Existence and Stability of Solutions of Fuzzy Fractional Stochastic Differential Equations with Fractional Brownian Motions

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The existence, uniqueness, and stability of solutions to fuzzy fractional stochastic differential equations (FFSDEs) driven by a fractional Brownian motion (fBm) with the Lipschitzian condition are investigated. Finally, we investigate the exponential stability of solutions.

1. Introduction

There appears to be confusion of various kinds in the modeling of several real world systems, such as trying to characterize a physical system and opinions on its parameters. To deal with this ambiguity, the fuzzy set theory will be used [1]. It is able to handle such linguistic statements mathematically using this theory, such as “large” and “less.” The capacity to investigate fuzzy differential equations (FDEs) in modeling numerous phenomena, including imprecision, is provided by a fuzzy set. In particular, the fuzzy stochastic differential equations (FSDEs), in instance, might be used to investigate a variety of economics and engineering problems that involve two types of uncertainty: randomness and fuzziness.

The fuzzy Itô stochastic integral was powered in [2, 3]. In [4, 5], the fuzzy stochastic integral is driven by the Wiener process as a fuzzy adapted stochastic process. In [6], Fei et al. studied the existence and uniqueness of solutions to the FSDEs under non-Lipschitzian condition. In [7], Jafari et al. study FSDEs driven by fBm. Jialu Zhu et al., in [8], prove existence of solutions to SDEs with fBm. Ding and Nieto [9] investigated analytical solutions of multitime-scale FSDEs driven by fBm. Vas’kovskii et al. [10] prove that the pth moments, p ≥ 1, of strong solutions of a mixed-type SDEs are driven by a standard Brownian motion and a fBm.

Despite the fact that some research exists on the problem of the uniqueness and existence of solutions to SDEs and FSDEs which are disturbed by Brownian motions or semimartingales [4, 11–15], a kind of the FFSDEs driven by an fBm has not been investigated. Agarwal et al. [16, 17] considered the concept of solution for FDEs with uncertainty and some results on FFDEs and optimal control nonlocal evolution equations. Recently, Zhou et al., in [18–20], gave some important works on the stability analysis of such SFDEs. Our results are inspired by the one in [21] where the existence and uniqueness results for the FSDEs with local martingales under the Lipschitzian condition are studied. The rest of this paper is given as follows. Section 2 summarizes the fundamental aspects. In Section 3, existence and uniqueness of solutions to the FFSDEs are proved. Moreover, the stability of solutions is studied in Section 4. Finally, in Section 5, a conclusion is given.

2. Preliminaries

This part introduces the notations, definitions, and background information that will be utilized throughout the article.

Let $K(\mathbb{R}^n)$ be the family of nonempty convex and compact subsets of $\mathbb{R}^n$. In $K(\mathbb{R}^n)$, the distance $d_H$ is defined by
\[ d_H(M, N) = \max \left( \sup_{m \in M} \| m - n \|, \sup_{n \in N} \| m - n \| \right), \quad M, N \in \mathbf{K}(\mathbb{R}^n). \]  

(1)

We denote by \( \mathcal{M}(\Omega, \mathcal{A}; \mathbf{K}(\mathbb{R}^n)) \) the family of \( \mathcal{A} \)-measurable multifunctions, taking value in \( \mathbf{K}(\mathbb{R}^n) \).

Definition 1 (see [21, 22]). A multifunction \( G \in \mathcal{M}(\Omega, \mathcal{A}; \mathbf{K}(\mathbb{R}^n)) \) is called \( \mathcal{L}^p \)-integrably bounded if \( \exists h \in \mathcal{L}^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^+ \) such that \( ||G|| \leq h \mathbb{P}\text{-a.e.} \), where

\[ \mathcal{L}^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbf{K}(\mathbb{R}^n)) = \{ G \in \mathcal{M}(\Omega, \mathcal{A}; \mathbf{K}(\mathbb{R}^n); ||G|| \in \mathcal{L}^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^+) \} \]  

(3)

