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Research Article

A New Notion of Fuzzy Local Function and Some Applications

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In this paper, a new notion of fuzzy local function called r-fuzzy local function is introduced, and some properties are given in a fuzzy ideal topological space (X, τ, ℓ) in Šostak sense. After that, the concepts of fuzzy upper (resp., lower) almost ℓ-continuous, weakly ℓ-continuous, and almost weakly ℓ-continuous multifunctions are introduced. Some properties and characterizations of these multifunctions along with their mutual relationships are discussed with the help of examples.

1. Introduction

Zadeh [1] introduced the basic idea of a fuzzy set as an extension of classical set theory. The basic notions of fuzzy sets have been improved and applied in different directions. Along this direction, we can refer [2–4]. A fuzzy multifunction (multivalued mapping) is a fuzzy set valued function [5–8]. The difference between fuzzy multifunctions and fuzzy functions has to do with the definition of an inverse image. For a fuzzy function, there is one inverse but for a fuzzy multifunction there are two types of inverses. By these two definitions of inverse, we can define the continuity of fuzzy multifunction. Fuzzy multifunctions have many applications, for instance, decision theory and artificial intelligence. Taha [9–11] introduced and studied the concepts of r-fuzzy ℓ-open, r-generalized fuzzy ℓ-open, and r-fuzzy δ-ℓ-open sets in a fuzzy ideal topological space (X, τ, ℓ) in Šostak sense [12]. Moreover, Taha [10, 11, 13] introduced and studied the concepts of fuzzy upper and lower α-ℓ-continuous (resp., β-ℓ-continuous, δ-ℓ-continuous, and generalized ℓ-continuous) multifunctions via fuzzy ideals [14].

We lay out the remainder of this article as follows. Section 2 contains some basic definitions and results that help in understanding the obtained results. In Section 3, we introduce and study the notion of r-fuzzy local function by using fuzzy ideal topological space and the definition of fuzzy difference between two fuzzy sets. Additionally, from the definition of r-fuzzy local function, we introduce a stronger form of fuzzy upper (resp., lower) precontinuous multifunctions [15], namely, fuzzy upper (resp., lower) ℓ-continuous multifunctions. In Sections 4, 5 and 6 we introduce the concepts of fuzzy upper (resp., lower) almost ℓ-continuous, weakly ℓ-continuous, and almost weakly ℓ-continuous multifunctions. Several properties of these new multifunctions are established. Finally, Section 7 gives some conclusions and suggests some future works.

2. Preliminaries

In this section, we present the basic definitions which we need in the next sections. Throughout this paper, X refers to an initial universe. The family of all fuzzy sets in X is denoted by I^X and for $\lambda \in I^X$, $\lambda^c(x) = 1 - \lambda(x)$ for all $x \in X$ (where $I = [0,1]$ and $(I^\circ = (0,1])$. For $t \in I$, $\underline{t}(x) = t$ for all $x \in X$. All other notations are standard notations of fuzzy set theory. Let us define the fuzzy difference between two fuzzy sets $\lambda, \mu \in I^X$ as follows:

$$\lambda \overline{\wedge} \mu = \begin{cases} \underline{0}, & \text{if } \lambda \leq \mu, \\ \lambda \wedge \mu^c, & \text{otherwise.} \end{cases} \tag{1}$$

Recall that a fuzzy idea ℓ on X [14] is a map ℓ: $I^X \longrightarrow I$ that satisfies the following conditions: (i) $\forall\lambda, \mu \in I^X$ and $\lambda \leq \mu \Rightarrow \ell(\mu) \leq \ell(\lambda)$; (ii) $\forall\lambda, \mu \in I^X \Rightarrow \ell(\lambda \vee \mu) \geq \ell(\lambda) \wedge \ell(\mu)$. Also, ℓ is called proper if $\ell(\underline{1}) = 0$ and there exists $\mu \in I^X$ such that $\ell(\mu) \rangle 0$. The simplest fuzzy ideals on X, ℓ_0, and ℓ_1 are defined as follows:

$$\ell_0(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

and $\ell_1(\lambda) = 1 \, \forall \, \lambda \in I^X$. If ℓ^1 and ℓ^2 are fuzzy ideals on X, we say that ℓ^1 is finer than ℓ^2 (ℓ^2 is coarser than ℓ^1), denoted by $\ell^2 \leq \ell^1$, iff $\ell^2(\lambda) \leq \ell^1(\lambda) \, \forall \, \lambda \in I^X$. Let τ be a fuzzy topology on X in Šostak sense, the triple (X, τ, ℓ) is called fuzzy ideal topological space.

A mapping F: $X \multimap Y$ is called a fuzzy multifunction [15, 16] iff $F(x) \in I^Y$ for each $x \in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y) = G_F(x, y)$ for any $(x, y) \in X \times Y$. Also, F is normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x, y_0) = 1$ and F is Crisp iff $G_F(x, y) = 1$ for each $x \in X$ and $y \in Y$. The upper inverse $F^u(\mu)$, the lower inverse $F^l(\mu)$ of $\mu \in I^Y$, and the image $F(\lambda)$ of $\lambda \in I^X$ are defined as follows: $F^u(\mu)(x) = \wedge_{y \in Y} [G_F^c(x, y) \vee \mu(y)]$, $F^l(\mu)(x) = \vee_{y \in Y} [G_F(x, y) \wedge \mu(y)]$, and $F(\lambda)(y) = \vee_{x \in X} [G_F(x, y) \wedge \lambda(x)]$. All definitions and properties of image, lower, and upper are found in [15].

3. Fuzzy Ideal and r-Fuzzy Local Function

Definition 1. Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I^\circ$. Then, the r-fuzzy local function λ_r^* of λ is defined as follows:

$$\lambda_r^* = \wedge\left\{\mu \in I^X \colon \ell(\lambda \overline{\wedge} \mu) \geq r, \tau(\mu^c) \geq r\right\}. \tag{3}$$

Remark 1

(1) If we take $\ell = \ell_0$ for each $\lambda \in I^X$, we have

$$\begin{aligned} \lambda_r^* &= \wedge\left\{\mu \in I^X \colon \lambda \leq \mu, \tau(\mu^c) \geq r\right\} \\ &= C_\tau(\lambda, r). \end{aligned} \tag{4}$$

(2) If we take $\ell = \ell_1$ (resp., $\ell(\lambda) \geq r$) for each $\lambda \in I^X$, we have $\lambda_r^* = \underline{0}$.

We will occasionally write λ_r^* or $\lambda_r^*(\ell)$ for $\lambda_r^*(\ell, \tau)$.

Theorem 1. *Let (X, τ, ℓ) be a fuzzy ideal topological space and ℓ^1 and ℓ^2 be two fuzzy ideals on X. Then, for any fuzzy sets $\lambda, \nu \in I^X$,*

(1) $\underline{0}_r^* = \underline{0}$
(2) If $\lambda \leq \nu \Rightarrow \lambda_r^* \leq \nu_r^*$
(3) If $\ell^2 \leq \ell^1 \Rightarrow \lambda_r^*(\ell^1) \leq \lambda_r^*(\ell^2)$
(4) $\lambda_r^* = C_\tau(\lambda_r^*, r) \leq C_\tau(\lambda, r)$
(5) $(\lambda_r^*)_r^* \leq \lambda_r^*$ and $(\lambda_r^*)^c \neq (\lambda^c)_r^*$
(6) $(\lambda \vee \nu)_r^* = \lambda_r^* \vee \nu_r^*$ and $(\lambda \wedge \nu)_r^* \leq \lambda_r^* \wedge \nu_r^*$
(7) If $\ell(\nu) \geq r \Rightarrow (\lambda \vee \nu)_r^* = \lambda_r^*$
(8) $\tau \leq \tau^*(\ell) \leq \tau^*(\ell_1)$

Proof

(1) From Definition 1, we have $\underline{0}_r^* = \underline{0}$.

(2) Suppose that $\lambda_r^* \nleq \nu_r^*$ if $\lambda \leq \nu$. By the definition of ν_r^*, there exists $\mu \in I^X$ with $\nu_r^* \leq \mu$, $\ell(\nu \overline{\wedge} \mu) \geq r$, and $\tau(\mu^c) \geq r$ such that $\lambda_r^* \nleq \mu$.

