( \mu \)-fuzzy filters of \( L \) is isomorphic with the set of all open sets in \( X^\mu \).

## 2. Preliminaries

We refer to Birkhoff [30] for the elementary properties of lattices.

**Definition 1** (see [2]). For any set \( S \) of a lattice \( L \), define \( S^\beta \) as follows:

\[
S^\beta = \{ x \in L : s \vee x = 1, \text{for all } s \in S \}. \tag{1}
\]

Here, \( S^\beta \) is called the coannihilator of \( S \). If \( S = \{ x \} \), we write \( (x)^\beta \) instead of \((\{x\})^\beta \). Then, clearly \( L^\beta = \{ 1 \} \) and \((1)^\beta = L \). For any subset \( S \) of a lattice \( L \), it is clear that \( S^\beta \) is a filter in \( L \).

**Lemma 1** (see [2]). For any \( x, y \in L \), the following conditions hold.

1. \( x \leq y \implies (x)^\beta \subseteq (y)^\beta \)
2. \( (x \wedge y)^\beta = (x)^\beta \cap (y)^\beta \)
3. \( (x \vee y)^\beta = (x)^\beta \cup (y)^\beta \)
4. \( (x)^\beta = L \) if and only if \( x = 1 \)

The set of all coannihilators denotes \( A^+(L) \). Each coannihilator is a coannihilator and filter, and hence, for two coannihilators \( (x)^\beta \) and \((y)^\beta \), their supremum and infimum in \( A^+(L) \) are

\[
(a \wedge b)^\beta = (a)^\beta \cap (b)^\beta,
\]

\[
(a \vee b)^\beta = (a)^\beta \cup (b)^\beta \quad \text{a}, \tag{2}
\]

respectively.

In a distributive lattice \( L \) with 1, the set of all coannihilators \( A^+(L) \) of \( L \) is a lattice \( (A^+(L), \wedge, \vee) \) and a sublattice of the Boolean algebra of coannihilator filters of \( L \).

For a filter \( F \) in \( L \),

\[
\mu(F) = \{ (x)^\beta : x \in F \} \tag{3}
\]

is an ideal in \( A^+(L) \) and the set

\[
\mu(F) = \{ x \in L : (x)^\beta \subseteq F \}
\]

is a filter of \( L \) when \( F \) is an ideal in \( A^+(L) \). A filter \( F \) of \( L \) is called a \( \mu \)-filter if \( \mu(F) = F \).

**Definition 2** (see [3]). Let \( X \) be any nonempty set. A mapping \( \mu : X \to [0, 1] \) is called a fuzzy subset of \( X \).

The unit interval \([0, 1]\) together with the operations \( \min \) and \( \max \) form a complete lattice satisfying the infinite meet distributive law. We often write \( \wedge \) for minimum or infimum and \( \vee \) for maximum or supremum. That is, for all \( \alpha, \beta \in [0, 1] \), we have \( \alpha \wedge \beta = \min \{ \alpha, \beta \} \) and \( \alpha \vee \beta = \max \{ \alpha, \beta \} \).

The characteristic function of any set \( A \) is defined as

\[
\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases} \tag{5}
\]

**Definition 3** (see [8]). Let \( \mu \) and \( \theta \) be fuzzy subsets of a set \( A \). Define the fuzzy subsets \( \mu \cup \theta \) and \( \mu \cap \theta \) of \( A \) as follows: for each \( x \in A \),

\[
(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \quad \text{and} \quad (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x). \tag{6}
\]

Then, \( \mu \cup \theta \) and \( \mu \cap \theta \) are called the union and intersection of \( \mu \) and \( \theta \), respectively.

For any collection \( \{ \mu_i : i \in I \} \) of fuzzy subsets of \( X \), where \( I \) is a nonempty index set, and the least upper bound \( \bigvee_{i \in I} \mu_i \) and the greatest lower bound \( \bigwedge_{i \in I} \mu_i \) of the \( \mu_i \)'s are given for each \( x \in X \),

\[
\left( \bigvee_{i \in I} \mu_i \right)(x) = \bigvee_{i \in I} \mu_i(x) \quad \text{and} \quad \left( \bigwedge_{i \in I} \mu_i \right)(x) = \bigwedge_{i \in I} \mu_i(x), \tag{7}
\]

respectively.

For each \( t \in [0, 1] \), the set

\[
\mu_t = \{ x \in A : \mu(x) \geq t \} \tag{8}
\]

is called the level subset of \( \mu \) at \( t \) [3].

**Definition 4** (see [27]). A fuzzy subset \( \mu \) of a lattice \( L \) is called a fuzzy ideal of \( L \) if, for all \( x, y \in L \), the following conditions are satisfied:

1. \( \mu(0) = 1 \)
2. \( \mu(x \vee y) \geq \mu(x) \wedge \mu(y) \)
3. \( \mu(x \wedge y) \geq \mu(x) \vee \mu(y) \)

**Definition 5** (see [27]). A fuzzy subset \( \mu \) of a lattice \( L \) is called a fuzzy filter of \( L \) if, for all \( x, y \in L \), the following condition is satisfied:

1. \( \mu(1) = 1 \)
2. \( \mu(x \vee y) \geq \mu(x) \vee \mu(y) \)
3. \( \mu(x \wedge y) \geq \mu(x) \wedge \mu(y) \)

In [27], Swamy and Raju observed the following:

1. A fuzzy subset \( \mu \) of a lattice \( L \) is a fuzzy ideal of \( L \) if and only if

\[
\mu(0) = 1 \quad \text{and} \quad \mu(x \vee y) = \mu(x) \wedge \mu(y), \quad \text{for all } x, y \in L. \tag{9}
\]

2. A fuzzy subset \( \mu \) of a lattice \( L \) is a fuzzy filter of \( L \) if and only if

\[
\mu(1) = 1 \quad \text{and} \quad \mu(x \wedge y) = \mu(x) \wedge \mu(y), \quad \text{for all } x, y \in L. \tag{10}
\]

Let \( \mu \) be a fuzzy subset of a lattice \( L \). The smallest fuzzy filter of \( L \) containing \( \mu \) is called a fuzzy filter of \( L \) induced by \( \mu \) and denoted by \( [\mu] \) and
Lemma 2 (see [23]). For any two fuzzy subsets \( \mu \) and \( \theta \) of a distributive lattice \( L \), we have

\[
(\mu \cdot \theta) = (\mu) \wedge (\theta).
\]  

The above result works dually.

