Research Article

Fuzzy Nano $\delta$-locally Closed Sets, Extremally Disconnected Spaces, Normal Spaces, and Their Application

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In this paper, we introduce fuzzy nano (resp. $\delta, \delta S, P$ and $Z$-) locally closed set and fuzzy nano (resp. $\delta, \delta S, P$ and $Z$-) extremally disconnected spaces in fuzzy nano topological spaces. Also, we introduce some new spaces called fuzzy nano (resp. $\delta, \delta S, P$ and $Z$-) normal spaces and strongly fuzzy nano (resp. $\delta, \delta S, P$ and $Z$-) normal spaces with the help of fuzzy nano (resp. $\delta, \delta S, P$ and $Z$-) open sets in fuzzy nano topological space. Numerical data is used to quantify the provided features. Furthermore, using fuzzy nano topological spaces, an algorithm for multiple attribute decision-making (MADM) with an application in medical diagnosis is devised.

1. Introduction

Through his significant theory on fuzzy sets, Zadeh [1] made the first effective attempt in mathematical modeling to contain non-probabilistic uncertainty, i.e. uncertainty that is not caused by randomness of an event. The study of fuzzy calculus plays a vital role in the field of mathematics due to its useful applications in variety of scientific domains including statistics, applied mathematics, dynamics and mathematical biology. Many applications of fuzzy mathematics can be found in engineering, bio-mathematics and basic sciences. A novel technique to solve the fuzzy system of equations has been presented by Mikaeilvand et al. [2]. Also many applications of fuzzy integral equations have been presented by various authors [3, 4]. A fuzzy set is one in which each element of the universe belongs to it, but with a value or degree of belongingness that falls between 0 and 1, and these values are referred to as the membership value of each element in that set. Chang [5] was the first to propose the concept of fuzzy topology later on.

Pawlak [6] introduces Rough set theory in 1992 as a substitute mathematical tool for describing reasoning and deciding how to handle vagueness and uncertainty. This theory uses equivalence relations to approximate sets, and it is used in conjunction with the principal non-statistical techniques to data analysis. Lower and upper
approximations are two definite sets that commonly characterise a rough set. The greatest definable set included inside the given collection of objects is the lower approximation, whereas the smallest definable set that contains the provided set is the upper approximation. Rough set concepts are frequently stated in broad terms using topological operations such as interior and closure, which are referred to as approximations.

Lellis Thivagar [7] introduced a new topology called nano topology in 2013, which is an extension of rough set theory. He also created Nano topological spaces, which are defined in terms of approximations and the boundary region of a subset of the universe using an equivalence relation. The Nano open sets are the constituents of a Nano topological space, while the Nano closed sets are their complements. The term "nano" refers to anything extremely small. Nano topology, then, is the study of extremely small surfaces. Nano topology is based on the concepts of approximations and indiscernibility relations. In addition, in [8], nano delta open sets in nano topological space were investigated.

This paper follows the definition of Lellis Thivagar et al. [9]. Generalizations of (fuzzy nano) open sets are a major topic in (fuzzy nano) topology. One of the important generalizations is a Z-open sets [10] which was studied in classical topology by El-Magharabi and Mubarki. Later on, many studies which investigated a nano topologies have been done such as nano M-open sets [11], nano Z-open sets [12], Z-closed sets in double fuzzy topological spaces [13, 14] and Z-open sets in a fuzzy nano topological spaces by Thangammal et al. [15].

Kuratowski and Sierpinski [16] explored the difference of two closed subsets of a n-dimensional Euclidean space in 1921, and the notion of a locally closed subset of a topological space was a key instrument in their work. Ganster and Reilly [17] defined LC-continuity in a topological space using locally closed sets in 1989.

Multiple attribute decision-making (MADM) is a decision-making process that takes into account the best possible options. Decisions were taken in medieaval times without taking into account data uncertainties, which could lead to a potential outcome. Inadequate outcomes have real-life consequences. If we deduced the consequence of obtained data without hesitancy, the results would be ambiguous, indeterminate, or incorrect. Without hesitation, I determined the result of the obtained data. MADM had a significant impact on Management, disease diagnosis, economics, and industry are examples of real-world problems. Each decision maker makes hundreds of decisions each time to carry out the key component. It should be a logical assessment of his or her job. MADM is a programme that helps you tackle difficult problems. For this, there are complex problems with a variety of parameters. The problem must be identified in MADM by defining viable alternatives, assessing each alternative against the criteria established by the decision-maker or community of decision-makers, and finally selecting the optimal alternative. To deal with the complications and complexity of MADM problems, a range of useful mathematical methods such as fuzzy sets, neutrosophic sets, and soft sets were developed.

