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Research Article

On Uncertainty Measures of the Interval-Valued Hesitant Fuzzy Set

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Interval-valued hesitant fuzzy sets (IVHFS), as a kind of decision information presenting tool which is more complicated and more scientific and more elastic, have an important practical value in multiattribute decision-making. There is little research on the uncertainty of IVHFS. The existing uncertainty measure cannot distinguish different IVHFS in some contexts. In my opinion, for an IVHFS, there should exist two types of uncertainty: one is the fuzziness of an IVHFS and the other is the nonspecificity of the IVHFS. To the best of our knowledge, the existing index to measure the uncertainty of IVHFS all are single indexes, which could not consider the two facets of an IVHFS. First, a review is given on the entropy of the interval-valued hesitant fuzzy set, and the fact that existing research cannot distinguish different interval-valued hesitant fuzzy sets in some circumstances is pointed out. With regard to the uncertainty measures of the interval-valued hesitant fuzzy set, we propose a two-tuple index to measure it. One index is used to measure the fuzziness of the interval-valued hesitant fuzzy set, and the other index is used to measure the nonspecificity of it. The method to construct the index is also given. The proposed two-tuple index can make up the fault of the existing interval-valued hesitant fuzzy set's entropy measure.

1. Introduction

In some real-life scenarios, we often need to make multicriteria decision-making, which is to sort some plans with several criteria and select the best one. One important stage in multicriteria decision-making is determining the membership degree of one alternative in regard to one certain evaluation term. The traditional method is a black and white problem. That is if the alternative meets the requirement of the evaluation term, then the membership degree is one; otherwise, the membership degree is zero. This kind of rule is simple to operate but is too absolute to lose a lot of information. In fact, the membership degree in a lot of circumstances is not a clear distinction between black and white, which otherwise have a certain degree of grey. In order to describe membership degree more perfectly, Zadeh creatively proposed the fuzzy sets (FS) theory based on sets theory [1]. In fuzzy sets, the information has some kind of uncertainty which has two dimensions. The first dimension is fuzziness which states that we cannot clearly define the degrees that one element is belonging to and not to a certain fuzzy set. De Luca and Termini proposed an entropy measure for FS which is not based on probability theory [2], and Liu developed the axiomatic definition of entropy for FS [3], both of which are important research studies on the fuzziness of FS. Fan and Ma had given some general results of the fuzzy entropy of FS based on the axiomatic definition of the fuzzy entropy of FS and distance measure of FS, and they generalized the fuzzy entropy formulation of FS proposed by De Luca and Termini [2]. The other aspect of the uncertainty of the FS is nonspecificity which measures the amount of information contained in the FS. Yager proposed several nonspecificity indexes to measures the degree that the FS only contains one element [5]. Garmendia et al. gave the general formulation for the nonspecificity measure of FS based on T-norms and negation operator [6].

There is one membership degree and nonmembership degree for each element in the FS. However, in some circumstances, it is more suitable to consider the hesitation degree. We assume that a committee is composed of ten experts, and the attitude of five of them is positive, that of three of them is negative, and two are abstained from voting. Then, the membership degree for the alternative to the feasible alternative set may be defined as 0.5, the nonmembership degree may be defined as 0.3, and the hesitation degree may be defined as 0.2. FS is not suited to be used in this kind of cases. Because of the universality of this kind of cases, Atanassov generalized FS to the intuitionistic fuzz set (IFS) [7]. Each element in the IFS has a membership degree, a nonmembership degree, and a hesitation degree, thus making IFS more suitable to deal with problems of fuzziness and uncertainty. Some research studies have been conducted on the quantification of the uncertainty of IFS. Xia and Xu proposed a new entropy and a new crossentropy of IFS, and they discussed the relation between them [8]. Huang developed two entropy measures for IFS based on the distance between two IFSs, which is simple to calculate and can give reliable results [9]. Huang and Yang gave the definition of fuzzy entropy based on probability theory [10]. Pal et al. pointed that there are two aspects associated with the uncertainty of IFS, which are fuzziness and nonspecificity, and existing studies cannot distinguish them [11].

Sometimes, in real decision-making, there is a hesitation among several membership degree values. We assume that several experts evaluate a plan on one attribute. Expert A thinks the membership degree of the plan that belongs to the attribute is 0.4, expert B thinks that the membership degree is 0.6, expert C thinks that the membership degree is 0.8, and they cannot reach an agreement, so how do we describe the evaluation result? FS and IFS both cannot be used in this circumstance. Hesitant fuzzy sets (HFS), proposed by Torra and Narukawa [12] and Torra [13], are more suitable in this kind of circumstance. The membership degree of every element in an HFS is a set, called the hesitant fuzzy element (HFE). HFS is an effective tool describing the hesitance degree of the decision maker, which is widely used in practical decision-making problems [14], so it is important to study uncertainty problems associated to the HFS. HFS is a new kind of information presentation tool, and there is little research on the uncertainty of it. Xu and Xia gave the axiom definition of the entropy for HFE, and they proposed several entropy formulations to measure the fuzziness degree of HFE [15]. Farhadina [16] pointed out that the entropy formulation of HFE proposed by Xu and Xia [15] gave the same value to several HFEs with different uncertainties intuitively. Singh and Ganie thought that the entropy formulation developed by Xu and Xia [15] cannot distinguish different HFEs in some circumstances and gave the same weights to attributes having different importance obviously, and they constructed creatively generalized hesitant fuzzy knowledge measure formulation which can be used to handle these two problems [17]. Zhao et al. [18] think that the entropy formula for the HFE introduced by Farhadinia [16] cannot differentiate different HFEs in some circumstances such as when two HFEs have the same distance to HFE $\{0.5\}$, and they gave the definition of binary entropy for HFS, with one entropy measuring the fuzziness of the HFE and the other measuring the nonspecificity. Wei et al. investigated the problem of how to apply different uncertainty facets of hesitant fuzzy linguistic term sets in different decision-making settings [19]. Xu et al. established the axiomatic definitions of fuzzy entropy and hesitancy entropy of weak probabilistic hesitant fuzzy elements [20]. Fang revisited the concept of uncertainty measures for probabilistic hesitant fuzzy information by comprehensively considering their fuzziness and hesitancy and proposed some novel entropy and cross-entropy measures for them [21]. Wei et al. focused on studying how to measure the uncertainty presented by the information of an extended hesitant fuzzy linguistic term set [22]. Fang developed some hybrid entropy and crossentropy measures of probabilistic linguistic term sets [23]. Wang et al. proposed an entropy measure of the Pythagorean fuzzy set by taking into account both Pythagorean fuzziness entropy in terms of membership and nonmembership degrees and Pythagorean hesitation entropy in terms of the hesitant degree [24]. Xu et al. modified the axiomatic definition of fuzzy entropy fuzzy sets (FSs), and the axiomatic definitions of fuzzy entropy and hesitancy entropy of intuitionist fuzzy sets (IFSs) and Pythagorean fuzzy sets (PFSs) are also revised [25]. In order to measure the uncertainly for type-2 fuzzy sets (T2FSs), the axiomatic framework of fuzzy entropy of T2FSs is established [26].

Chen et al. introduced the thought of interval number into HFS and proposed the definition of interval-valued hesitant fuzzy sets (IVHFS) which is a kind of generalization of HFS [27]. IVHFSs, as a kind of decision information presenting tool which is more complicated and more scientific and more elastic, have an important practical value in multiattribute decision-making [28, 29]. There is little research on the uncertainty of IVHFS. Farhadinia proposed the definition of entropy for IVHFS based on the distance between two IVHFSs, but the entropy formula of IVHFS cannot distinguish different IVHFSs in some contexts [16]. Pal et al. pointed out that there exist two types of uncertainty for an IFS, fuzzy-type uncertainty and nonspecificity-type uncertainty [11]. Zhao et al. thought that for an HFS, except for the fuzziness, there exists another kind of uncertainty, which is nonspecificity [18]. In my opinion, for an IVHFS, there exist two types of uncertainty, one is the fuzziness of an IVHFS, which is related to the departure of the IVHFS from its nearest script set, and the other is the nonspecificity of the IVHFS, which is related to the imprecise knowledge contained in the IVHFS. To the best of our knowledge, the existing index to measure the uncertainty of IVHFS are all single indexes, which could not consider the two facets of an IVHFS. Pal et al. stated that we cannot put forward any total measure of uncertainty for an HFE as we do not know how exactly these two types of uncertainty interact, so we also cannot put forward any total measure of uncertainty for an IVHFS as we do not know how exactly these two types of uncertainty of an IVHFS interact [13]. In view of that, this paper proposes an axiom frame which uses two-tuple entropy indexes to measure the uncertainty of the IVHFS. One entropy index is used to measure the IVHFS' fuzzy degree and the other to measure its unspecificity. The approaches to construct the two kinds of uncertainty measure are also given, and the two-tuple indexes can make up for the shortcomings of the existing entropy measures. The novelty of the paper lies in that, and to my knowledge, this is the first paper studying on the uncertainty measure of the interval-valued hesitant fuzzy set. Based on the two-tuple index, we can define distance measure, similarity measure, and design clustering algorithm to classify a set of interval-valued hesitant fuzzy sets. Distance measure has wide applications in decision-making, such as developing methods to reach consensus in a group, pattern recognition, and image processing [16]. So, this paper lays the foundation to develop distance measure and similarity measure between interval-valued hesitant fuzzy sets.