For any two fuzzy subsets \( \mu \) and \( \theta \) of a distributive lattice \( L \), we have

\[
[\mu + \theta] = [\mu] \wedge [\theta].
\]  

The binary operations “+” and “.” on the set of all fuzzy subsets of a distributive lattice \( L \) are as follows:

\[
(\mu + \theta)(x) = \text{Sup}[\mu(y) \wedge \theta(z); \quad y, z \in L, y \vee z = x] \quad \text{and} \quad (\mu \cdot \theta)(x) = \text{Sup}[\mu(y) \wedge \theta(z); \quad y, z \in L, y \wedge z = x].
\]

(14)

If \( \mu \) and \( \theta \) are fuzzy ideals of \( L \), then \( \mu \cdot \theta = \mu \wedge \theta = \mu \cap \theta \) and \( \mu + \theta = \mu \vee \theta \).

If \( \mu \) and \( \theta \) are fuzzy filters of \( L \), then \( \mu + \theta = \mu \wedge \theta \) and \( \mu \cdot \theta = \mu \vee \theta \).

The set of all fuzzy filters of \( L \) is denoted by \( \text{FF}(L) \).

\[
\mu(\theta)((x)^+) \cap \mu(\theta)((y)^+) = \text{Sup}[\theta(a): \quad (a)^+ = (x)^+, \quad a \in L] \\
\wedge \text{Sup}[\theta(b): \quad (b)^+ = (y)^+, \quad b \in L] \\
= \text{Sup}[\theta(a) \wedge \theta(b): \quad (a)^+ \vee (b)^+ = (x)^+ \vee (y)^+] \\
\leq \text{Sup}[\theta(a) \land \theta(b): \quad (a\land b)^+ = (x\land y)^+] \\
\leq \text{Sup}[\theta(c): \quad (c)^+ = (x\lor y)^+] = \mu(\theta)((x\lor y)^+). 
\]

(16)

Thus,

\[
\mu(\theta)((x)^+ \vee (y)^+) \geq \mu(\theta)((x)^+) \wedge \mu(\theta)((y)^+).
\]

On the other hand,

\[
\mu(\theta)((x)^+) = \text{Sup}[\theta(a): \quad (a)^+ = (x)^+] \\
\leq \text{Sup}[\theta(a \lor y): \quad (a \lor y)^+ = (x)^+ \lor (y)^+] \\
\leq \text{Sup}[\theta(c): \quad (c)^+ = (x \lor y)^+] \\
= \mu(\theta)((x \lor y)^+). 
\]

(17)

Similarly, \( \mu(\theta)((y)^+) \leq \mu(\theta)((x)^+ \land (y)^+) \). So,

\[
\mu(\theta)((x)^+ \land (y)^+) \geq \mu(\theta)((x)^+) \lor \mu(\theta)((y)^+). 
\]  

(18)

Hence, \( \mu(\theta) \) is a fuzzy ideal of \( A^+(L) \).

Lemma 3. Let \( \eta \) be a fuzzy ideal of \( A^+(L) \). Then, the fuzzy subset \( \mu(\eta) \) of \( L \) defined as \( \mu(\eta)(x) = \eta((x)^+) \) is a fuzzy filter of \( L \).

3. \( \mu \)-Fuzzy Filters

In this section, we introduce the concept of \( \mu \)-fuzzy filters in a distributive lattice with greatest element “1.” We study some basic properties of the class of \( \mu \)-fuzzy filters. We prove that the class of \( \mu \)-fuzzy filters forms a complete distributive lattice isomorphic to the class of fuzzy filters of \( A^+(L) \). We also show that there is a one-to-one correspondence between the set of all prime \( \mu \)-fuzzy filters of \( L \) and prime fuzzy ideals of \( A^+(L) \). Finally, we observe that every \( \mu \)-fuzzy filter is the intersection of all prime \( \mu \)-fuzzy filters containing it.

Throughout the rest of this paper, \( L \) stands for the distributive lattice with greatest element “1” unless otherwise mentioned.

\[\text{Theorem 1.} \quad \text{Let } \eta \text{ be a fuzzy filter of } L. \text{ Then, the fuzzy subset } \mu(\eta) \text{ of } A^+(L) \text{ defined by }\]

\[
\mu(\eta)((x)^+) = \text{Sup}[\eta(y): \quad (y)^+ = (x)^+, \quad y \in L] 
\]

is a fuzzy ideal of \( A^+(L) \).

Proof. Let \( \eta \) be a fuzzy filter of \( L \). Clearly, \( \mu(\eta)((1)^+) = 1 \).

For any \((x)^+, (y)^+ \in A^+(L)\),

\[
\mu(\eta)((x)^+ \lor (y)^+) \geq \mu(\eta)((x)^+) \land \mu(\eta)((y)^+). 
\]

Proof. Let \( \eta \) be a fuzzy ideal of \( A^+(L) \). Since \((1)^+\) is the smallest element of \( A^+(L) \), we get \( \mu(\eta)(1) = 1 \). For any \( x, y \in L \),

\[
\mu(\eta)(x \lor y) = \eta((x)^+) \land \eta((y)^+) = \mu(\eta)(x) \land \mu(\eta)(y). 
\]

(19)

Thus, \( \mu(\eta) \) is a fuzzy filter of \( L \).

Lemma 4. If \( \theta \) and \( \eta \) are fuzzy filters of \( L \), then \( \theta \subseteq \eta \) implies \( \mu(\theta) \leq \mu(\eta) \).

Lemma 5. If \( \theta \), \( \eta \) are fuzzy ideals of \( A^+(L) \), then \( \theta \subseteq \eta \) implies \( \mu(\theta) \leq \mu(\eta) \).

\[\text{Theorem 2.} \text{ The set } \text{FI}(A^+(L)) \text{ of all fuzzy ideals of } A^+(L) \text{ forms a complete distributive lattice, where the infimum and supremum of any family } \{\theta_j: \quad j \in J\} \text{ of fuzzy ideals are given by } \]

\[
\text{inf} \{\theta_j: \quad j \in J\} = \text{inf} \{\theta_j: \quad j \in J\} \text{ and } \sup \{\theta_j: \quad j \in J\} = \sup \{\theta_j: \quad j \in J\}. 
\]
Corollary 1. So, we proceed to show the following:

\[ \mu(\eta \cap \theta) \leq \mu(\eta) \cap \mu(\theta) \]
\[ \mu(\eta) \cap \mu(\theta) = \mu(\eta \cap \theta) \]
\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]
\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]
\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]

Proof. Let \( \eta \) and \( \theta \) be two fuzzy filters of \( L \). Then, by Lemma 4, we have \( \mu(\eta \cap \theta) \leq \mu(\eta) \cap \mu(\theta) \) and \( \mu(\eta) \cap \mu(\theta) \leq \mu(\eta \cap \theta) \). For any \( (x)^+ \in A^+(L) \),

\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]
\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]
\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]

Thus, \( \mu(\eta) \cap \mu(\theta) \leq \mu(\eta \cap \theta) \). So, \( \mu(\eta) \cap \mu(\theta) = \mu(\eta \cap \theta) \).