Zafer et al. [18] introduced and developed the MADM method based on rough fuzzy information. Several mathematicians have worked on correlation coefficients, similarity measurements, aggregation operators, topological spaces, and decision-making applications in this area. These structures feature better decision-making solutions and provide distinct formulas for diverse sets. It has a wide range of applications in domains such as medical diagnosis, pattern identification, social sciences, artificial intelligence, business, and multi-attribute decision making. The problems associated with these cases are interesting, and developing a hypothesis for them has prompted many scholars [19–21] to pay attention to them Motivation and objective. No investigation on fuzzy nano Z locally closed set, fuzzy nano Z extremely disconnected spaces, fuzzy nano Z normal spaces and strongly fuzzy nano Z normal spaces in fuzzy nano topological space has been reported in the fuzzy literature. We present this innovative notion of fuzzy nano topological space and apply it to the MADM issue based on the concepts of fuzzy sets [1], nano topological spaces [7], and neutrosophic nano topological space [9]. The enlarged and hybrid motivation and goal work is described in detail throughout the article. Under certain conditions, we ensure that other FShybrids systems are special FNts. Our proposed model and techniques are discussed in terms of their robustness, durability, superiority, and simplicity. This is the most prevalent model, and it is used to collect vast amounts of data in AI, engineering, and medical applications. Similar research can simply be duplicated in the future using alternative methodologies and hybrid structures.

The following is how this article is organised: Section 2 is devoted to discussing various fuzzy set theory and fuzzy nano topology definitions and results. In Section 3, we introduce the notion of fuzzy nano Z locally closed set and establish some of characteristics. The concept of fuzzy nano Z extremally disconnected spaces is introduced in fuzzy nano topological spaces and also gives some properties and theorems of such concepts in Section 4. In Sections 5 and 6, fuzzy nano Z normal space and strongly fuzzy nano Z normal spaces are introduced and proved many theorems. As a numerical example, in Sections 7 & 8, we devised a method for solving the MADM issue related to Medical Diagnosis utilising F. Ats. We also discussed the algorithms’ efficiency, advantage, consistency, and validity. In Section 9, the work’s conclusion is fundamentally summarised, and the next field of research is offered.

2. Preliminaries

This part explains the concepts and findings that we need to know in order to comprehend the manuscript.

Definition 1 (see [1]). A function \( f \) from \( X \) into the unit interval \( I \) is called a fuzzy set (briefly, \( F \) set) in \( X \).

Definition 2 (see [1]). If \( G \) and \( H \) are any two fuzzy subsets (briefly, \( F \) subs) of a set \( X \), then

(i) \( G \leq H \) iff \( \mu_G(l) \leq \mu_H(l), \forall l \in X \). (ii) \( G = H \), if \( G(l) = H(l), \forall l \in X \). (iii) \( (G \vee H)(l) = \max\{G(l), H(l)\}, \forall l \in X \). (iv) \( (G \wedge H)(l) = \min\{G(l), H(l)\}, \forall l \in X \).
Definition 3 (see [1]). The complement of a \( F \) subset \( G \) in \( X \), denoted by \( 1 - G \), is the \( F \) subset of \( X \) defined by \( 1 - G(l) \), \( \forall l \) in \( X \).

Definition 4 (see [9]). Let \( U \) be a non-empty set and \( R \) be an equivalence relation on \( U \). Let \( F \) be a \( F \) subset in \( U \) with the membership function \( \mu_F \). The fuzzy nano lower (upper) approximations and fuzzy nano boundary of \( F \) in the approximation \( (U, R) \) denoted by \( \overline{F}_R(F), \overline{F}_R(F) \) and \( \overline{F}_R(F) \) are respectively defined as follows: (i) \( \overline{F}_R(F) = \{ (l, \mu_B(A)(L))/Y \in [I]_R, l \in U \} \) (ii) \( \overline{F}_R(F) = \{ (l, \mu_B(A)(L))/Y \in [I]_R, l \in U \} \)

3. Fuzzy nano \( Z \) locally closed sets

The idea of fuzzy nano \( Z \) locally closed sets, which represents a class of generalisations of fuzzy nano \( Z \) open sets, is introduced in this section. The main features of fuzzy nano \( Z \) closed sets are established, as well as certain characterizations.

Definition 5. Let \((U, \tau_z(F))\) be a \( F \) nets with respect to \( F \) where \( F \) is a fuzzy subset of \( U \). Let \( S \) be a fuzzy subset of \( U \). Then fuzzy nano

(i) interior of \( S \) (briefly, \( F \) nets\( \int(S) \)) is represented as \( F \) nets\( \int(S) = \forall O: O \leq S \& O \) is a \( F \) nets set in \( U \).

(ii) closure of \( S \) (briefly, \( F \) nets\( cl(S) \)) is represented as \( F \) nets\( cl(S) = \forall L: S \leq L \& L \) is a \( F \) nets set in \( U \).

(iii) regular open (briefly, \( F \) nets\( ro \)) set if \( S = F \) nets\( int(F \) nets\( cl(S)) \).

(iv) regular closed (briefly, \( F \) nets\( rc \)) set if \( S = F \) nets\( int(F \) nets\( int(S)) \).

(v) \( \delta \) interior of \( S \) (briefly, \( F \) nets\( \delta int(S) \)) is represented as \( F \) nets\( \delta int(S) = \forall O: O \leq S \& O \) is a \( F \) nets set in \( U \).

