Research Article

A New Notion of Fuzzy Local Function and Some Applications

I. M. Taha and S. E. Abbas

Department Basic Sciences, Higher Institute of Engineering and Technology, Menoufia, Egypt
Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt

Correspondence should be addressed to I. M. Taha; imtaha2010@yahoo.com

Received 11 January 2022; Revised 18 April 2022; Accepted 3 May 2022; Published 13 June 2022

Academic Editor: Ferdinando Di Martino

Copyright © 2022 I. M. Taha and S. E. Abbas. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, a new notion of fuzzy local function called $r$-fuzzy local function is introduced, and some properties are given in a fuzzy ideal topological space $(X, \tau, \ell)$ in ˇSostak sense. After that, the concepts of fuzzy upper (resp., lower) almost $\ell$-continuous, weakly $\ell$-continuous, and almost weakly $\ell$-continuous multifunctions are introduced. Some properties and characterizations of these multifunctions along with their mutual relationships are discussed with the help of examples.

1. Introduction

Zadeh [1] introduced the basic idea of a fuzzy set as an extension of classical set theory. The basic notions of fuzzy sets have been improved and applied in different directions. Along this direction, we can refer [2–4]. A fuzzy multifunction (multivalued mapping) is a fuzzy set valued function [5–8]. The difference between fuzzy multifunctions and fuzzy functions has to do with the definition of an inverse image. For a fuzzy function, there is one inverse but for a fuzzy multifunction there are two types of inverses. By these two definitions of inverse, we can define the continuity of fuzzy multifunction. Fuzzy multifunctions have many applications, for instance, decision theory and artificial intelligence. Taha [9–11] introduced and studied the concepts of $r$-fuzzy $\ell$-open, $r$-generalized fuzzy $\ell$-open, and $r$-fuzzy $\delta\ell$-open sets in a fuzzy ideal topological space $(X, \tau, \ell)$ in ˇSostak sense [12]. Moreover, Taha [10, 11, 13] introduced and studied the concepts of fuzzy upper and lower $a\ell$-continuous (resp., $\beta\ell$-continuous, $\delta\ell$-continuous, and generalized $\ell$-continuous) multifunctions via fuzzy ideals [14].

We lay out the remainder of this article as follows. Section 2 contains some basic definitions and results that help in understanding the obtained results. In Section 3, we introduce and study the notion of $r$-fuzzy local function by using fuzzy ideal topological space and the definition of fuzzy difference between two fuzzy sets. Additionally, from the definition of $r$-fuzzy local function, we introduce a stronger form of fuzzy upper (resp., lower) precontinuous multifunctions [15], namely, fuzzy upper (resp., lower) $\ell$-continuous multifunctions. In Sections 4, 5 and 6 we introduce the concepts of fuzzy upper (resp., lower) almost $\ell$-continuous, weakly $\ell$-continuous, and almost weakly $\ell$-continuous multifunctions. Several properties of these new multifunctions are established. Finally, Section 7 gives some conclusions and suggests some future works.

2. Preliminaries

In this section, we present the basic definitions which we need in the next sections. Throughout this paper, $X$ refers to an initial universe. The family of all fuzzy sets in $X$ is denoted by $I^X$ and for $\lambda \in I^X$, $\lambda^c(x) = 1 - \lambda(x)$ for all $x \in X$ (where $I = [0, 1]$ and $I^c = (0, 1]$). For $t \in I$, $\xi(x) = t$ for all $x \in X$. All other notations are standard notations of fuzzy set theory. Let us define the fuzzy difference between two fuzzy sets $\lambda, \mu \in I^X$ as follows:

$$\lambda \overline{\triangle} \mu = \begin{cases} 0, & \text{if } \lambda \leq \mu, \\ \lambda \land \mu^c, & \text{otherwise.} \end{cases}$$
Recall that a fuzzy idea \( \ell \) on \( X \) [14] is a map \( \ell : I^X \rightarrow I \) that satisfies the following conditions: (i) \( \forall \lambda, \mu \in I^X \) and \( \lambda \leq \mu \Rightarrow \ell(\mu) \leq \ell(\lambda) \); (ii) \( \forall \lambda, \mu \in I^X \Rightarrow \ell(\lambda \wedge \mu) \geq \ell(\lambda) \wedge \ell(\mu) \). Also, \( \ell \) is called proper if \( \ell(1) = 0 \) and there exists \( \mu \in I^X \) such that \( \ell(\mu) > 0 \). The simplest fuzzy ideals on \( X \), \( \ell_0 \), and \( \ell_1 \) are defined as follows:

\[
\ell_0(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0, \\
0, & \text{otherwise},
\end{cases}
\tag{2}
\]

and \( \ell_1(\lambda) = 1 \forall \lambda \in I^X \). If \( \ell^1 \) and \( \ell^2 \) are fuzzy ideals on \( X \), we say that \( \ell^1 \) is finer than \( \ell^2 \) (\( \ell^2 \) is coarser than \( \ell^1 \)), denoted by \( \ell^1 \leq \ell^2 \), if \( \ell^2(\lambda) \leq \ell^1(\lambda) \forall \lambda \in I^X \). Let \( \tau \) be a fuzzy topology on \( X \) in Šostak sense, the triple \( (X, \tau, \ell) \) is called fuzzy ideal topological space.

A mapping \( F: X \rightarrow Y \) is called a fuzzy multifunction [15, 16] iff \( F(x) \in I^Y \) for each \( x \in X \). The degree of membership of \( y \) in \( F(x) \) is denoted by \( F(x)(y) = G_F(x, y) \) for any \( (x, y) \in X \times Y \). Also, \( F \) is normalized iff for each \( x \in X \), there exists \( y_0 \in Y \) such that \( G_F(x, y_0) = 1 \) and \( F^{-1}(y_0) \) is Crisp iff \( G_F(x, y) = 1 \) for each \( x \in X \) and \( y \in Y \). The upper inverse \( F^u(\mu) \), the lower inverse \( F_l(\mu) \) of \( \mu \in I^X \), and the image \( F(\lambda) \) of \( \lambda \in I^X \) are defined as follows:

\[
F^u(\mu)(x) = \bigwedge \{ G_F(x, y) \mu(y) \}, \\
F_l(\mu)(x) = \bigvee \{ G_F(x, y) \mu(y) \}, \\
F(\lambda)(y) = \bigvee \{ G_F(x, y) \lambda(x) \}.
\]

All definitions and properties of image, lower, and upper are found in [15].

3. Fuzzy Ideal and \( r \)-Fuzzy Local Function

Definition 1. Let \( (X, \tau, \ell) \) be a fuzzy ideal topological space, \( \lambda \subseteq I^X \) and \( r \in \Gamma' \). Then, the \( r \)-fuzzy local function \( \lambda_r \) of \( \lambda \) is defined as follows:

\[
\lambda_r = \bigwedge \{ \mu \in I^X : \ell(\lambda \wedge \mu) \geq r, \tau(\mu) \geq r \}.
\tag{3}
\]

Remark

(1) If we take \( \ell = \ell_0 \) for each \( \lambda \in I^X \), we have

\[
\lambda_r = \bigwedge \{ \mu \in I^X : \lambda \leq \mu, \tau(\mu) \geq r \} = C_\tau(\lambda, r).
\tag{4}
\]

(2) If we take \( \ell = \ell_1 \) (resp., \( \ell(\lambda) \geq r \)) for each \( \lambda \in I^X \), we have \( \lambda_r = 0 \).