Then, \( \mu(\eta \cap \theta) \leq \mu(\eta) \cap \mu(\theta) \). Thus, \( \mu(\eta \cap \theta) = \mu(\eta) \cap \mu(\theta) \). So, \( \mu \) is a homomorphism.

Corollary 1. For any two fuzzy filters \( \theta \) and \( \eta \) of \( L \), we have:

\[ \widetilde{\mu}(\eta \cap \theta) = \widetilde{\mu}(\eta) \cap \widetilde{\mu}(\theta) \]

Proof. For any \( x \in L \), \( \widetilde{\mu}(\eta \cap \theta)(x) = \mu(\eta \cap \theta)(x)^+ \)

Since \( \mu \) is a homomorphism, we have:

\[ \mu(\eta \cap \theta) = \mu(\eta) \cap \mu(\theta) \]

Lemma 6. For any fuzzy ideal \( \theta \) of \( A^+(L) \), \( \widetilde{\mu}(\theta) = \theta \).

Proof. Since \( \theta \) is a fuzzy ideal of \( A^+(L) \), by Lemma 3, \( \widetilde{\mu}(\theta) \) is a fuzzy filter of \( L \) and \( \widetilde{\mu}(\theta) \) is a fuzzy ideal of \( A^+(L) \). Now, we proceed to show \( \mu(\theta) = \theta \).

\[ \mu(\theta)(x)^+ = \sup(\theta(a)^+, (x)^+)^+ \]
\[ \mu(\theta)(x)^+ = \sup(\theta(a)^+, (x)^+)^+ \]

Thus, \( \mu(\theta) = \theta \).

Lemma 7. For any fuzzy filter \( \theta \) of \( L \), the map \( \theta \rightarrow \widetilde{\mu}(\theta) \) is a closure operator on \( \mathcal{F}(L) \). That is,

\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]
\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]
\[ \mu(\eta)(x)^+ \leq \sup(\eta(a)^+, \mu(\theta)(x)^+) \]

Thus, \( \mu(\eta) \cap \mu(\theta) \leq \mu(\eta \cap \theta) \). So, \( \mu(\eta) \cap \mu(\theta) = \mu(\eta \cap \theta) \).

Then, \( \theta \) is a fuzzy filter and \( \theta \leq \mu(\theta) \). To prove our claim, let \( t = \mu(\theta)(x) \). Then, for each \( \epsilon > 0 \), there is \( a \in L \) such that \( (x)^+ \leq (y)^+ \). Based on this, we have:

\[ (a)^+ = (x)^+ \]
\[ (b)^+ = (y)^+ \]

And if \( (a \land b)^+ = (x)^+ \), then, based on this, we have:

\[ (a \land b)^+ = (x)^+ \]

Therefore, we have:

\[ (a \land b)^+ = (x)^+ \]

And if \( (a \land b)^+ = (x)^+ \), then, based on this, we have:

\[ (a \land b)^+ = (x)^+ \]

Therefore, we have:

\[ (a \land b)^+ = (x)^+ \]

Thus, \( (a \land b)^+ = (x)^+ \) and \( (b \land c)^+ = (y)^+ \), we get:

\[ (a \land b)^+ = (x)^+ \]
(a)^+ = (x)^+ and \( \theta(a) > t - \varepsilon \), which implies \( a \in \theta_{t-\varepsilon} \). (a)^+ = (x)^+ and \( x \in \mu_{\theta_{t-\varepsilon}} = \theta_{t-\varepsilon} \). This shows that \( x \in \cap_{\varepsilon>0} \theta_{t-\varepsilon} = \theta \). Thus, \( \bar{\mu}(\theta) \subseteq \theta \). So, \( \theta \) is a \( \mu \)-fuzzy filter of \( L \).

**Corollary 2.** For a nonempty subset \( F \) of \( L \), \( F \) is a \( \mu \)-filter if and only if \( x_F \) is a \( \mu \)-fuzzy filter of \( L \).

**Theorem 5.** Let \( \theta \) be a fuzzy filter of \( L \). Then, \( \theta \) is a \( \mu \)-fuzzy filter if and only if, for each \( x, y \in L \), \((x)^+ = (y)^+\) implies \( \theta(x) = \theta(y) \).

**Proof.** Let \( \theta \) be a \( \mu \)-fuzzy filter of \( L \) and \( x, y \in L \) such that \((x)^+ = (y)^+\). Then,

\[
\theta(x) = \text{Sup}\{\theta(a): (a)^+ = (x)^+, a \in L\} = \text{Sup}\{\theta(a): (a)^+ = (y)^+, a \in L\} = \bar{\mu}(\theta)(y) = \theta(y).
\]

(26)

Conversely, suppose that for each \( x, y \in L \), \((x)^+ = (y)^+\) implies \( \theta(x) = \theta(y) \). For any \( x \in L \),

\[
\bar{\mu}(\theta)(x) = \text{Sup}\{\theta(a): (a)^+ = (x)^+, a \in L\} = \theta(x).
\]

(27)

Thus, \( \theta \) is a \( \mu \)-fuzzy filter of \( L \).