The paper is organized as follows: Section 2 introduces the concept of HFS, IVHFS, and the existing uncertainty measure of IVHFS. Section 3 proposes a two-tuple index and approaches to construct the two kinds of uncertainty measure and some theorems. The paper is concluded in Section 4, including the trends and directions of the IVHFS.

2. Preliminaries

Definition 1 (see [13]). An HFS M on the reference set X is defined in terms of a function $h_M(X)$ as follows:

$$M = \left\{ < x, h_M(x) > \mid x \in X \right\}, \quad (1)$$

where $h_M(x)$ is a set of several values in [0, 1], which represent possible membership degrees of the elements x of X to the set M. Based on the practical need, Chen et al. integrated the thought of interval number into HFS and proposed the interval-valued hesitant fuzzy sets [27].

Definition 2 An internal-valued hesitant fuzzy set $\widetilde{M}$ on the reference set X is defined as follows:

$$\widetilde{M} = \left\{ < x, h_{\widetilde{M}}(x) > \mid x \in X \right\}, \quad (2)$$

where $h_{\widetilde{M}}(x)$ is a set of several intervals of [0, 1], representing the membership degree of the element x in the reference set X to the IVHFS $\widetilde{M}$. Chen et al. called $\widetilde{M}$ as the interval-valued hesitant fuzzy element (IVHFE) [27].

Example 1. Let $X = \{x_1, x_2, x_3\}$ be a reference set, and $h_{\widetilde{M}}(x_1) = \{[0.1, 0.2], [0.3, 0.5], [0.4, 0.5]\}$, $h_{\widetilde{M}}(x_2) = \{[0.2, 0.4], [0.5, 0.8], [0.6, 0.7]\}$, and $h_{\widetilde{M}}(x_3) = \{[0.25, 0.45], [0.55, 0.85]\}$ are IVHFEs of $x_i \{i = 1, 2, 3\}$ to $\widetilde{M}$. Then,

$$\widetilde{M} = \left\{ \langle x_1, \{[0.1, 0.2], [0.3, 0.5], [0.4, 0.5]\} \rangle, \langle x_2, \{[0.2, 0.4], [0.5, 0.8], [0.6, 0.7]\} \rangle, \langle x_3, \{[0.25, 0.45], [0.55, 0.85]\} \rangle \right\}, \quad (3)$$

is an IVHFS. Let $\widetilde{H}$ be the set of all IVHFEs.

Based on the definition of the complement to an HFE α proposed by Torra and Narukawa [12], this paper defines the complement of an IVHFE $\widetilde{\alpha}$ as $\widetilde{\alpha}^C = \{\gamma \mid \gamma = [1, 1] - [a, b], [a, b] \in \widetilde{\alpha}\}$.

Definition 3 (see [15]). Given two IVHFSs $\widetilde{M}$ and $\widetilde{N}$, $h_{\widetilde{M}}^{\sigma(j)}(x_i) = [h_{\widetilde{M}}^{\sigma(j)L}(x_i), h_{\widetilde{M}}^{\sigma(j)U}(x_i)]$ and $h_{\widetilde{N}}^{\sigma(j)}(x_i) = [h_{\widetilde{N}}^{\sigma(j)L}(x_i), h_{\widetilde{N}}^{\sigma(j)U}(x_i)]$ denote the jth largest interval in $h_{\widetilde{M}}(x_i)$ and $h_{\widetilde{N}}(x_i)$, respectively. The interval-valued hesitant normalized Hamming distance is defined as follows:

$$d_{\text{ivhnh}}(M, N) = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{2l_{x_i}} \sum_{j=1}^{l_{x_i}} \left(\left| h_M^{\sigma(j)L}(x_i) - h_N^{\sigma(j)L}(x_i) \right| + \left| h_M^{\sigma(j)U}(x_i) - h_N^{\sigma(j)U}(x_i) \right| \right) \right]. \quad (4)$$

The generalized hybrid interval-valued hesitant weighted distance is defined as follows:

$$d_{\text{ghivhw}}(M, N) = \left[\sum_{i=1}^{n} w_i \left(\frac{1}{2l_{x_i}} \sum_{j=1}^{l_{x_i}} \left(\left| h_M^{\sigma(j)L}(x_i) - h_N^{\sigma(j)L}(x_i) \right|^{\lambda} + \left| h_M^{\sigma(j)U}(x_i) - h_N^{\sigma(j)U}(x_i) \right|^{\lambda} \right) \right) + \max_j \left\{ \left| h_M^{\sigma(j)L}(x_i) - h_N^{\sigma(j)L}(x_i) \right|^{\lambda}, \left| h_M^{\sigma(j)U}(x_i) - h_N^{\sigma(j)U}(x_i) \right|^{\lambda} \right\} \right]^{(/)}, \lambda > 0, \quad (5)$$

where $w_i\,(i=1,2,\ldots,n)$ is the weight of the element x_i with $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$.

Theorem 4 (see [15]) *Let Z: $[0,1]\longrightarrow[0,1]$ be a strictly monotone decreasing real function and d be a distance between IVHFSs. Then, for any IVHFS, $\widetilde{M}$ and $\widetilde{N}$*

$$\begin{aligned}S_d(\widetilde{M},\widetilde{N})&=\frac{Z(d(\widetilde{M},\widetilde{N}))-Z(1)}{Z(0)-Z(1)},\\ E_d^1(\widetilde{M})&=\frac{Z(2d(\widetilde{M},\{\widetilde{[0.5,0.5]}\}))-Z(1)}{Z(0)-Z(1)},\\ E_d^2(\widetilde{M})&=\frac{Z\left(2Z^{-1}\left(S_d(\widetilde{M},\{\widetilde{[0.5,0.5]}\})\right)\right)-Z(1)}{Z(0)-Z(1)}.\end{aligned}\tag{6}$$

Are, respectively, a similarity measure and two entropies for IVHFSs based on the corresponding distance d.

It is obvious that, for any IVHFS $\widetilde{M}$ and $\widetilde{N}$, if $d(\widetilde{M},\{\widetilde{[0.5,0.5]}\})=d(\widetilde{N},\{\widetilde{[0.5,0.5]}\})$, then we have $E_d^1(\widetilde{M})=E_d^1(\widetilde{N})$ and $E_d^2(\widetilde{M})=E_d^2(\widetilde{N})$, so we cannot differentiate $\widetilde{M}$ and $\widetilde{N}$ in this case. What is more, uncertainty can be considered of different types such as fuzziness and nonspecificity [30], and the index to measure the uncertainty of the interval-valued hesitant fuzzy set proposed by Farhadinia is a single index, which could not consider all facets of an interval-valued hesitant fuzzy set.

In view of that, this paper proposes an axiom frame in Section 3 which uses two-tuple entropy indexes to measure the uncertainty of the IVHFS. One entropy index is used to measure the IVHFS' fuzzy degree, and the other is used to measure its unspecificity.

3. Two-Tuple Entropy Measures for HFE

Definition 5. Let $\widetilde{E}_F,\widetilde{E}_{NS}$: $\widetilde{H}\longrightarrow[0,1]$ be two real functions. If $\widetilde{E}_F$ satisfies ($\widetilde{E}_F1$), ($\widetilde{E}_F2$), ($\widetilde{E}_F3$), and ($\widetilde{E}_F4$) and $\widetilde{E}_{NS}$ satisfies ($\widetilde{E}_{NS}1$), ($\widetilde{E}_{NS}2$), ($\widetilde{E}_{NS}3$), and ($\widetilde{E}_{NS}4$), we call $(\widetilde{E}_F,\widetilde{E}_{NS})$ as a two-tuple entropy measure of IVHFE $\widetilde{\alpha}$.