We will occasionally write \( \lambda_r^+ \) or \( \lambda_r^+(\ell) \) for \( \lambda_r^+(\ell, r) \).

Theorem 1. Let \( (X, \tau, \ell) \) be a fuzzy ideal topological space and \( \ell^1 \) and \( \ell^2 \) be two fuzzy ideals on \( X \). Then, for any fuzzy sets \( \lambda, \nu \subseteq I^X \):

\[
(1) \quad 0^+_\lambda = 0
\]

\[
(2) \quad \text{if } \lambda \leq \nu \Rightarrow \lambda^+_{\ell, \lambda} \leq \nu^+_{\ell, \lambda}
\]

\[
(3) \quad \text{if } \ell^1 \leq \ell^2 \Rightarrow \lambda^+_{\ell, \lambda} \leq \lambda^+_{\ell, \ell^2}
\]

\[
(4) \quad \lambda_r^+ = C_\tau(\lambda_r, r) \leq C_\tau(\lambda, r)
\]

\[
(5) \quad (\lambda_r^+)^\nu \leq \lambda_r^+ \quad \text{and} \quad (\lambda_r^+)^\nu \neq (\lambda_r^+) \nu
\]

\[
(6) \quad (\lambda \vee \nu)^+_{\ell} = \lambda^+_{\ell} \vee \nu^+_{\ell} \quad \text{and} \quad (\lambda \wedge \nu)^+_{\ell} \leq \lambda^+_{\ell} \wedge \nu^+_{\ell}
\]

\[
(7) \quad \text{if } \ell(\nu) \geq r \Rightarrow (\lambda \vee \nu)^+_{\ell} = \lambda^+_{\ell}
\]

\[
(8) \quad \tau^+ \in (\ell, \ell) \leq \tau^+ (\ell_1)
\]

Proof

(1) From Definition 1, we have \( 0^+_\lambda = 0 \).

(2) Suppose that \( \lambda^+_{\ell, \lambda} \leq \nu^+_{\ell, \lambda} \).

(3) Suppose that \( \lambda^+_{\ell, \lambda} \leq \nu^+_{\ell, \lambda} \).

(4) From Definition 2, we have \( \lambda^+_{\ell, \lambda} = C_\tau(\lambda^+_{\ell, \lambda}, r) \leq C_\tau(\lambda^+_{\ell, \lambda}, r) \).

(5) By (4), we have \( (\lambda^+_{\ell, \lambda})^\nu \leq C_\tau(\lambda^+_{\ell, \lambda}) \).

(6) Hence, \( \lambda^+_{\ell, \lambda} \leq (\lambda \vee \nu)^+_{\ell, \lambda} \).

(7) By (3), \( \tau^+ (\ell_0) \leq \tau^+ (\ell) \leq \tau^+ (\ell_1) \).

Example 1. Define \( \tau, \ell : I^X \rightarrow I \) as follows:

\[
\tau(x,y) = \begin{cases} 
x = y, & \text{if } x, y \in X \text{ and } x \leq y, \\
x = y, & \text{if } x, y \in X \text{ and } y \leq x \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y.
\end{cases}
\tag{5}
\]

\[
\ell(x,y) = \begin{cases} 
x = y, & \text{if } x, y \in X \text{ and } x \leq y, \\
x = y, & \text{if } x, y \in X \text{ and } y \leq x \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y, \\
x = y, & \text{if } x, y \in X \text{ and } x \neq y.
\end{cases}
\tag{6}
\]
In (X, τ, ℓ), \( 0 = (0.6, 0.1/2) \neq (0.6, 1/2) = 0 \) and \( 1 = (0.5, 1/2) \neq (0.5, 0) = 0 \).

**Lemma 1.** Let \((X, τ, ℓ)\) be a fuzzy ideal topological space, and \[\{λ_j; j ∈ J\} \subseteq I^X.\] Then, \(∀ j ∈ J \subseteq I^X, (λ_j)^∗ \subseteq (∨_{j ∈ J} λ_j)^∗ \) (resp., \((∧_{j ∈ J} λ_j)^∗ \subseteq (∧_{j ∈ J} λ_j)^∗ \)).

**Proof.** Since \(λ_j ≤ λ_j^∗\) for each \(j ∈ J\), by Theorem 2, we have \((λ_j)^∗ \subseteq (∨_{j ∈ J} λ_j)^∗ \) for each \(j ∈ J\). This implies \((∨_{j ∈ J} λ_j)^∗ \subseteq (∧_{j ∈ J} λ_j)^∗ \). Other case is similarly proved. □

**Definition 2.** Let \((X, τ, ℓ)\) be a fuzzy ideal topological space. Then, for each \(λ ∈ I^X \) and \(r ∈ I^I\), we define an operator 
\[
\text{Cl}^∗: I^X × I^I → I^X.
\]

The fuzzy topology generated by \(\text{Cl}^∗\) is denoted by \(τ^∗\), i.e., \(τ^∗(λ) = ∨(r; \text{Cl}^∗(λ^r, r) = λ^r) \) and \(τ ≤ τ^∗\). Now, if \(ℓ = ℓ_0\), then \(\text{Cl}^∗(λ, r) = λ^r = λ^r C_r(λ, r) = C_r(λ, r) \) for each \(λ ∈ I^X\). So, \(τ^∗ = τ\). Again if \(ℓ = ℓ_1\) (resp., \(ℓ(λ) ≥ r\)), then \(\text{Cl}^∗(λ, r) = λ\) for each \(λ ∈ I^X\).

**Theorem 2.** Let \((X, τ, ℓ)\) be a fuzzy ideal topological space. Then, for any fuzzy sets \(λ, ν ∈ I^X\) and \(r ∈ I^I\), the operator \(\text{Cl}^∗: I^X × I^I → I^X\) satisfies the following properties:

1. \(\text{Cl}^∗(0, r) = 0\)
2. \(λ ≤ \text{Cl}^∗(λ, r) ≤ C_r(λ, r)\)
3. If \(λ ≤ ν\), then \(\text{Cl}^∗(λ, r) ≤ \text{Cl}^∗(ν, r)\)
4. \(\text{Cl}^∗(λν^r, r) = \text{Cl}^∗(λ, r) ∨ \text{Cl}^∗(ν, r)\)
5. \(\text{Cl}^∗(λ∧ν^r, r) ≤ \text{Cl}^∗(λ, r) ∧ \text{Cl}^∗(ν, r)\)
6. \(\text{Cl}^∗(\text{Cl}^∗(λ, r), r) = \text{Cl}^∗(λ, r)\)

**Proof**

1. Hence, \(\text{Cl}^∗(0, r) = 0 ∨ 0^∗ = 0\) and \(0^∗ = 0\) implies \(\text{Cl}^∗(0, r) = 0\).

2. Hence, \(\text{Cl}^∗(λ, r) = λ ∨ λ^∗\) implies \(λ ≤ \text{Cl}^∗(λ, r)\). Since \(λ ≤ C_r(λ, r)\) and from Theorem 1(4), we have \(λ^∗ ≤ C_r(λ, r)\) implying \(\text{Cl}^∗(λ, r) ≤ C_r(λ, r)\). Thus, \(λ ≤ \text{Cl}^∗(λ, r) ≤ C_r(λ, r)\).