Let \( \mathbb{E}^n \) denote the set of the fuzzy \( x: \mathbb{R}^n \to [0, 1] \) such that \( [x]^a \in \mathbf{K}(\mathbb{R}^n) \), for every \( a \in [0, 1] \), where \([x]^a := \{ a \in \mathbb{R}^n: x(a) \geq a \} \), for \( a \in (0, 1) \), and \([x]^0 := \{ a \in \mathbb{R}^n: x(a) = 0 \} \). Let the metric be \( d_{co}(x, y) = \sup_{a \in [0, 1]} d_H([x]^a, [y]^a) \), in \( \mathbb{E}^n \), \( a \in \mathbb{R} \); we have \( d_{co}(x + z, y + z) = d_{co}(x, y) \), \( d_{co}(x + y, z + w) \leq d_{co}(x, z) + d_{co}(y, w) \), and \( d_{co}(ax, ay) = |a| d_{co}(x, y) \).

Definition 2 (see [23]). Let \( f: [c, d] \to \mathbb{E}^n \); the fuzzy Riemann–Liouville integral of \( f \) is given by

\[ (\mathcal{F}_c^a f)(u) = \frac{1}{\Gamma(a)} \int_c^u (u - v)^{a-1} f(v) dv. \]  

(4)

Definition 3 (see [23]). Let \( Df \in C([c, d], \mathbb{E}^n) \cap L([c, d], \mathbb{E}^n) \). The fuzzy fractional Caputo differentiability of \( f \) is given by

\[ \mathcal{C} \mathcal{D}_c^a f(u) = \mathcal{F}_c^{1-a} (Df)(u) = \frac{1}{\Gamma(1-a)} \int_c^u (u - v)^{1-a} (Df)(v) dv. \]  

(5)

Now, we define the Henry–Gronwall inequality [24], which can be used in the proof of our result.

Lemma 1. Let \( f, g: [0, T] \to \mathbb{R} \) be continuous functions. If \( g \) is nondecreasing and there exists constants \( K \geq 0 \) and \( \alpha > 0 \) as

\[ f(u) \leq g(u) + \int_0^u (u - v)^{\alpha-1} f(v) dv, \quad u \in [0, T], \]  

then

\[ f(u) \leq g(u) + \int_0^u \left[ \sum_{m=1}^{\infty} \frac{(KT(u))^m}{\Gamma(m\alpha + 1)} (u - v)^{m-1} f(v) \right] dv, \quad u \in [0, T]. \]  

(7)

Definition 4 (see [21, 22]).

A function \( f: \Omega \to \mathbb{E}^n \) is said fuzzy random variable if \([f]^a \) is an \( \mathcal{A} \)-measurable random variable \( \forall a \in [0, 1] \). A fuzzy random variable \( f: \Omega \to \mathbb{E}^n \) is said \( \mathcal{L}^p \)-integrably bounded, \( p \geq 1 \), if \([f]^a \in \mathcal{L}^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbf{K}(\mathbb{R}^n)) \), \( \forall a \in [0, 1] \). Let \( \mathcal{L}^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{E}^n) \) denote the set of all fuzzy random variables; they are \( \mathcal{L}^p \)-integrably bounded.

For the notion of an FBm, we referred to [25]. Let us define a sequence of partitions of \([a, b] \) by \( \{ \psi_m, m \in \mathbb{N} \} \) such that \( |\psi_m| \to 0 \) as \( m \to \infty \). If, in \( L^1(\Omega, \mathcal{A}, \mathbb{P}) \), \( \sum_{i=0}^{m-1} \phi(t_i) (B(t_i^m) - B(t_i^{m-1})) \) converge to the same limit for all this sequences \( \{ \psi_m, m \in \mathbb{N} \} \), then this limit is called a Stratonovich-type stochastic integral and denoted by \( \int_a^b \phi(s) dB \). Let \( f: [0, T], \) where \( 0 < T < \infty \).
Definition 6 (see [21, 22]).