(1) ($\widetilde{E}_F1$): $\widetilde{E}_F(\widetilde{\alpha})=0$ if and only if $\widetilde{\alpha}=\{[0,0]\}$ or $\widetilde{\alpha}=\{[1,1]\}$ or $\widetilde{\alpha}=\{[0,0],[1,1]\}$
(2) ($\widetilde{E}_F2$): $\widetilde{E}_F(\widetilde{\alpha})=1$ if and only if $\widetilde{\alpha}=\{[0.5,0.5]\}$;
(3) ($\widetilde{E}_F3$): $\widetilde{E}_F(\widetilde{\alpha})=\widetilde{E}_F(\widetilde{\alpha}^C)$
(4) ($\widetilde{E}_F4$): for $i,j=1,\ldots,l$, if $\widetilde{\beta}_{\sigma(i)}\prec[0.5,0.5]$ and $\widetilde{\alpha}_{\sigma(i)}\prec\widetilde{\beta}_{\sigma(i)}$ hold, or $\widetilde{\beta}_{\sigma(i)}\succ[0.5,0.5]$ and $\widetilde{\alpha}_{\sigma(i)}\succ\widetilde{\beta}_{\sigma(i)}$ is true, then we have $\widetilde{E}_F(\widetilde{\alpha})\le\widetilde{E}_F(\widetilde{\beta})$
(5) ($\widetilde{E}_{NS}1$): $\widetilde{E}_{NS}(\widetilde{\alpha})=0$ if and only if there exists $\alpha\in[0,1]$, such that $\widetilde{\alpha}=[a,a]$
(6) ($\widetilde{E}_{NS}2$): $\widetilde{E}_{NS}(\widetilde{\alpha})=1$ if and only if $\widetilde{\alpha}=\{[0,0],[1,1]\}$
(7) ($\widetilde{E}_{NS}3$): $\widetilde{E}_{NS}(\widetilde{\alpha})=\widetilde{E}_{NS}(\widetilde{\alpha}^C)$
(8) ($\widetilde{E}_{NS}4$): if for all $i,j=1,\ldots,l_{\widetilde{\alpha}}$, we have $|\widetilde{\alpha}_{\sigma(i)}-\widetilde{\alpha}_{\sigma(j)}|\le|\widetilde{\beta}_{\sigma(i)}-\widetilde{\beta}_{\sigma(j)}|$, then we obtain $\widetilde{E}_{NS}(\widetilde{\alpha})\le\widetilde{E}_{NS}(\widetilde{\beta})$

Note. The relationship of "$\prec$" and "$\succ$" in ($\widetilde{E}_F4$) is complemented according to the calculation rules of the interval number [28]. For example, we assume that $\widetilde{a}$ and $\widetilde{b}$ are two interval numbers, and if the possibility degree of $\widetilde{a}$ bigger than $\widetilde{b}$ is larger or equates to 0.5, we call $\widetilde{a}\succ\widetilde{b}$ is true; otherwise, if the possibility degree of $\widetilde{b}$ bigger than $\widetilde{a}$ is larger or equates to 0.5, we call $\widetilde{b}\succ\widetilde{a}$ is true. (2) In ($\widetilde{E}_{NS}4$), $\widetilde{\alpha}_{\sigma(i)}-\widetilde{\alpha}_{\sigma(j)}$ and $\widetilde{\beta}_{\sigma(i)}-\widetilde{\beta}_{\sigma(j)}$ are two interval numbers. If interval number $\widetilde{a}$ is [0.2,0.5], then we have $|[0.2,0.5]|=|(0.2-0.5)|=0.3$.

In Definition 3, a two-tuple $(\widetilde{E}_F,\widetilde{E}_{NS})$ is utilized to measure the uncertainty of IVHFE $\widetilde{\alpha}$. The fuzzy entropy $\widetilde{E}_F$ is used to measure the fuzziness degree of $\widetilde{\alpha}$, that is, the distance between $\widetilde{\alpha}$ and the crisp value which is closest to $\widetilde{\alpha}$. The nonspecificity entropy $\widetilde{E}_{NS}$ is used to measure the nonspecific degree of $\widetilde{\alpha}$, that is, the degree of which only contains one interval. Therefore, the two-tuple $(\widetilde{E}_F,\widetilde{E}_{NS})$ not only considers the fuzziness of a set which traditional entropy can measure, but it also quantifies the nonspecificity of a set, which is more reasonable [11].

3.1. *The Fuzzy Entropy $\widetilde{E}_F$ of IVHFE.*

The uncertainty of an IVHFE $\widetilde{\alpha}$ comprised fuzziness and nonspecificity. First, we study how to measure the fuzziness degree of an IVHFE. We will give some methods to construct the measure that can be used to quantify the fuzziness degree of an IVHFE. First, a general result is given as follows:

Theorem 6. *Let $\widetilde{I}$ be the set of all the subintervals contained in interval [0, 1]. $\widetilde{R}$: $\widetilde{I}^2\longrightarrow[0,1]$ is a mapping that possesses the properties as follows:*

(1) ($\widetilde{R}1$): $\widetilde{R}(\widetilde{x},\widetilde{y})=0$ if and only if $\widetilde{x}=\widetilde{y}=[0,0]$ or $\widetilde{x}=\widetilde{y}=[1,1]$ or $\widetilde{x}=[0,0]$ and $\widetilde{y}=[1,1]$
(2) ($\widetilde{R}2$): $\widetilde{R}(\widetilde{x},\widetilde{y})=1$ if and only if $\widetilde{x}=\widetilde{y}=[0.5,0.5]$
(3) ($\widetilde{R}3$): $\widetilde{R}(\widetilde{x},\widetilde{y})=\widetilde{R}(\widetilde{y},\widetilde{x})$ holds for any $\widetilde{x},\widetilde{y}\in\widetilde{I}$
(4) ($\widetilde{R}4$): $\widetilde{R}(\widetilde{x},\widetilde{y})=\widetilde{R}([1,1]-\widetilde{x},[1,1]-\widetilde{y})$ holds for any $\widetilde{x},\widetilde{y}\in\widetilde{I}$
(5) ($\widetilde{R}5$): if $[0,0]\prec\widetilde{x}_1\prec\widetilde{x}_2\prec[0.5,0.5]$ and $[0,0]\prec\widetilde{y}_1\prec\widetilde{y}_2\prec[0.5,0.5]$, we have $\widetilde{R}(\widetilde{x}_1,\widetilde{y}_1)\le\widetilde{R}(\widetilde{x}_2,\widetilde{y}_2)$; if $[0.5,0,5]\prec\widetilde{x}_1\prec\widetilde{x}_2\prec[1,1]$ and $[0.5,0,5]\prec\widetilde{y}_1\prec\widetilde{y}_2\prec[1,1]$, then we obtain $\widetilde{R}(\widetilde{x}_1,\widetilde{y}_1)\ge\widetilde{R}(\widetilde{x}_2,\widetilde{y}_2)$

Let $l_{\widetilde{\alpha}}$ be the number of intervals in $\widetilde{\alpha}$, then the mapping $\widetilde{E}_F$: $\widetilde{H}\longrightarrow[0,1]$ defined as follows meets the axioms ($\widetilde{E}_F1$) − ($\widetilde{E}_F4$):

$$\widetilde{E}_F(\widetilde{\alpha})=\frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}}+1)}\sum_{i=1}^{l_{\widetilde{\alpha}}}\sum_{j=i}^{l_{\widetilde{\alpha}}}\widetilde{R}\left(\widetilde{\alpha}_{\sigma(i)},\widetilde{\alpha}_{\sigma(j)}\right).\tag{7}$$

Proof. The proof of Theorem 6 is provided in Appendix A.

Note. From the proof of Theorem 4, we have $\widetilde{E}_F(\widetilde{\alpha})$ defined in (7) as the fuzzy entropy of IVHFE $\widetilde{\alpha}$. $\widetilde{E}_F\{[0,0],[1,1]\}=1/3[\widetilde{R}([0,0],\ [0,0])+\widetilde{R}([0,0],[1,1])+\widetilde{R}([1,1],[1,1])]=1/3\widetilde{R}([0,0],[1,1])\neq0$. But $\widetilde{E}_F([0,0])=\widetilde{R}([0,0],[0,0])=0$ and $\widetilde{E}_F[1,1]=\widetilde{R}[1,1],[1,1]=0$, that is,

$\widetilde{E}_F([0,0]) = \widetilde{E}_F([1,1]) = 0$, while $\widetilde{E}_F\{[0,0],[1,1]\} \neq 0$, which is reasonable. □

Theorem 7. *Let* $\overline{\overline{R}}: \widetilde{I}^2 \longrightarrow [0,1]$ *be a function. Function* $\widetilde{E}_F: \widetilde{H} \longrightarrow [0,1]$ *is defined as follows:*

$$\widetilde{E}_F(\widetilde{\alpha}) = \frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}}+1)} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \overline{\overline{R}}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right), \quad (8)$$

satisfying axiom $(\widetilde{E}_F 1) - (\widetilde{E}_F 4)$. *Then,* $\overline{\overline{R}}$ *possesses the following properties:*

(1) $(\overline{\overline{R}}1)$: $\overline{\overline{R}}([0,0],[0,0]) = 0$, $\overline{\overline{R}}([0,0],[1,1]) = 0$
(2) $(\overline{\overline{R}}2)$: $\overline{\overline{R}}([0.5,0.5],[0.5,0.5]) = 1$
(3) $(\overline{\overline{R}}3)$: *if for any* $\widetilde{x},\widetilde{y} \in \widetilde{I}$, $\overline{\overline{R}}(\widetilde{x},\widetilde{y}) = \overline{\overline{R}}(\widetilde{y},\widetilde{x})$, *then for any* $\widetilde{x},\widetilde{y} \in \widetilde{I}$, $\overline{\overline{R}}(\widetilde{x},\widetilde{y}) = \overline{\overline{R}}([1,1]-\widetilde{x},[1,1]-\widetilde{y})$
(4) $(\overline{\overline{R}}4)$: *if* $[0,0] \prec \widetilde{x}_1 \prec \widetilde{x}_2 \prec [0,5,0.5]$, *we have* $\overline{\overline{R}}(\widetilde{x}_1,\widetilde{x}_1) \leq \overline{\overline{R}}(\widetilde{x}_2,\widetilde{x}_2)$ *and if* $[0.5,0.5] \prec \widetilde{x}_1 \prec \widetilde{x}_2 \prec [1,1]$, *we obtain* $\overline{\overline{R}}(\widetilde{x}_1,\widetilde{x}_1) \geq \overline{\overline{R}}(\widetilde{x}_2,\widetilde{x}_2)$

Proof. The proof of Theorem 7 is provided in Appendix B. □

3.2. The Nonspecificity Entropy of IVHFE $\widetilde{E}_{NS}$. In this section, we investigate the other aspect of the uncertainty of the IVHFE, that is, nonspecificity. First, we proposed a new measure called nonspecificity used to measure the other aspect of uncertainty of the IVHFE.