3. From \(λ ≤ ν\) and Theorem 1(2), we have \(λ ∨ λ^∗ ≤ ν ∨ ν^∗\), i.e., \(\text{Cl}^∗(λ, r) ≤ \text{Cl}^∗(ν, r)\).

4. By the definition of \(\text{Cl}^∗\) and from Theorem 1(6), we have \(\text{Cl}^∗(λν^r, r) = (λν^r ∨ λν^∗^r) = (λν^r) ∨ (λν^∗^r) = (λν^r) ∨ (ν^∗^r) = \text{Cl}^∗(λ, r) ∨ \text{Cl}^∗(ν, r)\).

5. Hence, \(λ ∨ λ^∗ ≤ λ \) and \(λ ∨ λ^∗ ≤ ν\) imply \(\text{Cl}^∗(λν^r, r) ≤ \text{Cl}^∗(λ, r)\) and \(\text{Cl}^∗(λν^r, r) ≤ \text{Cl}^∗(ν, r)\). Thus, \(\text{Cl}^∗(λν^r, r) ≤ \text{Cl}^∗(λ, r) ∨ \text{Cl}^∗(ν, r)\).

6. From (2) and (3), we have \(\text{Cl}^∗(λ, r) ≤ \text{Cl}^∗(λ^r, r)\). Now, we show that \(\text{Cl}^∗(λ, r) ≤ \text{Cl}^∗(λ^r, r)\). By (4) and the definition of \(\text{Cl}^∗\), we have \(\text{Cl}^∗(λ^r, r) = \text{Cl}^∗(λ^r, r) ∨ \text{Cl}^∗(λ^r, r) = \text{Cl}^∗(λ^r, r) ∨ (λ^∗^r) = \text{Cl}^∗(λ^r, r) ∨ (λ^∗^r) ≤ \text{Cl}^∗(λ, r) ∨ λ^∗^r = \text{Cl}^∗(λ, r)\).

The following theorem is similarly proved, as in Theorem 2. □

**Theorem 3.** Let \((X, τ, ℓ)\) be a fuzzy ideal topological space. Then, for each \(λ ∈ I^X\) and \(r ∈ I^I\), we define an operator \(\text{int}^∗: I^X × I^I → I^X\) as follows:
\[
\text{int}^∗(λ, r) = λ ∧ ((λ^∗)^r).\]

For each \(λ, ν ∈ I^X\), the operator \(\text{int}^∗\) satisfies the following properties:

1. \(\text{int}^∗(0, r) = 0\)
2. \(I_≤(λ, r) ≤ \text{int}^∗(λ, r) ≤ λ\)
3. If \(λ ≤ ν\), then \(\text{int}^∗(λ, r) ≤ \text{int}^∗(ν, r)\)
4. \(\text{int}^∗(\text{int}^∗(λ, r), r) = \text{int}^∗(λ, r)\)
5. \(\text{int}^∗(λ ∧ ν, r) = \text{int}^∗(λ, r) ∧ \text{int}^∗(ν, r)\)
6. \(\text{int}^∗(λ, r) = I_≤(λ, r), if ℓ = ℓ_0\)
7. \(\text{int}^∗(λ^r, r) = (\text{Cl}^∗(λ, r))^c\)

**Definition 3.** A fuzzy multifunction \(F: (X, τ, ℓ) → (Y, η)\) is called

1. Fuzzy upper (resp., lower) \(ℓ\)-continuous at a fuzzy point \(x_0 ∈ \text{dom}(F)\) iff \(x_0 ∈ F^μ(μ)\) (resp., \(x_0 ∈ F^μ(μ)\)) for each \(μ ∈ I^I, η(μ) ≥ r\), there exists \(λ ∈ I^X\), \(r(λ) ≥ r\), and \(x_0 ∈ λ\) such that \(λ ∧ \text{dom}(F) ≤ [F^μ(μ)]^r\). (resp., \(λ ≤ [F^μ(μ)]^r\)).
2. Fuzzy upper \(ℓ\)-continuous (resp., fuzzy lower \(ℓ\)-continuous) iff it is fuzzy upper \(ℓ\)-continuous (resp., fuzzy lower \(ℓ\)-continuous) at every \(x_0 ∈ \text{dom}(F)\).

**Remark 2.** If \(F\) is normalized, then \(F\) is fuzzy upper \(ℓ\)-continuous at a fuzzy point \(x_0 ∈ \text{dom}(F)\) iff \(x_0 ∈ F^μ(μ)\) for each \(μ ∈ I^I, η(μ) ≥ r\) there exists \(λ ∈ I^X, r(λ) ≥ r\) and \(x_0 ∈ λ\) such that \(λ ≤ [F^μ(μ)]^r\).
Theorem 4. Let $F : (X, \tau, \ell) \rightarrow (Y, \eta)$ be a fuzzy multifunction (resp., normalized fuzzy multifunction). Then, $F$ is fuzzy lower (resp., upper) $\ell$-continuous iff $F(\mu) \subseteq L_1(F^{\mu}(\mu)^{\ast}_r)$ (resp., $F^{\mu}(\mu) \subseteq L_1(F^{\mu}(\mu)^{\ast}_r)$) for each $\mu \in I^X$ with $\eta(\mu) \geq r$ and $r \in I_\eta$.

Proof. ($\Rightarrow$) Let $x_{i} \in \text{dom}(F)$, $\mu \in I^{Y}$ with $\eta(\mu) \geq r$ and $x_{i} \in \lambda$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_{i} \in \lambda$ such that $\lambda \leq [F^{\mu}(\mu)]^{\ast}_r$. Thus, $x_{i} \in \lambda \leq [F^{\mu}(\mu)]^{\ast}_r$ and hence $F^{\mu}(\mu) \subseteq L_1([F^{\mu}(\mu)]^{\ast}_r)$. ($\Leftarrow$) Let $x_{i} \in \text{dom}(F)$, $\mu \in I^{Y}$ with $\eta(\mu) \geq r$ and $x_{i} \in \lambda$. Then, $F^{\mu}(\mu) \subseteq L_1([F^{\mu}(\mu)]^{\ast}_r)$ and hence $x_{i} \in \lambda \leq [F^{\mu}(\mu)]^{\ast}_r \subseteq [F^{\mu}(\mu)]^{\ast}_r$. Thus, $F$ is fuzzy lower $\ell$-continuous. Other case is similarly proved.