Hence, $\lambda \leq \nu$ implies $\lambda \overline{\wedge} \mu \leq \nu \overline{\wedge} \mu$ and $\ell(\lambda \overline{\wedge} \mu) \geq \ell(\nu \overline{\wedge} \mu) \geq r$. Hence, $\lambda_r^* \leq \mu$; it is a contradiction. Then, $\lambda_r^* \leq \nu_r^*$.

(3) Suppose that $\lambda_r^*(\ell^1) \nleq \lambda_r^*(\ell^2)$ if $\ell^2 \leq \ell^1$. By the definition of $\lambda_r^*(\ell^2)$, there exists $\mu \in I^X$ with $\lambda_r^*(\ell^2) \leq \mu$, $\ell^2(\lambda \overline{\wedge} \mu) \geq r$, and $\tau(\mu^c) \geq r$ such that $\lambda_r^*(\ell^1) \nleq \mu$. Hence, $\ell^2 \leq \ell^1$ implies $\ell^1(\lambda \overline{\wedge} \mu) \geq \ell^2(\lambda \overline{\wedge} \mu) \geq r$. Hence, $\lambda_r^*(\ell^1) \leq \mu$; it is a contradiction. Then, $\lambda_r^*(\ell^1) \leq \lambda_r^*(\ell^2)$.

(4) From Definition 2, we have $\lambda_r^* = C_\tau(\lambda_r^*, r)$. Since $\ell_0 \leq \ell$ for any fuzzy ideal ℓ, $\lambda_r^*(\ell) \leq \lambda_r^*(\ell_0) = C_\tau(\lambda, r)$. Thus, $\lambda_r^* = C_\tau(\lambda_r^*, r) \leq C_\tau(\lambda, r)$.

(5) By (4), we have $(\lambda_r^*)_r^* = C_\tau((\lambda_r^*)_r^*, r) \leq C_\tau(\lambda_r^*, r) = \lambda_r^*$. In general, the converse is not true, as shown by Example 1.

(6) Hence, λ and $\nu \leq \lambda \vee \nu$ implies $\lambda_r^* \leq (\lambda \vee \nu)_r^*$ and $\nu_r^* \leq (\lambda \vee \nu)_r^*$. Thus, $\lambda_r^* \vee \nu_r^* \leq (\lambda \vee \nu)_r^*$. Now, we show that $(\lambda \vee \nu)_r^* \leq \lambda_r^* \vee \nu_r^*$.

Suppose $(\lambda \vee \nu)_r^* \nleq \lambda_r^* \vee \nu_r^*$ implies $(\lambda \vee \nu)_r^* \nleq \lambda_r^*$ and $(\lambda \vee \nu)_r^* \nleq \nu_r^*$.

Case 1. Since $(\lambda \vee \nu)_r^* \nleq \lambda_r^*$, by the definition of λ_r^*, there exists $\mu_1 \in I^X$ with $\lambda_r^* \leq \mu_1$, $\ell(\lambda \overline{\wedge} \mu_1) \geq r$, and $\tau(\mu_1^c) \geq r$ such that $(\lambda \vee \nu)_r^* \nleq \mu_1$. Hence, $(\lambda \vee \nu) \overline{\wedge} \mu_1 = (\lambda \overline{\wedge} \mu_1) \vee (\nu \overline{\wedge} \mu_1)$ implies $\ell((\lambda \vee \nu) \overline{\wedge} \mu_1) \geq r$. Hence, $(\lambda \vee \nu)_r^* \leq \mu_1$, which is a contradiction. Thus, $(\lambda \vee \nu)_r^* \leq \lambda_r^*$.

Case 2. Since $(\lambda \vee \nu)_r^* \nleq \nu_r^*$, by the definition of ν_r^*, there exists $\mu_2 \in I^X$ with $\nu_r^* \leq \mu_2$, $\ell(\nu \overline{\wedge} \mu_2) \geq r$, and $\tau(\mu_2^c) \geq r$ such that $(\lambda \vee \nu)_r^* \nleq \mu_2$. Hence, $(\lambda \vee \nu) \overline{\wedge} \mu_2 = (\lambda \overline{\wedge} \mu_2) \vee (\nu \overline{\wedge} \mu_2)$ implies $\ell((\lambda \vee \nu) \overline{\wedge} \mu_2) \geq r$. Hence, $(\lambda \vee \nu)_r^* \leq \mu_2$, which is a contradiction. Thus, $(\lambda \vee \nu)_r^* \leq \nu_r^*$. Then, $(\lambda \vee \nu)_r^* \leq \lambda_r^* \vee \nu_r^*$.

Also, $\lambda \wedge \nu \leq \lambda$ and $\lambda \wedge \nu \leq \nu$ imply $(\lambda \wedge \nu)_r^* \leq \lambda_r^*$ and $(\lambda \wedge \nu)_r^* \leq \nu_r^*$. Thus, $(\lambda \wedge \nu)_r^* \leq \lambda_r^* \wedge \nu_r^*$.

(7) Hence, $\ell(\nu) \geq r$ implies $\nu_r^* = \underline{0}$. Thus, $(\lambda \vee \nu)_r^* = \lambda_r^* \vee \nu_r^* = \lambda_r^*$.

(8) By (3), $\tau^*(\ell_0) \leq \tau^*(\ell) \leq \tau^*(\ell_1)$, i.e., $\tau \leq \tau^*(\ell) \leq \tau^*(\ell_1)$. □

Example 1. Define τ, ℓ: $I^X \longrightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.5}, \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.8}, \\ 0, & \text{otherwise,} \end{cases}$$
$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \nu = \underline{0.5}, \\ \frac{2}{3}, & \text{if } \underline{0} < \nu < \underline{0.5}, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

In (X, τ, ℓ), $\underline{0} = (\underline{0.6}^*_{1/2})^*_{1/2} \neq \underline{0.6}^*_{1/2} = \underline{0.5}$ and $\underline{1} = (\underline{0.5}^*_{1/2})^c \neq (\underline{0.5}^c)^*_{1/2} = \underline{0}$.

Lemma 1. *Let (X, τ, ℓ) be a fuzzy ideal topological space, and $\{\lambda_j : j \in J\} \subset I^X$. Then, $\bigvee_{j\in J} (\lambda_j)^*_r \leq (\bigvee_{j\in J} \lambda_j)^*_r$ (resp., $(\bigwedge_{j\in J} \lambda_j)^*_r \leq \bigwedge_{j\in J} (\lambda_j)^*_r$).*

Proof. Since $\lambda_j \leq \bigvee_{j\in J} \lambda_j$ for each $j \in J$, by Theorem 1 (2), we have $(\lambda_j)^*_r \leq (\bigvee_{j\in J} \lambda_j)^*_r$ for each $j \in J$. This implies $\bigvee_{j\in J} (\lambda_j)^*_r \leq (\bigvee_{j\in J} \lambda_j)^*_r$. Other case is similarly proved. □

Definition 2. Let (X, τ, ℓ) be a fuzzy ideal topological space. Then, for each $\lambda \in I^X$ and $r \in I^\circ$, we define an operator $Cl^*: I^X \times I^\circ \longrightarrow I^X$ as follows:

$$Cl^*(\lambda, r) = \lambda \vee \lambda^*_r. \quad (6)$$

The fuzzy topology generated by Cl^* is denoted by τ^*, i.e., $\tau^*(\lambda) = \vee\{r : Cl^*(\lambda^c, r) = \lambda^c\}$ and $\tau \leq \tau^*$. Now, if $\ell = \ell_0$, then $Cl^*(\lambda, r) = \lambda \vee \lambda^*_r = \lambda \vee C_\tau(\lambda, r) = C_\tau(\lambda, r)$ for each $\lambda \in I^X$. So, $\tau^* = \tau$. Again if $\ell = \ell_1$ (resp., $\ell(\lambda) \geq r$), then $Cl^*(\lambda, r) = \lambda$ for each $\lambda \in I^X$.