If $l_{\widetilde{\alpha}}$ is one, let $\langle l_{\widetilde{\alpha}} \rangle$ take the value two; otherwise, if $l_{\widetilde{\alpha}}$ is equal or larger than two, let $\langle l_{\widetilde{\alpha}} \rangle$ take the value $l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}}-1)$. We give a general result as follows:

Theorem 8. *Let* $\widetilde{I}$ *be the set of all the subintervals of interval [0,1], and* $\widetilde{F}: \widetilde{I}^2 \longrightarrow [0,1]$ *is a map, which satisfies the following properties:*

(1) $(\widetilde{F}1)$: $\widetilde{F}(\widetilde{x},\widetilde{y}) = 0$ *if and only if there exists* $\alpha \in [0,1]$, *such that* $\widetilde{x} = \widetilde{y} = [a,a]$
(2) $(\widetilde{F}2)$: $\widetilde{F}(\widetilde{x},\widetilde{y}) = 1$ *if and only if* $\widetilde{x} = [1,1], \widetilde{y} = [0,0]$ *or* $\widetilde{x} = [0,0], \widetilde{y} = [1,1]$
(3) $(\widetilde{F}3)$: *for any* $\widetilde{x},\widetilde{y} \in \widetilde{I}$, *we have* $\widetilde{F}(\widetilde{x},\widetilde{y}) = \widetilde{F}(\widetilde{y},\widetilde{x})$
(4) $(\widetilde{F}4)$: $\widetilde{F}(\widetilde{x},\widetilde{y}) = \widetilde{F}([1,1]-\widetilde{y},[1,1]-\widetilde{x})$, *for any* $\widetilde{x},\widetilde{y} \in \widetilde{I}$
(5) $(\widetilde{F}5)$: *if* $\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{w} \in \widetilde{I}$ *and* $\widetilde{h}(\widetilde{x}-\widetilde{y}) \succ \widetilde{h}(\widetilde{z}-\widetilde{w})$, *then* $\widetilde{F}(\widetilde{x},\widetilde{y}) \geq \widetilde{F}(\widetilde{z},\widetilde{w})$

Then, the mapping $\widetilde{E}_{NS}: \widetilde{H} \longrightarrow [0,1]$ *is defined as follows, satisfying axioms* $(\widetilde{E}_{NS}1) - (\widetilde{E}_{NS}4)$:

$$\widetilde{E}_{NS}(\widetilde{\alpha}) = \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right). \quad (9)$$

Note. The absolute value of the intervals in $(\widetilde{F}4)$ is the width of the intervals. For example, if the interval $\widetilde{x}$ is [0.34, 0.59], then $|[0.34, 0.59]| = |0.59 - 0.34| = 0.25$.

Proof. The proof of Theorem 8 is provided in Appendix C. □

Theorem 9. *Let* $\widetilde{F}: \widetilde{I}^2 \longrightarrow [0,1]$ *be a function which satisfies for any* $\widetilde{x},\widetilde{y} \in \widetilde{H}$, *we have* $\overline{\overline{F}}(\widetilde{x},\widetilde{y}) = \overline{\overline{F}}(\widetilde{y},\widetilde{x})$. *The function* $\widetilde{E}_{NS}: \widetilde{H} \longrightarrow [0,1]$ *is defined as follows:*

$$\widetilde{E}_{NS}(\widetilde{\alpha}) = \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \overline{\overline{F}}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right), \quad (10)$$

Which meets axiom $(\widetilde{E}_{NS}1) - (\widetilde{E}_{NS}4)$. *Then,* $\overline{\overline{F}}$ *has the properties as follows:*

(1) $(\overline{\overline{F}}1)$: $\overline{\overline{F}}(\widetilde{x},\widetilde{y}) = 0$ *if and only if* $\widetilde{x} = \widetilde{y}$
(2) $(\overline{\overline{F}}2)$: $\overline{\overline{F}}(\widetilde{x},\widetilde{y}) = 1$ *if and only if* $\widetilde{x} = [1,1], \widetilde{y} = [0,0]$ *or* $\widetilde{x} = [0,0], \widetilde{y} = [1,1]$
(3) $(\overline{\overline{F}}3)$: *for any* $\widetilde{x},\widetilde{y} \in \widetilde{I}$, *we have* $\overline{\overline{F}}(\widetilde{x},\widetilde{y}) = \overline{\overline{F}}([1,1]-\widetilde{y},[1,1]-\widetilde{x})$
(4) $(\overline{\overline{F}}4)$: *for any* $\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{w} \in \widetilde{I}$, *where* $|\widetilde{x}-\widetilde{y}| \geq |\widetilde{z}-\widetilde{w}|$, *we obtain* $\overline{\overline{F}}(\widetilde{x},\widetilde{y}) \geq \overline{\overline{F}}(\widetilde{z},\widetilde{w})$

Proof. The proof of Theorem 9 is provided in Appendix D. □

Theorem 10. *Let* $\widetilde{g}: \widetilde{I} \longrightarrow [0,1]$ *is a mapping, which satisfies the following properties:*

(1) $(\widetilde{G}1)$: $\widetilde{g}(\widetilde{x}) = 0$ *if and only if* $\widetilde{x} = [0,0]$
(2) $(\widetilde{G}2)$: $\widetilde{g}(\widetilde{x}) = 1$ *if and only if* $\widetilde{x} = [1,1]$
(3) $(\widetilde{G}3)$: $\widetilde{g}(\widetilde{x})$ *is an increasing function*

Then, the mapping $\widetilde{E}_{NS}(\widetilde{\alpha})$ *defined by (9) satisfies* $(\widetilde{E}_{NS}1) - (\widetilde{E}_{NS}4)$:

$$\widetilde{E}_{NS}(\widetilde{\alpha}) = \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j\geq i}^{l_{\widetilde{\alpha}}} \widetilde{g}\left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right). \quad (11)$$

Note. We assume that $\widetilde{h}: [-1,1] \longrightarrow \widetilde{I}$ is a function which can generate a new interval by taking the absolution of the two endpoints and sorting the two numbers. For example, by assuming $\widetilde{x} = [-0.2, 0.16]$, then we have $\widetilde{h}(\widetilde{x}) = [0.16, 0.2]$.

Proof. The proof of Theorem 10 is provided in Appendix E.

Note. If we let $\widetilde{g}(\widetilde{x}) = (\widetilde{x}_1 + \widetilde{x}_2 - \widetilde{x}_1\widetilde{x}_2)(\widetilde{x}_1 + \widetilde{x}_2/2)$, then we can obtain easily $\widetilde{g}(\widetilde{x})$ which satisfies $(\widetilde{G}1)$ and $(\widetilde{G}2)$. We will prove $\widetilde{g}(\widetilde{x})$ satisfies $(\widetilde{G}3)$ as follows.

We assume that $[a_1,a_2] = \widetilde{x} \prec \widetilde{y} = [b_1,b_2]$, which is the possibility of $\widetilde{x}$ which is smaller than $\widetilde{y}$ and is larger than 0.5, and from Xu and Da [30], we obtain that $b_1 + b_2 > a_1 + a_2$. Then, we have.

$$\begin{aligned} \widetilde{g}(\widetilde{x}) &= (a_1 + a_2 - a_1a_2)\left(\frac{a_1+a_2}{2}\right) < \widetilde{g}(\widetilde{y}) \\ &= (b_1 + b_2 - b_1b_2)\left(\frac{b_1+b_2}{2}\right), \end{aligned} \quad (12)$$

and we gain that

$$\widetilde{E}_{NS}(\widetilde{\alpha}) = \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j\geq i}^{l_{\widetilde{\alpha}}} \widetilde{g}\left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right)$$

$$= \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j\geq i}^{l_{\widetilde{\alpha}}} \left(\left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right)_1 + \left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right)_2 - \left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right)_1 \left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)_2\right) * \right. \tag{13}$$

$$\left(\frac{\left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right)_1 + \left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right)_2}{2}\right),$$

satisfies $(\widetilde{E}_{NS}1) - (\widetilde{E}_{NS}4)$.