Remark 3

(1) Every fuzzy lower (resp., upper) $\ell$-continuous multifunction is fuzzy lower (resp., upper) precontinuous [15]

(2) If we take $\ell = \ell_0$, we have $F$ as fuzzy lower (resp., upper) $\ell$-continuous iff it is fuzzy lower (resp., upper) precontinuous.

(3) Fuzzy upper (resp., lower) $\ell$-continuity and fuzzy upper (resp., lower) semicontinuity [15] are independent notions, as shown by example 2.

Example 2. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, and $F : X \rightarrow Y$ be a fuzzy multifunction defined by $G_X(x_1, y_1) = 0.1$, $G_{x_2}(x_1, y_2) = 1$, $G_{x_1}(x_1, y_3) = 0.3$, $G_{x_1}(x_2, y_1) = 0.5$, $G_{x_2}(x_2, y_3) = 1.0$, and $G_{x_3}(x_2, y_2) = 1.0$. Define $\tau_1, \tau_2, \ell^1 : I^{\chi} \rightarrow I$ and $\eta : I^{\chi} \rightarrow I$ as follows:

\[
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.2, \\
\frac{3}{4}, & \text{if } \lambda = 0.3, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\tau_2(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.2, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\ell^1(\nu) = \begin{cases} 
1, & \text{if } \nu = 0, \\
\frac{2}{3}, & \text{if } 0 < \nu < 0.3, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\eta(\mu) = \begin{cases} 
1, & \text{if } \mu \in [0, 0.2], \\
\frac{1}{2}, & \text{if } \mu = 0.2, \\
0, & \text{otherwise}.
\end{cases}
\]

Then, $F : (X, \tau_1, \ell^1) \rightarrow (Y, \eta)$ is fuzzy upper (resp., lower) semicontinuous, but it is not fuzzy upper (resp., lower) $\ell$-continuous because $F^{\mu}(0.2) = 0.2$ (resp., $F^{\mu}(0.2) = 0.2$) and $0.2 \leq L_1(0.2, 0.2, 0.2, 0.2, 0.2)$.

(2) Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $F : X \rightarrow Y$ be a fuzzy multifunction defined by $G_X(x_1, y_1) = 0.1$, $G_{x_2}(x_1, y_2) = 1$, $G_{x_1}(x_1, y_3) = 0.3$, $G_{x_1}(x_2, y_1) = 0.5$, $G_{x_2}(x_2, y_3) = 1.0$, and $G_{x_2}(x_2, y_2) = 1.0$. Define $\tau_1, \tau_2, \ell^1 : I^{\chi} \rightarrow I$ and $\eta : I^{\chi} \rightarrow I$ as follows:

\[
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\theta, & \text{if } \lambda = 0.2, \\
\frac{3}{4}, & \text{if } \lambda = 0.3, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\tau_2(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.2, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\ell^1(\nu) = \begin{cases} 
1, & \text{if } \nu = 0, \\
\frac{2}{3}, & \text{if } 0 < \nu < 0.3, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\eta(\mu) = \begin{cases} 
1, & \text{if } \mu \in [0, 0.2], \\
\frac{1}{2}, & \text{if } \mu = 0.2, \\
0, & \text{otherwise}.
\end{cases}
\]

Corollary 1. Let $F : (X, \tau_1, \ell^1) \rightarrow (Y, \eta)$ and $G : (Y, \eta) \rightarrow (Z, \gamma)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $F \circ G$ is fuzzy lower (resp., upper) $\ell$-continuous.

4. Fuzzy Upper and Lower Almost $\ell$-Continuity

Definition 4. A fuzzy multifunction $F : (X, \tau) \rightarrow (Y, \eta, \ell)$ is called

(1) Fuzzy upper almost $\ell$-continuous at a fuzzy point $x_{i} \in \text{dom}(F)$ iff $x_{i} \in F^{\mu}(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_{i} \in \lambda$ such that $\lambda \leq F^{\mu}(\mu) \subseteq L_1(\eta(G^{\mu}(\mu), r), r)$.

(2) Fuzzy lower almost $\ell$-continuous at a fuzzy point $x_{i} \in \text{dom}(F)$ iff $x_{i} \in F^{\mu}(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_{i} \in \lambda$ such that $\lambda \leq F^{\mu}(\mu) \subseteq L_1(\eta(G^{\mu}(\mu), r), r)$.

(3) Fuzzy upper almost $\ell$-continuous (resp., fuzzy lower almost $\ell$-continuous) iff it is fuzzy upper almost $\ell$-continuous (resp., fuzzy lower almost $\ell$-continuous) at every $x_{i} \in \text{dom}(F)$.

Remark 4

(1) If $F$ is normalized, $F$ is fuzzy upper almost $\ell$-continuous at a fuzzy point $x_{i} \in \text{dom}(F)$ iff $x_{i} \in F^{\mu}(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq r$ and $x_{i} \in \lambda$ such that $\lambda \leq F^{\mu}(\eta(G^{\mu}(\mu), r), r)$.
(2) Fuzzy upper (lower) semicontinuity ⇒ fuzzy upper (lower) almost \( \ell \)-continuity ⇒ fuzzy upper (lower) almost continuity [17].

(3) Fuzzy upper (lower) almost \( \ell_r \)-continuity ⇔ fuzzy upper (lower) almost continuity.

Theorem 5. For a fuzzy multifunction \( F: (X, r) \to (Y, \eta, \ell) \), \( \mu \in \mathcal{F}^X \), and \( r \in I_{\omega} \), the following statements are equivalent:

1. \( F \) is fuzzy lower almost \( \ell \)-continuous
2. \( F^\mu(\mu) \leq I_\tau(F^\mu(I_\eta(Cl^r(\mu, r), r)), r) \), if \( \eta(\mu) \geq r \)
3. \( C_\tau(F^\mu(C_\eta(int^* (\mu, r), r)), r) \leq F^\mu(\mu) \), if \( \eta(\mu^*) \geq r \)

Proof

(1) \( \Rightarrow \) (2) Let \( x_\tau \in \text{dom}(F) \), \( \mu \in \mathcal{F}^X \) with \( \eta(\mu) \geq r \) and \( x_\tau \in F^\mu(\mu) \). Then, there exists \( \lambda \in \mathcal{F}^X \) with \( \tau(\lambda) \geq r \) and \( x_\tau \in \lambda \) such that \( \lambda \leq F^\mu(I_\eta(Cl^r(\mu, r), r)) \). Thus, \( x_\tau \in \lambda \leq I_\tau(F^\mu(I_\eta(Cl^r(\mu, r), r)), r) \), i.e., \( F^\mu(\mu) \leq I_\tau(F^\mu(I_\eta(Cl^r(\mu, r), r)), r) \). Hence, \( F \) is fuzzy lower almost \( \ell \)-continuous.