(vi) \( \delta \) closure of \( S \) (briefly, \( F \) nets\( \delta cl(S) \)) is represented as \( F \) nets\( \delta cl(S) = \forall L: S \leq L \& L \) is a \( F \) nets set in \( U \).

(vii) \( \delta \) open (briefly, \( F \) nets\( \delta o \)) set if \( S = F \) nets\( \delta int(S) \).

(viii) \( \delta \) semi-open (briefly, \( F \) nets\( \delta o \)) set if \( S \leq F \) nets\( cl(F \) nets\( \delta int(S)) \).

(ix) pre open (briefly, \( F \) nets\( po \)) set if \( S \leq F \) nets\( int(F \) nets\( cl(S)) \).

(x) \( \delta \) semi interior of \( S \) (briefly, \( F \) nets\( \delta int(S) \)) is represented as \( F \) nets\( \delta int(S) = \forall O: O \leq S \& O \) is a \( F \) nets\( \delta o \) set in \( U \).
Example 1. Assume $U = \{s_1, s_2, s_3, s_4\}$ and $U/R = \{\{s_1, s_2\}, \{s_3\}, \{s_4\}\}$. Let $S = \{\{s_1/0.2\}, \{s_2/0.3\}, \{s_3/0.4\}, \{s_4/0.1\}\}$ be a $\mathcal{F}$ sub of $U$.

\[
\mathcal{F}_\mathcal{U}^+(S) = \left\{\frac{s_1+s_2}{0.1}, \frac{s_2}{0.3}, \frac{s_3}{0.4}\right\},
\]
\[
\mathcal{F}_\mathcal{U}^-(S) = \left\{\frac{s_1+s_2}{0.2}, \frac{s_2}{0.3}, \frac{s_3}{0.4}\right\},
\]
\[
B_{\mathcal{F}_\mathcal{U}}(S) = \left\{\frac{s_1}{s_2}, \frac{s_2}{s_3}, \frac{s_3}{s_4}\right\}.\tag{1}
\]

Thus $\tau_\mathcal{U}(S) = \left\{0_{\mathcal{U}}, 1_{\mathcal{U}}, \mathcal{F}_\mathcal{U}^+(S), \mathcal{F}_\mathcal{U}^-(S) = B_{\mathcal{F}_\mathcal{U}}(S)\right\}$. Then

(i) $\mathcal{F}_\mathcal{U} \subseteq C = A \cap B = \{s_1, s_2/0.1\}, \{s_2/0.3\}, \{s_3/0.4\}\}$\& $\{s_1/0.1\}, \{s_2/0.3\}, \{s_3/0.4\}\}$

(ii) $\mathcal{F}_\mathcal{U}^+ \subseteq C = A \cup B = \{s_1/0.4, s_2/0.6, s_3/0.5\}\& \{s_1/0.3\}, \{s_2/0.7\}, \{s_3/0.4\}\}$

(iii) $\mathcal{F}_\mathcal{U} \subseteq C = A \cap B = \{s_1/0.4, s_2/0.6, s_3/0.5\}\& \{s_1/0.3\}, \{s_2/0.7\}, \{s_3/0.4\}\}$

(iv) $\mathcal{F}_\mathcal{U}^+ \subseteq C = A \cup B = \{s_1/0.4, s_2/0.6, s_3/0.5\}\& \{s_1/0.3\}, \{s_2/0.7\}, \{s_3/0.4\}\}$

(v) $\mathcal{F}_\mathcal{U} \subseteq C = A \cap B = \{s_1/0.4, s_2/0.6, s_3/0.5\}\& \{s_1/0.3\}, \{s_2/0.7\}, \{s_3/0.4\}\}$

Remark 2. The converse of the preceding proposition does not have to be true, as the following example demonstrates.

Example 2. In Example 1, $C = A \cap B = \{s_1/0.4, s_2/0.6, s_3/0.5\}$\& $\{s_1/0.3\}, \{s_2/0.7\}, \{s_3/0.4\}\}$, then $C$ is $\mathcal{F}_\mathcal{U} \subseteq C$ but not $\mathcal{F}_\mathcal{U}^+ \subseteq C$.

Theorem 1. Let $(U, \tau_\mathcal{U}(F))$ be a $\mathcal{F}$ set. Then $O$ is $\mathcal{F}_\mathcal{U} \subseteq C$ and $O \subseteq \mathcal{F}_\mathcal{U}^+ \subseteq C$ if and only if $O = B \cap \mathcal{F}_\mathcal{U} \subseteq C$ (resp. $O = B \cup \mathcal{F}_\mathcal{U} \subseteq C$) is extremally disconnected (briefly, FD) and therefore $O = B \cap \mathcal{F}_\mathcal{U} \subseteq C$ (resp. $O = B \cup \mathcal{F}_\mathcal{U} \subseteq C$) implies $O \subseteq B \cap \mathcal{F}_\mathcal{U} \subseteq C$ set $B$.