The function $f$ is called measurable if $[f]^n: J \times \Omega \to K(\mathbb{R}^n)$ is a $\mathcal{B}(J) \otimes \mathcal{A}^n$-measurable, for all $a \in [0, 1]$

The function $f: J \times \Omega \to \mathbb{E}^n$ is said to be non-anticipating if it is $\{\mathcal{A}^H\}_{\infty}^t$-adapted and measurable

Remark 2. The process $x$ is non-anticipating if and only if $x$ is measurable with respect to $\mathcal{N}: = \{A \in \mathcal{B}(J) \otimes \mathcal{A}^n \in \mathcal{A}^H, u \in J\}$, where, for $u \in J$, $A^n = \{v: (u, v) \in A\}$.

\[ E \sup_{a \in [0, 1]} \mathbb{E}\left(\left. \int_0^a f(u)du, \int_0^a g(u)du \right| \right) \leq t^{p-1} \int_0^T \mathbb{E}\left(\left. \int_0^1 f(u), g(u)du \right| \right). \] (10)

Proposition 4 (see [21, 22]). A fuzzy process $f: J \times \Omega \to \mathbb{E}^n$ is said $\mathcal{L}^p$-integrally bounded if $\exists h \in \mathcal{L}^p(\mathbb{K}(\mathbb{R}^n; \mathbb{R})/\mathcal{D}_\infty^H(f(s, v), \mathcal{D}_\infty^H(s, v) \leq h(s, v)$.

We denote by $\mathcal{L}^p(J \times \Omega, \mathbb{E}^n)$ the set of all $\mathcal{L}^p$-integrally bounded and nonanticipating fuzzy stochastic processes.

Proposition 1 (see [4]). For $f \in \mathcal{L}^p(J \times \Omega, \mathbb{E}^n)$ and $p \geq 1$, we have $J \times \Omega \ni (t, v) \to \int_0^t f(s, v)ds \in \mathcal{L}^p(\mathbb{K}(J \times \Omega, \mathbb{E}^n)$ and $\mathcal{D}_\infty^H$-continuous.

Proposition 2 (see [4]). For $f, g \in \mathcal{L}^p(J \times \Omega, \mathbb{E}^n)$ and $p \geq 1$, we have

\[ E\left(\left. \int_0^1 f(u), g(u)du \right| \right) \leq t^{p-1} \int_0^T \mathbb{E}\left(\left. \int_0^1 f(u), g(u)du \right| \right). \]

3. Main Result

Now, we investigate the FFSDEs driven by an fBm given by

\[ \begin{cases} \mathbb{P}^x(s) \mathbb{P}_x^0 = f(s, x(s))ds + \langle g(s) \mathbb{B}^H(s) \rangle, \\ x(0) = x_0, \end{cases} \] (14)

where

\[ \begin{aligned} f: J \times \Omega \times \mathbb{E}^n & \to \mathbb{E}^n, \\ g: J \to \mathbb{R}^n, \\ x_0: \Omega \to \mathbb{E}^n, \end{aligned} \] (15)

and $\mathcal{B}^H(s)_{\infty}^t$ is a fBm defined on $(\Omega, \mathcal{A}, \{\mathcal{A}^H\}_{\infty}^t, \mathcal{P})$ with Hirst index $H \in (1/2, 1)$.

Definition 8. A process $x: J \times \Omega \to \mathbb{E}^n$ is said to be a solution to equation (14) if the following holds:

(i) $x \in L^2(J \times \Omega, \mathcal{N}; \mathbb{E}^n)$.

(ii) $x$ is $\mathcal{D}_\infty^H$-continuous.

(iii) We have

\[ x(t) = x_0 + \frac{1}{\Gamma(a)} \int_0^t f(s, x(s))ds + \frac{1}{\Gamma(a)} \int_0^t g(s)g(s)ds. \] (16)

We will assume that all through this paper, $f: J \times \Omega \times \mathbb{E}^n \to \mathbb{E}^n$ is $\mathcal{B}_0 \otimes \mathcal{N}|\mathcal{B}_\infty - \mathcal{D}_\infty^H$-measurable. Let the following assumptions be introduced.