Theorem 2. *Let (X, τ, ℓ) be a fuzzy ideal topological space. Then, for any fuzzy sets $\lambda, \nu \in I^X$ and $r \in I^\circ$, the operator $Cl^*: I^X \times I^\circ \longrightarrow I^X$ satisfies the following properties:*

(1) $Cl^*(\underline{0}, r) = \underline{0}$
(2) $\lambda \leq Cl^*(\lambda, r) \leq C_\tau(\lambda, r)$
(3) If $\lambda \leq \nu$, then $Cl^*(\lambda, r) \leq Cl^*(\nu, r)$
(4) $Cl^*(\lambda \vee \nu, r) = Cl^*(\lambda, r) \vee Cl^*(\nu, r)$
(5) $Cl^*(\lambda \wedge \nu, r) \leq Cl^*(\lambda, r) \wedge Cl^*(\nu, r)$
(6) $Cl^*(Cl^*(\lambda, r), r) = Cl^*(\lambda, r)$

Proof

(1) Hence, $Cl^*(\underline{0}, r) = \underline{0} \vee \underline{0}^*_r$ and $\underline{0}^*_r = \underline{0}$ implies $Cl^*(\underline{0}, r) = \underline{0}$.

(2) Hence, $Cl^*(\lambda, r) = \lambda \vee \lambda^*_r$ implies $\lambda \leq Cl^*(\lambda, r)$. Since $\lambda \leq C_\tau(\lambda, r)$ and from Theorem 1(4), we have $\lambda^*_r \leq C_\tau(\lambda, r)$ implying $Cl^*(\lambda, r) \leq C_\tau(\lambda, r)$. Thus, $\lambda \leq Cl^*(\lambda, r) \leq C_\tau(\lambda, r)$.

(3) From $\lambda \leq \nu$ and Theorem 1 (2), we have $\lambda \vee \lambda^*_r \leq \nu \vee \nu^*_r$, i.e., $Cl^*(\lambda, r) \leq Cl^*(\nu, r)$.

(4) By the definition of Cl^* and from Theorem 1 (6), we have $Cl^*(\lambda \vee \nu, r) = (\lambda \vee \nu) \vee (\lambda \vee \nu)^*_r = (\lambda \vee \nu) \vee (\lambda^*_r \vee \nu^*_r) = (\lambda \vee \lambda^*_r) \vee (\nu \vee \nu^*_r) = Cl^*(\lambda, r) \vee Cl^*(\nu, r)$.

(5) Hence, $\lambda \wedge \nu \leq \lambda$ and $\lambda \wedge \nu \leq \nu$ imply $Cl^*(\lambda \wedge \nu, r) \leq Cl^*(\lambda, r)$ and $Cl^*(\lambda \wedge \nu, r) \leq Cl^*(\nu, r)$. Thus, $Cl^*(\lambda \wedge \nu, r) \leq Cl^*(\lambda, r) \wedge Cl^*(\nu, r)$.

(6) From (2) and (3), we have $Cl^*(\lambda, r) \leq Cl^*(Cl^*(\lambda, r), r)$. Now, we show that $Cl^*(\lambda, r) \geq Cl^*(Cl^*(\lambda, r), r)$. By (4) and the definition of Cl^*, we have $Cl^*(Cl^*(\lambda, r), r) = Cl^*(\lambda \vee \lambda^*_r, r) = Cl^*(\lambda, r) \vee Cl^*(\lambda^*_r, r) = Cl^*(\lambda, r) \vee \lambda^*_r \vee (\lambda^*_r)^*_r = Cl^*(\lambda, r) \vee (\lambda^*_r)^*_r \leq Cl^*(\lambda, r) \vee \lambda^*_r = Cl^*(\lambda, r)$.

The following theorem is similarly proved, as in Theorem 2. □

Theorem 3. *Let (X, τ, ℓ) be a fuzzy ideal topological space. Then, for each $\lambda \in I^X$ and $r \in I^\circ$, we define an operator $int^*: I^X \times I^\circ \longrightarrow I^X$ as follows:*

$$\text{int}^*(\lambda, r) = \lambda \wedge ((\lambda^c)^*_r)^c. \quad (7)$$

For each $\lambda, \nu \in I^X$, the operator int* satisfies the following properties:

(1) $\text{int}^*(\underline{1}, r) = \underline{1}$
(2) $I_\tau(\lambda, r) \leq \text{int}^*(\lambda, r) \leq \lambda$
(3) If $\lambda \leq \nu$, then $\text{int}^*(\lambda, r) \leq \text{int}^*(\nu, r)$
(4) $\text{int}^*(\text{int}^*(\lambda, r), r) = \text{int}^*(\lambda, r)$
(5) $\text{int}^*(\lambda \wedge \nu, r) = \text{int}^*(\lambda, r) \wedge \text{int}^*(\nu, r)$
(6) $\text{int}^*(\lambda, r) = I_\tau(\lambda, r)$, if $\ell = \ell_0$
(7) $\text{int}^*(\lambda^c, r) = (Cl^*(\lambda, r))^c$

Definition 3. A fuzzy multifunction $F: (X, \tau, \ell) \multimap (Y, \eta)$ is called

(1) Fuzzy upper (resp., lower) ℓ-continuous at a fuzzy point $x_t \in \text{dom}(F)$ iff $x_t \in F^u(\mu)$ (resp., $x_t \in F^l(\mu)$) for each $\mu \in I^Y$, $\eta(\mu) \geq r$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq r$, and $x_t \in \lambda$ such that $\lambda \wedge \text{dom}(F) \leq [F^u(\mu)]^*_r$ (resp., $\lambda \leq [F^l(\mu)]^*_r$).

(2) Fuzzy upper ℓ-continuous (resp., fuzzy lower ℓ-continuous) iff it is fuzzy upper ℓ-continuous (resp., fuzzy lower ℓ-continuous) at every $x_t \in \text{dom}(F)$.

Remark 2. If F is normalized, then F is fuzzy upper ℓ-continuous at a fuzzy point $x_t \in \text{dom}(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in I^Y$, $\eta(\mu) \geq r$ there exists $\lambda \in I^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq [F^u(\mu)]^*_r$.

Theorem 4. *Let* F: $(X,\tau,\ell)\multimap(Y,\eta)$ *be a fuzzy multifunction (resp., normalized fuzzy multifunction). Then, F is fuzzy lower (resp., upper) ℓ-continuous iff* $F^l(\mu)\le I_\tau([F^l(\mu)]_r^*,r)$ *(resp.,* $F^u(\mu)\le I_\tau([F^u(\mu)]_r^*,r)$*) for each* $\mu\in I^Y$ *with* $\eta(\mu)\ge r$ *and* $r\in I_o$.

Proof. (⇒) Let $x_t\in \mathrm{dom}(F)$, $\mu\in I^Y$ with $\eta(\mu)\ge r$ and $x_t\in F^l(\mu)$. Then, there exists $\lambda\in I^X$ with $\tau(\lambda)\ge r$ and $x_t\in\lambda$ such that $\lambda\le[F^l(\mu)]_r^*$. Thus, $x_t\in\lambda\le I_\tau([F^l(\mu)]_r^*,r)$ and hence $F^l(\mu)\le I_\tau([F^l(\mu)]_r^*,r)$.

(⇐) Let $x_t\in \mathrm{dom}(F)$, $\mu\in I^Y$ with $\eta(\mu)\ge r$ and $x_t\in F^l(\mu)$. Then, $F^l(\mu)\le I_\tau([F^l(\mu)]_r^*,r)$ and hence $x_t\in I_\tau([F^l(\mu)]_r^*,r)\le[F^l(\mu)]_r^*$. Thus, F is fuzzy lower ℓ-continuous. Other case is similarly proved. □

Remark 3

(1) Every fuzzy lower (resp., upper) ℓ-continuous multifunction is fuzzy lower (resp., upper) precontinuous [15]

(2) If we take $\ell=\ell_0$, we have F as fuzzy lower (resp., upper) ℓ-continuous iff it is fuzzy lower (resp., upper) precontinuous

(3) Fuzzy upper (resp., lower) ℓ-continuity and fuzzy upper (resp., lower) semicontinuity [15] are independent notions, as shown by example 2.

