Synchronization in Finite Time of Fuzzy Neural Networks with Hybrid Delays and Uncertain Nonlinear Perturbations

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1. Introduction

Since Yang and Yang [1, 2] proposed fuzzy cellular neural networks (FCNNs) on the basis of traditional cellular neural networks (CNNs) in 1996, many researchers have performed extensive work on this topic due to their application in image processing and pattern recognition, see [3–6]. However, in the realization of neural networks, the emergence of time delays is unavoidable due to the limitation of velocity information. For one thing, discrete time delays often occur due to the limited switching speed of neurons and amplifiers [7–10]. For another, neural networks usually have spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths, the propagation velocity distribution of these paths usually results in the propagation distribution delays [11–13]. The presence of time delays can lead to instability, chaos, oscillation, and other performance degradations of the nervous system. In this case, considering the combination of discrete time delays and distributed time delays is of great significance to the study of neural networks.

Synchronization plays an essential role in practical applications such as biological systems, secure communications, and image protection. Therefore, synchronization has become a hot spot studying the dynamic behavior of neural networks. So far, many types of synchronization for fuzzy neural networks have been proposed, for instance, exponential synchronization, antisynchronization, projective synchronization, and adaptive synchronization, for example, see [14–19]. However, in practical applications, we are not only interested in the synchronization performance of the system but also more concerned with the convergence time of the system. Finite-time synchronization has the best convergence time in neural networks compared with infinite-time synchronization. That is because finite-time synchronization has better robustness and anti-interference ability. Hence, many scholars are interested in the finite-time synchronization of fuzzy neural networks and have studied it extensively [13, 20–26]. Abdurahman et al. [20] and Wang [21] investigated the finite-time synchronization problem of FCNNs with time-varying delays or time-varying coefficients and proportional delays based on the finite-time...
stability theory, and inequality techniques, and some criteria of finite-time synchronization for the addressed network are derived. In [22], Duan et al. studied the finite-time synchronization of delayed FCNNs with discontinuous activations. Under the framework of differential inclusions, by utilizing the discontinuous state feedback control method and constructing Lyapunov functionals, new finite-time synchronization criteria for the considered networks are established. In the same year, Tang et al. [13] further considered the finite-time cluster synchronization issue for coupled FCNNs with Markovian switching topology, discontinuous activation functions, proportional leakage, and time-varying unbounded delays, and novel quantization controllers without the sign function are designed to avoid the chattering and save communication resources. Several sufficient conditions are derived to guarantee the finite-time cluster synchronization by constructing new Lyapunov–Krasovskii functionals and utilizing M-matrix methods. What’s more, Jian and Duan [23] researched the synchronization in finite time of fuzzy neural networks with time-varying coefficients and proportional leakage.

Based on finite-time stability theory and some inequality techniques, several criteria are established to ensure that the drive–response systems can synchronize in finite time. Compared with linear matrix inequality, the finite-time synchronization criterion in this paper is easy to verify with the parameters of the system and easy to implement in practice.

The designed state feedback controller and the adaptive controller can not only eliminate the influence of time delays and uncertain nonlinear perturbations but also change their form according to the change of system state or perturbation to achieve a better control effect.

2. Preliminaries

In this paper, we concern with the fuzzy neural networks with hybrid delays and uncertain nonlinear perturbations described by

\[
\dot{x}_k(t) = -c_k x_k(t) + \sum_{i=1}^{n} a_{ki} f_i(x_i(t)) + \sum_{i=1}^{n} b_{ki} \nu_i + \sum_{i=1}^{n} S_{ki} \nu_i \\
+ \nu_i H_k \nu_i + \nu_i \nu_i + \sum_{i=1}^{n} a_{ki} f_i(x_i(t - \tau_i(t))) \\
+ \nu_i \nu_i + \sum_{i=1}^{n} b_{ki} \int_{t-\sigma_i(t)}^{t} f_i(x_i(s))ds \\
+ \nu_i \nu_i + \sum_{i=1}^{n} S_{ki} \int_{t-\sigma_i(t)}^{t} f_i(x_i(s))ds + \delta_k(t),
\]

(1)

where \( k \in \mathcal{F} \overset{\Delta}{=} \{1, 2, \ldots, n\}, n \geq 2 \); \( n \) corresponds to the number of neurons in the delayed network system; \( x_k(t) \) denotes the state of the \( k \)th unit at moment \( t \); \( c_k \) represents the passive decay rates to the state of \( k \)th unit; \( a_{ki} \) and \( b_{ki} \) are the connection weight of the feedback template and feed-forward template; \( a_{ki} , \overline{a}_{ki} , \overline{b}_{ki} , \) and \( \overline{S}_{ki} \) denote the connection weights of the elements of the discrete fuzzy feedback MIN template, the discrete fuzzy feedback MAX template, the distributed fuzzy feedback MIN template, and the distributed fuzzy feedback MAX template, respectively; \( S_{ki} \) and \( H_{ki} \) are the elements of the fuzzy feed-forward MIN and the fuzzy feed-forward MAX template; \( \nu_i \) and \( \delta_k(t) \) denote the fuzzy AND and fuzzy OR operations, respectively; \( \nu_i \) presents input and bias of the \( i \)th neuron; \( f_i(\cdot) \) denotes the signal activation function of the \( i \)th neuron at moment \( t \); \( \tau_i(t) \) and \( \sigma_i(t) \) correspond to the discrete time-varying delay and distributed time-varying delay and satisfy \( 0 \leq \tau_i(t) \leq \tau_i \leq 0 \leq \sigma_i(t) \leq \sigma_i \leq \sigma \), where \( \tau = \max_{i \in \mathcal{F}} \{\tau_i\} \) and \( \sigma = \max_{i \in \mathcal{F}} \{\sigma_i\} \) are nonnegative constants; \( \delta_k(t) \) is the disturbance signal.

Motivated by the above discussions, we study the finite-time synchronization issue of fuzzy neural networks with hybrid delays and uncertain nonlinear perturbations. The main contributions of this article are as follows:

(a) A new general neural network model is formulated, and the model assembles fuzzy cellular neural networks, discrete time delays, distributed time delays, and uncertain nonlinear perturbations. Compared with the existing literature on neural networks [20, 21, 31, 33, 34], this model is more general and extensive.

(b) Some algebraic sufficient criteria are established to ensure that the drive–response system can be synchronized in finite time. Compared with linear matrix inequality, the finite-time synchronization criterion in this paper is easy to verify with the parameters of the system and easy to implement in practice.

(c) The designed state feedback controller and the adaptive controller can not only eliminate the influence of time delays and uncertain nonlinear perturbations but also change their form according to the change of system state or perturbation to achieve a better control effect.
As the drive system; then, the corresponding response system is described as follows:

\[
y_k(t) = -c_k y_k(t) + \sum_{i=1}^{n} a_{ki} f_i(y_i(t)) + \sum_{i=1}^{n} b_{ki} v_i \\
+ \lambda_n \int_{t-\tau(t)}^{t} f_i(y_i(t)) \, ds + \sigma(t) + u_k(t),
\]

where \( \dot{\delta}(t) = \delta(t, y(t), y(t - \tau(t)), \int_{t-\tau(t)}^{t} M(t) \, ds) \) represents the nonlinear perturbations to system (2), and \( u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \) is the suitable controller to be designed for realizing synchronization of the drive-response system. The other parameters are the same as those defined in system (1).

We define \( \epsilon_k(t) = y_k(t) - x_k(t) \) as the synchronization error of system (1) and (2). Then, subtracting (1) from (2) yields the following error system:

\[
\dot{\epsilon}_k(t) = -c_k \epsilon_k(t) + \sum_{i=1}^{n} a_{ki} f_i(\epsilon_i(t)) + \lambda_n \int_{t-\tau(t)}^{t} f_i(\epsilon_i(t)) \, ds + \sigma(t) + \delta(t) + u_k(t),
\]

where \( \dot{\epsilon}_k(t) = \dot{\delta}(t) \).

For research purposes, we make some basic assumptions and introduce the definition and the lemmas that we need to use.