(2) \( \Rightarrow \) (3) Let \( \mu \in \mathcal{F}^X \) with \( \eta(\mu^*) \geq r \). Then, by (2), \( F^\mu(\mu^*) \leq I_\tau(F^\mu(I_\eta(Cl^r(\mu^*, r), r)), r) = (C_\tau(F^\mu(C_\eta(int^* (\mu, r), r)), r))^\tau \). Thus, \( C_\tau(F^\mu(C_\eta(int^* (\mu, r), r)), r) \leq F^\mu(\mu) \). (9)

(3) \( \Rightarrow \) (1) Let \( x_\tau \in \text{dom}(F) \), \( \mu \in \mathcal{F}^X \) with \( \eta(\mu) \geq r \) and \( x_\tau \in F^\mu(\mu) \). Then, by (3), \( (I_\tau(F^\mu(I_\eta(Cl^r(\mu, r), r)), r))^\tau \leq F^\mu(\mu^*) = (F^\mu(\mu))^\tau \), i.e., \( F^\mu(\mu) \leq I_\tau(F^\mu(I_\eta(Cl^r(\mu, r), r)), r) \). Thus, \( x_\tau \in I_\tau(F^\mu(I_\eta(Cl^r(\mu, r), r)), r) \leq F^\mu(I_\tau(Cl^r(\mu, r), r)) \). Hence, \( F \) is fuzzy lower almost \( \ell \)-continuous.

The following theorem is similarly proved as in Theorem 5. \( \Box \)

Theorem 6. For a normalized fuzzy multifunction \( F: (X, r) \to (Y, \eta, \ell) \), \( \mu \in \mathcal{F}^X \) and \( r \in I_{\omega} \), the following statements are equivalent:

1. \( F \) is fuzzy upper almost \( \ell \)-continuous
2. \( F^u(\mu) \leq I_\tau(F^u(I_\eta(Cl^r(\mu, r), r)), r) \), if \( \eta(\mu) \geq r \)
3. \( C_\tau(F^u(C_\eta(int^* (\mu, r), r)), r) \leq F^u(\mu) \), if \( \eta(\mu^*) \geq r \)

Example 3. Define a fuzzy multifunction \( F: X \to Y \) as in example 2, \( \tau_1, \tau_2: I^X \to I \) and \( \eta_1, \eta_2, \ell^1, \ell^2: I^X \to I \) as follows:

\[
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{3}{4}, & \text{if } \lambda = 0.3, \\
0, & \text{otherwise}, 
\end{cases}
\]

\[
\tau_2(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.5, \\
0, & \text{otherwise}, 
\end{cases}
\]

\[
\eta_1(\mu) = \begin{cases} 
1, & \text{if } \mu \in [0, 1], \\
\frac{1}{2}, & \text{if } \mu = 0.2, \\
0, & \text{otherwise}, 
\end{cases}
\]

\[
\eta_2(\mu) = \begin{cases} 
1, & \text{if } \mu \in [0, 1], \\
\frac{1}{2}, & \text{if } \mu = 0.5, \\
0, & \text{otherwise}, 
\end{cases}
\]

Then,

(1) \( F: (X, \tau_1) \to (Y, \eta_1, \ell^1) \) is fuzzy upper (resp., lower) almost \( \ell \)-continuous, but it is not fuzzy upper (resp., lower) semicontinuous because \( 0.2 = F^u(0.2) \leq I_{\tau_1}(F^u(I_\eta(Cl^r(0.2, 1/2), 1/2)), 1/2) = 0.3 \) and \( 0.3 = F^u(0.3) \leq I_{\tau_1}(F^u(I_\eta(Cl^r(0.3, 1/2), 1/2)), 1/2) = 0.3 \), but \( 0.2 \neq F_\tau^u(0.2, 1/2) = 0 \).

(2) \( F: (X, \tau_2) \to (Y, \eta_2, \ell^2) \) is fuzzy upper (resp., lower) almost continuous, but it is not fuzzy upper (resp., lower) almost \( \ell \)-continuous because \( 0.4 = F^u(0.4) \leq I_{\tau_2}(F^u(I_\eta(Cl^r(0.4, 1/2), 1/2)), 1/2) = 0.5 \) and \( F^u(0.5) \leq I_{\tau_2}(F^u(I_\eta(Cl^r(0.5, 1/2), 1)), 1/2) = 0.5 \), but \( 0.4 = F^u(0.4) \neq I_{\tau_2}(F^u(I_\eta(Cl^r(0.4, 1/2), 1/2)), 1/2) = 0.5 \).
Theorem 7. For a fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \Gamma$, and $r \in \Gamma$, the following statements are equivalent:

1. $F$ is fuzzy lower almost $\ell$-continuous.
2. $\tau((F^0(\mu))^\ell) \geq r$ if $\mu = I_\eta(C_\ell^\mu(\mu, r), r)$.
3. $\tau((I_\eta(C_\ell^\mu(\mu, r), r))) \geq r$ if $\eta(\mu) \geq r$.

Proof

(1) $\Rightarrow$ (2) If $\mu = I_\eta(C_\ell^\mu(\mu, r), r)$, then $\eta(\mu) \geq r$. By Theorem 5, $F^0(\mu) \leq I_\eta(F^0(I_\eta(C_\ell^\mu(\mu, r), r), r) = I_\tau(F^0(\mu), r)$. Thus, $\tau(F^0(\mu))^\ell \geq r$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Let $x_\mu \in \operatorname{dom}(F)$, $\mu \in \Gamma^\ell$ with $\eta(\mu) \geq r$ and $x_\mu \in F^\ell(\mu)$. Then, by (3) and $\mu \leq I_\eta(C_\ell^\mu(\mu, r), r)$, $\tau(F^0(I_\eta(C_\ell^\mu(\mu, r), r))) \geq r$, and $x_\mu \in F^\ell(I_\eta(C_\ell^\mu(\mu, r), r), r)$. Thus, $F$ is fuzzy lower almost $\ell$-continuous.

The following theorems are similarly proved as in Theorem 7.\qed

Theorem 8. For a fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \Gamma$, and $r \in \Gamma$, the following statements are equivalent:

1. $F$ is fuzzy lower almost $\ell$-continuous.
2. $\tau((F^0(\mu))^\ell) \geq r$ if $\mu = C_\eta(C_\ell^\mu(\mu, r), r)$.
3. $\tau((F^0(\eta(C_\ell^\mu(\mu, r), r)))^\ell) \geq r$ if $\eta(\mu) \geq r$.

Theorem 9. For a normalized fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \Gamma^\ell$, and $r \in \Gamma$, the following statements are equivalent:

1. $F$ is fuzzy upper almost $\ell$-continuous.
2. $\tau((F^0(\mu))^\ell) \geq r$ if $\mu = I_\eta(C_\ell^\mu(\mu, r), r)$.
3. $\tau((I_\eta(C_\ell^\mu(\mu, r), r)))^\ell \geq r$ if $\eta(\mu) \geq r$.

Theorem 10. For a normalized fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \Gamma^\ell$, and $r \in \Gamma$, the following statements are equivalent:

1. $F$ is fuzzy upper almost $\ell$-continuous.
2. $\tau((F^0(\mu))^\ell) \geq r$ if $\mu = C_\eta(C_\ell^\mu(\mu, r), r)$.
3. $\tau((F^0(\eta(C_\ell^\mu(\mu, r), r)))^\ell) \geq r$ if $\eta(\mu) \geq r$.