Proof. Let $O$ be a $\mathcal{F}_\mathcal{U} \subseteq C$ set. Since $B \cap \mathcal{F}_\mathcal{U} \subseteq C$ and $B \cup \mathcal{F}_\mathcal{U} \subseteq C$ set in $X$. Since $O \subseteq C, \mathcal{F}_\mathcal{U} \subseteq C \subseteq \mathcal{F}_\mathcal{U} \subseteq C$ and so $B \cap \mathcal{F}_\mathcal{U} \subseteq C \subseteq O \subseteq B \cap \mathcal{F}_\mathcal{U} \subseteq C$ set $B$. Conversely, assume $O = B \cap \mathcal{F}_\mathcal{U} \subseteq C$. Since $\mathcal{F}_\mathcal{U} \subseteq C$ is $\mathcal{F}_\mathcal{U} \subseteq C$ set and $B$ is $\mathcal{F}_\mathcal{U} \subseteq C$ set.

The rest of the cases are the same.

4. Fuzzy nano Z extremally disconnected space

In this section, we introduce fuzzy nano $Z$ extremally disconnected space and we obtain several characterizations based on fuzzy set.

Definition 10. Let $(U, \tau_\mathcal{U}(F))$ be a $\mathcal{F}$ set is called fuzzy nano (resp. $\mathcal{F}_\mathcal{U} \subseteq C$, $\mathcal{F}_\mathcal{U} \subseteq C$) set if $\mathcal{F}_\mathcal{U} \subseteq C$ set. The following are similar.

Example 3. In Example 1, $\mathcal{F}_\mathcal{U} \subseteq C$ closure of every $\mathcal{F}_\mathcal{U} \subseteq C$ set in $U$ is $\mathcal{F}_\mathcal{U} \subseteq C$ set in $U$.

Remark 3. Every $\mathcal{F}_\mathcal{U} \subseteq C$ set on space is $\mathcal{F}_\mathcal{U} \subseteq C$ set (resp. $\mathcal{F}_\mathcal{U} \subseteq C$ set, $\mathcal{F}_\mathcal{U} \subseteq C$ set, $\mathcal{F}_\mathcal{U} \subseteq C$ set) of $U$.

Theorem 2. Let $(U, \tau_\mathcal{U}(F))$ be a $\mathcal{F}$ set. Then the following are similar.

(i) $U$ is $\mathcal{F}_\mathcal{U} \subseteq C$ set.

(ii) $\mathcal{F}_\mathcal{U} \subseteq C$ set, for each $\mathcal{F}_\mathcal{U} \subseteq C$ set of $U$.

(iii) $\mathcal{F}_\mathcal{U} \subseteq C$ set (resp. $\mathcal{F}_\mathcal{U} \subseteq C$ set, $\mathcal{F}_\mathcal{U} \subseteq C$ set) of $U$.
(iv) \( B = (\mathcal{F} N Z c l(A))^\circ \) implies \( \mathcal{F} N Z c l(B) = (\mathcal{F} N Z c l(A))^\circ \) for each pair of \( \mathcal{F} N Z o \) set \( A \& B \) of \( U \).

Proof. (i) \( \Rightarrow \) (ii) Let \( A \) be a \( \mathcal{F} N Z c \) set of \( U \). Then \( A^\circ \) is \( \mathcal{F} N Z c \) set of \( U \). Since \( U \) is \( \mathcal{F} N Z E x D C \) on space, \( \mathcal{F} N Z c l(A^\circ) \) is \( \mathcal{F} N Z o \) set. But \( \mathcal{F} N Z o \) set \( \mathcal{F} N Z c l(A)^\circ = (\mathcal{F} N Z i n t(A))^\circ \). Therefore \( \mathcal{F} N Z i n t(A) \) is \( \mathcal{F} N Z c \) set. (ii) \( \Rightarrow \) (iii) Suppose that \( A \) is a \( \mathcal{F} N Z o \) set of \( U \). Then \( \mathcal{F} N Z c l(\mathcal{F} N Z c l(A)^\circ) = \mathcal{F} N Z c l(\mathcal{F} N Z i n t(A))^\circ \). By assumption, \( \mathcal{F} N Z i n t(A)^\circ \) is a \( \mathcal{F} N Z c \) set of \( U \). So, \( \mathcal{F} N Z c l(\mathcal{F} N Z c l(A)^\circ) = \mathcal{F} N Z c l(\mathcal{F} N Z i n t(A))^\circ = (\mathcal{F} N Z c l(A))^\circ \).

(iii) \( \Rightarrow \) (iv) Let \( A \) and \( B \) be \( \mathcal{F} N Z o \) sets of \( U \). We put \( B = (\mathcal{F} N Z c l(A))^\circ \). From the assumption, \( \mathcal{F} N Z c l(B) = (\mathcal{F} N Z c l(\mathcal{F} N Z c l(A)))^\circ = (\mathcal{F} N Z c l(A))^\circ \). (iv) \( \Rightarrow \) (i) Let \( A \) be a \( \mathcal{F} N Z o \) set of \( U \). Let \( B = (\mathcal{F} N Z c l(A))^\circ \). From the assumption, we obtain that \( \mathcal{F} N Z c l(B) = (\mathcal{F} N Z c l(A))^\circ \).