(3) If $x_0$ is $\mathcal{A}_0$-measurable, we have

\[ E\mathcal{D}_\infty^H(x_0, \mathbb{0}) < \infty. \] (17)

(4) For $f(s, \mathbb{0})$ and $g$, we have

\[ \max\{\mathcal{D}_\infty^H(f(s, \mathbb{0}), \mathbb{0}), \|g\|\} \leq C, \] (18)

for every $s \in J$.

(5) For all $z, w \in \mathbb{E}^n$,

\[ \mathcal{D}_\infty^H(f(s, z), f(s, w)) \leq \mathcal{D}_\infty^H(z, w), \] (19)
where \( c \) is equal to one in (\( \mathcal{H} 2 \)).

Let us now introduce the main theorem in this part.

**Theorem 1.** Under assumptions (\( \mathcal{H} 1 \))–(\( \mathcal{H} 3 \)) and \( x_0 \in L^2(\Omega, \mathcal{A}_0, \mathbb{P}; E^n) \), the equation (14) has a unique solution \( x(t) \).

\[
x_n(t) = x_0 + \frac{1}{\Gamma(n)} \int_0^t f(s, x_{n-1}(s)) \, ds + \frac{1}{\Gamma(n)} \int_0^t g(s) \, dB^H(s).
\]

It is clear that \( x_n \) are in \( L^2(f \times \Omega, \mathbf{N}; E^n) \) and \( d_{\alpha} \)-continuous. Indeed, we have \( x_0 \in L^2(f \times \Omega, \mathbf{N}; E^n) \) and \( x_0 \) is \( d_{\alpha} \)-continuous.

**Proof.** The method of successive approximations will be used to demonstrate the existence of a solution to (1). Therefore, define a sequence \( x_0; f \times \Omega \rightarrow E^n \) as follows:

\[
x_0(t) = x_0, \quad (20)
\]

and for \( n = 1, \ldots, \)

\[
K_1(t) = \sup_{0 \leq u \leq t} E_{\alpha}^2 \left( \frac{1}{\Gamma(n)} \int_0^u f(s, x_0(s)) + \frac{1}{\Gamma(n)} \int_0^u g(s) \, dB^H(s), \theta \right)
\]

\[
\leq 2 \sup_{0 \leq u \leq t} \left[ E_{\alpha}^2 \left( \frac{1}{\Gamma(n)} \int_0^u f(s, x_0(s)) + \frac{1}{\Gamma(n)} \int_0^u g(s) \, dB^H(s), \theta \right) \right]^{2}
\]

\[
K_n(t) = \sup_{0 \leq u \leq t} E_{\alpha}^2 \left( \frac{1}{\Gamma(n)} \int_0^u f(s, x_{n-1}(s)) + \frac{1}{\Gamma(n)} \int_0^u g(s) \, dB^H(s), \theta \right)
\]

where \( l_1 = 4cT d_{\alpha}^2(x_0, \theta) + 4Tc + 2c^2TJc \). Moreover, similarly, we have

\[
K_{n+1}(t) = \sup_{0 \leq u \leq t} E_{\alpha}^2 \left( \frac{1}{\Gamma(n)} \int_0^u f(s, x_0(s)) + \frac{1}{\Gamma(n)} \int_0^u g(s) \, dB^H(s), \theta \right)
\]

\[
\leq 2 \sup_{0 \leq u \leq t} \left[ E_{\alpha}^2 \left( \frac{1}{\Gamma(n)} \int_0^u f(s, x_{n-1}(s)) + \frac{1}{\Gamma(n)} \int_0^u g(s) \, dB^H(s), \theta \right) \right]^{2}
\]
Thus, we obtain
\[ K_n(t) \leq \frac{l_1}{l_2} \left( \frac{t \alpha}{n!} \right)^n, \quad \forall t \in J, \ n \in \mathbb{N}, \] (24)
where \( l_2 = 2T_c \).