Theorem 11. Let $F: (X, \tau) \rightarrow (Y, \eta, \ell)$ be a fuzzy multifunction. Then, $F$ is fuzzy upper almost $\ell$-continuous if and only if $\tau((F^0(\mu))^\ell) \geq r$ if $\mu = C_\eta(C_\ell^\mu(\mu, r), r)$.

Proof. ($\Rightarrow$) Let $\mu \in \Gamma^\ell$ with $\mu = C_\eta(C_\ell^\mu(\mu, r), r)$ and $\eta(\mu) \geq r$. Then, $v = C_\eta(C_\ell^\mu(\mu, v), r)$, where $v = C_\eta(C_\ell^\mu(\mu, v), r) \cdot v$ (say). By Theorem 8, $\tau((F^0(v))^\ell) \geq r$ and thus $C_\eta(F^0(\mu, r), r) \leq C_\eta(F^0(v), r) = F^0(C_\eta(C_\ell^\mu(\mu, v), r), r)$.

(1) Fuzzy upper weakly $\ell$-continuous at a fuzzy point $x_\mu \in \operatorname{dom}(F)$ if $x_\mu \in F^\ell(\mu)$ for each $\mu \in \Gamma$ and $\eta(\mu) \geq r$, there exists $\lambda \in \Gamma^\ell$ such that $\lambda \cdot \mu \geq (F^0(\mu))^\ell(\mu, r)$.

(2) Fuzzy lower weakly $\ell$-continuous at a fuzzy point $x_\mu \in \operatorname{dom}(F)$ if $x_\mu \in F^\ell(\mu)$ for each $\mu \in \Gamma$ and $\eta(\mu) \geq r$, there exists $\lambda \in \Gamma^\ell$ such that $\lambda \cdot \mu \geq (F^0(\mu))^\ell(\mu, r)$.

(3) Fuzzy upper weakly $\ell$-continuous (resp., fuzzy lower weakly $\ell$-continuous) iff it is fuzzy upper weakly $\ell$-continuous (resp., fuzzy lower weakly $\ell$-continuous) at every $x_\mu \in \operatorname{dom}(F)$.\qed
Remark 5

1. If $F$ is normalized, $F$ is fuzzy upper weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom}(F)$ iff $x_i \in F^u(\mu)$ for each $\mu \in I^F$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^x$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^u(C^\tau(\mu, r))$.

2. Fuzzy upper (lower) almost $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) weakly $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) weakly continuity [17].

3. Fuzzy upper (lower) weakly $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) weakly continuity.

Theorem 15. A fuzzy multifunction $F : (X, \tau) \rightarrow (Y, \eta, \ell)$ is fuzzy lower weakly $\ell$-continuous iff $F^u(\mu) \subseteq I_\tau(F^u(C^\tau(\mu, r), r))$ for each $\mu \in I^F$, $\eta(\mu) \geq r$, and $r \in I_\tau$.

Proof. ($\Rightarrow$) Let $x_i \in \text{dom}(F)$, $\mu \in I^F$ with $\eta(\mu) \geq r$ and $x_i \in F^u(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^u(C^\tau(\mu, r))$. Thus, $x_i \in \lambda \leq I_\tau(F^u(C^\tau(\mu, r), r))$ and hence $F^u(\mu) \subseteq I_\tau(F^u(C^\tau(\mu, r), r))$.

($\Leftarrow$) Let $x_i \in \text{dom}(F)$, $\mu \in I^F$ with $\eta(\mu) \geq r$ and $x_i \in F^u(\mu)$. Then, $x_i \in F^u(\mu) \subseteq I_\tau(F^u(C^\tau(\mu, r), r))$. Thus, $x_i \in I_\tau(F^u(C^\tau(\mu, r), r)) \subseteq F^u(C^\tau(\mu, r), r)$. Hence, $F$ is fuzzy lower weakly $\ell$-continuous.

The following theorem is similarly proved as in Theorem 15. $\square$

Theorem 16. A normalized fuzzy multifunction $F : (X, \tau) \rightarrow (Y, \eta, \ell)$ is fuzzy upper weakly $\ell$-continuous iff $F^u(\mu) \subseteq I_\tau(F^u(C^\tau(\mu, r), r))$ for each $\mu \in I^F$ with $\eta(\mu) \geq r$ and $r \in I_\tau$.

Example 4. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \rightarrow Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.8, G_F(x_1, y_2) = 0.3, G_F(x_1, y_3) = 0.3, G_F(x_2, y_1) = 0.1, G_F(x_2, y_2) = 1.0, G_F(x_2, y_3) = 0.1, G_F(x_3, y_1) = 0.2, G_F(x_3, y_2) = 0.2, G_F(x_3, y_3) = 1.0$. Define $\mu_1 \in I^X$ and $\mu_2 \in I^F$ as follows: $\mu_1 = \{x_1/0.4, x_2/0.1, x_3/0.2\}$ and $\mu_2 = \{y_1/0.4, y_2/0.1, y_3/0.2\}$. Define $\tau : I^X \rightarrow I$ and $\eta, \ell^1, \ell^2 : I^Y \rightarrow I$ as follows:

\[
\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in [0, 1] \\ \frac{1}{2}, & \text{if } \lambda = 0.5 \\ 0, & \text{otherwise} \end{cases},
\]

\[
\eta(\mu) = \begin{cases} 1, & \text{if } \mu \in [0, 1] \\ \frac{3}{4}, & \text{if } \mu = \mu_2 \\ 0, & \text{otherwise} \end{cases}.
\]

Then,

1. $F : (X, \tau) \rightarrow (Y, \eta, \ell^1)$ is fuzzy upper (resp., lower) weakly $\ell$-continuous, but it is not fuzzy upper (resp., lower) weakly $\ell$-continuous because $\mu_1 = F^u(\mu_2) \subseteq I_\tau(F^u(C^\tau(\mu_2, 1/2)), 1/2) = 0.5$, but $\mu_1 = F^u(\mu_2) \subseteq I_\tau(F^u(C^\tau(\mu_2, 1/2)), 1/2) = 0$.

2. $F : (X, \tau) \rightarrow (Y, \eta, \ell^2)$ is fuzzy upper (resp., lower) weakly $\ell$-continuous, but it is not fuzzy upper (resp., lower) weakly $\ell$-continuous because $\mu_1 = F^u(\mu_2) \subseteq I_\tau(F^u(C^\tau(\mu_2, 1/2)), 1/2) = 0.5$, but $\mu_1 = F^u(\mu_2) \subseteq I_\tau(F^u(I^\tau(C^\tau(\mu_2, 1/2)), 1/2)), 1/2) = 0$.

Theorem 17. A fuzzy multifunction $F : (X, \tau) \rightarrow (Y, \eta, \ell)$ is fuzzy lower weakly $\ell$-continuous iff $C_\tau(F^u(\text{int}^\tau(\mu, r)), r) \leq F^u(\mu)$ for each $\mu \in I^F$ with $\eta(\mu) \geq r$.