Example 2. Let $X=\{x_1,x_2\}$, $Y=\{y_1,y_2,y_3\}$, and F: $X\multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1)=0.1$, $G_F(x_1,y_2)=1.0$, $G_F(x_1,y_3)=0.3$, $G_F(x_2,y_1)=0.5$, $G_F(x_2,y_2)=0.1$, and $G_F(x_2,y_3)=1.0$. Define $\tau_1,\tau_2,\ell^1,\ell^2$: $I^X\longrightarrow I$ and η: $I^Y\longrightarrow I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \lambda=\underline{0.2},\\ \frac{3}{4}, & \text{if } \lambda=\underline{0.3},\\ 0, & \text{otherwise},\end{cases}$$

$$\tau_2(\lambda)=\begin{cases}1, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \lambda=\underline{0.3},\\ 0, & \text{otherwise},\end{cases}$$

$$\ell^1(\nu)=\begin{cases}1, & \text{if } \nu=\underline{0},\\ \frac{2}{3}, & \text{if } \underline{0}<\nu<\underline{0.3},\\ 0, & \text{otherwise},\end{cases}$$

$$\ell^2(\nu)=\begin{cases}1, & \text{if } \nu=\underline{0},\\ \frac{1}{2}, & \text{if } \underline{0}<\nu<\underline{0.2},\\ 0, & \text{otherwise},\end{cases}$$

$$\eta(\mu)=\begin{cases}1, & \text{if } \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \mu=\underline{0.2},\\ 0, & \text{otherwise}.\end{cases} \quad (8)$$

Then,

(1) F: $(X,\tau_1,\ell^1)\multimap(Y,\eta)$ is fuzzy upper (resp., lower) semicontinuous, but it is not fuzzy upper (resp., lower) ℓ-continuous because $F^u(\underline{0.2})=\underline{0.2}$ (resp., $F^l(\underline{0.2})=\underline{0.2}$) and $\underline{0.2}\le I_{\tau_1}(\underline{0.2},1/2)=\underline{0.2}$, but $\underline{0.2}\nleq I_{\tau_1}(\underline{0.2}^*_{1/2},1/2)=\underline{0}$

(2) F: $(X,\tau_2,\ell^2)\multimap(Y,\eta)$ is fuzzy upper (resp., lower) ℓ-continuous, but it is not fuzzy upper (resp., lower) semicontinuous because $F^u(\underline{0.2})=\underline{0.2}$ (resp., $F^l(\underline{0.2})=\underline{0.2}$) and $\underline{0.2}\le I_{\tau_2}(\underline{0.2}^*_{1/2},1/2)=\underline{0.3}$, but $\underline{0.2}\nleq I_{\tau_2}(\underline{0.2},1/2)=\underline{0}$

Corollary 1. *Let* F: $(X,\tau,\ell)\multimap(Y,\eta)$ *and* H: $(Y,\eta)\multimap(Z,\gamma)$ *be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then,* $H^\circ F$ *is fuzzy lower (resp., upper) ℓ-continuous if F is fuzzy lower (resp., upper) ℓ-continuous and H is fuzzy lower (resp., upper) semicontinuous.*

4. Fuzzy Upper and Lower Almost ℓ-Continuity

Definition 4. A fuzzy multifunction F: $(X,\tau)\multimap(Y,\eta,\ell)$ is called

(1) Fuzzy upper almost ℓ-continuous at a fuzzy point $x_t\in\mathrm{dom}(F)$ iff $x_t\in F^u(\mu)$ for each $\mu\in I^Y$ and $\eta(\mu)\ge r$, there exists $\lambda\in I^X$ with $\tau(\lambda)\ge r$ and $x_t\in\lambda$ such that $\lambda\wedge\mathrm{dom}(F)\le F^u(I_\eta(Cl^*(\mu,r),r))$.

(2) Fuzzy lower almost ℓ-continuous at a fuzzy point $x_t\in\mathrm{dom}(F)$ iff $x_t\in F^l(\mu)$ for each $\mu\in I^Y$ and $\eta(\mu)\ge r$, there exists $\lambda\in I^X$ with $\tau(\lambda)\ge r$ and $x_t\in\lambda$ such that $\lambda\le F^l(I_\eta(Cl^*(\mu,r),r))$.

(3) Fuzzy upper almost ℓ-continuous (resp., fuzzy lower almost ℓ-continuous) iff it is fuzzy upper almost ℓ-continuous (resp., fuzzy lower almost ℓ-continuous) at every $x_t\in\mathrm{dom}(F)$.

Remark 4

(1) If F is normalized, F is fuzzy upper almost ℓ-continuous at a fuzzy point $x_t\in\mathrm{dom}(F)$ iff $x_t\in F^u(\mu)$ for each $\mu\in I^Y$ and $\eta(\mu)\ge r$, there exists $\lambda\in I^X$, $\tau(\lambda)\ge r$ and $x_t\in\lambda$ such that $\lambda\le F^u(I_\eta(Cl^*(\mu,r),r)$.

(2) Fuzzy upper (lower) semicontinuity ⇒ fuzzy upper (lower) almost ℓ-continuity ⇒ fuzzy upper (lower) almost continuity [17].

(3) Fuzzy upper (lower) almost ℓ_0-continuity ⇔ fuzzy upper (lower) almost continuity.

Theorem 5. *For a fuzzy multifunction* $F: (X,\tau) \multimap (Y,\eta,\ell)$, $\mu \in I^Y$, *and* $r \in I_o$, *the following statements are equivalent:*

(1) F is fuzzy lower almost ℓ-continuous

(2) $F^l(\mu) \le I_\tau(F^l(I_\eta(Cl^*(\mu,r),r)),r)$, *if* $\eta(\mu) \ge r$

(3) $C_\tau(F^u(C_\eta(int^*(\mu,r),r)),r) \le F^u(\mu)$, *if* $\eta(\mu^c) \ge r$

Proof

(1) ⇒ (2) Let $x_t \in \text{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \ge r$ and $x_t \in F^l(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \ge r$ and $x_t \in \lambda$ such that $\lambda \le F^l(I_\eta(Cl^*(\mu,r),r))$. Thus, $x_t \in \lambda \le I_\tau(F^l(I_\eta(Cl^*(\mu,r),r)),r)$, i.e., $F^l(\mu) \le I_\tau(F^l(I_\eta(Cl^*(\mu,r),r)),r)$.

(2) ⇒ (3) Let $\mu \in I^Y$ with $\eta(\mu^c) \ge r$. Then, by (2), $(F^u(\mu))^c = F^l(\mu^c) \le I_\tau(F^l(I_\eta(Cl^*(\mu^c,r),r)),r) = (C_\tau(F^u(C_\eta(\text{int}^*(\mu,r),r)),r))^c$. Thus,

$$C_\tau\Big(F^u\Big(C_\eta(\text{int}^*(\mu,r),r)\Big),r\Big) \le F^u(\mu). \quad (9)$$

(3) ⇒ (1) Let $x_t \in \text{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \ge r$ and $x_t \in F^l(\mu)$. Then, by (3), $(I_\tau(F^l(I_\eta(Cl^*(\mu,r),r)),r))^c = C_\tau(F^u(C_\eta(\text{int}^*(\mu^c,r),r)),r) \le F^u(\mu^c) = (F^l(\mu))^c$, i.e., $F^l(\mu) \le I_\tau(F^l(I_\eta(Cl^*(\mu,r),r)),r)$. Thus, $x_t \in I_\tau(F^l(I_\eta(Cl^*(\mu,r),r)),r) \le F^l(I_\eta(Cl^*(\mu,r),r))$. Hence, F is fuzzy lower almost ℓ-continuous.