(i) (A1) For each \( k \in \mathcal{K} \), there exist a nonnegative constant \( F_k \) such that

\[
|f_k(x) - f_k(y)| \leq F_k |x - y|, \text{ for all } x, y \in \mathbb{R}.
\]

(ii) (A2) For each \( k \in \mathcal{K} \), there exist nonnegative constants \( A_k, B_k, C_k, \) and \( D_k \) such that

\[
|\delta_k(t) - \dot{\delta}_k(t)| \leq A_k \sum_{i=1}^{n} |\epsilon_i(t)| + B_k \sum_{i=1}^{n} |\epsilon_i(t - \tau_i(t))| + C_k \int_{t-\tau(t)}^{t} |\epsilon_i(s)| \, ds + D_k.
\]

**Definition 1** (see [36]). The neural network (2) is said to be synchronized with (1) in finite time if, for a suitable designed controller, there exists an initial state vector error value \( \psi(s) \), such that \( \|\epsilon(0)\| = 0 \) and \( \|\epsilon(t)\| = \|y(t) - x(t)\| = 0 \) for \( t > T \), where the \( T \) is called the setting time.
Lemma 1 (see [37]). We assume that there exist a continuous, positive-definite function $V(t) : \mathbb{R} \to \mathbb{R}^+$, constants $\alpha, \beta > 0$, $0 < \eta < 1$, and an open neighborhood $\mathcal{V}' \subset \mathcal{V}$ of the origin such that

$$
V(t) \leq -\alpha V^\eta(t) - \beta V(t), \quad \forall t \in \mathcal{V}'[0].
$$

(6)

Then, the origin of system is finite-time stable. The settling time $T$ satisfies

$$
T \leq t_0 + \frac{\ln(1 + \beta/\alpha V^{1-\eta}(t_0))}{\beta(1-\eta)}.
$$

(7)

Moreover, if $\mathcal{V}' = \mathbb{R}^+$, $V(t)$ is proper and radially unbounded, then the origin is globally finite-time stable.

Lemma 2 (see [38]). We assume that a continuous, positive-definite function $V(t)$ and constants $\alpha > 0, 0 < \eta < 1$ satisfy the following differential inequality

$$
V(t) \leq -\alpha V^\eta(t), \quad \forall t \geq t_0, V(t_0) \geq 0.
$$

(8)

Then, for any given $t_0$, $V(t)$ satisfies the following inequality

$$
V^{1-\eta}(t) \leq V^{1-\eta}(t_0) - \alpha (1-\eta)(t-t_0), \quad t_0 \leq t \leq T,
$$

(9)

and $V(t) \equiv 0$ for all $t \geq T$ with the settling time $T$ given by

$$
T = t_0 + V^{1-\eta}(t_0)/\alpha (1-\eta).
$$

(10)

Lemma 3 (see [39]). Let $e(t)$ be a solution of the error system, which is defined on $[0, T]$, $T \in (0, +\infty)$. Then, function $|e(t)|$ is absolutely continuous and

$$
d/dt|e(t)| = v(t)^T \dot{c} = \sum_{k=1}^{n} v_k(t) \dot{c}_k(t) \text{ for any } t \in [0, T],
$$

(11)

where $v_k(t) = \text{sign}(c_k(t))$, if $c_k(t) \neq 0$, while $v_k(t)$ can be arbitrarily chosen in $[-1, 1]$, if $c_k(t) = 0$.

Lemma 4 (see [1]). If suppose $x$ and $\bar{x}$ are two states of neural networks (1), then we have

$$
\left| h_{r_1} \alpha_{k_1} f_i(x_i) - h_{r_2} \alpha_{k_2} f_i(\bar{x}_i) \right| \leq \sum_{i=1}^{n} \alpha_i \left| f_i(x_i) - f_i(\bar{x}_i) \right|,
$$

(12)

$$
\left| v_{r_1} \alpha_{k_1} f_i(x_i) - v_{r_2} \alpha_{k_2} f_i(\bar{x}_i) \right| \leq \sum_{i=1}^{n} \alpha_i \left| f_i(x_i) - f_i(\bar{x}_i) \right|.
$$

Lemma 5 (see [20]). If suppose $n$ is a positive integer and $a_1, a_2, \ldots, a_n$ and $0 < q < p$ are real numbers, then the following inequality holds

$$
\left| a_1 \right|^q + \left| a_2 \right|^q + \ldots + \left| a_n \right|^q \geq \left( \left| a_1 \right|^p + \left| a_2 \right|^p + \ldots + \left| a_n \right|^p \right)^{q/p}.
$$

(13)

Lemma 6 (see [40]). Let $a \geq 0, b_k \geq 0 (k = 1, 2, \ldots, l)$, then

$$
a \prod_{k=1}^{l} b_k^{q_k} \leq \frac{1}{r} \left( a^{q_1} + \sum_{k=1}^{l} q_k b_k^{q_k} \right),
$$

(14)

where $q_k \geq 0 (k = 1, 2, \ldots, l)$ are some constants, $\sum_{k=1}^{l} q_k = r$, and $r \geq 1$.

3. Main Results

In this section, two different types of controllers should be designed for the finite-time stability of the zero solution of the error system (3), which is equivalent to the finite-time synchronization between system (1) and system (2). Specifically, a state feedback controller is first designed for the finite-time synchronization problem. Then, an adaptive controller is considered based on the state feedback controller so that the control strength can be adjusted automatically. At the same time, several sufficient conditions that the considered system can achieve finite-time synchronization are obtained through rigorous mathematical proofs.

3.1. State Feedback Control. State feedback is to multiply each state variable of the system by the corresponding feedback coefficient and feedback to the input end, which is added with the reference input as the control input of the controlled system. The basic structure of state feedback control system is shown in Figure 1. In order to realize the finite-time synchronization of fuzzy neural networks with hybrid delays and uncertain nonlinear perturbations, we design a new state feedback controller $u_k$ for the response system.

The aforementioned controller $u_k$ is designed in the following form:

$$
u_k(t) = -\eta_k c_k(t) - \text{sign}(e_k(t))(\xi_k + \lambda |e_k(t)|^\mu) - \sum_{i=1}^{n} \gamma_k \text{sign}(e_k(t)) |e_i(t - \tau_i(t))| - \sum_{i=1}^{n} \omega_k \text{sign}(e_k(t)) \int_{t-\sigma_i(t)}^{t} |e_i(s)| ds,
$$

(15)

where $k, i \in \mathcal{F}$, the positive constants $\eta_k, \xi_k, \gamma_k, \omega_k$ are the gain coefficients to be determined, $\lambda$ is a tunable positive constant, and the real number $\mu$ satisfies $0 < \mu < 1$.

Theorem 1. Suppose that the assumptions (A1) and (A2) are satisfied, then the neural network (1) and (2) can be synchronized in a finite time under controller (15) if for any $k \in \mathcal{F}$, and the following conditions hold:

$$
d = \min_{k \in \mathcal{F}} \left[ c_k + \eta_k |a_{kk}| F_k - A_k - \mathcal{A}_k - \mathcal{B}_k \right] > 0, \quad \xi_k \geq D_k,
$$

(16)

$$
\gamma_k \geq |a_{kk}| F_i + |b_{kk}| F_i + B_k, \quad \omega_k \geq |b_{kk}| F_i + |b_{kk}| F_i + C_k.
$$

(17)
where \( \mathcal{A}_k = \frac{1}{p} \sum_{i=1}^{n} \alpha_i \mathcal{F}_k^{(i)} \) and feedback coefficient response system time can be estimated as

\[
T = \frac{\ln(1 + d/l\nu^{1/n}p(0))}{d(1 - \mu)},
\]

in which \( V(0) = \sum_{k=1}^{n} |\epsilon_k(0)|^p \), and \( p \geq 1 \) is a positive integer.

**Proof.** define the following Lyapunov function:

\[
V(t) = \sum_{k=1}^{n} |\epsilon_k(t)|^p.
\]

Taking the derivative of \( V(t) \) along the trajectories of system (3) and using Lemma 3, we obtain that

\[
\dot{V}(t) = \sum_{k=1}^{n} p |\epsilon_k(t)|^{p-1} \nu_k(t) \dot{\epsilon}_k(t)
\]

\[
\leq - p \sum_{k=1}^{n} |\epsilon_k(t)|^p + \sum_{i=1}^{n} \sum_{k=1}^{n} |\epsilon_k(t)|^{p-1} |\alpha_{ki} f_i(\epsilon_k(t))|
\]

\[
+ \sum_{i=1}^{n} |\epsilon_k(t)|^{p-1} |\alpha_{ki} f_i(\epsilon_k(t))| + \sum_{i=1}^{n} |\epsilon_k(t)|^{p-1} |\beta_{ki} f_i(\epsilon_k(t))| + \sum_{i=1}^{n} |\epsilon_k(t)|^{p-1} |\gamma_i(\epsilon_k(t))|,
\]

\[
- \sum_{i=1}^{n} |\epsilon_k(t)|^{p-1} |\alpha_{ki} f_i(\epsilon_k(t))| + \sum_{i=1}^{n} |\epsilon_k(t)|^{p-1} |\beta_{ki} f_i(\epsilon_k(t))| + \sum_{i=1}^{n} |\epsilon_k(t)|^{p-1} |\gamma_i(\epsilon_k(t))|
\]

\[
\leq p \sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} |\epsilon_i(t)|^{p-1} \alpha_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} |\epsilon_i(t)|^{p-1} \beta_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} \alpha_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} \beta_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \gamma_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1}
\]

\[
\leq \sum_{k=1}^{n} |\epsilon_k(t)|^{p-1} \nu_k(t) \dot{\epsilon}_k(t) + \sum_{i=1}^{n} |\epsilon_i(t)|^{p-1} \alpha_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} |\epsilon_i(t)|^{p-1} \beta_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} \alpha_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} \beta_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \gamma_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1}
\]

\[
\leq \sum_{k=1}^{n} |\epsilon_k(t)|^{p-1} \nu_k(t) \dot{\epsilon}_k(t) + \sum_{i=1}^{n} |\epsilon_i(t)|^{p-1} \alpha_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} |\epsilon_i(t)|^{p-1} \beta_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} \alpha_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \sum_{i=1}^{n} \beta_{ki} f_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1} + \gamma_i(\epsilon_k(t)) |\epsilon_i(t)|^{p-1}
\]