Hence, from Chebyshev’s inequality and (24), we obtain
\[ \mathbb{P} \left( \sup_{u \in J} d_{\infty}^2 (x_n(u), x_{n-1}(u)) > \frac{1}{4^3} \right) \leq \frac{l_1}{l_2} \left( \frac{4l_1 T_n \alpha}{n!} \right)^n. \] (25)

Since the series \( \sum_{t=1}^{\infty} (4l_1 T_n)^2/n! \) converges, according to Borel–Cantelli lemma, we obtain
\[ \mathbb{P} \left( \sup_{u \in J} d_{\infty}^2 (x_n(u), x_{n-1}(u)) > \frac{1}{2^3} \right) = 0. \] (26)

Thus, the sequence \( \{x_n(\cdot, \cdot)\} \) is uniformly convergent to \( \bar{x}(\cdot, \cdot) \): \( J \to \mathbb{R}^n \) for \( u \in \Omega_c \), where \( \Omega_c \in \mathcal{A} \) and \( \mathbb{P}(\Omega_c) = 1 \). Then,
\[ \lim_{n \to \infty} \sup_{(t, u) \in J} E d_{\infty}^2 (x_n(t), \bar{x}(t)) = 0. \] (27)

Let us define \( x: J \times \Omega \to E^n \) as follows:
\[ x(\cdot, \cdot) = \begin{cases} \bar{x}(\cdot, \cdot), & \text{if } u \in \Omega_c, \\ \text{freely chosen}, & \text{if } u \in \Omega \setminus \Omega_c. \end{cases} \] (28)

We can observe that, for each \( 0 \leq \alpha \leq 1 \) and \( t \in J \), we have
\[ \lim_{n \to \infty} d_H \left( [x_n(\cdot, t)], [x_{n-1}(\cdot, t)] \right) = 0. \] (29)

Then, \( [x(\cdot, \cdot)]^{\alpha} : \Omega \to K(\mathbb{R}^n) \) is \( \mathcal{A}_f \)-measurable. Hence, \( x \) is nonanticipating. By (27), we have
\[ \lim_{n \to \infty} \sup_{t \in J} E d_{\infty}^2 (x_n(t), x(t)) = 0, \] (30)
which shows that \( \exists \lambda > 0 \) independent of \( n \in \mathbb{N} \) such that
\[ \sup_{t \in J} E d_{\infty}^2 (x_n(t), x(t)) \leq \lambda. \] (31)

Since \( x_n \in L^2 (J \times \Omega, \mathcal{A}_f, \mathbb{P}; E^n) \), we have \( x_n(t) \in L^2 (\Omega, \mathcal{A}, \mathbb{P}; E^n) \). In addition, we can prove that \( x \in L^2 (J \times \Omega, \mathcal{A}_f, \mathbb{P}; E^n) \).

Indeed, for all \( n \in \mathbb{N} \) and \( t \in J \), let us denote
\[ \psi_n(t) = \sup_{0 \leq s \leq t} E d_{\infty}^2 (x_n, 0). \] (32)

Then, we obtain
\[ \psi_n(t) \leq 3E d_{\infty}^2 (x_n, 0) + 3 \sup_{0 \leq s \leq t} E d_{\infty}^2 \left( \frac{1}{\Gamma(\alpha)} \int_0^s (u-s)^{\alpha-1} f(s, x_{n-1}(s))ds, 0 \right) + 3E \sup_{0 \leq s \leq t} \left\| \frac{1}{\Gamma(\alpha)} \int_0^s (u-s)^{\alpha-1} g(s)dB^H(s) \right\|^2. \] (33)