The following theorem is similarly proved as in Theorem 5. □

Theorem 6. *For a normalized fuzzy multifunction* $F: (X,\tau) \multimap (Y,\eta,\ell)$, $\mu \in I^Y$ *and* $r \in I_o$, *the following statements are equivalent:*

(1) F is fuzzy upper almost ℓ-continuous

(2) $F^u(\mu) \le I_\tau(F^u(I_\eta(Cl^*(\mu,r),r)),r)$, *if* $\eta(\mu) \ge r$

(3) $C_\tau(F^l(C_\eta(int^*(\mu,r),r)),r) \le F^l(\mu)$, *if* $\eta(\mu^c) \ge r$

Example 3. Define a fuzzy multifunction $F: X \multimap Y$ as in example 2, $\tau_1, \tau_2: I^X \longrightarrow I$ and $\eta_1, \eta_2, \ell^1, \ell^2: I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_1(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \underline{0.2}, \\ \frac{2}{3}, & \text{if } \mu = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_2(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \underline{0.5}, \\ \frac{3}{4}, & \text{if } \mu = \underline{0.4}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\ell^1(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{2}{3}, & \text{if } \underline{0} \langle \nu \langle \underline{0.2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\ell^2(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{2}{3}, & \text{if } \underline{0} < \nu \le \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases} \quad (10)$$

Then,

(1) $F: (X,\tau_1) \multimap (Y,\eta_1,\ell^1)$ is fuzzy upper (resp., lower) almost ℓ-continuous, but it is not fuzzy upper (resp., lower) semicontinuous because $\underline{0.2} = F^u(\underline{0.2}) \le I_{\tau_1}(F^u(I_{\eta_1}(Cl^*(\underline{0.2},1/2),1/2)),1/2) = \underline{0.3}$ and $\underline{0.3} = F^u(\underline{0.3}) \le I_{\tau_1}(F^u(I_{\eta_1}(Cl^*(\underline{0.3},1/2),1/2)),1/2) = \underline{0.3}$, but $\underline{0.2} \nleq I_{\tau_1}(\underline{0.2},1/2) = \underline{0}$,

(2) $F: (X,\tau_2) \multimap (Y,\eta_2,\ell^2)$ is fuzzy upper (resp., lower) almost continuous, but it is not fuzzy upper (resp., lower) almost ℓ-continuous because $\underline{0.4} = F^u(\underline{0.4}) \le I_{\tau_2}(F^u(I_{\eta_2}(C_{\eta_2}(\underline{0.4},1/2),1/2)),1/2) = \underline{0.5}$ and $F^u(\underline{0.5}) \le I_{\tau_2}(F^u(I_{\eta_2}(C_{\eta_2}(\underline{0.5},1/2),1/2)),1/2) = \underline{0.5}$, but $\underline{0.4} = F^u(\underline{0.4}) \nleq I_{\tau_2}(F^u(I_{\eta_2}(Cl^*(\underline{0.4},1/2),1/2)),1/2) = \underline{0}$.

Theorem 7. *For a fuzzy multifunction F: $(X,\tau)\multimap(Y,\eta,\ell)$, $\mu\in I^Y$, and $r\in I_o$, the following statements are equivalent:*

(1) F is fuzzy lower almost ℓ-continuous

(2) $\tau(F^l(\mu))\geq r$, if $\mu=I_\eta(Cl^*(\mu,r),r)$

(3) $\tau(F^l(I_\eta(Cl^*(\mu,r),r)))\geq r$, if $\eta(\mu)\geq r$

Proof

(1) ⇒ (2) If $\mu=I_\eta(Cl^*(\mu,r),r)$, then $\eta(\mu)\geq r$. By Theorem 5 (2), $F^l(\mu)\leq I_\tau(F^l(I_\eta(Cl^*(\mu,r),r)),r)=I_\tau(F^l(\mu),r)$. Thus, $\tau(F^l(\mu))\geq r$.

(2) ⇔ (3) is obvious.

(3) ⇒ (1) Let $x_t\in\mathrm{dom}(F)$, $\mu\in I^Y$ with $\eta(\mu)\geq r$ and $x_t\in F^l(\mu)$. Then, by (3) and $\mu\leq I_\eta(Cl^*(\mu,r),r)$, $\tau(F^l(I_\eta(Cl^*(\mu,r),r)))\geq r$, and $x_t\in F^l(\mu)\leq F^l(I_\eta(Cl^*(\mu,r),r))$. Thus, F is fuzzy lower almost ℓ-continuous.

The following theorems are similarly proved as in Theorem 7. □

Theorem 8. *For a fuzzy multifunction F: $(X,\tau)\multimap(Y,\eta,\ell)$, $\mu\in I^Y$, and $r\in I_o$, the following statements are equivalent:*

(1) F is fuzzy lower almost ℓ-continuous

(2) $\tau((F^u(\mu))^c)\geq r$ if $\mu=C_\eta(int^*(\mu,r),r)$

(3) $\tau((F^u(C_\eta(int^*(\mu,r),r)))^c)\geq r$ if $\eta(\mu^c)\geq r$

Theorem 9. *For a normalized fuzzy multifunction F: $(X,\tau)\multimap(Y,\eta,\ell)\,\mu\in I^Y$ and $r\in I_o$, the following statements are equivalent:*

(1) F is fuzzy upper almost ℓ-continuous

(2) $\tau(F^u(\mu))\geq r$ if $\mu=I_\eta(Cl^*(\mu,r),r)$

(3) $\tau(F^u(I_\eta(Cl^*(\mu,r),r)))\geq r$ if $\eta(\mu)\geq r$

Theorem 10. *For a normalized fuzzy multifunction F: $(X,\tau)\multimap(Y,\eta,\ell)$, $\mu\in I^Y$, and $r\in I_o$, the following statements are equivalent:*

(1) F is fuzzy upper almost ℓ-continuous

(2) $\tau((F^l(\mu))^c)\geq r$ if $\mu=C_\eta(int^*(\mu,r),r)$

(3) $\tau((F^l(C_\eta(int^*(\mu,r),r)))^c)\geq r$ if $\eta(\mu^c)\geq r$

Theorem 11. *Let F: $(X,\tau)\multimap(Y,\eta,\ell)$ be a fuzzy multifunction. Then, F is fuzzy lower almost ℓ-continuous iff $C_\tau(F^u(\mu),r)\leq F^u(C_\eta(\mu,r))$ for any $\mu\in I^Y$ with $\mu\leq C_\eta(int^*(\mu,r),r)$ and $r\in I^\circ$.*

Proof. (⇒) Let F be a fuzzy lower almost ℓ-continuous. Then, for any $\mu\in I^Y$ with $\mu\leq C_\eta(\mathrm{int}^*(\mu,r),r)=\nu$ (say), where $\nu=C_\eta(\mathrm{int}^*(\nu,r),r)$. By Theorem 8, $\tau((F^u(\nu))^c)\geq r$ and thus $C_\tau(F^u(\mu),r)\leq C_\tau(F^u(\nu),r)=F^u(C_\eta(\mathrm{int}^*(\nu,r),r))\leq F^u(C_\eta(\mu,r))$.

(⇐) Let $\mu\in I^Y$ with $\mu=C_\eta(\mathrm{int}^*(\mu,r),r)$. Then, $\mu\leq C_\eta(\mathrm{int}^*(\mu,r),r)$ and $C_\tau(F^u(\mu),r)\leq F^u(C_\eta(\mu,r))=F^u(\mu)$. Therefore, we obtain $\tau((F^u(\mu))^c)\geq r$. Thus, by Theorem 8, F is fuzzy lower almost ℓ-continuous.

11. The following theorem is similarly proved as in Theorem. □

Theorem 12. *Let F: $(X,\tau)\multimap(Y,\eta,\ell)$ be a normalized fuzzy multifunction. Then, F is fuzzy upper almost ℓ-continuous iff $C_\tau(F^l(\mu),r)\leq F^l(C_\eta(\mu,r))$ for any $\mu\in I^Y$ with $\mu\leq C_\eta(int^*(\mu,r),r)$ and $r\in I^\circ$.*

Theorem 13. *Let $\{F_i$: $(X,\tau)\multimap(Y,\eta,\ell)\}_{i\in\Gamma}$ be a family of fuzzy lower almost ℓ-continuous. Then, $\cup_{i\in\Gamma}F_i$ is fuzzy lower almost ℓ-continuous.*

Proof. Let $\mu\in I^Y$. Then, $(\cup_{i\in\Gamma}F_i)^l(\mu)=\vee_{i\in\Gamma}(F_i^l(\mu))$. Since $\{F_i\}_{i\in\Gamma}$ is a family of fuzzy lower almost ℓ-continuous, $\tau(F_i^l(\mu))\geq r$ for any $\mu=I_\eta(Cl^*(\mu,r),r)$ and $i\in\Gamma$. Then, $\tau(\cup_{i\in\Gamma}F_i)^l(\mu))=\tau(\vee_{i\in\Gamma}(F_i^l(\mu))\geq\wedge_{i\in\Gamma}\tau(F_i^l(\mu))\geq r$. Hence, $\cup_{i\in\Gamma}F_i$ is fuzzy lower almost ℓ-continuous. □