(20)
According to (A1) and Lemma 4, we have
\[ |f_i(e_i(t))| \leq F_i |e_i(t)|, \quad (21) \]
\[ \left| \bigwedge_{i=1}^{n} a_{k_i} f_i(e_i(t - \tau_i(t))) \right| \leq \sum_{i=1}^{n} |a_{k_i}| F_i |e_i(t - \tau_i(t))|, \quad (22) \]
\[ \left| \bigvee_{i=1}^{n} \alpha_{k_i} f_i(e_i(t - \tau_i(t))) \right| \leq \sum_{i=1}^{n} |\alpha_{k_i}| F_i |e_i(t - \tau_i(t))|, \quad (23) \]
\[ \| \bigwedge_{i=1}^{n} \beta_{k_i} \int_{t - \sigma_i(t)}^{t} f_i(e_i(s))ds \| \leq \sum_{i=1}^{n} |\beta_{k_i}| F_i \int_{t - \sigma_i(t)}^{t} |e_i(s)|ds, \quad (24) \]
\[ \| \bigvee_{i=1}^{n} \bar{\beta}_{k_i} \int_{t - \sigma_i(t)}^{t} f_i(e_i(s))ds \| \leq \sum_{i=1}^{n} |\bar{\beta}_{k_i}| F_i \int_{t - \sigma_i(t)}^{t} |e_i(s)|ds. \quad (25) \]

Substituting inequalities (21–25) into (20), it produces

\[ \dot{V}(t) \leq - p \sum_{k=1}^{n} (c_k + \eta_k) |e_k(t)|^p + p \sum_{k=1}^{n} \sum_{i=1}^{n} |a_{k_i}| F_i |e_k(t)|^{p-1} |e_i(t)| \\
+ p \sum_{k=1}^{n} \sum_{i=1}^{n} \left( |a_{k_i}| F_i + |\alpha_{k_i}| F_i - \gamma_{k_i} \right) |e_k(t)|^{p-1} |e_i(t - \tau_i(t))| \\
+ p \sum_{k=1}^{n} \sum_{i=1}^{n} \left( |\beta_{k_i}| F_i + |\bar{\beta}_{k_i}| F_i - \omega_{k_i} \right) |e_k(t)|^{p-1} \int_{t - \sigma_i(t)}^{t} |e_i(s)|ds \\
+ p \sum_{k=1}^{n} |e_k(t)|^{p-1} v_k(t) (\delta_k^+ (t) - \delta_k^- (t)) - p \sum_{k=1}^{n} \xi_k |e_k(t)|^{p-1} - p \sum_{k=1}^{n} |e_k(t)|^{p-1} \mu. \quad (26) \]

Based on Lemma 6, we can infer

\[ p \sum_{k=1}^{n} \sum_{i=1}^{n} |a_{k_i}| F_i |e_k(t)|^{p-1} |e_i(t)| = p \sum_{k=1}^{n} |a_{k_k}| F_k |e_k(t)|^p + p \sum_{k=1}^{n} \sum_{i,j \neq k} |a_{k_i}| F_i |e_k(t)|^{p-1} |e_i(t)| \]
\[ = p \sum_{k=1}^{n} |a_{k_k}| F_k |e_k(t)|^p + p \sum_{k=1}^{n} \sum_{i,j \neq k} \left( |a_{k_i}| F_i |e_i(t)| + \left( \prod_{l=1}^{L_{i}} a_{k_l} \right) F_i^{(l_{i+1})} |e_i(t)| \right) \left( \prod_{l=1}^{L_{i}} F_i^{(l_{i+1})} \right)^q |e_k(t)|^q \]
\[ \leq p \sum_{k=1}^{n} |a_{k_k}| F_k |e_k(t)|^p + \sum_{k=1}^{n} \sum_{i,j \neq k} \left( |a_{k_i}| F_i^{(l_{i+1})} |e_i(t)|^p + \sum_{l=1}^{L_{i}} q_l |a_{k_l}| F_i^{(l_{i+1})} |e_i(t)|^q \right) \]
\[ = p \sum_{k=1}^{n} |a_{k_k}| F_k |e_k(t)|^p + \sum_{k=1}^{n} \left( \sum_{i,j \neq k} |a_{k_i}| F_i^{(l_{i+1})} |e_k(t)|^p + \sum_{l=1}^{L_{i}} q_l |a_{k_l}| F_i^{(l_{i+1})} |e_k(t)|^q \right) \]
\[ = p \sum_{k=1}^{n} |a_{k_k}| F_k + \sum_{k=1}^{n} |\phi_k| |e_k(t)|^p. \quad (27) \]

Similarly, it follows from (A2) and Lemma 6 that
\[ p \sum_{k=1}^{n} |e_k(t)|^{p-1} v_k(t) (\delta_k^p(t) - \delta_k^p(t)) \leq p \sum_{k=1}^{n} \sum_{i=1}^{n} A_k |e_k(t)|^{p-1} |e_i(t)| + p \sum_{k=1}^{n} \sum_{i=1}^{n} B_k |e_k(t)|^{p-1} |e_i(t - \tau_i(t))| + p \sum_{k=1}^{n} \sum_{i=1}^{n} C_k |e_k(t)|^{p-1} \int_{t-\tau_i(t)}^{t} |e_i(s)| \, ds + p \sum_{k=1}^{n} |D_k e_k(t)|^{p-1} \leq p \sum_{k=1}^{n} A_k |e_k(t)|^{p} + \sum_{k=1}^{n} \left( \sum_{i=1,i \neq k}^{n} \sum_{j=1,j \neq k}^{n} \sum_{l=1}^{n} \omega_i |A_k|^{p-1} |e_k(t)|^{p} \right) + p \sum_{k=1}^{n} \sum_{i=1}^{n} B_k |e_k(t)|^{p-1} |e_i(t - \tau_i(t))| + p \sum_{k=1}^{n} \sum_{i=1}^{n} C_k |e_k(t)|^{p-1} \int_{t-\tau_i(t)}^{t} |e_i(s)| \, ds \]

Substituting (27) and (28) into (26) yields

\[ \dot{V}(t) \leq -p \sum_{k=1}^{n} \left( e_k + \eta_k |a_{kk}| F_k - A_k - \xi_k - \beta_k \right) |e_k(t)|^{p} + p \sum_{k=1}^{n} \sum_{i=1}^{n} \left( a_{ki} + \tilde{a}_{ki} + B_k - y_{ki} \right) |e_k(t)|^{p-1} |e_i(t - \tau_i(t))| + p \sum_{k=1}^{n} \sum_{i=1}^{n} \left( \beta_{ki} + \tilde{\beta}_{ki} + C_k - \omega_{ki} \right) |e_k(t)|^{p-1} \int_{t-\tau_i(t)}^{t} |e_i(s)| \, ds - p \sum_{k=1}^{n} |e_k(t)|^{p-1+\mu}. \]  

By Lemma 5 and 0 < \mu < 1, one can derive that

\[ \sum_{k=1}^{n} |e_k(t)|^{p-1+\mu} \leq \left( \sum_{k=1}^{n} |e_k(t)|^{p} \right)^{p-1+\mu/p}. \]  

In view of

\[ d = \min_{k \in \mathcal{F}} \left\{ e_k + \eta_k |a_{kk}| F_k - A_k - \xi_k - \beta_k \right\} > 0, \xi_k \geq D_k, \]

\[ y_{ki} \geq |a_{ki}| F_i + |\tilde{a}_{ki}| F_i + B_k, \omega_{ki} \geq |\beta_{ki}| F_i + |\tilde{\beta}_{ki}| F_i + C_k, \]

and combining (29) and (30), we can conclude that

\[ \dot{V}(t) \leq -p \, dV(t) - p \sum_{k=1}^{n} |e_k(t)|^{p-1+\mu} \leq -p \, dV(t) - p \left( \sum_{k=1}^{n} |e_k(t)|^{p} \right)^{p-1+\mu/p} = -p \, dV(t) - pA \left( V(t) \right)^{p-1+\mu/p} (t). \]  

According to Lemma 1 and (32), the error system (3) will converge to zero within \( T = \ln (1 + d/\lambda V^{1-\mu/p}(0))/d(1-\mu) \). Therefore, the response system (2) is synchronized with the drive system (1) in finite time \( T \) under the controller (15). The proof of Theorem 1 is completed.