Proof. ($\Rightarrow$) Let $\mu \in I^F$ with $\eta(\mu) \geq r$. Then, by Theorem 15, $F^u(\mu) \subseteq I_\tau(F^u(C^\tau(\mu, r), r)) = C_\tau(F^u(\text{int}^\tau(\mu, r)), r)$. Thus, $C_\tau(F^u(\text{int}^\tau(\mu, r)), r) \leq F^u(\mu)$.

($\Leftarrow$) Let $x_i \in \text{dom}(F)$, $\mu \in I^F$ with $\eta(\mu) \geq r$ and $x_i \in F^u(\mu)$. Then, $I_\tau(F^u(C^\tau(\mu, r), r)) = C_\tau(F^u(\text{int}^\tau(\mu, r)), r) \subseteq F^u(\mu)$.

The following theorem is similarly proved as in Theorem 17. $\square$

Theorem 18. A fuzzy multifunction $F : (X, \tau) \rightarrow (Y, \eta, \ell)$ is fuzzy upper weakly $\ell$-continuous multifunction and $F(\lambda) \leq I^\ell(X^\tau(\mu, r), r)$ for each $\lambda \in I^X$, then $F$ is fuzzy upper almost $\ell$-continuous.

Proof. Let $x_i \in \text{dom}(F)$, $\mu \in I^F$, $\eta(\mu) \geq r$, and $x_i \in F^u(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^u(C^\tau(\mu, r), r) \subseteq F^u(\mu)$. Since $F(\lambda) \leq I^\ell(X^\tau(\mu, r), r) \subseteq F^u(C^\tau(\mu, r), r)$, $\lambda \leq F^u(\mu)$.

Theorem 19. If $F : (X, \tau) \rightarrow (Y, \eta, \ell)$ is a normalized fuzzy upper weakly $\ell$-continuous multifunction and $F(\lambda) \leq I^\ell(X^\tau(\mu, r), r)$ for each $\lambda \in I^X$, then $F$ is fuzzy upper almost $\ell$-continuous.

Proof. Let $x_i \in \text{dom}(F)$, $\mu \in I^F$, $\eta(\mu) \geq r$, and $x_i \in F^u(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^u(C^\tau(\mu, r), r) \subseteq F^u(\mu)$. Since $F(\lambda) \leq I^\ell(X^\tau(\mu, r), r) \subseteq F^u(C^\tau(\mu, r), r)$, $\lambda \leq F^u(\mu)$.
Corollary 3. Let $F: (X, r) \rightarrow (Y, \eta, \ell)$ and $H: (Y, \eta) \rightarrow (Z, \gamma)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $H \circ F$ is fuzzy lower (resp., upper) weakly $\ell$-continuous if $F$ is fuzzy lower (resp., upper) weakly $\ell$-continuous and $H$ is fuzzy lower (resp., upper) semicontinuous.

Theorem 20. Let $F: (X, r) \rightarrow (Y, \eta, \ell)$ be a fuzzy lower weakly $\ell$-continuous multifunction. Then, $F^\mu (\mu) \subseteq I_1 (F^\mu (Cl^\mu (\mu, r)), r)$ for any $\mu \in I^Y$ with $\mu \subseteq I_\eta (Cl^\mu (\mu, r), r)$ and $r \in I^r$.

Proof. Let $F$ be a fuzzy lower weakly $\ell$-continuous and $\mu \subseteq I_\eta (Cl^\mu (\mu, r), r)$. Then, if $x_i \in E^\mu (\mu) \subseteq F^\mu (I_\eta (Cl^\mu (\mu, r), r))$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \subseteq F^\mu (Cl^\mu (\eta(\lambda), r), r) \subseteq F^\mu (Cl^\mu (\mu, r), r)$. Then, $\lambda \subseteq I_\tau (F^\mu (Cl^\mu (\mu, r), r), r)$ and $F^\mu (\mu) \subseteq I_\tau (F^\mu (Cl^\mu (\mu, r), r), r)$.

The following theorem is similarly proved as in Theorem 20.

Theorem 21. Let $F: (X, r) \rightarrow (Y, \eta, \ell)$ be a normalized fuzzy upper weakly $\ell$-continuous multifunction. Then, $F^\mu (\mu) \subseteq I_1 (F^\mu (Cl^\mu (\mu, r), r), r)$ for any $\mu \in I^Y$ with $\mu \subseteq I_\eta (Cl^\mu (\mu, r), r)$ and $r \in I^r$.

6. Fuzzy Upper and Lower Almost Weakly $\ell$-Continuity

Definition 6. A fuzzy multifunction $F: (X, r) \rightarrow (Y, \eta, \ell)$ is called

(1) Fuzzy upper almost weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom}(F)$ iff $x_i \in F^\mu (\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \subseteq \text{dom}(F) \subseteq C_\tau (F^\mu (Cl^\mu (\mu, r), r), r)$.

(2) Fuzzy lower almost weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom}(F)$ iff $x_i \in F^\mu (\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \subseteq C_\tau (F^\mu (Cl^\mu (\mu, r), r), r)$.

(3) Fuzzy upper almost weakly $\ell$-continuous (resp., fuzzy lower almost weakly $\ell$-continuous) iff it is fuzzy upper almost weakly $\ell$-continuous (resp., fuzzy lower almost weakly $\ell$-continuous) at every $x_i \in \text{dom}(F)$.

Remark 6

(1) If $F$ is normalized, then $F$ is fuzzy upper almost weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom}(F)$ iff $x_i \in F^\mu (\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \subseteq C_\tau (F^\mu (Cl^\mu (\mu, r), r), r)$.

(2) Fuzzy upper (lower) weakly $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) almost weakly $\ell$-continuity.

(3) Fuzzy upper (lower) almost weakly $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) almost weakly continuity [17].

Theorem 22. For a fuzzy multifunction $F: (X, r) \rightarrow (Y, \eta, \ell)$, $\mu \in I^Y$, and $r \in I^r$, the following statements are equivalent:

(1) $F$ is fuzzy lower almost weakly $\ell$-continuous
(2) $F^\mu (\mu) \subseteq I_1 (C_r (F^\mu (Cl^\mu (\mu, r), r), r), r)$ if $\eta(\mu) \geq r$
(3) $C_r (I_1 (F^\mu (int^\mu (\mu, r)), r), r) \subseteq F^\mu (\mu)$ if $\eta(\mu) \geq r$

Proof.

(1) $\Rightarrow$ (2) Let $x_i \in \text{dom}(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_i \in F^\mu (\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \subseteq C_r (F^\mu (Cl^\mu (\mu, r), r), r)$.

(2) $\Rightarrow$ (3) Let $\mu \in I^Y$ with $\eta(\mu) \geq r$. Then, by (2), $F^\mu (\mu) = F^\mu (\mu) \subseteq I_1 (C_r (F^\mu (Cl^\mu (\mu, r), r), r), r) = C_r (I_1 (F^\mu (int^\mu (\mu, r)), r), r)$.