Theorem 14. *Let F_1 and F_2: $(X,\tau)\multimap(Y,\eta,\ell)$ be two normalized fuzzy upper almost ℓ-continuous. Then, $F_1\cup F_2$ is fuzzy upper almost ℓ-continuous.*

Proof. Let $\mu\in I^Y$. Then, $(F_1\cup F_2)^u(\mu)=F_1^u(\mu)\wedge F_2^u(\mu)$. Since F_1 and F_2 are two normalized fuzzy upper almost ℓ-continuous, $\tau(F_i^u(\mu))\geq r$ for any $\mu=I_\eta(Cl^*(\mu,r),r)$ and $i\in\{1,2\}$. Then, $\tau((F_1\cup F_2)^u(\mu))=\tau(F_1^u(\mu)\wedge F_2^u(\mu))\geq\tau(F_1^u(\mu))\wedge\tau(F_2^u(\mu))\geq r$. Hence, $F_1\cup F_2$ is fuzzy upper almost ℓ-continuous. □

Corollary 2. *Let F: $(X,\tau)\multimap(Y,\eta,\ell)$ and H: $(Y,\eta)\multimap(Z,\gamma)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $H^\circ F$ is fuzzy lower (resp., upper) almost ℓ-continuous if F is fuzzy lower (resp., upper) almost ℓ-continuous and H is fuzzy lower (resp., upper) semicontinuous.*

5. Fuzzy Upper and Lower Weakly ℓ-Continuity

Definition 5. A fuzzy multifunction F: $(X,\tau)\multimap(Y,\eta,\ell)$ is called

(1) Fuzzy upper weakly ℓ-continuous at a fuzzy point $x_t\in\mathrm{dom}(F)$ iff $x_t\in F^u(\mu)$ for each $\mu\in I^Y$ and $\eta(\mu)\geq r$, there exists $\lambda\in I^X$ with $\tau(\lambda)\geq r$ and $x_t\in\lambda$ such that $\lambda\wedge\mathrm{dom}(F)\leq F^u(Cl^*(\mu,r))$.

(2) Fuzzy lower weakly ℓ-continuous at a fuzzy point $x_t\in\mathrm{dom}(F)$ iff $x_t\in F^l(\mu)$ for each $\mu\in I^Y$ and $\eta(\mu)\geq r$, there exists $\lambda\in I^X$ with $\tau(\lambda)\geq r$ and $x_t\in\lambda$ such that $\lambda\leq F^l(Cl^*(\mu,r))$.

(3) Fuzzy upper weakly ℓ-continuous (resp., fuzzy lower weakly ℓ-continuous) iff it is fuzzy upper weakly ℓ-continuous (resp., fuzzy lower weakly ℓ-continuous) at every $x_t\in\mathrm{dom}(F)$.

Remark 5

(1) If F is normalized, F is fuzzy upper weakly ℓ-continuous at a fuzzy point $x_t \in \operatorname{dom}(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(Cl^*(\mu, r))$.

(2) Fuzzy upper (lower) almost ℓ-continuity $\Rightarrow$ fuzzy upper (lower) weakly ℓ-continuity $\Rightarrow$ fuzzy upper (lower) weakly continuity [17].

(3) Fuzzy upper (lower) weakly ℓ_0-continuity $\Leftrightarrow$ fuzzy upper (lower) weakly continuity.

Theorem 15. *A fuzzy multifunction* $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ *is fuzzy lower weakly* ℓ*-continuous iff* $F^l(\mu) \leq I_\tau(F^l(Cl^*(\mu, r)), r)$ *for each* $\mu \in I^Y$, $\eta(\mu) \geq r$, *and* $r \in I_o$.

Proof. ($\Rightarrow$) Let $x_t \in \operatorname{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(Cl^*(\mu, r))$. Thus, $x_t \in \lambda \leq I_\tau(F^l(Cl^*(\mu, r)), r)$ and hence $F^l(\mu) \leq I_\tau(F^l(Cl^*(\mu, r)), r)$.

($\Leftarrow$) Let $x_t \in \operatorname{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then, $x_t \in F^l(\mu) \leq I_\tau(F^l(Cl^*(\mu, r)), r)$. Thus, $x_t \in I_\tau(F^l(Cl^*(\mu, r)), r) \leq F^l(Cl^*(\mu, r))$. Hence, F is fuzzy lower weakly ℓ-continuous.

The following theorem is similarly proved as in Theorem 15. □

Theorem 16. *A normalized fuzzy multifunction* $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ *is fuzzy upper weakly* ℓ*-continuous iff* $F^u(\mu) \leq I_\tau(F^u(Cl^*(\mu, r)), r)$ *for each* $\mu \in I^Y$ *with* $\eta(\mu) \geq r$ *and* $r \in I_o$.

Example 4. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $F\colon X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.8$, $G_F(x_1, y_2) = 0.3$, $G_F(x_1, y_3) = 0.3$, $G_F(x_2, y_1) = 0.1$, $G_F(x_2, y_2) = 1.0$, $G_F(x_2, y_3) = 0.1$, $G_F(x_3, y_1) = 0.1$, $G_F(x_3, y_2) = 0.2$, $G_F(x_3, y_3) = 1.0$. Define $\mu_1 \in I^X$ and $\mu_2 \in I^Y$ as follows: $\mu_1 = \{x_1/0.4, x_2/0.1, x_3/0.2\}$ and $\mu_2 = \{y_1/0.4, y_2/0.1, y_3/0.2\}$. Define $\tau\colon I^X \longrightarrow I$ and $\eta, \ell^1, \ell^2\colon I^Y \longrightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \dfrac{1}{2}, & \text{if } \lambda = \underline{0.5}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \dfrac{3}{4}, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\ell^1(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \dfrac{1}{2}, & \text{if } \underline{0} \langle \nu \leq \underline{0.4}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\ell^2(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \dfrac{1}{2}, & \text{if } \underline{0} < \nu \leq \underline{0.2}, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Then,

(1) $F\colon (X, \tau) \multimap (Y, \eta, \ell^1)$ is fuzzy upper (resp., lower) weakly continuous, but it is not fuzzy upper (resp., lower) weakly ℓ-continuous because $\mu_1 = F^u(\mu_2) \leq I_\tau(F^u(C_\eta(\mu_2, 1/2)), 1/2) = \underline{0.5}$, but $\mu_1 = F^u(\mu_2) \nleq I_\tau(F^u(Cl^*(\mu_2, 1/2)), 1/2) = \underline{0}$.

(2) $F\colon (X, \tau) \multimap (Y, \eta, \ell^2)$ is fuzzy upper (resp., lower) weakly ℓ-continuous, but it is not fuzzy upper (resp. lower) almost ℓ-continuous because $\mu_1 = F^u(\mu_2) \leq I_\tau(F^u(Cl^*(\mu_2, 1/2)), 1/2) = \underline{0.5}$, but $\mu_1 = F^u(\mu_2) \nleq I_\tau(F^u(I_\eta(Cl^*(\mu_2, 1/2), 1/2)), 1/2) = \underline{0}$.

Theorem 17. *A fuzzy multifunction* $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ *is fuzzy lower weakly* ℓ*-continuous iff* $C_\tau(F^u(int^*(\mu, r)), r) \leq F^u(\mu)$ *for each* $\mu \in I^Y$ *with* $\eta(\mu^c) \geq r$.

Proof.. ($\Rightarrow$) Let $\mu \in I^Y$ with $\eta(\mu^c) \geq r$. Then, by Theorem 15, $(F^u(\mu))^c = F^l(\mu^c) \leq I_\tau(F^l(Cl^*(\mu^c, r)), r) = (C_\tau(F^u(\text{int}^*(\mu, r)), r))^c$. Thus, $C_\tau(F^u(\text{int}^*(\mu, r)), r) \leq F^u(\mu)$.