**Theorem 6.** A fuzzy filter \( \theta \) of \( L \) is a \( \mu \)-fuzzy filter if and only if

\[
\wedge_{a \in (x)^+} \theta(a) \geq \theta(x), \text{ for all } x \in L.
\]

(28)

**Proof.** Suppose a fuzzy filter \( \theta \) of \( L \) is a \( \mu \)-fuzzy filter. Then, by Theorem 4, every level subset is a \( \mu \)-filter of \( L \). Let \( x \in L \) such that \( \theta(x) = t \). Since \( \theta \) is a \( \mu \)-filter of \( L \), then \((x)^+ \subseteq \theta \), which implies \( a \in \theta \) for all \( a \in (x)^+ \). Thus, \( \theta(a) \geq \theta(x) \) for each \( a \in (x)^+ \). So,

\[
\wedge_{a \in (x)^+} \theta(a) \geq \theta(x), \text{ for all } x \in L.
\]

(29)

Suppose conversely that the condition holds. To prove \( \theta \) is a \( \mu \)-fuzzy filter, it suffices to show that \( \bar{\mu}(\theta) \subseteq \theta \). For any \( x \in L \), \( \bar{\mu}(\theta)(x) = \text{Sup}\{\text{Sup}(\theta(b): (b)^+ = (x)^+)\} \). If \((b)^+ = (x)^+\), then \( x \in (b)^+ \). By the assumption, \( \theta(x) \geq \theta(b) \) for each \( (b)^+ = (x)^+ \). This shows that \( \theta(x) \) is an upper bound of \( \{\theta(b): (b)^+ = (x)^+\} \). Thus, \( \bar{\mu}(\theta)(x) \leq \theta(x) \) for all \( x \in L \). So, \( \mu(\theta) \subseteq \theta \). Hence, \( \theta \) is a \( \mu \)-fuzzy filter of \( L \).

Let us denote the set of all \( \mu \)-fuzzy filters of \( L \) by \( FF_\mu(L) \).

**Theorem 7.** The class \( (FF_\mu(L), \wedge, \vee) \) of all \( \mu \)-fuzzy filters of \( L \) forms a complete distributive lattice with respect to set inclusion.

**Proof.** Clearly, \( (FF_\mu(L), \subseteq) \) is a partially ordered set. For \( \eta, \theta \in FF_\mu(L) \), define

\[
\eta \wedge \theta = \eta \cap \theta, \quad \eta \vee \theta = \bar{\mu}(\eta \vee \theta).
\]

(30)

Then, clearly \( \eta \wedge \theta, \eta \vee \theta \in FF_\mu(L) \). We need to show \( \eta \vee \theta \) is the least upper bound of \( \{\eta, \theta\} \). Since \( \eta \subseteq \eta \cup \theta \subseteq \eta \vee \theta \), \( \eta \vee \theta \) is an upper bound of \( \{\mu, \theta\} \). Let \( \lambda \) be any upper bound for \( \eta, \theta \) in \( FF_\mu(L) \). Then, \( \eta \vee \theta \subseteq \lambda \), which implies that \( \bar{\mu}(\eta \vee \theta) \leq \bar{\mu}(\lambda) = \lambda \). Therefore, \( \bar{\mu}(\eta \vee \theta) \) is the supremum of both \( \{\eta, \theta\} \) in \( FF_\mu(L) \).

Hence, \( (FF_\mu(L), \wedge, \vee) \) is a lattice.

We now prove the distributivity. Let \( \eta, \theta, \lambda \in FF_\mu(L) \). Then,

\[
\eta \vee (\theta \cap \lambda) = \bar{\mu}(\eta \vee (\theta \cap \lambda)) = \bar{\mu}(\eta \vee (\theta \cap (\eta \vee \lambda))) = \bar{\mu}(\eta \vee \theta) \cap \bar{\mu}(\eta \vee \lambda) = (\eta \vee \theta) \cap (\eta \vee \lambda).
\]

(31)

Thus, \( FF_\mu(L) \) is a distributive lattice.

Now, we proceed to show the completeness. Since \( \{1\} \) and \( L \) are \( \mu \)-filters, \( \chi[1] \) and \( \chi[\eta] \) are least and greatest elements of \( FF_\mu(L) \), respectively. Let \( \{\theta_i : i \in I\} \subseteq FF_\mu(L) \). Then, \( \cap_{i \in I} \theta_i \) is a \( \mu \)-fuzzy filter of \( L \) and \( \cap_{i \in I} \theta_i \subseteq \mu(\cap_{i \in I} \theta_i) \).

\[
\eta \in I \implies \bar{\mu}(\cap_{i \in I} \theta_i) \subseteq \theta_i.
\]

(32)

Thus, \( \bar{\mu}(\cap_{i \in I} \theta_i) = \cap_{i \in I} \theta_i \). So, \( FF_\mu(L), \wedge, \vee \) is a complete distributive lattice.

**Theorem 8.** The set \( FF_\mu(L) \) is isomorphic to the lattice of fuzzy ideals of \( A^*(L) \).

**Proof.** Define

\[
f: FF_\mu(L) \to FI(A^*(L)),
\]

\[
f(\eta) = \mu(\eta), \quad \forall \theta \in FF_\mu(L).
\]

(33)

Let \( \eta, \theta \in FF_\mu(L) \) and \( f(\eta) = f(\theta) \). Then, \( \mu(\eta) = \mu(\theta) \). Thus, \( \bar{\mu}(\eta) = \bar{\mu}(\theta) \). So, \( \eta = \theta \). Hence, \( f \) is one to one.

Let \( \lambda \in FI(A^*(L)) \). Then, by Lemma 3, \( \bar{\mu}(\lambda) \) is a fuzzy filter of \( L \). Now, we proceed to show that \( \bar{\mu}(\lambda) \) is a \( \mu \)-fuzzy filter of \( L \). Let \( x \in L \). Then, \( \bar{\mu}(\lambda)(x) = \bar{\mu}(\lambda)(x)^+ \). Thus, by Lemma 6, we get that \( \bar{\mu}(\lambda)(x)^+ = \mu(\eta)(x) \). So, \( \bar{\mu}(\lambda) = \bar{\mu}(\lambda) \). Thus, for each \( \lambda \in FI(A^*(L)) \), \( f(\bar{\mu}(\lambda)) = \lambda \). Therefore, \( f \) is onto.

Now, for any \( \eta, \theta \in FI_\mu(L) \), \( f(\eta \vee \theta) = f(\bar{\mu}(\eta \vee \theta)) = \mu(\eta \vee \theta) = \mu(\eta) \vee \mu(\theta) = f(\eta) \vee f(\theta) \). Similarly, \( f(\eta \wedge \theta) = f(\eta) \cap f(\theta) \). Therefore, \( f \) is an isomorphism of \( FF_\mu(L) \) onto the lattice of fuzzy filters of \( A^*(L) \).

**Theorem 9.** The following are equivalent for each non-constant \( \mu \)-fuzzy filter \( \lambda \) of \( L \).

1. For all \( \eta, \theta \in FF(L) \), \( \theta \cap \eta \leq \lambda \) or \( \theta \leq \lambda \) or \( \eta \leq \lambda \).