Remark 4. The Theorem 2 also holds for \( \mathcal{F} N Z N o r \) and \( \mathcal{F} N Z c \) sets.

Theorem 3. Let \((U, \tau_\mathcal{F}(F))\) be a \( \mathcal{F} M t s \) is \( \mathcal{F} N Z E x D C \) on space if and only if \( \mathcal{F} N Z c l(A) = \mathcal{F} N Z i n t(\mathcal{F} N Z c l(A)) \), \( \mathcal{F} N Z c l(B) = \mathcal{F} N Z c l(\mathcal{F} N Z c l(B))^\circ = (\mathcal{F} N Z c l(A))^\circ = \mathcal{F} N Z c l(\mathcal{F} N Z c l(A))^\circ = (\mathcal{F} N Z c l(A))^\circ \) for each \( \mathcal{F} N Z o \) set. Thus \( \mathcal{F} N Z c l(A) \) is a \( \mathcal{F} N Z o \) set of \( U \).

Proof. Let \( A \) be a \( \mathcal{F} N Z o \) set in \( \mathcal{F} N Z E x D C \) on space. Then \( \mathcal{F} N Z c l(A) = \mathcal{F} N Z o \) set. This implies \( \mathcal{F} N Z c l(A) = \mathcal{F} N Z i n t(\mathcal{F} N Z c l(A)) \). Conversely, let \( A \) be a \( \mathcal{F} N Z o \) set and \( \mathcal{F} N Z c l(A) = \mathcal{F} N Z i n t(\mathcal{F} N Z c l(A)) \). Hence \( \mathcal{F} N Z c l(A) \) is \( \mathcal{F} N Z o \) set of \( U \).

The rest of the cases are the same.

5. Fuzzy nano Z normal spaces

In this section, we first present fuzzy nano Z normal spaces and scrutinize their essential properties.

Definition 11. Let \((U, \tau_\mathcal{F}(F))\) be a \( \mathcal{F} M t s \) is said to be fuzzy nano (resp. \( \delta \), \( \delta S \), \( P \) and \( Z \)) normal (briefly, \( \mathcal{F} N Z N o r \) (resp. \( \mathcal{F} N Z N o r \) and \( \mathcal{F} N Z c \) and \( \mathcal{F} N Z c \) and \( \mathcal{F} N Z N o r \)) normal if for any two disjoint \( \mathcal{F} M c \) (resp. \( \mathcal{F} N Z N o r \) and \( \mathcal{F} N Z c \) and \( \mathcal{F} N Z c \) and \( \mathcal{F} N Z o \)) sets \( A \& B \) of \( \mathcal{F} N Z o \) sets \( A \& B \) respectively, then there exists disjoint \( \mathcal{F} N Z o \) such that \( A \leq L \) and \( B \leq M \).

Proposition 3. Every \( \mathcal{F} N Z N o r \) is \( \mathcal{F} N Z N o r \).

Proof. Let \( U \) be a \( \mathcal{F} N \) set. Then for any two disjoint \( \mathcal{F} M c \) sets \( A \& B \), then there exists disjoint \( \mathcal{F} N Z N o r \) sets \( A \& B \) respectively.
F∗NZo. Now F∗NZc(M) ≤ HZ implies that HZ ≤ (F∗NZc(M))Φ. Also M∩(F∗NZc(M))Φ ≤ F∗NZc(M)∩(F∗NZc(M))Φ = 0N. That is M and (F∗NZc(M))Φ are disjoint F∗NZc sets containing HZ and HZ respectively. This shows that (U, τF (F)) is F∗NZNor.

Theorem 6. For a F∗NZts (U, τF (F)), then the following are comparable: (i) U is F∗NZNor. (ii) For any two F∗NZ sets L & M whose union is 1N, ∃ F∗NZc subsets A of L & B of M whose union is also U.

Proof. (i) ⇒ (ii): Let L & M be two F∗NZc sets in a F∗NZNor space U ⊇ 1N = LυM. Then L, M are disjoint F∗NZc sets. Since U is F∗NZNor, then ∃ disjoint F∗NZc sets G1 & G2 ⊆ L = G1 and M = G2. Let A = G1 and B = G2. Then A & B are F∗NZc subsets of L & M respectively ⊆ AvB = 1N. This proves (ii). (ii) ⇒ (i): Let A & B be disjoint F∗NZc sets in U. Then A and B are F∗NZc sets whose union is 1N. By (ii), there exists disjoint F∗NZc sets F1 & F2 ⊆ F1 ⊆ A, F2 ⊆ B & F1 ∨ F2 = 1N. Then F1 & F2 are disjoint F∗NZ sets containing A and B respectively. Therefore U is F∗NZNor.

Theorem 7. Let f: (U1, τF (F1)) → (U2, τF (F2)) be a function. (i) If f is injective, F∗NZIr, F∗NZo & U1 is F∗NZNor then U2 is F∗NZNor. (ii) If f is F∗NZIr, F∗NZc and U2 is F∗NZNor then U1 is F∗NZNor.