($\Leftarrow$) Let $x_t \in \operatorname{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then, $(I_\tau(F^l(Cl^*(\mu, r)), r))^c = C_\tau(F^u(\text{int}^*(\mu^c, r)), r) \leq F^u(\mu^c) = F^l(\mu)^c$. Hence, $F^l(\mu) \leq I_\tau(F^l(Cl^*(\mu, r)), r)$, i.e., $x_t \in I_\tau(F^l(Cl^*(\mu, r)), r) \leq F^l(Cl^*(\mu, r))$. Thus, F is fuzzy lower weakly ℓ-continuous.

The following theorem is similarly proved as in Theorem 17. □

Theorem 18. *A fuzzy multifunction* $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ *is fuzzy upper weakly* ℓ*-continuous iff* $C_\tau(F^l(\text{int}^*(\mu, r)), r) \leq F^l(\mu)$ *for each* $\mu \in I^Y$ *with* $\eta(\mu^c) \geq r$.

Theorem 19. *If* $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ *is a normalized fuzzy upper weakly* ℓ*-continuous multifunction and* $F(\lambda) \leq I_\eta(Cl^*(F(\lambda), r), r)$ *for each* $\lambda \in I^X$, *then* F *is fuzzy upper almost* ℓ*-continuous.*

Proof. Let $x_t \in \operatorname{dom}(F)$, $\mu \in I^Y$, $\eta(\mu) \geq r$, and $x_t \in F^u(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(Cl^*(\mu, r))$, $F(\lambda) \leq F(F^u(Cl^*(\mu, r))) \leq Cl^*(\mu, r)$. Since $F(\lambda) \leq I_\eta(Cl^*(F(\lambda), r), r) \leq I_\eta(Cl^*(\mu, r), r)$, $\lambda \leq F^u(F(\lambda)) \leq F^u(I_\eta(Cl^*(\mu, r), r))$. Then, F is fuzzy upper almost ℓ-continuous. □

Corollary 3. *Let $F: (X,\tau) \multimap (Y,\eta,\ell)$ and $H: (Y,\eta) \multimap (Z,\gamma)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $H^\circ F$ is fuzzy lower (resp., upper) weakly ℓ-continuous if F is fuzzy lower (resp., upper) weakly ℓ-continuous and H is fuzzy lower (resp., upper) semicontinuous.*

Theorem 20. *Let $F: (X,\tau) \multimap (Y,\eta,\ell)$ be a fuzzy lower weakly ℓ-continuous multifunction. Then, $F^l(\mu) \leq I_\tau(F^l(Cl^*(\mu,r)),r)$ for any $\mu \in I^Y$ with $\mu \leq I_\eta(Cl^*(\mu,r),r)$ and $r \in I^\circ$.*

Proof. Let F be a fuzzy lower weakly ℓ-continuous and $\mu \leq I_\eta(Cl^*(\mu,r),r)$. Then, if $x_t \in F^l(\mu) \leq F^l(I_\eta(Cl^*(\mu,r),r))$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(Cl^*(I_\eta(Cl^*(\mu,r),r),r)) \leq F^l(Cl^*(\mu,r))$. Thus, $\lambda \leq I_\tau(F^l(Cl^*(\mu,r)),r)$ and $F^l(\mu) \leq I_\tau(F^l(Cl^*(\mu,r)),r)$.

The following theorem is similarly proved as in Theorem 20. □

Theorem 21. *Let $F: (X,\tau) \multimap (Y,\eta,\ell)$ be a normalized fuzzy upper weakly ℓ-continuous multifunction. Then, $F^u(\mu) \leq I_\tau(F^u(Cl^*(\mu,r)),r)$ for any $\mu \in I^Y$ with $\mu \leq I_\eta(Cl^*(\mu,r),r)$ and $r \in I^\circ$.*

6. Fuzzy Upper and Lower Almost Weakly ℓ-Continuity

Definition 6. A fuzzy multifunction $F: (X,\tau) \multimap (Y,\eta,\ell)$ is called

(1) Fuzzy upper almost weakly ℓ-continuous at a fuzzy point $x_t \in \mathrm{dom}(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge \mathrm{dom}(F) \leq C_\tau(F^u(Cl^*(\mu,r)),r)$.

(2) Fuzzy lower almost weakly ℓ-continuous at a fuzzy point $x_t \in \mathrm{dom}(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq C_\tau(F^l(Cl^*(\mu,r)),r)$.

(3) Fuzzy upper almost weakly ℓ-continuous (resp., fuzzy lower almost weakly ℓ-continuous) iff it is fuzzy upper almost weakly ℓ-continuous (resp., fuzzy lower almost weakly ℓ-continuous) at every $x_t \in \mathrm{dom}(F)$.

Remark 6

(1) If F is normalized, then F is fuzzy upper almost weakly ℓ-continuous at a fuzzy point $x_t \in \mathrm{dom}(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq C_\tau(F^u(Cl^*(\mu,r)),r)$

(2) Fuzzy upper (lower) weakly ℓ-continuity ⇒ fuzzy upper (lower) almost weakly ℓ-continuity ⇒ fuzzy upper (lower) almost weakly continuity [17]

(3) Fuzzy upper (lower) almost weakly ℓ_0-continuity ⇔ fuzzy upper (lower) almost weakly continuity

Theorem 22. *For a fuzzy multifunction $F: (X,\tau) \multimap (Y,\eta,\ell)$, $\mu \in I^Y$, and $r \in I_o$, the following statements are equivalent:*

(1) F is fuzzy lower almost weakly ℓ-continuous

(2) $F^l(\mu) \leq I_\tau(C_\tau(F^l(Cl^*(\mu,r)),r),r)$ if $\eta(\mu) \geq r$

(3) $C_\tau(I_\tau(F^u(int^*(\mu,r)),r),r) \leq F^u(\mu)$ if $\eta(\mu^c) \geq r$

Proof

(1) ⇒ (2) Let $x_t \in \mathrm{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq C_\tau(F^l(Cl^*(\mu,r)),r)$. Thus, $x_t \in I_\tau(C_\tau(F^l(Cl^*(\mu,r)),r),r)$, i.e., $F^l(\mu) \leq I_\tau(C_\tau(F^l(Cl^*(\mu,r)),r),r)$.

(2) ⇒ (3) Let $\mu \in I^Y$ with $\eta(\mu^c) \geq r$. Then, by (2), $F^u(\mu)^c = F^l(\mu^c) \leq I_\tau(C_\tau(F^l(Cl^*(\mu^c,r)),r),r) = (C_\tau(I_\tau(F^u(\mathrm{int}^*(\mu,r)),r),r))^c$. Thus,

$$C_\tau(I_\tau(F^u(\mathrm{int}^*(\mu,r)),r),r) \leq F^u(\mu). \quad (12)$$

(3) ⇒ (1) Let $x_t \in \mathrm{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then, by (3), $(I_\tau(C_\tau(F^l(Cl^*(\mu,r)),r),r))^c = C_\tau(I_\tau(F^u(\mathrm{int}^*(\mu^c,r)),r),r) \leq F^u(\mu^c) = F^l(\mu)^c$, i.e., $F^l(\mu) \leq I_\tau(C_\tau(F^l(Cl^*(\mu,r)),r),r)$. Therefore, $x_t \in I_\tau(C_\tau(F^l(Cl^*(\mu,r)),r),r) \leq C_\tau(F^l(Cl^*(\mu,r)),r)$. Thus, F is fuzzy lower almost weakly ℓ-continuous.