If there is no uncertain nonlinear perturbation in the drive system (1) and the response system (2), that is, \( \delta_k^p(t) = \delta_k^p(t) = 0 \), then the Assumption 2 is no longer needed and the control parameter \( \xi_k \) in the state feedback controller (15) designed can also be removed. At the same time, Theorem 1 can be simplified. Therefore, we have the following result.

\[ \text{Corollary 1.} \quad \text{suppose that the assumption (A1) and } \delta_k^p(t) = \delta_k^p(t) = 0 \text{ are satisfied, then under the following controller:} \]

\[ u_k(t) = -\eta_k e_k(t) - \lambda \text{sign}(e_k(t)) |e_k(t)|^{\mu} - \sum_{i=1}^{n} \gamma_{ki} \text{sign}(e_k(t)) |e_i(t - \tau_i(t))| - \sum_{i=1}^{n} \omega_{ki} \text{sign}(e_k(t)) \int_{t-\tau_i(t)}^{t} |e_i(s)| \, ds, \]

the response system (2) can be synchronized with drive system (1) in finite time \( T \) if for any \( k \in \mathcal{F} \), and the following conditions hold:

\[ d = \min_{k \in \mathcal{F}} \left\{ e_k + \eta_k |a_{kk}| F_k - A_k - \xi_k - \beta_k \right\} > 0, \]

\[ y_{ki} \geq a_{ki} F_i + \tilde{a}_{ki} F_i, \omega_{ki} \geq \beta_{ki} F_i + \tilde{\beta}_{ki} F_i, \]

where \( \lambda > 0 \), \( \eta_k > 0 \), and \( \gamma_{ki} > 0 \) are positive constants.
When the drive-response system is considered without discrete time delay (or without distributed time delay), that is, $a_{ki} = a_{ki} = 0$ or $\beta_{ki} = \beta_{ki} = 0$, $k, i \in \mathcal{J}$ in (1) and (2), the influence of discrete time delay (or distributed time delay) cannot be considered in the uncertain nonlinear perturbation. Therefore, through the similar discussion to Theorem 1, the following two corollaries can be drawn.

**Corollary 2.** Suppose that the assumptions (A1) and (A2) are satisfied, when the drive-response system without discrete time delays is concerned, that is, $a_{ki} = a_{ki} = 0$, $k, i \in \mathcal{J}$ in (1) and (2); under the following controller:

$$u_k(t) = -\eta_k e_k(t) - \text{sign}(e_k(t))\left(\xi_k + \lambda|e_k(t)|^\mu\right)$$

$$- \sum_{i=1}^{n} \omega_i \text{sign}(e_i(t)) \int_{t-\tau_{i}(t)}^{t} |e_i(s)|ds,$$

(35)

the response system (2) is synchronized with the drive system (1) in finite time $\bar{T}$ if (16) and the following condition hold:

$$\omega_{ki} \geq \tilde{\beta}_{ki} F_i + \tilde{\beta}_{ki} F_i + C_k,$$

(36)

The response system (2) is synchronized with the drive system (1) in finite time $\bar{T}$ if (16) and the following condition hold:

$$\omega_{ki} \geq \tilde{\beta}_{ki} F_i + \tilde{\beta}_{ki} F_i + C_k.$$

**Corollary 3.** Suppose that the assumptions (A1) and (A2) are satisfied, when the drive-response system without distributed time delays is concerned, that is, $\beta_{ki} = \beta_{ki} = 0$, $k, i \in \mathcal{J}$ in (1) and (2); under the following controller:

$$u_k(t) = -\eta_k e_k(t) - \text{sign}(e_k(t))\left(\xi_k + \lambda|e_k(t)|^\mu\right)$$

$$- \sum_{i=1}^{n} \gamma_i\text{sign}(e_i(t))|e_i(t-\tau_i(t))|,$$

(37)

the response system (2) is synchronized with the drive system (1) in finite time $\bar{T}$ if (16) and the following condition hold:

$$\gamma_{ki} \geq a_{ki} F_i + \tilde{a}_{ki} F_i + B_k.$$

(38)

### 3.2. Adaptive Control

It is well known that adaptive controller can automatically change the control intensity and obtain satisfactory results in practical application. Therefore, by applying adaptive technology to state feedback controller, we provide an adaptive controller $\eta_k$ such that the drive-response system can be synchronized in a finite time.

The adaptive controller and the update rules be expressed by:

$$\eta_k(t) = \frac{\sqrt{2} V^{1/2}(0)}{\delta_k},$$

(41)

where $V(0) = 1/2 \sum_{k=1}^{n} |e_k(0)|^2 + 1/2 \sum_{k=1}^{n} (\eta_k(0) - \eta_k^2)/\rho_k + 1/2 \sum_{k=1}^{n} (\xi_k(0) - \xi_k^2)/\gamma_k$, $\delta_k = \min_{k \in \mathcal{J}} \{\delta_k, \sqrt{V_k}, 1/\sqrt{\delta_k}, 1/\sqrt{\gamma_k}\}$.

**Proof.** Define the following Lyapunov function:

$$\bar{V}(t) = \frac{1}{2} \sum_{k=1}^{n} |e_k(t)|^2 + \frac{1}{2} \sum_{k=1}^{n} \frac{(\eta_k(t) - \eta_k)^2}{\rho_k}$$

(42)

Taking the derivative of $\bar{V}(t)$ along the trajectories of system (3), applying Lemma 3 and inequalities (21–25), one obtains

$$\bar{V}(t) = \frac{1}{2} \sum_{k=1}^{n} (\xi_k(t) - \xi_k(t))^2.$$
\[ 
\tilde{V}(t) = \sum_{k=1}^{n} |e_k(t)|u_k(t)\dot{e}_k(t) + \sum_{k=1}^{n} \frac{n_k(t) - \eta_k(t)}{\rho_k} + \sum_{k=1}^{n} \bar{\xi}_k(t) + \tilde{\xi}_k(t)
\]

\[ 
= \sum_{k=1}^{n} |e_k(t)|u_k(t) \left[ -c_k e_k(t) + \sum_{i=1}^{n} a_{ki} f_i(e_i(t)) + \lambda_{i=1}^{n} a_{ki} f_i(e_i(t - \tau_i(t))) \right] + \lambda_{i=1}^{n} \beta_{ki} \int_{t-\sigma_i(t)}^{t} f_i(e_i(s)) ds + \left( \delta_k^+(t) - \delta_k^-(t) \right)
\]

\[ 
+ \lambda_{i=1}^{n} \beta_{ki} \int_{t-\sigma_i(t)}^{t} f_i(e_i(s)) ds - \eta_k(t) e_k(t) - \bar{\xi}_k(t) \text{sign}(e_k(t))
\]

\[ 
- \sum_{i=1}^{n} \gamma_i \text{sign}(e_k(t)) |e_i(t - \tau_i(t))| - \sum_{i=1}^{n} \omega_i \text{sign}(e_k(t)) \int_{t-\sigma_i(t)}^{t} |e_i(s)| ds \right]
\]

\[ 
+ \sum_{k=1}^{n} \frac{\eta_k(t) - \eta_k(t)}{\rho_k} \left( \rho_k |e_k(t)|^2 - \text{sign}(\eta_k(t) - \eta_k(t)) + \sum_{k=1}^{n} \frac{\bar{\xi}_k(t) - \tilde{\xi}_k(t)}{\theta_k} (\theta_k |e_k(t)| - \text{sign}(\bar{\xi}_k(t) - \tilde{\xi}_k(t))) \right)
\]

\[ 
\leq \sum_{k=1}^{n} -c_k |e_k(t)|^2 + \sum_{i=1}^{n} |a_{ki}| f_i |e_k(t)||e_i(t)| + \sum_{i=1}^{n} \left( |a_{ki}| f_i + |\bar{a}_{ki}| f_i - \omega_i \right) |e_k(t)||e_i(t - \tau_i(t))|
\]

\[ 
+ \sum_{i=1}^{n} \left( |\beta_{ki}| f_i + |\bar{\beta}_{ki}| f_i - \omega_i \right) |e_k(t)| \int_{t-\sigma_i(t)}^{t} |e_i(s)| ds + \sum_{k=1}^{n} \gamma_i \text{sign}(e_k(t)) |e_i(t - \tau_i(t))|
\]

\[ 
- \sum_{k=1}^{n} (\bar{\xi}_k - D_k) |e_k(t)| - \sum_{k=1}^{n} \left( \frac{\eta_k(t) - \eta_k(t)}{\rho_k} - \sum_{k=1}^{n} \frac{\bar{\xi}_k(t) - \tilde{\xi}_k(t)}{\theta_k} \right)
\]