Thus, $C_r (I_1 (F^\mu (int^\mu (\mu, r)), r), r) \subseteq F^\mu (\mu)$.

6. Fuzzy Upper and Lower Almost Weakly $\ell$-Continuity

Theorem 23. For a normalized fuzzy multifunction $F: (X, r) \rightarrow (Y, \eta, \ell)$, $\mu \in I^Y$ and $r \in I^r$, the following statements are equivalent:

(1) $F$ is fuzzy upper almost weakly $\ell$-continuous
(2) $F^\mu (\mu) \subseteq I_1 (C_r (F^\mu (Cl^\mu (\mu, r), r), r), r)$ if $\eta(\mu) \geq r$
(3) $C_r (I_1 (F^\mu (int^\mu (\mu, r)), r), r) \subseteq F^\mu (\mu)$ if $\eta(\mu) \geq r$

Example 5. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, and $F: X \rightarrow Y$ be a fuzzy multifunction defined by $G_F (x_1, y_1) = 0.8$, $G_F (x_1, y_2) = 0.3$, $G_F (x_1, y_3) = 0.3$, $G_F (x_2, y_1) = 0.1$, $G_F (x_2, y_2) = 1.0$, $G_F (x_2, y_3) = 0.1$, $G_F (x_3, y_1) = 0.2$, $G_F (x_3, y_2) = 1.0$. Define $\mu_1 \in I^Y$ and $\mu_2 \in I^Y$ as follows: $\mu_1 = [x_1/0.4, x_2/0.1, x_3/0.2]$ and $\mu_2 = [y_1/0.4, y_2/0.1, y_3/0.2]$. Define $\tau_1, \tau_2: I^X \rightarrow I$ and $\eta, \ell: I^Y \rightarrow I$ as follows:

For $\lambda \subseteq [0, 1]$, $\tau_1 (\lambda) = \begin{cases} 1, & \text{if } \lambda = 0.5 \\ \frac{1}{2}, & \text{otherwise} \end{cases}$

For $\lambda \subseteq [0, 1]$, $\tau_2 (\lambda) = \begin{cases} 1, & \text{if } \lambda = 0.5 \\ \frac{1}{2}, & \text{otherwise} \end{cases}$
Fu

inuous and fuzzy upper almost \( \ell \)-continuous. \( \ell \)-continuous and fuzzy lower almost \( \ell \)-continuous. Then, 

\[ \eta(\mu) = \begin{cases} 1, & \text{if } \mu \in [0, 1], \\ 3/4, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise}, \\ \end{cases} \quad (13) \]

Then, 

1. \( F: (X, \tau) \to (Y, \eta, \ell) \) is fuzzy upper (resp., lower) almost weakly \( \ell \)-continuous, but it is not fuzzy upper (resp., lower) almost weakly \( \ell \)-continuous

2. \( F: (X, \tau) \to (Y, \eta, \ell) \) is fuzzy upper (resp., lower) almost weakly \( \ell \)-continuous, but it is not fuzzy upper (resp., lower) weakly \( \ell \)-continuous

**Theorem 24.** Let \( F: (X, \tau) \to (Y, \eta, \ell) \) be a normalized fuzzy multifunction and \( F \) be a fuzzy upper almost weakly \( \ell \)-continuous and fuzzy lower almost \( \ell \)-continuous. Then, \( F \) is fuzzy upper weakly \( \ell \)-continuous.

Proof. Let \( \mu \in I^Y \) with \( \eta(\mu) \geq r \) and \( F \) be a fuzzy upper almost weakly \( \ell \)-continuous. Then, by Theorem 23 (2), 

\[ F^\mu(\mu) \leq \mu(C_\ell(C^\ell(\mu, r), r), r). \]

Since 

\[ C_\ell(\mu, r) = C_\ell(\text{int}^* C_\ell(\mu, r), r), \]

it follows from Theorem 8 (2) that 

\[ \tau((F^\mu(C_\ell(\mu, r)))) \geq r, \quad \tau((F^\ell(C^\ell(\mu, r))))) \geq r, \quad \text{and} \quad F^\mu(\mu) \leq \mu(C_\ell(C^\ell(\mu, r), r)). \]

Thus, by Theorem 16, \( F \) is fuzzy upper weakly \( \ell \)-continuous.

24. The following theorem is similarly proved as in Theorem 25.

**Theorem 25.** Let \( F: (X, \tau) \to (Y, \eta, \ell) \) be a normalized fuzzy multifunction and \( F \) be a fuzzy lower almost weakly \( \ell \)-continuous and fuzzy upper almost \( \ell \)-continuous. Then, \( F \) is fuzzy lower weakly \( \ell \)-continuous.

**Corollary 4.** Let \( F: (X, \tau) \to (Y, \eta, \ell) \) and \( H: (Y, \eta) \to (Z, \gamma) \) be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, \( H \circ F \) is fuzzy lower (resp., upper) almost weakly \( \ell \)-continuous if \( F \) is fuzzy lower (resp., upper) almost weakly \( \ell \)-continuous and \( H \) is fuzzy lower (resp., upper) semicontinuous.

**7. Conclusion and Future Work**

In the present paper, based on fuzzy operators \( \alpha, \beta, \delta \), \( \theta \), \( \delta_\ell \rightarrow I^X \times \Gamma \rightarrow I^X \) and \( \delta, \theta, \delta_\ell \rightarrow I^F \times \Gamma \rightarrow I^F \), we can give a generalized form of fuzzy (resp., normalized fuzzy) multifunction as \( F: (X, \tau, \ell) \to (Y, \eta, \ell) \) is fuzzy lower (resp., upper) \( (\alpha, \beta, \delta, \theta, \delta_\ell, \ell^2) \)-continuous multifunction if for every \( \mu \in I^Y \) with \( \eta(\mu) \geq r \) and \( r \in I^\ell \), 

\[ \ell^1[\alpha(F^\delta(\delta(\mu, r), r))] \leq \eta(\mu) (\text{resp., } \ell^1[\alpha(F^\delta(\delta(\mu, r), r))] \leq \eta(\mu)). \]

It is clear that

1. Fuzzy lower (resp., upper) \( (id_X, I, \ell_0) \)-continuous is fuzzy lower (resp., upper) almost \( \ell^2 \)-continuous multifunction

2. Fuzzy lower (resp., upper) \( (id_X, I, \ell_0) \)-continuous is fuzzy lower (resp., upper) weakly \( \ell^2 \)-continuous multifunction

3. Fuzzy lower (resp., upper) \( (id_X, I, \ell_1) \)-continuous is fuzzy lower (resp., upper) almost weakly \( \ell^2 \)-continuous multifunction

In the upcoming work, we will define some new separation axioms on fuzzy ideal topological spaces. Also, we shall discuss the concepts given here in the frames of fuzzy soft \( r \)-minimal spaces [3]. We hope that this work will contribute fuzzy ideal topological structure studies.

**Data Availability**

No data were used to support the findings of this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