By the triangle inequality, \( (\mathcal{H}1) \sim (\mathcal{H}3) \), and Propositions 2 and 3, we have
\[ \psi_n(t) \leq 3E d_{\infty}^2 (x_n, 0) + \frac{6t}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ E d_{\infty}^2 (f(s, x_{n-1}(s)), f(s, 0)) + E d_{\infty}^2 (f(s, 0), 0) \right\} ds \]
\[ + \frac{3cT_H}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g(s)\|^2 ds, \]
\[ \leq 3E d_{\infty}^2 (x_n, 0) + \frac{6ct}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E d_{\infty}^2 (x_{n-1}(s), 0) ds + \frac{6ct^{\alpha+1}}{\Gamma(\alpha + 1)} + \frac{3t^{\alpha+2}c^2 T_H}{\Gamma(\alpha + 1)}. \] (34)

We obtain
\[ \psi_n(t) \leq A_1 + A_2 \int_0^t (t-s)^{\alpha-1} \psi_{n-1}(s) ds, \] (35)
where \( A_1 = 3E d_{\infty}^2 (x_n, 0) + (6ct^{\alpha+1} + (6ct^{\alpha+1} + (3t^{\alpha+2}c^2 T_H)) \Gamma(\alpha + 1)) \) and \( A_2 = 6ct/\Gamma(\alpha) \).

According to Lemma 1 and Remark 1, there exist a constant \( M_{A_1} > 0 \) independent of \( A_1 \), such that
\[ \psi_n(t) \leq M_{A_1} A_1. \] (36)

Due to \( (\mathcal{H}1), (31), \) and (36), we obtain
\[ \sup_{0 \leq s \leq t} E d_{\infty}^2 (x_n, 0) \leq 2 \sup_{0 \leq s \leq t} E d_{\infty}^2 (x_n(s), 0) + 2 \sup_{0 \leq s \leq t} E d_{\infty}^2 (x_n(s), 0) \leq 2\lambda + 2M_{A_1} A_1 < \infty, \] (37)
which implies
\[ \int_{0}^{T} \mathbb{E}_{\infty}^{2}(x(s), \tilde{0}) ds \leq T \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), \tilde{0}) < \infty. \]  \hfill (38)

Thus, we get \( x \in L^{2}(f \times \Omega, \mathbb{N}; \mathbb{E}^{n}) \).

On the contrary, we have
\[ \lim_{n \to \infty} I_{3} = 0 \text{ and } I_{2} = 0. \]

For the uniqueness of a solution, suppose that \( x, z : J \times \Omega \to \mathbb{E}^{n} \) are solutions to equation (14). We denote by \( K(t) = \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), z(t)) \). So, for each \( t \in J \), we obtain
\[ K(t) \leq \frac{tc}{\Gamma(a)} \int_{0}^{t} \mathbb{E}_{\infty}^{2}(x(s), z(s)) \frac{ds}{(t-s)^{1-a}} \]
\[ \leq \frac{Tc}{\Gamma(a)} \int_{0}^{t} \frac{K(s)}{(t-s)^{1-a}} ds. \]  \hfill (42)

Thus, by Lemma 1, we have, for \( t \in J \), \( K(t) \equiv 0 \), which implies
\[ \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), z(t)) \leq 0. \]  \hfill (43)

\section{4. Stability Result}

In this part, we examine the stability of the solution with respect to initial values by using Henry–Gronwall inequality. Indeed, let \( x \) and \( z \) denote the solutions of the following FFSDEs:

\[ \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), x_{0}) + \frac{1}{\Gamma(a)} \int_{0}^{t} f(s, x(s)) \frac{ds}{(t-s)^{1-a}} \]
\[ + \frac{1}{\Gamma(a)} \int_{0}^{t} g(s) \frac{ds}{(t-s)^{1-a}} = 0. \]  \hfill (39)

Indeed, we observe
\[ \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), x_{n}(t)) \]
\[ \leq \left[ \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), x_{n}(t)) + \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), x_{0}) + \frac{1}{\Gamma(a)} \int_{0}^{t} f(s, x_{n}(s)) \frac{ds}{(t-s)^{1-a}} \right. \]
\[ + \frac{1}{\Gamma(a)} \int_{0}^{t} g(s) \frac{ds}{(t-s)^{1-a}} = I_{1} + I_{2} + I_{3}, \]  \hfill (40)

where \( \lim_{n \to \infty} I_{1} = 0 \) and \( I_{2} = 0. \) For \( I_{3} \), by using Propositions 2 and 3, \((\mathcal{H} 3)\), and (30), we have
\[ \lim_{n \to \infty} I_{3} \leq \lim_{n \to \infty} \left( \frac{t^{a-1}e}{\Gamma(a+1)} \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(u), x_{n-1}(u)) du \right) = 0. \]  \hfill (41)

Hence, we get (39), which implies (16) holds. Hence, from definition (8), \( x(t) \) is a solution to equation (14).