Note. $(\widetilde{h}(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}))_1$ and $(\widetilde{h}(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}))_2$ represent the first and second numbers of the interval $\widetilde{h}(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)})$, respectively. □

4. Conclusions

To the best of our knowledge, there are a few research studies on the uncertainty of IVHFE and most of them cannot differentiate different IVHFE in some situations. This paper proposed a two-tuple entropy model to quantify the uncertainty of IVHFE. We use one index to measure its fuzziness degree and the other index to measure its nonspecificity. For nonspecificity entropy, we gave some methods to construct this index and represent some examples to illustrate the effectiveness of it. With regard to fuzzy entropy, due to the difficulty in the comparison of the interval number, we failed to give construction approaches, which is the important problem that we are going to focus in the near future. Furthermore, the theoretical frame of this paper can be used to quantify the uncertainty of more generalized fuzzy sets. For example, Fu and Zhao proposed the concept of the hesitant intuition fuzzy set, integrating the advantages of both hesitant fuzzy set and intuition fuzzy set [31], and Zhu et al. proposed the concept of the dual hesitant fuzzy set (DHFS) [32], as an extension of HFS to deal with the hesitant fuzzy set both for membership degree and nonmembership degree. Ren et al. introduced the normal wiggly hesitant fuzzy sets (NWHFS) as an extension of the hesitant fuzzy set [33]. So, how to apply the theoretical frame of this paper in the hesitant intuition fuzzy situation, the dual hesitant fuzzy situation, and the normal wiggly hesitant fuzzy information environment is an important topic. Based on the proposed two-tuple index, the fuzzy knowledge measure and accuracy measure can be developed further which can be used in pattern analysis and multiple attribute decision-making [34].

Based on the two-tuple entropy measure, the experts can construct interval-valued hesitant fuzzy preference relations in group decision-making problems. In order to guarantee that decision makers are nonrandom and logical and obtain reasonable decision results that are accepted by most decision makers, we can consider individual consistency control in consensus reaching processes for group decision-making problems [35]. Due to increasingly complicated decision conditions and relatively limited knowledge of decision makers, decision makers may provide incomplete interval-valued hesitant fuzzy preference relations, so how to apply the new two-tuple measure in an incomplete environment is also an important research topic. To fully consider the properties of social network evolution and improve the efficiency of consensus reaching process in group decision-making, Dong et al. introduced the concept of the local world opinion derived from individuals' common friends and then proposed an individual and local world opinion-based opinion dynamics (OD) model [36]. As future work, the study of the OD model based on social network could be extended to interval-valued hesitant fuzzy preference relations in group decision-making problems [37]. Besides, how to apply the two-tuple index proposed in this paper to the OD model is an interesting research topic.

Appendix

A. Proof of Theorem 6

Theorem 11. *Let $\widetilde{I}$ be the set of all the subintervals contained in interval [0,1]. $\widetilde{R}$: $\widetilde{I}^2 \longrightarrow [0,1]$ is a mapping that possesses the properties as follows:*

(1) $(\widetilde{R}1)$: $\widetilde{R}(\widetilde{x},\widetilde{y}) = 0$ if and only if $\widetilde{x} = \widetilde{y} = [0,0]$ or $\widetilde{x} = \widetilde{y} = [1,1]$ or $\widetilde{x} = [0,0]$ and $\widetilde{y} = [1,1]$

(2) $(\widetilde{R}2)$: $\widetilde{R}(\widetilde{x},\widetilde{y}) = 1$ if and only if $\widetilde{x} = \widetilde{y} = [0.5,0.5]$

(3) $(\widetilde{R}3)$: $\widetilde{R}(\widetilde{x},\widetilde{y}) = \widetilde{R}(\widetilde{y},\widetilde{x})$ holds for any $\widetilde{x},\widetilde{y}\in\widetilde{I}$

(4) $(\widetilde{R}4)$: $\widetilde{R}(\widetilde{x},\widetilde{y}) = \widetilde{R}([1,1]-\widetilde{x},[1,1]-\widetilde{y})$ holds for any $\widetilde{x},\widetilde{y}\in\widetilde{I}$

(5) $(\widetilde{R}5)$: if $[0,0]\prec\widetilde{x}_1\prec\widetilde{x}_2\prec[0.5,0.5]$ and $[0,0]\prec\widetilde{y}_1\prec\widetilde{y}_2\prec[0.5,0.5]$, we have $\widetilde{R}(\widetilde{x}_1,\widetilde{y}_1)\leq\widetilde{R}(\widetilde{x}_2,\widetilde{y}_2)$; if $[0.5,0,5]\prec\widetilde{x}_1\prec\widetilde{x}_2\prec[1,1]$ and $[0.5,0,5]\prec\widetilde{y}_1\prec\widetilde{y}_2\prec[1,1]$, then we obtain $\widetilde{R}(\widetilde{x}_1,\widetilde{y}_1)\geq\widetilde{R}(\widetilde{x}_2,\widetilde{y}_2)$.

Let $l_{\widetilde{\alpha}}$ be the number of intervals in $\widetilde{\alpha}$, then the mapping $\widetilde{E}_F$: $\widetilde{H} \longrightarrow [0,1]$ defined as follows meets the axioms $(\widetilde{E}_F1) - (\widetilde{E}_F4)$:

$$\widetilde{E}_F(\widetilde{\alpha}) = \frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}}+1)} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{R}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right). \tag{A.1}$$

Proof. We assume that $\widetilde{E}_F(\widetilde{\alpha})$ is defined as equation (A.1).

(1) First, we will prove that $(\widetilde{E}_F 1)$ is true. If $\widetilde{\alpha} = [0,0]$, then from (A.1), we have $\widetilde{E}_F\{[0,0]\} = \widetilde{R}([0,0], [0,0]) = 0$; if $\widetilde{\alpha} = [1,1]$, then $\widetilde{E}_F([1,1]) = \widetilde{R}([1,1], [1,1]) = 0$. If $\widetilde{\alpha} = \{[0,0], [1,1]\}$, we obtain

$$\widetilde{E}_F(\widetilde{\alpha}) = \widetilde{E}_F\{[0,0], [1,1]\} = \frac{(\widetilde{R}([0,0], [0,0]) + \widetilde{R}([1,1], [1,1]) + \widetilde{R}([0,0], [1,1]))}{3} = 0. \tag{A.2}$$

Adequacy if proved. Next, we will prove the necessity. If $\widetilde{E}_F(\widetilde{\alpha}) = 0$, then from equation (A.1), we have for any $i, j = 1, \ldots, l_{\widetilde{\alpha}}$, $\widetilde{R}(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}) = 0$.

From $(\widetilde{R}1)$, we have for any $i = 1, \ldots, l_{\widetilde{\alpha}}$, $\widetilde{\alpha}_{\sigma(i)} = [0,0]$, or $\widetilde{\alpha}_{\sigma(i)} = [1,1]$, that is, $\widetilde{\alpha} = [0,0]$ or $\widetilde{\alpha} = [1,1]$ or $\widetilde{\alpha} = \{[0,0], [1,1]\}$, necessity is proved. So, $(\widetilde{E}_F 1)$ holds.

(2) Now, we will prove that $(\widetilde{E}_F 2)$ is true. First, as of sufficiency. If $\widetilde{\alpha} = \{[0.5, 0.5]\}$, then from (3) and $(\widetilde{R}2)$, we obtain $\widetilde{E}_F(\widetilde{\alpha}) = \widetilde{E}_F[0.5, 0.5] = 1$. The adequacy is proved. We then will prove the necessity. If $\widetilde{E}_F(\widetilde{\alpha}) = 1$, then from (A.1), we have for any $i, j = 1, \ldots, l_{\widetilde{\alpha}}$, $\widetilde{R}(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}) = 1$. From $(\widetilde{R}2)$, we obtain for any $i, j = 1, \ldots, l_{\widetilde{\alpha}}$, $\widetilde{\alpha}_{\sigma(i)} = [0.5, 0.5]$, that is, $\widetilde{\alpha} = \{[0.5, 0.5]\}$, so the necessity is proved. Therefore, $\widetilde{E}_F(\widetilde{\alpha})$ defined in (3) meets $(\widetilde{E}_F 2)$.