It can be derived from Lemma 6 and (27) that

\[ 
\sum_{k=1}^{n} \sum_{i=1}^{n} |a_{ki}| f_i |e_k(t)||e_i(t)| \leq \sum_{k=1}^{n} |a_{kk}| f_k |e_k(t)|^2 + \frac{1}{2} \sum_{k=1}^{n} \left( \sum_{i=1, i \neq k}^{n} \left| a_{ik} \right|^2 \Delta_{(t)} \Delta_{(t+1)} \Delta_{(t+1)} \Delta_{(t+1)} \Delta_{(t+1)} \right) |e_k(t)|^2
\]

\[ 
= \sum_{k=1}^{n} \left( |a_{kk}| f_k + \mathscr{S}_k \right) |e_k(t)|^2.
\]

Similarly, according to (A2) and (28), we get

\[ 
\sum_{k=1}^{n} |e_k(t)| u_k(t) \left( \delta^+(t) - \delta^-(t) \right) \leq \sum_{k=1}^{n} A_k |e_k(t)|^2 + \frac{1}{2} \sum_{k=1}^{n} \left( \sum_{i=1, i \neq k}^{n} A_{ik} \Delta_{(t)} \Delta_{(t+1)} \Delta_{(t+1)} \Delta_{(t+1)} \Delta_{(t+1)} \right) |e_k(t)|^2 + \sum_{k=1}^{n} B_k |e_k(t)| e_i(t - \tau_i(t))
\]

\[ 
+ \sum_{k=1}^{n} \sum_{i=1}^{n} C_k |e_k(t)| \int_{t-\sigma_i(t)}^{t} |e_i(s)| ds + \sum_{k=1}^{n} D_k |e_k(t)| = \sum_{k=1}^{n} (A_k + \mathscr{S}_k) |e_k(t)|^2
\]

\[ 
+ \sum_{k=1}^{n} \sum_{i=1}^{n} B_k |e_k(t)||e_i(t - \tau_i(t))| + \sum_{k=1}^{n} \sum_{i=1}^{n} C_k |e_k(t)| \int_{t-\sigma_i(t)}^{t} |e_i(s)| ds + \sum_{k=1}^{n} D_k |e_k(t)|.
\]

Substituting (44) and (45) into (43), we have
\[ \dot{V}(t) \leq \sum_{k=1}^{n} \left( c_k + \eta_k \left| a_{sk} \right| F_k - A_k - \omega_f \right) \left| e_k(t) \right|^2 + \sum_{k=1}^{n} \left( \left| \beta_k \right| F_i + \left| \beta_k \right| F_i + C_k - \omega_k \right) \left| e_k(t) \right| e_i(t-\tau_i(t)) \right] + \sum_{k=1}^{n} \left( \left| \beta_k \right| F_i + \left| \beta_k \right| F_i + C_k - \omega_k \right) \int_{t-\sigma_i(t)}^{t} e_i(s) \, ds - \sum_{k=1}^{n} \left( \xi_k - D_k \right) \left| e_k(t) \right| - \sum_{k=1}^{n} \eta_k \left| e_k(t) \right| - \sum_{k=1}^{n} \left| \xi_k(t) - \xi_k \right| - \sum_{k=1}^{n} \left| \xi_k(t) - \xi_k \right| - \sum_{k=1}^{n} \left| \xi_k(t) - \xi_k \right| \right] (46) \]

By virtue of
\[ \eta_k \geq \left| a_{sk} \right| F_k + A_k + \omega_f + \omega_k - c_k, \xi_k \geq D_k, Y_{ki} \]
\[ \geq \left| a_{sk} \right| F_k + A_k + \omega_f + \omega_k \left| \beta_k \right| F_i + \left| \beta_k \right| F_i + C_k. (47) \]

Let \( \theta_k = \xi_k - D_k \) and choose \( \theta_k > 0 \), it is derived from Lemma 5 that
\[ \dot{V}(t) \leq -\sum_{k=1}^{n} \theta_k \left| e_k(t) \right| - \sum_{k=1}^{n} \eta_k \left| e_k(t) \right| - \sum_{k=1}^{n} \left| \xi_k(t) - \xi_k \right| \]
\[ \leq -\sum_{k=1}^{n} \left| e_k(t) \right|^2 + \sum_{k=1}^{n} \eta_k \left| e_k(t) \right| - \sum_{k=1}^{n} \left| \xi_k(t) - \xi_k \right| \]
\[ \leq -\sum_{k=1}^{n} \left| e_k(t) \right|^2 + \sum_{k=1}^{n} \eta_k \left| e_k(t) \right| - \sum_{k=1}^{n} \left| \xi_k(t) - \xi_k \right| \]
\[ = -\sqrt{2} \theta \dot{V}(t). (48) \]

According to Lemma 2 and (48), the error system (3) will converge to zero within \( \dot{V} = \sqrt{2} \theta \dot{V}(0) / \sqrt{2} \theta \). Therefore, the response system (2) is synchronized with the drive system (1) in finite time \( \bar{T} \) under the controller (39). The proof of Theorem 2 is completed.

**Remark 1.** Compared with traditional state feedback controller \( u_k(t) = -\eta_k e_k(t) \), the state feedback controller (15) includes many compensation terms. Such a novel controller can effectively deal with the finite-time synchronization problems. From Theorem 1, it can be seen that control parameters \( \eta_k, \xi_k, Y_{ki} \), and \( \omega_{ki} \) play an important role in the finite-time synchronization of drive system (1) and response system (2). In addition, the settling time \( \bar{T} \) is inversely proportional to the tunable constant \( \lambda \), which means that a great \( \lambda \) results in short time \( \bar{T} \).

**Remark 2.** It is obvious from Theorem 2 that the settling time of finite-time synchronization can be effectively adjusted by control parameter \( \xi_k, \rho_k, \) and \( \eta_k \) of the adaptive controller (35). Besides, adding \( \sum_{i=1}^{n} \gamma_{ki} \text{sign}(e_i(t)) |e_i(t-\tau_i(t))| \sum_{i=1}^{n} \omega_{ki} \text{sign}(e_i(t)) \int_{t-\sigma_i(t)}^{t} |e_i(s)| ds \) into the adaptive controller (35) can further eliminate the influence of time delays and perturbations upon the states of fuzzy neural networks with hybrid delays and uncertain nonlinear perturbations.

**Table 2: Model comparisons.**

<table>
<thead>
<tr>
<th>Models</th>
<th>[20]</th>
<th>[31]</th>
<th>[41]</th>
<th>[32]</th>
<th>[34]</th>
<th>This paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy terms</td>
<td>( \checkmark )</td>
<td>( \times )</td>
<td>( \times )</td>
<td>( \times )</td>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
<td>Time-varying delays</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>Distributed time-varying delays</td>
<td>( \times )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>Perturbation terms</td>
<td>( \times )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
</tbody>
</table>

**Remark 3.** If we make the perturbation term \( \delta_k(t) = I_k \) in the system, then many previous neural network models are the special cases of system (1). It is also clear from Table 2 that the model of this paper is more general. Where “\( \checkmark \)” means that the model contains this component, “\( \times \)” means that the model does not contain this component. Moreover, since there is no finite-time synchronization of the fuzzy neural network as in form (1), the two different forms of controllers designed in this paper can successfully achieve the finite-time synchronization of the considered drive-response system. This shows that the finite-time synchronization results obtained in this paper are new.

**Remark 4.** Since controllers (15), (33), (35), (37) and (39) contain discrete symbolic functions, some undesirable chattering may occur when synchronization is implemented. To prevent this from happening, the methods in [20] are used, and the continuous \( \tanh(\cdot) \) function can be used instead of the discontinuous symbolic function. For example, the control law (15) can be redesigned as
\[ u_k(t) = -\eta_k e_k(t - \tanh(\omega_{ki} e_k(t)) \xi_k) \leq D_k, Y_{ki} \]
\[ \leq \left| a_{sk} \right| F_k + A_k + \omega_f \left| \beta_k \right| F_i + \left| \beta_k \right| F_i + C_k. (47) \]

\[ \leq -\sum_{k=1}^{n} \gamma_{ki} \tanh(\omega_{ki} e_k(t)) e_i(t - \tau_i(t)) \]
\[ -\sum_{k=1}^{n} \omega_{ki} \tanh(\omega_{ki} e_k(t)) \left| e_i(t - \tau_i(t)) \right| |e_i(s)| ds, (49) \]

where \( \omega_{ki} > 0 \) and \( k \in \mathcal{K} \).

**4. Illustrative Examples**

Here, three numerical examples are provided to demonstrate that our obtained theoretical results are effectiveness.