Theorem 8. If given a pair of disjoint F∗NZc sets A, B of (U, τF (F)), there is a F∗NZc sets function f : f(A) = 0N and f(B) = 1N, then (U, τF (F)) is F∗NZNor.

Theorem 9. Let f: (U1, τF (F1)) → (U2, τF (F2)) be a function. If f is a F∗NZts, F∗NZc bijection of a F∗NZ space U1 into a space U2 and if every F∗NZc set in U2 is F∗NZc, then U2 is F∗NZReg.

Proof. Let M1 and M2 be F∗NZc sets in U2. Then by assumption, M2 is F∗NZc in U2. Since f is a F∗NZts bijection, f−1(M1) and f−1(M2) is a F∗NZc set in U1. Since U1 is F∗NZNor, there exist disjoint F∗NZo sets L1 and L2 in U1 such that f−1(M1) ⊆ L1 and f−1(M2) ⊆ L2. Since f is F∗NZc, f(L1) and f(L2) are disjoint F∗NZo sets in U2 containing M1 and M2 respectively. Hence U2 is F∗NZNor.

Remark 5. Theorems 4, 5, 6, 7, 8 & 9 are also holds for F∗NZNor and F∗NZc sets.

6. Strongly fuzzy nano Z normal spaces

The principles of strongly fuzzy nano Z normal spaces are introduced in this section. We describe each of these notions and show how they are related to one another.

Definition 12. A F∗NZts(U, τF (F)) is said to be strongly fuzzy nano Z (resp. δ, δδ, P) normal (briefly, S∗NZNor (resp. F∗NZNor and F∗NZc and F∗NZc)) if for every pair of disjoint F∗NZc sets A & B in U, there are disjoint F∗NZo (resp. F∗NZNor and F∗NZc) sets L and M in U containing A & B respectively.

Example 4. In Example 1, A = \{<s1, s1/0.8>, <s2/0.7>, <s3/0.6>\}, B = \{<s1/0.9>, <s2/0.7>, <s3/0.6>\} are F∗NZc sets. L = \{<s1/0.8>, <s2/0.7>, <s3/0.7>\}, M = \{<s1, s1/0.9>, <s2/0.7>, <s3/0.7>\} are F∗NZc sets in U containing A & B respectively.

Theorem 10. Let (U, τF (F)) – b ± √b² – 4ac/2a be a F∗NZts. Every F∗NZNor space is S∗NZNor.

Proof. Suppose U is F∗NZNor. Let A & B be disjoint F∗NZc sets in U. Then A & B are F∗NZc in U. Since U is F∗NZNor, ∃ disjoint F∗NZo sets L & M containing A and B respectively. Since, every F∗NZo is F∗NZc, L and M are F∗NZc in U. This implies that U is S∗NZNor.

Theorem 11. In a F∗NZts (U, τF (F)), the following are comparable: (i) U is S∗NZNor. (ii) ∀ F∗NZc set F in U and every F∗NZo set containing F, there exists a F∗NZc set M containing ∪ F∗NZc(M) = L. (iii) For each pair of disjoint F∗NZc sets M1 & M2, in U, there exists a F∗NZo set L containing M1 ∪ F∗NZc(L)M1 = 0N.

Proof. (i) ⇒ (ii): Let L be a F∗NZo set containing the F∗NZc set F. Then H = L is a F∗NZc set disjoint from F. Since U is S∗NZNor, ∃ disjoint F∗NZo sets M and W containing F & H respectively. Then F∗NZc(W) is disjoint from H, since if y ∈ H, the set W is a F∗NZc set containing y disjoint from M. Hence F∗NZc(W) ≤ L. (ii) ⇒ (iii): Let M1 & M2 be disjoint F∗NZc sets in U. Then M1 ∪ F∗NZc(L) containing M1. (ii) there exists a F∗NZo set L containing M1 ∪ F∗NZc(L)M1 = 0N. This proves (iii). (iii) ⇒ (i): Let M1 & M2 be the disjoint F∗NZc sets in U. By (iii), there exists a F∗NZo set L containing M1 ∪ F∗NZc(L)M1 = 0N. Take M = F∗NZc(L). Then L and M are disjoint F∗NZc sets containing M1 and M2 respectively. Thus U is S∗NZNor.

Theorem 12. For a F∗NZts(U, τF (F)), then the following are comparable: (i) U is S∗NZNor. (ii) For any two F∗NZo sets L & M whose union is 1N, ∃ F∗NZc subsets M1 of L and M2 of M whose union is also 1N.

Proof. (i) ⇒ (ii): Let L & M be two F∗NZo sets in a S∗NZNor space U ⊇ 1N = LυM. Then L, M are disjoint F∗NZc sets. Since U is S∗NZNor, ∃ disjoint F∗NZo sets G1 & G2 ⊆ L = G1 and M = G2. Let M1 = G1 and M2 = G2. Then M1 & M2 are F∗NZc subsets of L and M respectively ⊆ M1 ∪ M2 = 1N. This proves (ii). (ii) ⇒ (i): Let M1 & M2 be disjoint F∗NZc sets in U. Then M1 & M2 are F∗NZo sets whose union is U. By (ii), there exists F∗NZc sets F1 and F2 ⊆ F1 ⊆ M1, F2 ⊆ M2 and F1 ∨ F2 = 1N. Then F1 and F2 are disjoint F∗NZo sets containing M1 and M2 respectively. Therefore U is S∗NZNor.