(34)

2. For any fuzzy points \( x_\eta \) and \( y_\theta \) of \( L \),
\[x_y + y_β ≤ λ \Rightarrow x_y ≤ λ \text{ or } y_β ≤ λ.\]  
(35)

(3) For all \(θ, η \in FF_μ(L),\)

\[θ ∩ η ≤ λ \Rightarrow θ ≤ λ \text{ or } η ≤ λ.\]  
(36)

Proof

1⇒2. Let \(x, y ∈ L\) such that \(x_y + y_β ≤ λ\). Then, \([x_y] ∩ [y_β] ≤ λ\). Since \(L\) is a distributive lattice, dual of Lemma 2, we have \([x_y] ∩ [y_β] ≤ λ\). Since \([x_y]\) and \([y_β]\) are fuzzy filters of \(L\), by the assumption, \([x_y] \subseteq λ\) or \([y_β] \subseteq λ\). This shows that \(x_y ≤ λ\) or \(y_β ≤ λ\).

2⇒3. Let \(θ, η ∈ FF_μ(L)\) such that \(θ ∩ η ≤ λ\). Now, we need to show \(θ ≤ λ\) or \(η ≤ λ\). Suppose not. Then, \(θ ≥ λ\) and \(η ≥ λ\), which implies there exist \(x, y ∈ L\) such that \(θ(x) > λ(x)\) and \(η(y) > λ(y)\). Put \(θ(x) = γ\) and \(η(y) = β\). Then, \(x_γ θ\) and \(y_β θ\). Since \(x_y + y_β ≤ θ ∩ η\), we have \(x_y \subseteq λ\) or \(y_β \subseteq λ\). By the assumption, we get that \(x_y ≤ λ\) or \(y_β ≤ λ\), which is a contradiction. Thus, \(θ ≤ λ\) or \(η ≤ λ\).

3⇒1. Suppose \(θ, η ∈ FF(L)\) such that \(θ ∩ η ≤ λ\). Then, by Corollary 1 we have \(μ(θ) \cap μ(η) ≤ λ\). Since \(μ(θ)\) and \(μ(η)\) are fuzzy filters, by the assumption, we get that \(μ(θ) ≤ λ\) or \(μ(η) ≤ λ\), which implies \(θ ≤ λ\) or \(η ≤ λ\).

Definition 7. By a prime \(μ\)-fuzzy filter, we mean a nonconstant \(μ\)-fuzzy filter of \(L\) satisfying (1) and hence all of the conditions of Theorem 9.

We have proved in Theorem 8 that there is an order isomorphism between the class of fuzzy filters and the set of fuzzy ideals of \(A^*(L)\). Now, we show that there is an isomorphism between the prime \(μ\)-fuzzy filters and the prime fuzzy ideals of the lattice of coannihilators.

Theorem 10. There is an isomorphism between the prime \(μ\)-fuzzy filters and the prime fuzzy ideals of the lattice of coannihilators.

Proof. By Theorem 8, the map \(f\) is an isomorphism from \(FF_μ(L)\) into \(FI(A^*(L))\). Let \(σ\) be a prime \(μ\)-fuzzy filter of \(L\). Then, \(μ(σ) ∈ FI(A^*(L))\). Now, we prove \(μ(σ)\) is a prime fuzzy ideal of \(FI(A^*(L))\). Let \(θ, η \in FI(A^*(L))\) such that \(θ ∩ η ≤ μ(σ)\). Since \(f\) is onto, there exist \(λ, μ ∈ FF_μ(L)\) such that \(f(μ) = θ\) and \(f(λ) = η\). Thus, \(μ(λ) ∩ μ(η) ≤ μ(σ)\). Since \(μ\) is an isotope, we have \(μ(λ ∩ μ(η)) ≤ μ(λ)\). Thus, \(λ ∩ μ(η) ≤ μ(σ)\).

Since \(σ\) is a prime fuzzy filter, either \(λ = η\) or \(η = \sigma\). This shows that either \(μ(λ) ≤ σ\) or \(μ(η) ≤ σ\). Thus, \(θ ≤ μ(σ)\) or \(η ≤ μ(σ)\). Hence, \(μ(σ)\) is a prime fuzzy ideal of \(A^*(L)\).

Conversely, suppose that \(θ\) is a prime fuzzy ideal in \(A^*(L)\). Since \(f\) is onto, there exists a \(μ\)-fuzzy filter \(σ\) in \(FF_μ(L)\) such that \(θ = μ(σ)\). Let \(η, λ ∈ FF_μ(L)\) such that \(η ∩ λ = σ\). Since \(μ\) is an isotope, we get \(μ(η ∩ λ) ≤ μ(σ) = θ\). Thus, \(μ(η) ∩ μ(λ) ≤ μ(σ)\). Since \(μ(σ)\) is a prime fuzzy ideal of \(A^*(L)\), either \(μ(η) ≤ μ(σ)\) or \(μ(λ) ≤ μ(σ)\). This implies \(η ≤ μ(σ)\) or \(λ ≤ μ(σ)\). Since \(σ\) is a \(μ\)-fuzzy filter, we get \(η ≤ σ\) or \(λ ≤ σ\). Thus, \(σ\) is prime fuzzy filter in \(FF_μ(L)\). So, the prime \(μ\)-fuzzy filters correspond to prime fuzzy ideals of \(A^*(L)\).

Theorem 11. Let \(θ\) be a \(μ\)-fuzzy filter of \(L\) and \(η\) be a fuzzy ideal of \(L\) such that \(θ ∩ η ≤ α\), \(α ∈ [0, 1)\). Then, there exists a prime \(μ\)-fuzzy filter \(λ\) of \(L\) such that \(θ ≤ λ\) and \(λ ∩ η ≤ α\).

Proof. Put \(P = \{a ∈ FF_μ(L) : a ≤ η \text{ and } η ∩ a ≤ α\}\). Since \(θ ∈ P\), \(P\) is nonempty, and it forms a poset together with the inclusion ordering of fuzzy sets. Let \(α = \{θ_i\}_{i∈I}\) be any chain in \(P\). Then, clearly \(∪_{i∈I}θ_i\) is a \(μ\)-fuzzy filter. Since \(θ_i ∩ η ≤ α\) for each \(i ∈ I\), we get that \((∪_{i∈I}θ_i) ∩ η ≤ a\). Thus, \(∪_{i∈I}θ_i ∈ α\).

By applying Zorn's lemma, we get a maximal element, say \(σ ∈ P\); that is, \(σ\) is a \(μ\)-fuzzy filter of \(L\) such that \(θ ≤ σ\) and \(σ ∩ η ≤ a\).