**Example 1.** We consider the following 2-D neural network model as the drive system:
\[ \dot{x}_k(t) = -c_k x_k(t) + \sum_{i=1}^{n} a_{ki} f_i(x_i(t)) + \sum_{i=1}^{n} b_{ki} y_i + \lambda \sum_{i=1}^{n} S_k V_i \]
\[ + \sqrt{\sigma_{ki}} W_k V_i + \delta_k(t) \]
\[ + \lambda \sum_{i=1}^{n} a_{ki} f_i(x_i(t - \tau_i(t))) + \sqrt{\sigma_{ki}} W_k f_i(x_i(t - \tau_i(t))) \]
\[ + \lambda \sum_{i=1}^{n} b_{ki} \int_{t-\tau_i(t)}^{t} f_i(x_i(s)) ds \]
\[ + \sqrt{\sigma_{ki}} W_k \int_{t-\tau_i(t)}^{t} f_i(x_i(s)) ds, (50) \]

for \( k \in \{1,2\} \), where \( x(t) = (x_1(t), x_2(t))^T \). Other parameters of system (50) are as follows:
Figure 2: (a) Trajectories of \( x_1(t) \) and \( y_1(t) \) for drive-response system without controller; (b) trajectories of \( x_2(t) \) and \( y_2(t) \) for drive-response system without controller.

\[
c_1 = 1.2, c_2 = 1
\]
\[
f_i(x) = \tanh(x), F_i = 1, i = 1, 2
\]
\[
(a_{ki})_{2\times2} = \begin{pmatrix} 1.2 & -1.5 \\ -0.1 & 2.5 \end{pmatrix}, (b_{ki})_{2\times2} = (S_{ki})_{2\times2} = (H_{ki})_{2\times2}
\]
\[
= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}
\]
\[
(a_{ki})_{2\times2} = \begin{pmatrix} -0.1 & -0.4 \\ -1.5 & -1.8 \end{pmatrix}, (\bar{a}_{ki})_{2\times2} = \begin{pmatrix} -0.2 & 1.2 \\ -1.6 & -2.5 \end{pmatrix}
\]
\[
(b_{ki})_{2\times2} = \begin{pmatrix} -1.2 & 0.1 \\ 2.8 & -1 \end{pmatrix}, (\bar{b}_{ki})_{2\times2} = \begin{pmatrix} -1.8 & 0.2 \\ 2.5 & -1.5 \end{pmatrix}
\]
\[
v_1 = v_2 = 1, \tau_i(t) = \sigma_i(t) = 1
\]
\[
\delta^c_1(t) = 0.1 \sum_{i=1}^2 x_i(t) + 0.1 \sum_{i=1}^2 x_i(t - 1) + 0.1 \sum_{i=1}^2 x_i(t - 1) + 0.1
\]
\[
\delta^c_2(t) = 0.2 \sum_{i=1}^2 x_i(t) + 0.2 \sum_{i=1}^2 x_i(t - 1) + 0.2 \sum_{i=1}^2 x_i(t - 1) + 0.2
\]
\[
\zeta = 1, A_1 = B_1 = C_1 = D_1 = 0.1, A_2 = B_2 = C_2 = D_2 = 0.2.
\]

The corresponding response system is depicted as follows:

\[
\dot{y}_k(t) = -c_k y_k(t) + \sum_{i=1}^2 a_{ki} f_i(y_i(t)) + \sum_{i=1}^2 b_{ki} v_i
\]
\[
+ \lambda^2 \sum_{i=1}^2 \bar{a}_{ki} f_i(y_i(t - \tau_i(t)))
\]
\[
+ \nu^2 \sum_{i=1}^2 \bar{b}_{ki} f_i(y_i(t - \tau_i(t)))
\]

for \( k \in \{1, 2\} \), in which the parameters in system (52) are the same as the drive system (50). If \( u_1(t) = u_2(t) \equiv 0 \), in response system (52), then state trajectories and synchronization error trajectories of (50)–(52) with the initial conditions \( \varphi(s) = (1.8, -2.8)^T \), \( \varphi(s) = (-2.6, 1.5)^T \), and \( s \in [-1, 0] \) are presented in Figures 2 and 3. It can be clearly seen that system (50) and system (52) are not synchronized. Therefore, to enable them to synchronize in a finite time, the controller is given by

\[
u_k(t) = -\eta_k e_k(t) - \tanh(e_k(t))(\xi_k + 2|e_k(t)|^{0.8}) - \sum_{i=1}^2 y_i \tanh(e_i(t))|e_i(t - \tau_i(t))|
\]

(53)

where \( e_k(t) = y_k(t) - x_k(t) \) for \( k = 1, 2 \). By simple calculations, we can select the following control gains
It is obvious that all the conditions for Theorem 1 are completely satisfied. Therefore, system (52) is synchronized with system (50) in finite time $T = 2.74$. The state trajectories of the drive-response system under the controller (53) are shown in Figure 4. The synchronization error trajectories between the drive system (50) and the corresponding response system (52) under the controller (53) are presented in Figure 5. The validity of our obtained Theorem 1 can be seen intuitively in Figures 4 and 5.

Example 2. We consider the following 3-D neural network model as the drive system:

$$
\eta_1 = 2, \eta_2 = 8.5, \xi_1 = 0.12, \xi_2 = 0.25
\left( y_{ki} \right)_{2 \times 2} = \left(\begin{array}{cc} 0.5 & 1.8 \\ 3.4 & 4.6 \end{array}\right), \left( w_{ki} \right)_{2 \times 2} = \left(\begin{array}{cc} 3.2 & 0.5 \\ 4.6 & 2.8 \end{array}\right)
$$

$$p = 2, \alpha^{(l)}_{ki} = \beta^{(l)}_{ki} = \gamma^{(l)}_{ki} = 0.5, l = 1, 2, q_1 = w_1 = 1.
$$

(54)

$$
\dot{x}_k(t) = -c_k x_k(t) + \sum_{i=1}^{3} a_{ki} f_i(x_i(t)) + \sum_{i=1}^{3} b_{ki} v_i + \lambda_{i=1}^{3} s_{ki} v_i
+ \nu_{i=1}^{3} H_{ki} v_i + \delta_{ki}^2(t)
+ \lambda_{i=1}^{3} a_{ki} f_i(x_i(t - \tau_i(t)))
+ \nu_{i=1}^{3} \bar{a}_{ki} f_i(x_i(t - \tau_i(t)))
+ \nu_{i=1}^{3} \bar{\beta}_{ki} \int_{t - \sigma_i(t)}^{t} f_i(x_i(s)) ds
+ \nu_{i=1}^{3} \bar{\gamma}_{ki} \int_{t - \sigma_i(t)}^{t} f_i(x_i(s)) ds,
$$

(55)

for $k \in \{1, 2, 3\}$, where $x(t) = (x_1(t), x_2(t), x_3(t))^T$ and the other coefficients and functions are taken as.
\( e_1(t), e_2(t) \)

\[
\begin{align*}
    c_1 &= 1.8, c_2 = 1.5, c_3 = 1.2, \\
    f_i(x) &= \frac{1}{2} (|x + 1| - |x - 1|), F_i = 1, i = 1, 2, 3, \\
    (a_{ki})_{3 \times 3} &= \begin{pmatrix} 0.8 & -2 & -1.2 \\ -1.6 & 0.6 & 2.8 \\ -2.5 & -2.2 & 0.4 \end{pmatrix}, (b_{ki})_{3 \times 3} = (S_{ki})_{3 \times 3} = (H_{ki})_{3 \times 3} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{pmatrix}, \\
    (a_{ki})_{3 \times 3} &= \begin{pmatrix} -1.8 & -1.5 & -0.7 \\ -2.5 & -1.6 & -2.8 \\ -0.3 & -1.4 & -1.2 \end{pmatrix}, (\bar{a}_{ki})_{3 \times 3} = \begin{pmatrix} -2 & -1.2 & -1.5 \\ -1.4 & -2.2 & -2.6 \\ -0.4 & -1.6 & -1 \end{pmatrix}, \\
    (\beta_{ki})_{3 \times 3} &= \begin{pmatrix} -2.1 & 2.5 & 2.6 \\ 0.2 & -1.2 & -2.8 \\ 1.8 & 1.6 & -1.4 \end{pmatrix}, (\bar{\beta}_{ki})_{3 \times 3} = \begin{pmatrix} -1.2 & 2.6 & 1.4 \\ 1 & -1.8 & -2.6 \\ 1.4 & 1.8 & -1.6 \end{pmatrix}, \\
    \nu_1 &= \nu_2 = \nu_3 = 1, \tau_i(t) = |\sin t|, \sigma_i(t) = |\cos t|, \\
    \delta_i^x(t) &= \delta_i^y(t) = \delta_i^z(t) = 0.3 \sum_{i=1}^{3} x_i(t) + 0.3 \sum_{i=1}^{3} x_i(t - |\sin t|) + 0.3 \sum_{i=1}^{3} \int_{t-|\cos t|}^{t} x_i(s)ds + 0.3, \\
    \zeta &= 1, A_k = B_k = C_k = D_k = 0.3.
\end{align*}
\]