Theorem 13. Let h: (U1, τF (F1)) → (U2, τF (F2)) be a function. (i) If h is injective, F∗NZts, F∗NZO and U1 is
St\(\mathcal{F}N\)ZNor then \(U_2\) is \(\mathcal{F}N\)ZNor. (ii) If \(h\) is \(\mathcal{F}N\)ZIrr, \(\mathcal{F}N\)ZO and \(U_2\) is \(\mathcal{F}N\)ZNor then \(U_1\) is \(\mathcal{F}N\)ZNor.

Proof. (i) Suppose \(U_1\) is \(\mathcal{F}N\)ZNor. Let \(M_1\) and \(M_2\) be disjoint \(\mathcal{F}N\)c sets in \(U_2\). Since \(h\) is \(\mathcal{F}N\)Cts, \(h^{-1}(M_1)\) and \(h^{-1}(M_2)\) are \(\mathcal{F}N\)c in \(U_1\). Since \(U_1\) is \(\mathcal{F}N\)ZNor, \(\exists\) disjoint \(\mathcal{F}N\)Zo sets \(L\) and \(M\) in \(U_1\) such that \(\sum h^{-1}(M_i) \leq L\) and \(h^{-1}(M_2) \leq M\). Now \(h^{-1}(M_2) \leq L\Rightarrow M_1 \leq h(L)\) and \(h^{-1}(M_2) \leq M\Rightarrow M_2 \leq h(M)\). Hence, since \(h\) is a \(\mathcal{F}N\)ZO map, \(h(L)\) and \(h(M)\) are \(\mathcal{F}N\)Zo in \(U_2\). Also \(L \subseteq M \Rightarrow h(L) \subseteq h(M) = 0\). Thus \(h(L)\) and \(h(M)\) are disjoint \(\mathcal{F}N\)Zo sets in \(U_2\) containing \(M_1\) and \(M_2\) respectively. Thus, \(U_2\) is \(\mathcal{F}N\)ZNor. (ii) Suppose \(U_2\) is \(\mathcal{F}N\)ZNor. Let \(M_1\) and \(M_2\) be disjoint \(\mathcal{F}N\)c sets in \(U_1\). Since \(h\) is \(\mathcal{F}N\)ZIrr and \(\mathcal{F}N\)ZC, \(h(M_1)\) and \(h(M_2)\) are \(\mathcal{F}N\)c in \(U_2\). Since \(U_2\) is \(\mathcal{F}N\)ZNor, there exist disjoint \(\mathcal{F}N\)Zo sets \(L\) and \(M\) in \(U_2\) such that \(\sum h^{-1}(M_i) \leq L\) and \(h^{-1}(M_2) \leq M\). That is, \(M_1 \leq h^{-1}(L)\) and \(M_2 \leq h^{-1}(M)\). Since \(h\) is \(\mathcal{F}N\)ZIrr, \(h^{-1}(L)\) and \(h^{-1}(M)\) are disjoint \(\mathcal{F}N\)Zo. Thus \(U_1\) is \(\mathcal{F}N\)ZNor.

Remark 6. Theorems 10, 11, 12 & 13 are also holds for \(\mathcal{F}N\)ZNor and \(\mathcal{F}N\)ZC sets.

7. Fuzzy score function

We provide a fuzzy scoring function for decision-making problems using fuzzy information in this part, which is based on a methodical approach.

Definition 13. Let \(S: M \rightarrow [0,1]\). The Fuzzy score function (in short, \(\mathcal{FSF}\)) is \(S(M) = 1/k \sum_{h=1}^{k} \mu_M \) that represents the average of positiveness of the fuzzy component \(\mu_M\).

The specific technique to deal with selecting the correct qualities and alternatives in a decision-making situation utilising fuzzy sets is proposed in the following fundamental steps.

Step 1: Problem field selection: Consider the universe of discourse (set of objects) \(m\), the set of alternatives \(n\), the set of decision attributes \(p\).

Step 2: Construct a fuzzy matrix of alternative verses objects and object verses decision attributes. Calculation Part:

Step 3: Frame the in-discriminability relation \(R\) on \(m\).

Step 4: Construct the fuzzy nano topologies \(\tau_j\) and \(\nu_k\).

Step 5: Find the score values by Definition 1 each of the entries of the \(\mathcal{F}A\)'s. Conclusion part:

Step 6: Organize the fuzzy score values of the alternatives \(\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n\) and the attributes \(\nu_1 \leq \nu_2 \leq \cdots \leq \nu_p\). Choose the attribute \(\nu_p\) for the alternative \(\tau_1\) and \(\nu_{p-1}\) for the alternative \(\tau_2\) etc. If \(n < p\), then ignore \(\nu_k\), where \(k = 1,2,\ldots,n-p\).