Now, we proceed to show \(σ\) is a prime fuzzy filter. Assume that \(σ\) is not prime fuzzy filter. Let \(γ_1 ∩ γ_2 ⊆ σ\) such that \(γ_1 ∩ η ≤ a\) and \(γ_2 ∩ η ≤ a\), and then both \(γ_1\) and \(γ_2\) are \(μ\)-fuzzy filters of \(L\) properly containing \(σ\). Since \(σ\) is maximal in \(P\), we get \(σ_1, σ_2 ∉ P\). Thus, \(σ_1 ∩ η ≤ a\) and \(σ_2 ∩ η ≤ a\). This implies there exist \(x, y ∈ L\) such that \((σ_1 ∩ η)(x) > a\) and \((σ_2 ∩ η)(y) > a\), which implies \((σ_1 ∩ η)(x)(x ∧ y) > a\) or \((μ(σ ∩ σ_1)(x ∧ y)) > a\) or \((x ∧ y) η(σ_1 ∩ η)(x ∧ y) > a\). This shows that \((θ ∩ η)(x ∧ y) > a\). This is a contradiction. Thus, \(σ\) is a prime fuzzy filter of \(L\).

Corollary 3. Let \(θ\) be a \(μ\)-fuzzy filter of \(L\), \(a ∈ L\) and \(a ∈ [0, 1)\). If \(θ(a) ≤ a\), then there exists a prime \(μ\)-fuzzy filter \(η\) of \(L\) such that \(θ ≤ η\) and \(η(α) ≤ α\).

Proof. Put \(P = \{a ∈ FF_μ(L) : η ≤ a\}\). Since \(θ ∈ P\), \(P\) is nonempty, and it forms a poset together with the inclusion ordering of fuzzy sets. Let \(α = \{θ_i\}_{i∈I}\) be any chain in \(P\). Clearly, \(∪_{i∈I}θ_i\) is a \(μ\)-fuzzy filter. Since \(θ_i ∩ η ≤ a\) for each \(i ∈ I\), \(a\) is an upper bound of \(θ_i\). Thus, \(∪_{i∈I}θ_i(α) ≤ a\). Hence, \(∪_{i∈I}θ_i ∉ P\). By applying Zorn's lemma, we get a maximal element, say \(σ ∈ P\); that is, \(σ\) is a \(μ\)-fuzzy filter of \(L\) such that \(θ ≤ σ\) and \(σ(a) ≤ a\).

Now, we proceed to show \(σ\) is a prime fuzzy filter. Assume that \(σ\) is not prime fuzzy filter. Let \(γ_1 ∩ γ_2 ⊆ σ\) and \(γ_1 ∩ η ≤ a\) and \(γ_2 ∩ η ≤ a\), then both \(γ_1\) and \(γ_2\) are \(μ\)-fuzzy filters of \(L\) properly containing \(σ\). Since \(σ\) is maximal in \(P\), we get \(σ_1, σ_2 ∉ P\). Thus, \(σ_1 ∩ η ≤ a\) and \(σ_2 ∩ η ≤ a\). Now, \((σ_1 ∩ η)(a) ∩ (σ_2 ∩ η)(a) = \(μ(σ_1 ∩ η)(a) ∩ (γ_1 ∨ γ_2) ∩ (γ_1 ∨ η)(a) = (σ_1 ∩ η)(a) = σ(a) > a\). This is a contradiction. Hence, \(σ\) is prime fuzzy filter.

Corollary 4. Any \(μ\)-fuzzy filter of \(L\) is the intersection of all prime \(μ\)-fuzzy filters containing it.

Proof. Let \(θ\) be a proper \(μ\)-fuzzy filter of \(L\). Consider the following.
\( \lambda = \cap \{ \eta : \eta \text{ is a prime } \mu - \text{fuzzy filter and } \theta \subseteq \eta \} \). 

(37)

Clearly, \( \theta \subseteq \lambda \). Assume that \( \lambda \nsubseteq \theta \). Then, there is \( a \in L \) such that \( \lambda(a) > \theta(a) \). Let \( \theta(a) = a \). Consider the set

\[
\mathcal{P} = \{ \eta \in \mathcal{F}_\mu(L) : \theta \subseteq \eta \text{ and } \eta(a) \leq a \} .
\]

(38)

By the above corollary, we can find a prime \( \mu \)-fuzzy filter \( \gamma \) of \( L \) such that \( \theta \subseteq \gamma \) and \( \gamma(a) \leq a \). This implies \( \lambda \nsubseteq \gamma \). This shows that \( \lambda \nsubseteq a \), which is a contradiction. Thus, \( \lambda \subseteq \theta \). So, \( \lambda = \theta \).

4. The Space of Prime \( \mu \)-Fuzzy Filters

In this section, we study the space of prime \( \mu \)-fuzzy filters of a distributive lattice and some properties of the space also.

Let \( X_\mu \) be the set of all prime \( \mu \)-fuzzy filters of a distributive lattice. Let \( V(\theta) = \{ \eta \in X_\mu : \theta \subseteq \eta \} \), where \( \theta \) is a fuzzy subset of \( L \) and \( X(\theta) = \{ \eta \in X_\mu : \theta \subseteq \eta \text{ and } \eta \notin V(\theta) \} \). We let \( \mu_* = \mu_1 \), i.e., \( \mu_* = \{ x \in L : \mu(x) = 1 \} \).

Lemma 8. For any fuzzy filters \( \lambda \) and \( \theta \) of \( L \), we have

1. \( \lambda \subseteq \theta \Rightarrow X(\lambda) \subseteq X(\theta) \)
2. \( X(\lambda \lor \theta) = X(\lambda) \cup X(\theta) \)
3. \( X(\lambda \land \theta) = X(\lambda) \cap X(\theta) \)

Proof.

\begin{enumerate}
  \item Let \( \lambda \subseteq \theta \) and \( \eta \in X(\lambda) \). Then, \( \lambda \theta \eta \) and \( \theta \eta \theta \). Thus, \( \eta \in X(\theta) \).
  \item Since \( \lambda, \theta \subseteq \lambda \lor \theta \), by (1), we have \( X(\lambda) \cup X(\theta) \subseteq X(\lambda \lor \theta) \). Now, we proceed to show the other inclusion; let \( \eta \in X(\lambda \lor \theta) \). Then, \( \lambda \lor \theta \eta \). This shows that either \( \lambda \theta \eta \) or \( \theta \eta \theta \). So, \( \eta \in X(\lambda) \cup X(\theta) \). Hence, \( X(\lambda \lor \theta) = X(\lambda) \cup X(\theta) \).
  \item By (1), we have \( X(\lambda \land \theta) \subseteq X(\lambda) \cap X(\theta) \). On the other hand, let \( \eta \in X(\lambda) \cap X(\theta) \). Then, \( \lambda \theta \eta \) and \( \theta \eta \theta \). Since \( \eta \) is a prime fuzzy filter, we get that \( \lambda \theta \eta \). This shows that \( \eta \in X(\lambda \land \theta) \). Thus, \( X(\lambda \land \theta) = X(\lambda) \cap X(\theta) \).
\end{enumerate}

Lemma 9. Let \( \theta \) be a fuzzy subset of \( L \). Then, \( X(\theta) = X(\{ \theta \}) \).