(3) Now, we will prove that $(\widetilde{E}_F 3)$ is true. From (A.1), we have

$$\widetilde{E}_F\left(\widetilde{\alpha}^C\right) = \frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}} + 1)} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j \geq i}^{l_{\widetilde{\alpha}}} \widetilde{R}\left(\widetilde{\alpha}^C_{\sigma(i)}, \widetilde{\alpha}^C_{\sigma(j)}\right). \tag{A.3}$$

Because $\widetilde{\alpha}^C_{\sigma(i)} = [1,1] - \widetilde{\alpha}_{\sigma(l_{\widetilde{\alpha}} - i + 1)}$ is true for any $i = 1, \ldots, l_{\widetilde{\alpha}}$, then

$$\widetilde{E}_F\left(\widetilde{\alpha}^C\right) = \frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}} + 1)} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j \geq i}^{l_{\widetilde{\alpha}}} \widetilde{R}\left([1,1] - \widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}} - i + 1\right)}, [1,1] - \widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}} - j + 1\right)}\right). \tag{A.4}$$

From $(\widetilde{R}3)$ and $(\widetilde{R}4)$, we obtain

$$\begin{aligned}\widetilde{E}_F\left(\widetilde{\alpha}^C\right) &= \frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}} + 1)} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j \geq i}^{l_{\widetilde{\alpha}}} \widetilde{R}\left(\widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}} - i + 1\right)}, \widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}} - j + 1\right)}\right) \\ &= \frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}} + 1)} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j \geq i}^{l_{\widetilde{\alpha}}} \widetilde{R}\left(\widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}} - j + 1\right)}, \widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}} - i + 1\right)}\right) = \widetilde{E}_F(\widetilde{\alpha}).\end{aligned} \tag{A.5}$$

That is, $\widetilde{E}_F(\widetilde{\alpha})$ defined in (A.1) meets $(\widetilde{E}_F 3)$

(4) Now, we will prove that $(\widetilde{E}_F 4)$ is true. If for any $i = 1, \ldots, l_{\widetilde{\alpha}}$, we have $\widetilde{\beta}_{\sigma(i)} \prec [0.5, 0.5]$, and if $\widetilde{\alpha}_{\sigma(i)} \prec \widetilde{\beta}_{\sigma(i)}$, then from $(\widetilde{R}5)$, we obtain for any $i, j = 1, \ldots, l_{\widetilde{\alpha}}$, $j \geq i$, $\widetilde{R}(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}) \leq \widetilde{R}(\widetilde{\beta}_{\sigma(i)}, \widetilde{\beta}_{\sigma(j)})$. Therefore, from (A.1), $\widetilde{E}_F(\widetilde{\alpha}) \leq \widetilde{E}_F(\widetilde{\beta})$ is true. Similarly, if $\widetilde{\beta}_{\sigma(i)} \succ [0.5, 0.5]$ and $\widetilde{\alpha}_{\sigma(i)} \succ \widetilde{\beta}_{\sigma(i)}$ holds, from $(\widetilde{R}5)$, we have for any $i, j = 1, \ldots, l_{\widetilde{\alpha}}$, $j \geq i$, $\widetilde{R}(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}) \leq \widetilde{R}(\widetilde{\beta}_{\sigma(i)}, \widetilde{\beta}_{\sigma(j)})$. From (A.1), we gain $\widetilde{E}_F(\widetilde{\alpha}) \leq \widetilde{E}_F(\widetilde{\beta})$. So, $\widetilde{E}_F(\widetilde{\alpha})$ defined in (3) satisfies $(\widetilde{E}_F 4)$. □

B. Proof of Theorem 7

Theorem 12. *Let* $\overline{\widetilde{R}}: \widetilde{I}^2 \to [0,1]$ *be a function. Function* $\widetilde{E}_F: \widetilde{H} \to [0,1]$ *is defined as follows:*

$$\widetilde{E}_F(\widetilde{\alpha}) = \frac{2}{l_{\widetilde{\alpha}}(l_{\widetilde{\alpha}} + 1)} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \overline{\widetilde{R}}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right), \tag{A.6}$$

satisfying axiom $(\widetilde{E}_F 1) - (\widetilde{E}_F 4)$. *Then,* $\overline{\widetilde{R}}$ *possesses the following properties:*

$(\overline{\widetilde{R}}1)$: $\overline{\widetilde{R}}([0,0], [0,0]) = 0$, $\overline{\widetilde{R}}([0,0], [1,1]) = 0$

($\overline{\overline{R}}2$): $\overline{\overline{R}}([0.5,0.5],[0.5,0.5]) = 1$

($\overline{\overline{R}}3$): *if for any* $\tilde{x},\tilde{y}\in\tilde{I}$, $\overline{\overline{R}}(\tilde{x},\tilde{y}) = \overline{\overline{R}}(\tilde{y},\tilde{x})$, *then for any* $\tilde{x},\tilde{y}\in\tilde{I}$, $\overline{\overline{R}}(\tilde{x},\tilde{y}) = \overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y})$

($\overline{\overline{R}}4$): *if* $[0,0]\prec\tilde{x}_1\prec\tilde{x}_2\prec[0,5,0.5]$, *we have* $\overline{\overline{R}}(\tilde{x}_1,\tilde{x}_1)\le\overline{\overline{R}}(\tilde{x}_2,\tilde{x}_2)$; *if* $[0.5,0.5]\prec\tilde{x}_1\prec\tilde{x}_2\prec[1,1]$, *we obtain* $\overline{\overline{R}}(\tilde{x}_1,\tilde{x}_1)\ge\overline{\overline{R}}(\tilde{x}_2,\tilde{x}_2)$

Proof

(1) We assume that $\tilde{E}_F(\tilde{\alpha})$ is defined in (6). From ($\tilde{E}_F1$), we have if $\tilde{E}_F(\tilde{\alpha}) = 0$, then $\tilde{\alpha} = [0,0]$ or $\tilde{\alpha} = [1,1]$ or $\tilde{\alpha} = \{[0,0],[1,1]\}$. If $\tilde{\alpha} = [0,0]$, $0 = \tilde{E}_F(\tilde{\alpha}) = \overline{\overline{R}}([0,0],[0,0])$, so $\overline{\overline{R}}([0,0],[0,0]) = 0$ is true.
If $\tilde{\alpha} = [1,1]$, we obtain $0 = \tilde{E}_F(\tilde{\alpha}) = \overline{\overline{R}}([1,1],[1,1])$, so $\overline{\overline{R}}([1,1],[1,1]) = 0$ is true. If $\tilde{\alpha} = \{[0,0],[1,1]\}$, we have $0 = \tilde{E}_F(\tilde{\alpha}) = \overline{\overline{R}}([0,0],[1,1])$, so function $\overline{\overline{R}}$ satisfies property ($\overline{\overline{R}}1$).

(2) Let $\tilde{\alpha} = [0,0]$, from (6), we have $\tilde{E}_F([0.5,0.5]) = \overline{\overline{R}}([0.5,0.5],[0.5,0.5])$, and because $\tilde{E}_F(\tilde{\alpha})$ meets ($\tilde{E}_F2$), so $\overline{\overline{R}}([0.5,0.5],[0.5,0.5]) = 1$, which is $\tilde{E}_F(\tilde{\alpha})$ meets ($\overline{\overline{R}}2$).

(3) If for any $\tilde{x},\tilde{y}\in\tilde{I}$, we have $\overline{\overline{R}}(\tilde{x},\tilde{y}) = \overline{\overline{R}}(\tilde{y},\tilde{x})$. We assume that there exists $\tilde{x},\tilde{y}\in\tilde{I}$, which meets $\overline{\overline{R}}(\tilde{x},\tilde{y})\ne\overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y})$, and without loss of generality, we suppose that $\overline{\overline{R}}(\tilde{x},\tilde{y})<\overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y})$. Let $\tilde{\alpha} = \{\tilde{x},\tilde{y}\}$, and we assume that $\tilde{x}\prec\tilde{y}$. From ($\tilde{E}_F3$), we gain $\tilde{E}_F(\tilde{\alpha}^C) = \tilde{E}_F\{[1,1]-\tilde{x},[1,1]-\tilde{y}\}$

$$\begin{aligned}
&= \frac{1}{3}(\overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{x}) + \overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y}) + \overline{\overline{R}}([1,1]-\tilde{y},[1,1]-\tilde{y})),\\
&= \frac{1}{3}\Big(\tilde{E}_F([1,1]-\tilde{x}) + \overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y}) + \tilde{E}_F([1,1]-\tilde{y})\Big),\\
&= \frac{1}{3}\Big(\tilde{E}_F(\tilde{x}) + \overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y}) + \tilde{E}_F(\tilde{y})).
\end{aligned} \tag{A.7}$$

While $\tilde{E}_F(\tilde{\alpha}) = \tilde{E}_F\{\tilde{x},\tilde{y}\}$

$$= \frac{1}{3}(\overline{\overline{R}}(\tilde{x},\tilde{x}) + \overline{\overline{R}}(\tilde{x},\tilde{y}) + \overline{\overline{R}}(\tilde{y},\tilde{y})) = \frac{1}{3}\Big(\tilde{E}_F(\tilde{x}) + \overline{\overline{R}}(\tilde{x},\tilde{y}) + \tilde{E}_F(\tilde{y})). \tag{A.8}$$

From the previous assumption, $\overline{\overline{R}}(\tilde{x},\tilde{y})<\overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y})$, we have $\tilde{E}_F(\tilde{\alpha})<\tilde{E}_F(\tilde{\alpha}^C)$, which is in contradiction to the fact that $\tilde{E}_F(\tilde{\alpha})$ constructed in (4) satisfies ($\tilde{E}_F3$).
Therefore, if for any $\tilde{x},\tilde{y}\in\tilde{I}$, we have $\overline{\overline{R}}(\tilde{x},\tilde{y}) = \overline{\overline{R}}(\tilde{y},\tilde{x})$, then for any $\tilde{x},\tilde{y}\in\tilde{I}$, $\overline{\overline{R}}(\tilde{x},\tilde{y}) = \overline{\overline{R}}([1,1]-\tilde{x},[1,1]-\tilde{y})$ is true. That is, $\overline{\overline{R}}$ satisfies ($\overline{\overline{R}}3$).