7.1. Numerical example. New medical breakthroughs have expanded the number of data available to clinicians, which includes vulnerabilities. The process of grouping multiple sets of symptoms under a single term of illness is extremely challenging in medical diagnosis. In this section, we use a medical diagnosis problem to demonstrate the usefulness and applicability of the above-mentioned approach.

Step 1: Problem field selection: Consider the following tables, which provide information from five patients who were consulted by physicians, Patient 1 (Pat 1), Patient 2 (Pat 2), Patient 3 (Pat 3), Patient 4 (Pat 4), Patient 5 (Pat 5) and symptoms are Weight gain (Wg), Tiredness (Td), Myalgia (Mi), Swelling of legs (Sl), Mensus Problem (Mp). We need to find the patient and to find the disease such as Lymphedema, Insomnia, Hypothyroidism, Menarche, Arthritis of the patient. The data in Tables 1 and 2 are explained by the membership, the indeterminacy and the non-membership functions of the patients and diseases respectively.

Step 2: Construct the in-discriminability relation for the correlation between the symptoms is given as \(U/R = \{(Wg),\{Mi\},\{Td\},\{Sl\},\{Mp\}\}\).

Step 3: From fuzzy nano topologies for \(\tau_j\) and \(\nu_k\):

(i) \(\tau_1 = \{0,1,0,0,0,9,0,3,0,2\}\).

(ii) \(\tau_2 = \{0,1,0,0,8,0,1,0,4,0,3\}\).

(iii) \(\tau_3 = \{0,1,0,0,8,0,3,0,2,0,4\}\).

(iv) \(\tau_4 = \{0,1,0,0,3,0,1,0,2,0,4,0,9\}\).

(v) \(\tau_5 = \{0,1,0,0,3,0,6,0,4,0,7\}\).

(i) \(\nu_1 = \{0,1,0,2,0,7,0,9\}\).

(ii) \(\nu_2 = \{0,1,0,0,9,0,2\}\).

(iii) \(\nu_3 = \{0,1,0,0,9,0,1,0,2\}\).

(iv) \(\nu_4 = \{0,1,0,0,6,0,1,0,2,0,9\}\).

(v) \(\nu_5 = \{0,1,0,0,1,0,9,0,4,0,3\}\).

Step 5: Find fuzzy score functions: (i) \(\mathcal{FSF}(\tau_1) = 0.48\).

(ii) \(\mathcal{FSF}(\tau_2) = 0.4333\).

(iii) \(\mathcal{FSF}(\tau_3) = 0.45\).

(iv) \(\mathcal{FSF}(\tau_4) = 0.4143\).

(v) \(\mathcal{FSF}(\tau_5) = 0.5\).

(i) \(\mathcal{FSF}(\nu_1) = 0.56\).

(ii) \(\mathcal{FSF}(\nu_2) = 0.525\).

(iii) \(\mathcal{FSF}(\nu_3) = 0.44\).

(iv) \(\mathcal{FSF}(\nu_4) = 0.4667\).

(v) \(\mathcal{FSF}(\nu_5) = 0.45\).

Step 6: Final decision: Arrange fuzzy nano score values for the alternatives \(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\) and the attributes \(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\).
2, 3, 4, 5 in ascending order. We get the following sequences \( \tau_5 \leq \tau_1 \leq \tau_3 \leq \tau_2 \leq \tau_4 \) and \( \nu_5 \leq \nu_3 \leq \nu_4 \leq \nu_2 \leq \nu_3 \). Thus the patient Pat\textsuperscript{5} suffers from Hypothyroidism, the patient Pat\textsuperscript{6} suffers from Arthritis, the patient Pat\textsuperscript{5} suffers from Menarche, the patient Pat\textsuperscript{5} suffers from Lymphedema and the patient Pat\textsuperscript{5} suffers from Insomnia. The results are presented in Figures 2 and 3.

### 8. Final thoughts and future work

This paper adds to the growing body of knowledge about fuzzy nano topological spaces. The obtained results show that most of the offered concepts’ nano topological features are kept in the framework of fuzzy nano topologies, implying that some topological prerequisites are unnecessary. Because the study’s limitations are relaxed, exploring nano topological notions using fuzzy nano topologies has a benefit. On the other hand, by extending fuzzy nano \( Z \) locally closed sets, a few characteristics of particular topological concepts are partially lost. We will finish introducing the main fuzzy nano topological concepts using fuzzy nano \( Z \) open sets, such as fuzzy nano \( Z \) locally continuous, respective mappings and homemorphisms, separation axioms, compactness and connectedness in fuzzy nano topological spaces, in this work. Our study plan also includes testing the concepts and results presented here with various generalisations of fuzzy nano \( Z \) open sets, such as fuzzy nano \( e \) open and fuzzy nano \( Z' \) open sets. Furthermore, we will use these expansions of fuzzy nano \( Z \) open sets to present new types of rough approximations and apply them to improve set accuracy metrics.

### Data Availability

Data used to support this study are included within this paper.

### Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

### References