Proof. To prove our claim, it suffices to show \( X(\{ \theta \}) \subseteq X(\theta) \). Let \( \eta \in X(\{ \theta \}) \). Then, \( \theta \eta \). We need to show \( \theta \eta \theta \). Suppose not. Then, \( \theta \subseteq \eta \), which implies that \( \{ \theta \} \subseteq \eta \), which is a contradiction. Thus, \( \theta \eta \theta \). So, \( X(\theta) = X(\{ \theta \}) \).

Theorem 12. Let \( x, y \in L \) and \( \beta \in (0, 1] \). Then

1. \( X(x \land y) = X(x) \cup X(y) \)
2. \( X(x \lor y) = X(x) \cap X(y) \)
3. \( \bigcup_{x \in L, \beta \in (0, 1]} X(x) = X_\mu \)

Proof.

\begin{enumerate}
  \item If \( \lambda \in X(x) \cup X(y) \), then either \( x \theta \lambda \) or \( y \theta \lambda \). This shows that \( \beta \lambda(x) \) or \( \beta \lambda(y) \). Thus, \( \beta \lambda(x) \cap \lambda(y) = \lambda(x \land y) \). So, \( (x \land y) \theta \lambda \). Hence, \( \lambda \in X((x \land y) \theta) \).
  \item On the other hand, let \( \lambda \in X(x \land y) \). Then, \( \beta \lambda(x \land y) = \lambda(x) \land \lambda(y) \). This implies either \( x \theta \lambda \) or \( y \theta \lambda \). Thus, \( \lambda \in X(x) \cup X(y) \).
  \item If \( \lambda \in X(x) \cap X(y) \), then \( x \theta \lambda \) and \( y \theta \lambda \). This implies \( \beta \lambda(x) \) and \( \beta \lambda(y) \). This shows that \( x \land y \notin \lambda \). Since \( \lambda \) is prime fuzzy filter, card \( \lambda(x) = 0 \). Hence, \( x \land y \neq \lambda \). This implies \( \beta \lambda(x \land y) \). Thus, \( (x \land y) \theta \lambda \). Hence, \( X(x) \cap X(y) \subseteq X((x \land y) \theta) \).
  \item Conversely, let \( \lambda \in X((x \land y) \theta) \). Then, \( (x \land y) \theta \lambda \). Thus, \( \beta \lambda(x \land y) \). This implies \( \beta \lambda(x) \) and \( \beta \lambda(y) \). This shows \( x \theta \lambda \) and \( y \theta \lambda \). So, \( \lambda \in X(x) \cup X(y) \).
\end{enumerate}
Theorem 13. The collection $\mathcal{T} = \{X(\theta): \theta$ is a fuzzy filter of $L\}$ is a topology on $X_\mu$.

Proof. First, we define two fuzzy subsets $\eta_1, \eta_2$ of $L$ as follows: $\eta_1(x) = 0$ and $\eta_2(x) = 1$ for all $x \in L$. Then, $[\eta_1]$ and $[\eta_2]$ are fuzzy filters of $L$. Since $[\eta_1] \subseteq \lambda$, for all $\lambda \in X_\mu$, we get that $V([\eta_1]) = X_\mu$. This shows that $X([\eta_1]) = \phi$. Since each $\lambda \in X_\mu$ is nonconstant, $\eta_2 \not\subseteq \lambda$ for all $\lambda \in X_\mu$. So $X([\eta_2]) = X_\mu$. Hence, $\phi, X_\mu \in \mathcal{T}$.

Next, let $X(\theta_1), X(\theta_2) \in \mathcal{T}$. Then, by Lemma 8, we get that $X(\theta_1) \cap X(\theta_2) = X(\theta_1 \cap \theta_2)$. This shows that $\mathcal{T}$ is closed under finite intersection.

Now, we proceed to show that $\mathcal{T}$ is closed under arbitrary union. Let $\{\theta_i; i \in I\}$ be any family of fuzzy filters of $L$. Then, by Lemma 11 we have

$$\bigcap_{i \in I} V(\theta_i) = V\left(\bigcup_{i \in I} \theta_i\right),$$

which implies $\bigcup_{i \in I} X(\theta_i) = X\left(\bigcup_{i \in I} \theta_i\right)$. Thus, by Lemma 9, we get that

$$X\left(\bigcup_{i \in I} \theta_i\right) = X\left(\bigcap_{i \in I} \theta_i\right).$$

So, $\mathcal{T}$ is closed under arbitrary union. Therefore, $\mathcal{T}$ is a topology on $X_\mu$. The space $(X_\mu, \mathcal{T})$ will be called the space of prime $\mu$-fuzzy filters in $L$.

In the above theorem, we proved that the family of $X(\theta)$ is a topology on $X_\mu$. In the following result, we show that the set of all open sets of the form $X(x_\beta)$ is a basis for the topology on $X_\mu$.

Theorem 14. The collection $\mathcal{B} = \{X(x_\beta): x \in L, \beta \in (0, 1]\}$ forms a base for some topology $X_\mu$.

Proof. Let $X(\theta)$ be any open set in $X_\mu$ and $\lambda \in X(\theta)$. Then, $\theta \not\subseteq \lambda$ and there is $x \in L$ such that $\theta(x) > \lambda(x)$. Put $\theta(x) = \beta$; then $x_\beta \leq \theta$ and $\lambda \in X(x_\beta)$. To show $X(x_\beta) \subseteq X(\theta)$, let $\eta \in X(x_\beta)$. Then, $x_\beta \not\subseteq \eta$ and $\theta(x) > \eta(x)$. This shows that $\eta \in X(\theta)$. Thus, $\lambda \in X(x_\beta) \subseteq X(\theta)$. Hence, for any open set $X(\theta)$ in $X_\mu$, we can find $X(x_\beta) \in \mathcal{B}$ such that $X(x_\beta) \subseteq X(\theta)$. Therefore, $\mathcal{B}$ is a base for $\mathcal{T}$.