For the uniqueness of a solution \( x \), suppose that \( x, z : J \times \Omega \to \mathbb{E}^{n} \) are solutions to equation (14). We denote by \( K(t) = \sup_{t \in J} \mathbb{E}_{\infty}^{2}(x(t), z(t)) \). So, for each \( t \in J \), we obtain
\[ K(t) \leq \frac{tc}{\Gamma(a)} \int_{0}^{t} \mathbb{E}_{\infty}^{2}(x(s), z(s)) \frac{ds}{(t-s)^{1-a}} \]
\[ \leq \frac{Tc}{\Gamma(a)} \int_{0}^{t} \frac{K(s)}{(t-s)^{1-a}} ds. \]  \hfill (42)

Thus, by Lemma 1, we have, for \( t \in J \), \( K(t) \equiv 0 \), which implies
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\section{4. Stability Result}

In this part, we examine the stability of the solution with respect to initial values by using Henry–Gronwall inequality. Indeed, let \( x \) and \( z \) denote the solutions of the following FFSDEs:
\[ K(t) \leq 2E d^2_\infty(x_0, z_0) + \frac{2}{\Gamma(\alpha)} \sup_{s \in [0,t]} E d^2_\infty \left( \int_0^t f(s, x(s)) \, ds, \int_0^t f(s, z(s)) \, ds \right) \]

\[ \leq 2E d^2_\infty(x_0, z_0) + \frac{2Tc}{\Gamma(\alpha)} \int_0^t E d^2_\infty(x(s), z(s)) \, ds \]

\[ \leq 2E d^2_\infty(x_0, z_0) + \frac{2Tc}{\Gamma(\alpha)} \int_0^t \sup_{u \in (0,t]} E d^2_\infty(x(u), z(u))(t-s)^{-\alpha} \, du \]

\[ = 2E d^2_\infty(x_0, z_0) + \frac{2Tc}{\Gamma(\alpha)} \int_0^t \frac{K(s)}{(t-s)^{1-\alpha}} \, ds \]

\[ = \lambda_0 + \lambda_1 \int_0^t \frac{K(s)}{(t-s)^{1-\alpha}} \, ds. \] (47)

Then, according to Lemma 1 and Remark 1, there exist a constant \( M_{\lambda_1} > 0 \) independent of \( \lambda_0 \) such that

\[ K(t) \leq \lambda_0 M_{\lambda_1}, \quad \forall t \in J. \] (48)

Then, \( \lambda_0 = 0 \) if \( x_0 \overset{\text{IP}}{=} z_0 \). Therefore, we know that \( x(t) = z(t) \).

Finally, we examine the exponential stability of solutions to the FFSDEs which disturbed an FBM with respect to \( f \) and \( g \). Thus, let \( x \) and \( x_n \) denote solutions to the following FFSDEs:

\[ \begin{align*}
\frac{C\partial_t}{\Gamma(\alpha)} x(s) &= f(s, x(s)) + \langle g(s) dB^H(s) \rangle, \\
\lambda x(0) &= x_0,
\end{align*} \] (49)

\[ \begin{align*}
\frac{C\partial_t}{\Gamma(\alpha)} x_n(s) &= f_n(s, x_n(s)) + \langle g_n(s) dB^H(s) \rangle, \\
x_n(0) &= x_0,
\end{align*} \] (50)

respectively.