The corresponding response system is depicted as follows:

\[
\begin{align*}
    \dot{y}_k(t) &= -c_ky_k(t) + \sum_{i=1}^{3} a_{ki}f_i(y_i(t)) + \sum_{i=1}^{3} b_{ki}\nu_i + \lambda^{3}_{i=1} S_{ki}\nu_i + \nu^{3}_{i=1} H_{ki}\nu_i + \delta_i^x(t) \\
    &+ \lambda^{3}_{i=1} a_{ki}f_i(y_i(t - \tau_i(t))) + \nu^{3}_{i=1} \bar{a}_{ki}f_i(y_i(t - \tau_i(t))) + \lambda^{3}_{i=1} \beta_{ki} \int_{t-\sigma_i(t)}^{t} f_i(y_i(s))ds \\
    &+ \nu^{3}_{i=1} \bar{\beta}_{ki} \int_{t-\sigma_i(t)}^{t} f_i(y_i(s))ds + u_k(t),
\end{align*}
\]

**Figure 5:** Trajectories of \( e_1(t) \) and \( e_2(t) \) for drive-response system under controller (53).
for \( k \in \{1, 2, 3\} \), in which the parameters in system (57) are the same as the drive system (55). If \( u_1(t) = u_2(t) = u_3(t) \equiv 0 \) in response system (57), then state trajectories and error trajectories of (55)–(57) with the initial conditions \( \phi(s) = (0.6, 0.4, -0.2)^T \), \( \phi(s) = (-0.5, -0.1, 0.3)^T \), and \( s \in [-1, 0] \) are presented in Figure 6. It can be clearly seen that system (55) and system (57) are not synchronized. Therefore, to enable them to synchronize in a finite time, the controller is given by

\[
\begin{aligned}
    u_k(t) &= -\eta_k(t)e_k(t) - \xi_k(t)\text{sign}(e_k(t)) \\
    - \sum_{i=1}^3 \gamma_k \text{sign}(e_k(t))|e_i(t - \tau_i(t))| \\
    - \sum_{i=1}^3 \omega_k \text{sign}(e_k(t)) \int_{t-\sigma_i(t)}^t |e_i(s)| \, ds \\
    \dot{\eta}_k(t) &= |e_k(t)|^2 - \text{sign}(\eta_k(t) - \eta_k) \\
    \dot{\xi}_k(t) &= |e_k(t)| - \text{sign}(\xi_k(t) - \xi_k),
\end{aligned}
\]  

(58)

Figure 6: (a) Trajectories of \( x_1(t) \) and \( y_1(t) \) for drive-response system without controller; (b) trajectories of \( x_2(t) \) and \( y_2(t) \) for drive-response system without controller; (c) trajectories of \( x_3(t) \) and \( y_3(t) \) for drive-response system without controller; (d) trajectories of \( e_1(t), e_2(t) \) and \( e_3(t) \) for drive-response system without controller.
where $c_k(t) = y_k(t) - x_k(t)$ for $k = 1, 2, 3$. By simple calculations, we can select

$$
\eta_1 = 3.5, \eta_2 = 4.2, \eta_3 = 4.5, \xi_1 = \xi_2 = \xi_3 = 1.3
$$

$$(y_{ki})_{3 \times 3} = \begin{pmatrix}
4.2 & 3.2 & 2.6 \\
4.4 & 4.3 & 5.8 \\
0.8 & 3.4 & 2.4
\end{pmatrix},
(\omega_{ki})_{3 \times 3} = \begin{pmatrix}
3.8 & 5.6 & 4.5 \\
1.6 & 3.4 & 5.8 \\
3.6 & 3.8 & 3.4
\end{pmatrix}
$$

$$
\rho_k = \gamma_k = 1,
$$

$$
p = 2, \delta_{ki}^{(l)} = \epsilon_{ki}^{(l)} = 0.5, l = 1, 2, q_1 = w_1 = 1.
$$

(59)

It is easy to verify that the above parameters satisfy all the conditions of Theorem 2. Hence, the neural networks (55)–(57) can be synchronized with finite time $\mathbb{T} = 3.46$. The trajectories of the control gains of controller (58) with the initial conditions $\eta(s) = (1.5, 2.4, 3.2)$ and $\xi(s) = (0.3, 0.8, 1), s \in [-1, 0]$ are shown in Figure 7. And the state trajectories and error trajectories of the drive-response system are presented in Figure 8 under the adaptive controller (58). The effectiveness of our obtained Theorem 2 can be seen intuitively in Figure 8.

**Example 3.** If $a_{ki} = \tilde{a}_{ki} = 0$, $b_{ki} = \tilde{b}_{ki} = 0$, and $\delta_k(t) = I_k$, then system (1) is transformed into the following neural networks model:

$$
\dot{x}_k(t) = -c_kx_k(t) + \sum_{i=1}^{2} a_{ki}f_i(x_i(t)) + I_k,
$$

(60)

for $k \in \{1, 2\}$, where $x(t) = (x_1(t), x_2(t))^T$. The parameters $c_k, a_{ki}$, and $f_i$ of system (60) are the same as those of system (50) and $I_k = 0$. Taking system (60) as the drive system, the response system has the following form:

$$
y_k(t) = -c_ky_k(t) + \sum_{i=1}^{2} a_{ki}f_i(y_i(t)) + I_k + u_k(t),
$$

(61)

for $k \in \{1, 2\}$, in which the parameters in system (61) are the same as the drive system (60). Meanwhile, the initial conditions for the drive system (60) and response system (61) are the same as in Example 1. If $u_1(t) = u_2(t) \equiv 0$ in response system (61), then state trajectories and synchronization error trajectories of (60) and (61) are shown in Figures 9 and 10. It is obvious that the systems are not synchronized. Therefore, to enable them to synchronize in a finite time, we change the controller (15) to the following form:

$$
u_k(t) = -\eta_k c_k(t) - \lambda \tanh(c_k(t))\|c_k(t)\|^\mu,
$$

(62)

where $c_k(t) = y_k(t) - x_k(t)$ for $k = 1, 2$. The values of $\eta, \lambda, \mu, p, q, l, i, k$, and $F$ are the same as in Example 1. In the case, the condition in Theorem 1 is satisfied. Therefore, system (61) is synchronized with system (60) in finite time $\mathbb{T} = 2.60$. The state trajectories of the drive-response system under the controller (62) are described in Figure 11. The synchronization error trajectories between the drive system (60) and the corresponding response system (61) under the controller (62) are presented in Figure 12.

The validity of Theorem 1 and Theorem 2 we have obtained are evident from Example 1 and Example 2. It can be seen from Example 3 that the controller designed in this paper can change its form according to the change of the state or disturbance of the system so that the considered system can achieve finite-time synchronization. In addition, the convergence time is shortened when the complexity of the system is reduced. Therefore, the controller designed by this paper is more general and practical.
5. Discussion

Based on the numerical simulations given in Section 4, it can be intuitively seen that both controllers designed in this paper can achieve finite-time synchronization of the considered drive-response system. Among them, since the internal characteristics of the system can be fully reflected by the state variables, the state feedback is beneficial to improving the control performance of the system. However, in the actual system, the introduced unknown feedback gain needs to be solved, but most of the obtained feedback gains are large and difficult to achieve. The adaptive controller can avoid large control gains and is also suitable when the system parameters are unknown or the model is uncertain, but its cost is very high. Therefore, in practical applications, the designer should select the corresponding controller according to their needs to achieve better control effects.

5.1. Comparative Analysis. In Section 3, finite-time synchronization of system (1) is realized by designing state feedback controller and adaptive controller. In the existing research, many scholars also use feedback control to realize finite-time synchronization of neural networks. Abdurahman et al. [20] realized the finite-time synchronization of the following neural networks based on the theory of finite-time convergence via controller $u_k(t) = -\eta_k e_k(t) - \xi_k \text{sign}(e_k(t)) - \lambda \text{sign}(e_k(t)) |e_k(t)|^\mu$. 