Theorem 15. The space $X_\mu$ is a $T_0$-space.

Proof. Take any two different elements $\eta$ and $\theta$ in $X_\mu$. Then, either $\eta \not\subseteq \theta$ or $\theta \not\subseteq \eta$. Without loss of generality, we can assume that $\eta \not\subseteq \theta$. Then, $\theta \in X(\eta)$ and $\eta \notin X(\eta)$. Thus, $X_\mu$ is a $T_0$-space.

Theorem 16. For any fuzzy filter $\eta$ of $L$, $X(\eta) = X(\widetilde{\mu_\eta}(\eta))$.

Proof. For any fuzzy filter $\eta$ of $L$, we have $\eta \subseteq \mu_\eta(\eta)$ and $X(\eta) \subseteq X(\widetilde{\mu_\eta}(\eta))$. Now we proceed to show the other inclusion; let $\theta \in X(\widetilde{\mu_\eta}(\eta))$. Then, $\mu_\eta(\eta) \not\subseteq \theta$. Suppose $\theta \notin X(\eta)$; then $\eta \subseteq \theta$. This implies $\mu_\eta(\eta) \subseteq \mu_\eta(\theta) = \theta$, which is impossible. Thus, $\theta \in X(\eta)$ and hence $X(\eta) = X(\widetilde{\mu_\eta}(\eta))$.

In the following result, we show that there is a one-to-one correspondence between the class of $\mu$-fuzzy filters and the lattice of all open sets in $X_\mu$.

Theorem 17. The lattice $FF_\mu(L)$ is isomorphic with the lattice of all open sets in $X_\mu$.

Proof. The lattice of all open sets in $X_\mu$ is $(\mathcal{T}, \cap, \cup)$. Define the mapping

$$f: FF_\mu(L) \rightarrow \mathcal{T} \quad \text{by} \quad f(\lambda) = X(\lambda), \quad \text{for all} \ \lambda \in X_\mu.$$ (43)

Since $X(\lambda) = X(\widetilde{\mu_\lambda}(\lambda))$ and $\mu_\lambda(\lambda)$ is a $\mu$-fuzzy filter, every open subset of $X_\mu$ is of the form $X(\theta)$ for some $\theta \in FF_\mu(L)$. This shows that the map is onto.

Let $f(\lambda) = f(\theta)$. Now, we need to show $\lambda = \theta$. Suppose not. Then, $\lambda \not\subseteq \theta$, which implies that there is $x \in L$ such that either $\lambda(x) < \theta(x)$ or $\theta(x) < \lambda(x)$. Without loss of generality, we can assume that $\lambda(x) < \theta(x)$. Put $\lambda(x) = \beta$. Then by Corollary 3, we can find a prime $\mu$-fuzzy filter $\eta$ such that $\lambda \subseteq \eta$ and $\theta \not\subseteq \eta$. Thus, $\eta \notin X(\lambda)$ and $\theta \notin X(\lambda)$. This is a contradiction. Hence, $\lambda = \theta$.

Now, we show that $f$ is a homomorphism. Let $\lambda, \theta \in FF_\mu(L)$. Then,

$$f(\lambda \lor \theta) = f(\widetilde{\mu_{\lambda \lor \theta}}(\lambda \lor \theta)) = X(\lambda \lor \theta) = f(\lambda) \lor f(\theta).$$

Similarly, $f(\lambda \land \theta) = f(\lambda) \land f(\theta)$. This shows that $f$ is a homomorphism. Hence, $f$ is an isomorphism.

For any fuzzy subset $\eta$ of $L$, $X(\eta) = \{\eta \in X_\mu: \eta \not\subseteq \eta\}$ is an open set of $X_\mu$ and $V(\eta) = \{\eta \in X_\mu: \eta \subseteq \eta\} = X_\mu - V(\eta)$ is a closed set of $X_\mu$. In the following result, we prove the closure of a fuzzy set.

Theorem 18. For any family $\mathcal{F} \subseteq X_\mu$, closure of $\mathcal{F}$ is given by $\overline{\mathcal{F}} = V(\bigcap_{\lambda \in \mathcal{F}} \lambda)$.

Proof. We know that closure of $\mathcal{F}$ is the smallest closed set containing $\mathcal{F}$. To prove our claim, it is enough to show that $V(\bigcap_{\lambda \in \mathcal{F}} \lambda)$ is the smallest closed set containing $\mathcal{F}$. Since the set of all $\mu$-fuzzy filter is a complete distributive lattice, $\bigcap_{\lambda \in \mathcal{F}} \lambda$ is a $\mu$-fuzzy filter and $V(\bigcap_{\lambda \in \mathcal{F}} \lambda)$ is a closed set in $X_\mu$.

If $\eta \in \mathcal{F}$; then $\bigcap_{\lambda \in \mathcal{F}} \lambda \subseteq \eta$. Thus, $\eta \in V(\bigcap_{\lambda \in \mathcal{F}} \lambda)$. This implies that $\mathcal{F} \subseteq V(\bigcap_{\lambda \in \mathcal{F}} \lambda)$. Let $V(\theta)$ be any closed set in $X_\mu$ containing $\mathcal{F}$. Then, $\theta \subseteq \lambda$, for each $\lambda \in \mathcal{F}$. Thus, $\theta \subseteq \bigcap_{\lambda \in \mathcal{F}} \lambda$ and $V(\bigcap_{\lambda \in \mathcal{F}} \lambda) \subseteq V(\theta)$. So, $V(\bigcap_{\lambda \in \mathcal{F}} \lambda)$ is the smallest closed set containing $\mathcal{F}$. Hence, $\overline{\mathcal{F}} = V(\bigcap_{\lambda \in \mathcal{F}} \lambda)$.

5. Conclusion

In this work, we studied the concept of $\mu$-fuzzy filters of a distributive lattice. We proved that the set of all $\mu$-fuzzy filters of a distributive lattice forms a complete distributive lattice isomorphic to the set of all fuzzy ideals of $A^*(L)$. We observed that every $\mu$-fuzzy filter is the intersection of all $\mu$-fuzzy filters containing it. We also studied the space of all prime $\mu$-fuzzy filters in a distributive lattice. Our future work will focus on $\alpha$-fuzzy ideals of a C-algebra.

Data Availability

No data were used to support this study.
Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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