**Proposition 6.** Suppose that \( x_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{E}^n) \) and \( f, f_n : J \times \Omega \times \mathbb{E}^n \rightarrow \mathbb{E}^n \), \( g, g_n : J \rightarrow \mathbb{R}^n \) fulfill \((\mathcal{H}1)-(\mathcal{H}3)\). Furthermore, assume that

\[ \lim_{n \to \infty} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} E d^2_\infty((t,x), f_n(t,x)(t,x)) \, dt \right) = 0, \] (51)

\[ \lim_{n \to \infty} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} \| g_n(s) - g(s) \|^2 \, ds \right) = 0. \] (52)

Then, we have

\[ \lim_{n \to \infty} \left( E \sup_{t \in J} E d^2_\infty(x(t), x_n(t)) \right) = 0, \] (53)

where \( x, x_n : J \times \Omega \rightarrow \mathbb{E}^n \) are solutions of equations (49) and (50), respectively.

**Proof.** According to Theorem 1, the solutions \( x \) and \( x_n \) are unique and exist. From Propositions 3 and 4, we deduce that, for every \( t \in J \),

\[ \sup_{0 \leq u \leq t} E d^2_\infty(x(u), x_n(u)) \leq 2 \sup_{0 \leq u \leq t} E d^2_\infty \left( \langle \frac{1}{\Gamma(\alpha)} \int_0^u f(s, x(s)) \, ds, \frac{1}{\Gamma(\alpha)} \int_0^u f_n(s, x_n(s)) \, ds \rangle \right) \]

\[ + 2 \sup_{0 \leq u \leq t} E d^2_\infty \left( \langle \frac{1}{\Gamma(\alpha)} \int_0^u g(s) \, ds \, dB^H(s) \rangle, \langle \frac{1}{\Gamma(\alpha)} \int_0^u g_n(s) \, ds \, dB^H(s) \rangle \right) \]

\[ \leq \sup_{0 \leq u \leq t} E d^2_\infty \left( \langle \frac{1}{\Gamma(\alpha)} \int_0^u f(s, x(s)) \, ds, \frac{1}{\Gamma(\alpha)} \int_0^u f_n(s, x_n(s)) \, ds \rangle \right) \]

\[ + 2 \sup_{0 \leq u \leq t} E d^2_\infty \left( \langle \frac{1}{\Gamma(\alpha)} \int_0^u g(s) \, ds \, dB^H(s) \rangle, \langle \frac{1}{\Gamma(\alpha)} \int_0^u g_n(s) \, ds \, dB^H(s) \rangle \right) \]
\[ q^4 \sup_{0 \leq s \leq t} \mathbb{E} \mathcal{D}^2_{\alpha}(f_n(s, x(s))) \leq 4 \mathbb{E} \mathcal{D}^2_{\alpha}(f_n(s, x(s))) \]

\[ + 4 \mathbb{E} \mathcal{D}^2_{\alpha}(f_n(s, x(s))) \leq 4 \mathbb{E} \mathcal{D}^2_{\alpha}(f_n(s, x(s))) \]

\[ \leq 4 \int_0^t \mathbb{E} \mathcal{D}^2_{\alpha}(f_n(s, x(s))) \, ds, \]

where

\[ \beta_1^n = \frac{4T}{\Gamma(\alpha)} \int_0^t \mathbb{E} \mathcal{D}^2_{\alpha}(f_n(s, x(s))) \, ds, \]

\[ \beta_2 = 4cT/\Gamma(\alpha). \] From Lemma 2 and Remark 1, \( \exists M_{\beta_1} > 0 \) is independent of \( \beta_1^n \) such that

\[ \sup_{u \in [0, t]} \mathbb{E} \mathcal{D}^2_{\alpha}(f_n(s, x(s))) \leq \beta_1^n M_{\beta_2}. \]

Hence, from (51) and (52), we get \( \lim_{n \rightarrow \infty} \beta_1^n = 0. \)

5. Conclusions

In this study, we have proved the existence and uniqueness of solutions to FFSDEs under the Lipschitzian coefficient. On the contrary, the stability of solutions to the FFSDEs is analyzed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