(4) If there exists $\tilde{x}_1,\tilde{x}_2$, then we subject to $[0,0]\prec\tilde{x}_1\prec\tilde{x}_2\prec[0.5,0.5]$ and $\overline{\overline{R}}(\tilde{x}_1,\tilde{x}_1)>\overline{\overline{R}}(\tilde{x}_2,\tilde{x}_2)$. From (4), $\tilde{E}_F(\{\tilde{x}_1\}) = \overline{\overline{R}}(\tilde{x}_1,\tilde{x}_1)>\overline{\overline{R}}(\tilde{x}_2,\tilde{x}_2) = \tilde{E}_F(\{\tilde{x}_2\})$, so we obtain $\tilde{E}_F(\{\tilde{x}_1\})>\tilde{E}_F(\{\tilde{x}_2\})$, which contradicts ($\tilde{E}_F4$). So, for any $\tilde{x},\tilde{y}\in\tilde{I}$, if $[0,0]\prec\tilde{x}_1\prec\tilde{x}_2\prec[0.5,0.5]$ is true, then we have $\tilde{E}_F(\{\tilde{x}_1\})<\tilde{E}_F(\{\tilde{x}_2\})$ is true. Similarly, we can obtain if $[0.5,0.5]\prec\tilde{x}_1\prec\tilde{x}_2\prec[1,1]$ is satisfied, then $\overline{\overline{R}}(\tilde{x}_1,\tilde{x}_1)\ge\overline{\overline{R}}(\tilde{x}_2,\tilde{x}_2)$ is true. That is, $\overline{\overline{R}}$ satisfies ($\overline{\overline{R}}4$).

From (1)–(4), we have that Theorem 7 is true. □

C. Proof of Theorem 8

Theorem 13. *Let $\tilde{I}$ be the set of all the subintervals of interval [0,1] and $\tilde{F}$: $\tilde{I}^2\to[0,1]$ is a map, which satisfies the following properties:*

($\tilde{F}1$): $\tilde{F}(\tilde{x},\tilde{y}) = 0$ *if and only if there exists* $\alpha\in[0,1]$, *such that* $\tilde{x} = \tilde{y} = [a,a]$

($\tilde{F}2$): $\tilde{F}(\tilde{x},\tilde{y}) = 1$ *if and only if* $\tilde{x} = [1,1]$, $\tilde{y} = [0,0]$ *or* $\tilde{x} = [0,0]$, $\tilde{y} = [1,1]$

($\tilde{F}3$): *for any* $\tilde{x},\tilde{y}\in\tilde{I}$, *we have* $\tilde{F}(\tilde{x},\tilde{y}) = \tilde{F}(\tilde{y},\tilde{x})$

($\tilde{F}4$): $\tilde{F}(\tilde{x},\tilde{y}) = \tilde{F}([1,1]-\tilde{y},[1,1]-\tilde{x})$, *for any* $\tilde{x},\tilde{y}\in\tilde{I}$

($\tilde{F}5$): *if* $\tilde{x},\tilde{y},\tilde{z},\tilde{w}\in\tilde{I}$ *and* $\tilde{h}(\tilde{x}-\tilde{y})\succ\tilde{h}(\tilde{z}-\tilde{w})$, *then* $\tilde{F}(\tilde{x},\tilde{y})\ge\tilde{F}(\tilde{z},\tilde{w})$

Then, the mapping $\widetilde{E}_{NS}$: $\widetilde{H} \longrightarrow [0,1]$ *is defined as follows which satisfies axioms* $(\widetilde{E}_{NS}1)-(\widetilde{E}_{NS}4)$.

$$\widetilde{E}_{NS}(\widetilde{\alpha}) = \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right). \tag{A.9}$$

Proof

(1) If $\widetilde{E}_{NS}(\widetilde{\alpha}) = 0$, from equation (A.9), for any $i, j = 1, \dots, l_{\widetilde{\alpha}}, j \geq i$, we have $\widetilde{F}(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}) = 0$. From $(\widetilde{F}1)$, we obtain, for any $i, j = 1, \dots, l_{\widetilde{\alpha}}, j \geq i$, $\widetilde{\alpha}_{\sigma(i)} = \widetilde{\alpha}_{\sigma(j)}$, and there exists $\alpha \in [0,1]$, which satisfies $\widetilde{\alpha}_{\sigma(i)} = \widetilde{\alpha}_{\sigma(j)} = [a,a]$, so $\widetilde{\alpha} = [a,a]$. Conversely, if there exists $\alpha \in [0,1]$, such that $\widetilde{\alpha} = [a,a]$, then from (A.9) and $(\widetilde{F}1)$, we gain $\widetilde{E}_{NS}(\widetilde{\alpha}) = 0$. That is, $\widetilde{E}_{NS}$ defined by (A.9) satisfies $(\widetilde{E}_{NS}1)$.

(2) If $\widetilde{E}_{NS}(\widetilde{\alpha}) = 1$, then from (A.9), for any $i, j = 1, \dots, l_{\widetilde{\alpha}}, j > i$, we have $\widetilde{F}(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}) = 1$. From $(\widetilde{F}2)$, for any $i, j = 1, \dots, l_{\widetilde{\alpha}}$, $\widetilde{\alpha}_{\sigma(i)} = [0,0]$ or $\widetilde{\alpha}_{\sigma(i)} = [1,1]$. Then, we obtain $\widetilde{\alpha} = \{[0,0],[1,1]\}$. Conversely, if $\widetilde{\alpha} = \{[0,0],[1,1]\}$, from (A.9) and $(\widetilde{F}2)$, we have $\widetilde{E}_{NS}(\widetilde{\alpha}) = \widetilde{F}([0,0],[1,1]) = 1$. Therefore, $\widetilde{E}_{NS}$ defined by (A.9) satisfies $(\widetilde{E}_{NS}2)$.

(3) $\widetilde{E}_{NS}(\widetilde{\alpha}^C) = 2/\langle l_{\widetilde{\alpha}} \rangle \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}(\widetilde{\alpha}^C_{\sigma(i)}, \widetilde{\alpha}^C_{\sigma(j)}) = 2/\langle l_{\widetilde{\alpha}} \rangle \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}([1,1] - \widetilde{\alpha}_{\sigma(l_{\widetilde{\alpha}}-i+1)}, [1,1] - \widetilde{\alpha}_{\sigma(l_{\widetilde{\alpha}}-j+1)}) \circ$ From $(\widetilde{F}3)$ and $(\widetilde{F}4)$, we have

$$\begin{aligned}
\widetilde{E}_{NS}\left(\widetilde{\alpha}^C\right) &= \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}\left(\widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}-i+1}\right)}, \widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}-j+1}\right)}\right), \\
&= \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}\left(\widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}-j+1}\right)}, \widetilde{\alpha}_{\sigma\left(l_{\widetilde{\alpha}-i+1}\right)}\right), \\
&= \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right), \\
&= \widetilde{E}_{NS}(\widetilde{\alpha}).
\end{aligned} \tag{A.10}$$

That is, $\widetilde{E}_{NS}(\widetilde{\alpha}) = \widetilde{E}_{NS}(\widetilde{\alpha}^C)$ is true.

(4) If for any $i, j = 1, \dots, l_{\widetilde{\alpha}}$, we have $|\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}| \leq |\widetilde{\beta}_{\sigma(i)} - \widetilde{\beta}_{\sigma(j)}|$. From equation (A.9) and $(\widetilde{F}5)$, we obtain $\widetilde{E}_{NS}(\widetilde{\alpha}) = 2/\langle l_{\widetilde{\alpha}} \rangle \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}) \leq 2/\langle l_{\widetilde{\alpha}} \rangle \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \widetilde{F}(\widetilde{\beta}_{\sigma(i)}, \widetilde{\beta}_{\sigma(j)}) = \widetilde{E}_{NS}(\widetilde{\beta})$.