![Figure 8](a) Trajectories of $x_1(t)$ and $y_1(t)$ for drive-response system under the controller (58); (b) trajectories of $x_2(t)$ and $y_2(t)$ for drive-response system under the controller (58); (c) trajectories of $x_3(t)$ and $y_3(t)$ for drive-response system under the controller (58); (d) trajectories of $e_1(t), e_2(t)$, and $e_3(t)$ for drive-response system under the controller (58).
If we make $\beta_{ki} = \tilde{\beta}_{ki} = 0$ and replace the perturbation term $\delta_{ki}(t)$ with $I_k$ in system (1), and make $b_{ki} = 0$ in the above system, then the two systems are identical. At the same time, finite-time synchronization of the transformed system can be achieved by changing the form of the controller (15). It is worth noting that the FCNNs considered in [20] have no distributed delay, and its results cannot be adopted in the fuzzy neural networks with distributed delays. Therefore, the results obtained in this paper are more general. Besides, in [42–44], the authors put forward some criteria to ensure the finite-time synchronization of complex networks and coupled neural networks according to matrix inequalities. Compared with the sufficient criterion of the matrix inequalities obtained by them, several criteria in this paper are easier to verify with parameters and avoid the complicated calculation of the matrix inequalities. Compared with [20, 21, 27, 31, 36, 45, 46], the delay-independent feedback controllers they designed are not suitable for model (1), while the feedback controller (15) can change its form according to the state of the system or the change of

$$\dot{x}_k(t) = -c_k x_k(t) + \sum_{i=1}^{n} a_{ki} f_i(x_i(t)) + \sum_{i=1}^{n} b_{ki} f_i(x_i(t - \tau_i(t))) + \sum_{i=1}^{n} b_{ki} v_i + \sum_{i=1}^{n} S_{ki} v_i + v_j^p H_{ki} v_i$$

$$+ \sum_{i=1}^{n} \alpha_{ki} f_i(x_i(t - \tau_i(t))) + v_j^p \bar{a}_{ki} f_i(x_i(t - \tau_i(t))) + I_k.$$  

(63)
perturbation to achieve finite-time synchronization of the models studied by them. In addition, for some uncertain systems and systems with unknown parameters, the adaptive controller (35) can more effectively achieve finite-time synchronization of these systems. Overall, the controller designed in this paper is more flexible and suitable for more complex systems.

5.2. Application in Secure Communication. Secure communication using synchronization between chaotic systems (chaotic secure communication, for short) is a new concept of secure communication [47]. In secure communication, message encryption is performed when two people want to communicate with each other over an insecure communication channel, but a third person cannot identify the message by listening. Therefore, one of the main objectives of chaotic secure communication is to protect information from eavesdropping and interception. It is worth noting that we can regard the communication network as a chaotic neural network, in which the transmitter and receiver correspond to the drive system and the response system, respectively. We apply the feedback controller synchronization designed in Theorem 1 to make the receiver track the sender accurately in finite time. An information signal \( h(t) \) containing the message to be transmitted can be masked by a chaotic signal \( x(t) \) and recovered using finite-time synchronization. Different strategies can be used to make the actual transmitted signal \( p(t) \) as broadband as possible. In general, three strategies are involved in secure communications with chaos [48]:

1. Signal masking, where \( p(t) = x(t) + \pi h(t) \);
2. Modulation, where \( p(t) = x(t)h(t) \);
3. Masking and modulation, where \( p(t) = x(t)[1 + \pi h(t)] \).

For the sake of illustration, we only use signal masking here, which is \( p(t) = x(t) + \pi h(t) \). For another, we can extract messages from receivers in a communication network through finite-time synchronization. The proposed communication system consisting of a transmitter and a receiver is shown in Figure 13. Masking techniques are proposed as follows. The transmitter is described as

![Figure 11](image1.png)

Figure 11: (a) Trajectories of \( x_1(t) \) and \( y_1(t) \) for drive-response system under controller (62); (b) trajectories of \( x_2(t) \) and \( y_2(t) \) for drive-response system under controller (62).

![Figure 12](image2.png)

Figure 12: Trajectories of \( e_1(t) \) and \( e_2(t) \) for drive-response system under controller (62).
\( \dot{x}_1(t) = -c_1x_1(t) + \sum_{i=1}^{2} a_{ii}f_i(x_i(t)) + \sum_{i=1}^{2} b_{ii}v_i + \lambda_{i=1}^{2}S_{ii}v_i + \nu_{i=1}^{2}H_{ii}v_i + \delta_i^x(t) \\
+ \lambda_{i=1}^{2}a_{ii}f_i(x_i(t - \tau_i(t))) + \nu_{i=1}^{2}a_{ii}f_i(x_i(t - \tau_i(t))) + \lambda_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(x_i(s))\,ds \\
+ \nu_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(x_i(s))\,ds + h_1(t), \tag{64} \)

\( \dot{x}_2(t) = -c_2x_2(t) + \sum_{i=1}^{2} a_{ii}f_i(x_i(t)) + \sum_{i=1}^{2} b_{ii}v_i + \lambda_{i=1}^{2}S_{ii}v_i + \nu_{i=1}^{2}H_{ii}v_i + \delta_i^x(t) \\
+ \lambda_{i=1}^{2}a_{ii}f_i(x_i(t - \tau_i(t))) + \nu_{i=1}^{2}a_{ii}f_i(x_i(t - \tau_i(t))) + \lambda_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(x_i(s))\,ds \\
+ \nu_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(x_i(s))\,ds + h_2(t), \)

where the parameters are the same as in Example 1. \( h_1(t) = \pi h(t) \) is the information message, and we set \( \pi = 0.05 \) makes the energy of information signal much less than chaotic carrier signal. In order not to lose generality, we make \( h_2(t) = 0 \). Accordingly, the receiver is stated as

\( \dot{y}_1(t) = -c_1y_1(t) + \sum_{i=1}^{2} a_{ii}f_i(y_i(t)) + \sum_{i=1}^{2} b_{ii}v_i + \lambda_{i=1}^{2}S_{ii}v_i + \nu_{i=1}^{2}H_{ii}v_i + \delta_i^y(t) \\
+ \lambda_{i=1}^{2}a_{ii}f_i(y_i(t - \tau_i(t))) + \nu_{i=1}^{2}a_{ii}f_i(y_i(t - \tau_i(t))) + \lambda_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(y_i(s))\,ds \\
+ \nu_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(y_i(s))\,ds + u_1(t) - y_1(t) + P_1(t), \tag{65} \)

\( \dot{y}_2(t) = -c_2y_2(t) + \sum_{i=1}^{2} a_{ii}f_i(y_i(t)) + \sum_{i=1}^{2} b_{ii}v_i + \lambda_{i=1}^{2}S_{ii}v_i + \nu_{i=1}^{2}H_{ii}v_i + \delta_i^y(t) \\
+ \lambda_{i=1}^{2}a_{ii}f_i(y_i(t - \tau_i(t))) + \nu_{i=1}^{2}a_{ii}f_i(y_i(t - \tau_i(t))) + \lambda_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(y_i(s))\,ds \\
+ \nu_{i=1}^{2}b_{ii} \int_{t-\sigma_i(t)}^{t} f_i(y_i(s))\,ds + u_2(t) - y_2(t) + P_2(t), \)

where \( P_1(t) = p(t) = x_1(t) + \pi h(t) \) is the transmission signal of the system. \( P_2(t) = x_2(t) \). The transmitted information signal can be recovered by transformation \( r(t) = \pi^{-1}[x_1(t) + \pi h(t) - y_1(t)] \). We use the same functions and parameters as in Example 1, and the information signal is arbitrarily selected as \( h(t) = \sin(t) \). The information signal \( h(t) \), transmitted signal \( p(t) \), recovered signal \( r(t) \), and error between the information signal \( h(t) \)
and the recovered signal $r(t)$ are depicted in Figure 14. The simulation results show that the information signal can be recovered in finite time under the feedback controller (15).

### 6. Conclusion

In this paper, we discuss the finite-time synchronization of a class of fuzzy neural networks with hybrid delays and uncertain nonlinear perturbations. Through the application of famous finite-time stability theory, differential inequality techniques, and the analysis approach, useful state feedback controller and adaptive controller are involved, and several new algebraic sufficient criteria are derived to realize finite-time synchronization. In addition, the system studied in this paper can contain many previous neural networks, so the system in this paper is more general. Finally, the main results obtained are as follows:

(a) Some new algebraic sufficient criteria are proposed to ensure finite-time synchronization between the drive system and the response system, and the settling time is also estimated.

(b) The designed state feedback controller and adaptive controller not only realize the finite-time synchronization of the drive-response system but also can change their forms according to the change of the state or perturbation of the system to obtain better a control effect.

(c) The studied model and the designed controller can be effectively used in other practical applications such as secure communication.

It is well known that the settling time for finite-time synchronization depends essentially on the initial conditions of the system. However, in many practical applications, it is difficult or impossible to obtain accurate values for the initial conditions of systems such as industrial systems, robots, and biological models, which greatly limits practical applications. While in [49], Polyakov proposed the concept of fixed-time stability and pointed out that the time of fixed-time synchronization is independent of the initial synchronization error, and the settling time can be estimated in advance. From this perspective, fixed-time synchronization has more advantages. In some circumstances, there are several abrupt...
changes at certain moments in physical systems since instantaneous disturbances, which are called impulsive effects [50]. The influence of impulsive phenomena may appear in some other fields, such as automatic control system and artificial intelligence. Therefore, for future research, we will consider fixed-time synchronization of fuzzy neural networks with time delays and impulsive effects.

Data Availability

No data were used to support this research.

Conflicts of Interest

The authors declared that they have no conflicts of interest.

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