The following theorem is similarly proved as in Theorem 22. □

Theorem 23. *For a normalized fuzzy multifunction $F: (X,\tau) \multimap (Y,\eta,\ell)$, $\mu \in I^Y$ and $r \in I_o$, the following statements are equivalent:*

(1) F is fuzzy upper almost weakly ℓ-continuous

(2) $F^u(\mu) \leq I_\tau(C_\tau(F^u(Cl^*(\mu,r)),r),r)$ if $\eta(\mu) \geq r$

(3) $C_\tau(I_\tau(F^l(int^*(\mu,r)),r),r) \leq F^l(\mu)$ if $\eta(\mu^c) \geq r$

Example 5. Let $X = \{x_1,x_2,x_3\}$, $Y = \{y_1,y_2,y_3\}$, and $F: X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1) = 0.8, G_F(x_1,y_2) = 0.3, G_F(x_1,y_3) = 0.3, G_F(x_2,y_1) = 0.1, G_F(x_2,y_2) = 1.0, G_F(x_2,y_3) = 0.1, G_F(x_3,y_1) = 0.1, G_F(x_3,y_2) = 0.2, G_F(x_3,y_3) = 1.0$. Define $\mu_1 \in I^X$ and $\mu_2 \in I^Y$ as follows: $\mu_1 = \{x_1/0.4, x_2/0.1, x_3/0.2\}$ and $\mu_2 = \{y_1/0.4, y_2/0.1, y_3/0.2\}$. Define $\tau_1,\tau_2: I^X \longrightarrow I$ and $\eta,\ell: I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{3}{4}, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise}, \end{cases},$$

$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} \langle \nu \leq \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases} \quad (13)$$

Then,

(1) $F\colon (X, \tau_2) \multimap (Y, \eta, \ell)$ is fuzzy upper (resp., lower) almost weakly continuous, but it is not fuzzy upper (resp., lower) almost weakly ℓ-continuous

(2) $F\colon (X, \tau_1) \multimap (Y, \eta, \ell)$ is fuzzy upper (resp., lower) almost weakly ℓ-continuous, but it is not fuzzy upper (resp., lower) weakly ℓ-continuous

Theorem 24. *Let $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ be a normalized fuzzy multifunction and F be a fuzzy upper almost weakly ℓ-continuous and fuzzy lower almost ℓ-continuous. Then, F is fuzzy upper weakly ℓ-continuous.*

Proof.. Let $\mu \in I^Y$ with $\eta(\mu) \geq r$ and F be a fuzzy upper almost weakly ℓ-continuous. Then, by Theorem 23 (2), $F^u(\mu) \leq I_\tau(C_\tau(F^u(Cl^*(\mu, r)), r), r)$. Since $C_\eta(\mu, r) = C_\eta(\text{int}^*(C_\eta(\mu, r), r), r)$, it follows from Theorem 8 (2) that $\tau((F^u(C_\eta(\mu, r)))^c) \geq r$, $\tau((F^u(Cl^*(\mu, r)))^c) \geq r$, and $F^u(\mu) \leq I_\tau(F^u(Cl^*(\mu, r)), r)$. Thus, by Theorem 16, F is fuzzy upper weakly ℓ-continuous.

24. The following theorem is similarly proved as in Theorem. □

Theorem 25. *Let $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ be a normalized fuzzy multifunction and F be a fuzzy lower almost weakly ℓ-continuous and fuzzy upper almost ℓ-continuous. Then, F is fuzzy lower weakly ℓ-continuous.*

Corollary 4. *Let $F\colon (X, \tau) \multimap (Y, \eta, \ell)$ and $H\colon (Y, \eta) \multimap (Z, \gamma)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $H^\circ F$ is fuzzy lower (resp., upper) almost weakly ℓ-continuous if F is fuzzy lower (resp., upper) almost weakly ℓ-continuous and H is fuzzy lower (resp., upper) semicontinuous.*

7. Conclusion and Future Work

In the present paper, based on fuzzy operators α, β, $id_X\colon I^X \times I^\circ \longrightarrow I^X$, and $\theta, \theta^*, \partial, id_Y\colon I^Y \times I^\circ \longrightarrow I^Y$, we can give a generalized form of fuzzy (resp., normalized fuzzy) multifunction as $F\colon (X, \tau, \ell^1) \multimap (Y, \eta, \ell^2)$ is fuzzy lower (resp., upper) $(\alpha, \beta, \theta, \partial, \ell^1, \ell^2)$-continuous multifunction if for every $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $r \in I_o$, $\ell^1[\alpha(F^l(\partial(\mu, r)), r) \overline{\wedge} \beta(F^l(\theta(\mu, r)), r)] \geq \eta(\mu)$ (resp., $\ell^1[\alpha(F^u(\partial(\mu, r)), r) \overline{\wedge} \beta(F^u(\theta(\mu, r)), r)] \geq \eta(\mu)$). It is clear that

(1) Fuzzy lower (resp., upper) $(id_X, I_\tau, I_\eta(Cl^*), id_Y, \ell_0^1, \ell^2)$-continuous is fuzzy lower (resp., upper) almost ℓ^2-continuous multifunction

(2) Fuzzy lower (resp., upper) $(id_X, I_\tau, Cl^*, id_Y, \ell_0^1, \ell^2)$-continuous is fuzzy lower (resp., upper) weakly ℓ^2-continuous multifunction

(3) Fuzzy lower (resp., upper) $(id_X, I_\tau(C_\tau), Cl^*, id_Y, \ell_0^1, \ell^2)$-continuous is fuzzy lower (resp., upper) almost weakly ℓ^2-continuous multifunction

In the upcoming work, we will define some new separation axioms on fuzzy ideal topological spaces. Also, we shall discuss the concepts given here in the frames of fuzzy soft r-minimal spaces [3]. We hope that this work will contribute fuzzy ideal topological structure studies.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic," Theory and Applications*, Prentice-Hall, New Jersey, 1995.

[3] I. M. Taha, "Compactness on fuzzy soft r-minimal spaces," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 21, no. 3, pp. 251–258, 2021.

[4] I. M. Taha, "On upper and lower generalized semi-continuous fuzzy multifunctions," *The Journal of Mathematics and Computer Science*, vol. 25, no. 03, pp. 251–258, 2021.

[5] J. Albryc ht and M. MatŁoka, "On fuzzy multi-valued functions Part 1: introduction and general properties," *Fuzzy Sets and Systems*, vol. 12, no. 1, pp. 61–69, 1984.

[6] N. S. Papageorgiou, "Fuzzy topology and fuzzy multifunctions," *Journal of Mathematical Analysis and Applications*, vol. 109, no. 2, pp. 397–425, 1985.

[7] E. Tsiporkova-Hristoskova, B. De Baets, and E. Kerre, "A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings," *Fuzzy Sets and Systems*, vol. 85, no. 1, pp. 93–108, 1997.

[8] E. Tsiporkova, B. De Baets, and E. Kerre, "Continuity of fuzzy multivalued mappings," *Fuzzy Sets and Systems*, vol. 94, no. 3, pp. 335–348, 1998.

[9] I. M. Taha, "On r-fuzzy ℓ-open sets and continuity of fuzzy multifunctions via fuzzy ideals," *Journal of Mathematical and Computational Science*, vol. 10, no. 6, pp. 2613–2633, 2020.

[10] I. M. Taha, "On r-generalized fuzzy ℓ-closed sets: properties and applications," *Journal of Mathematics*, vol. 2021, Article ID 4483481, 8 pages, 2021.

[11] I. M. Taha, "-fuzzy δ-ℓ-open sets and fuzzy upper (lower) δ-ℓr-continuity via fuzzy idealization," *The Journal of Mathematics and Computer Science*, vol. 25, no. 1, pp. 1–9, 2022.

[12] A. P. Šostak, "On a fuzzy topological structure," in *Proceedings of the 13th winter School on Abstract Analysis, Section of Topology*, pp. 89–103, Circolo Matematico di Palermo, Palermo, 1985.
[13] I. M. Taha, "On fuzzy upper and lower α-ℓ-continuity and their decomposition," *Journal of Mathematical and Computational Science*, vol. 11, no. 1, pp. 427–441, 2021.
[14] A. A. Ramadan, Y. C. Kim, and M. K. Ei-Gayyar, "Smooth ideals," *Journal of Korean Institute of Intelligent Systems*, vol. 12, no. 1, pp. 90–95, 2002.
[15] S. E. Abbas, M. A. Hebeshi, and I. M. Taha, "On fuzzy upper and lower semi-continuous multifunctions," *Journal of Fuzzy Mathematics*, vol. 22, no. 4, pp. 951–962, 2014.
[16] I. Beg, "Continuity of fuzzy multifunctions," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 12, no. 1, pp. 17–22, 1999.
[17] S. E. Abbas, M. A. Hebeshi, and I. M. Taha, "On upper and lower almost weakly continuous fuzzy multifunctions," *Iranian Journal of Fuzzy Systems*, vol. 12, no. 1, pp. 101–114, 2015.