That is, $\widetilde{E}_{NS}(\widetilde{\alpha}) \leq \widetilde{E}_{NS}(\widetilde{\beta})$ is true. So, $\widetilde{E}_{NS}$ defined by equation (A.9) satisfies $(\widetilde{E}_{NS}4)$. From (1)–(4), we have Theorem 8 as true. □

D. Proof of Theorem 9

Theorem 14. *Let* $\overline{\widetilde{F}}$: $\widetilde{I}^2 \rightarrow [0,1]$ *be a function which satisfies the condition that for any* $\widetilde{x}, \widetilde{y} \in \widetilde{H}$, *we have* $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = \overline{\widetilde{F}}(\widetilde{y}, \widetilde{x})$. *The function* $\widetilde{E}_{NS}$: $\widetilde{H} \rightarrow [0,1]$ *is defined as follows:*

$$\widetilde{E}_{NS}(\widetilde{\alpha}) = \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j=i}^{l_{\widetilde{\alpha}}} \overline{\widetilde{F}}\left(\widetilde{\alpha}_{\sigma(i)}, \widetilde{\alpha}_{\sigma(j)}\right), \tag{A.11}$$

Which meets axiom $(\widetilde{E}_{NS}1)-(\widetilde{E}_{NS}4)$. *Then,* $\overline{\widetilde{F}}$ *has the properties as follows:*

$(\overline{\widetilde{F}}1)$: $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 0$ *if and only if* $\widetilde{x} = \widetilde{y}$

$(\overline{\widetilde{F}}2)$: $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 1$ *if and only if* $\widetilde{x} = [1,1], \widetilde{y} = [0,0]$ *or* $\widetilde{x} = [0,0], \widetilde{y} = [1,1]$

$(\overline{\widetilde{F}}3)$: *for any* $\widetilde{x}, \widetilde{y} \in \widetilde{I}$, *we have* $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = \overline{\widetilde{F}}([1,1] - \widetilde{y}, [1,1] - \widetilde{x})$

$(\overline{\widetilde{F}}4)$: *for any* $\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{w} \in \widetilde{I}$, *where* $|\widetilde{x} - \widetilde{y}| \geq |\widetilde{z} - \widetilde{w}|$, *we obtain* $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) \geq \overline{\widetilde{F}}(\widetilde{z}, \widetilde{w})$

Proof

(1) First, we will prove the necessity. If there exists $\widetilde{x} \neq \widetilde{y}$, such that $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 0$. Let IVHFE $\widetilde{\alpha} = \{\widetilde{x}, \widetilde{y}\}$. From equation (A.11), we have $\widetilde{E}_{NS}(\widetilde{\alpha}) = \overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 0$. Because $\widetilde{E}_{NS}(\widetilde{\alpha})$ satisfies $(\widetilde{E}_{NS}1)$, so $\widetilde{x} = \widetilde{y}$, which is contradictory. That is, if $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 0$ is true, then we obtain $\widetilde{x} = \widetilde{y}$; therefore, necessity is true. Conversely, if $\widetilde{x} = \widetilde{y}$, let IVHFE $\widetilde{\alpha} = \{\widetilde{x}, \widetilde{y}\}$. From equation (A.11), we have $\widetilde{E}_{NS}(\widetilde{\alpha}) = \overline{\widetilde{F}}(\widetilde{x}, \widetilde{y})$. Due to the fact that $\widetilde{E}_{NS}(\widetilde{\alpha})$ satisfies $(\widetilde{E}_{NS}1)$, we obtain $\widetilde{E}_{NS}(\widetilde{\alpha}) = 0$, so $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 0$. The adequacy is proved.

(2) First, we will prove the necessity. If $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 1$, without loss of generality, we assume that $\widetilde{x} \prec \widetilde{y}$ (we note that $\widetilde{x}$ and $\widetilde{y}$ are two intervals, and based on the rules of comparison of the intervals due to Xu and Da (2002) [21], if the possibility degree of $\widetilde{x}$ is smaller than $\widetilde{y}$ that is bigger than 0.5, then we define the relationship between $\widetilde{x}$ and $\widetilde{y}$ as $\widetilde{x} \prec \widetilde{y}$). Let IVHFE $\widetilde{\alpha} = \{\widetilde{x}, \widetilde{y}\}$. From equation (A.11), we have $\widetilde{E}_{NS}(\widetilde{\alpha}) = \overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 1$. Because $\widetilde{E}_{NS}(\widetilde{\alpha})$ satisfies $(\widetilde{E}_{NS}2)$, we have $\widetilde{\alpha} = \{[0,0],[1,1]\}$. If we assume $\widetilde{x} \prec \widetilde{y}$, then $\widetilde{x} = [0,0]$ and $\widetilde{y} = [1,1]$. If we assume $\widetilde{x} \succ \widetilde{y}$, then we obtain $\widetilde{x} = [1,1]$ and $\widetilde{y} = [1,1]$, which shows the necessity is proved. Conversely, if $\widetilde{x} = [0,0]$ and $\widetilde{y} = [1,1]$. Let IVHFE $\widetilde{\alpha} = \{\widetilde{x}, \widetilde{y}\}$. Because $\widetilde{E}_{NS}(\widetilde{\alpha})$ satisfies $(\widetilde{E}_{NS}2)$, we have $\widetilde{E}_{NS}(\widetilde{\alpha}) = 1$. From equation (A.11), we obtain $\widetilde{E}_{NS}(\widetilde{\alpha}) = \overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = \overline{\widetilde{F}}([0,0],[1,1])$. Therefore, $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 1$. Similarly, if we assume that $\widetilde{x} = [1,1]$, $\widetilde{y} = [0,0]$, we can obtain $\overline{\widetilde{F}}(\widetilde{x}, \widetilde{y}) = 1$. The adequacy is proved.

(3) If exist $\widetilde{x}, \widetilde{y} \in \widetilde{I}$, such that $\overline{\overline{F}}(\widetilde{x}, \widetilde{y}) \neq \overline{\overline{F}}([1,1] - \widetilde{y}, [1,1] - \widetilde{x})$. Without loss of generality, we assume that $\widetilde{x} \prec \widetilde{y}$, and $\overline{\overline{F}}(\widetilde{x}, \widetilde{y}) < \overline{\overline{F}}([1,1] - \widetilde{y}, [1,1] - \widetilde{x})$. Let IVHFE $\widetilde{\alpha} = \{\widetilde{x}, \widetilde{y}\}$. From equation (A.11), we have $\widetilde{E}_{NS}(\widetilde{\alpha}^C) = \overline{\overline{F}}([1,1] - \widetilde{y}, [1,1] - \widetilde{x})$. Because $\widetilde{E}_{NS}(\widetilde{\alpha})$ satisfies $(\widetilde{E}_{NS}3)$, we obtain

$\overline{\overline{F}}(\widetilde{x}, \widetilde{y}) = \widetilde{E}_{NS}(\widetilde{\alpha}) = \widetilde{E}_{NS}(\widetilde{\alpha}^C) = \overline{\overline{F}}([1,1] - \widetilde{y}, [1,1] - \widetilde{x})$. So, $\overline{\overline{F}}(\widetilde{x}, \widetilde{y}) = \overline{\overline{F}}([1,1] - \widetilde{y}, [1,1] - \widetilde{x})$ holds, which is a contradiction. Therefore, for any $\widetilde{x}, \widetilde{y} \in \widetilde{I}$, $\overline{\overline{F}}(\widetilde{x}, \widetilde{y}) = \overline{\overline{F}}([1,1] - \widetilde{y}, [1,1] - \widetilde{x})$ holds. That is, $\overline{\overline{F}}$ satisfies $(\overline{\overline{F}}3)$.

(4) If there exist $\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{w} \in \widetilde{I}$, then we have $|\widetilde{x} - \widetilde{y}| \geq |\widetilde{z} - \widetilde{w}|$ and $\overline{\overline{F}}(\widetilde{x}, \widetilde{y}) < \overline{\overline{F}}(\widetilde{z}, \widetilde{w})$. Let IVHFE $\widetilde{\alpha} = \{\widetilde{x}, \widetilde{y}\}$ and IVHFE $\widetilde{\beta} = \{\widetilde{z}, \widetilde{w}\}$. Without loss of generality, we assume that $\widetilde{x} \prec \widetilde{y}$ and $\widetilde{z} \prec \widetilde{w}$. From equation (A.11), we have $\widetilde{E}_{NS}(\widetilde{\alpha}) = \overline{\overline{F}}(\widetilde{x}, \widetilde{y})$ and $\widetilde{E}_{NS}(\widetilde{\beta}) = \overline{\overline{F}}(\widetilde{z}, \widetilde{w})$. It follows that $\widetilde{E}_{NS}(\widetilde{\alpha}) < \widetilde{E}_{NS}(\widetilde{\beta})$. Due to the fact that $\widetilde{E}_{NS}(\widetilde{\alpha})$ satisfies $(\widetilde{E}_{NS}4)$, we have $\widetilde{E}_{NS}(\widetilde{\alpha}) \geq \widetilde{E}_{NS}(\widetilde{\beta})$, which is a contradiction. So, for any $\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{w} \in \widetilde{I}$, if $|\widetilde{x} - \widetilde{y}| \geq |\widetilde{z} - \widetilde{w}|$ holds, then we have $\overline{\overline{F}}(\widetilde{x}, \widetilde{y}) \geq \overline{\overline{F}}(\widetilde{z}, \widetilde{w})$. That is, $\overline{\overline{F}}$ satisfies $(\overline{\overline{F}}4)$. □

E. Proof of Theorem 10

Theorem 15. *Let $\widetilde{g}$: $\widetilde{I} \to [0,1]$ is a mapping, which satisfies the following properties:*

$(\widetilde{G}1)$: *$\widetilde{g}(\widetilde{x}) = 0$ if and only if $\widetilde{x} = [0,0]$*

$(\widetilde{G}2)$: *$\widetilde{g}(\widetilde{x}) = 1$ if and only if $\widetilde{x} = [1,1]$*

$(\widetilde{G}3)$: *$\widetilde{g}(\widetilde{x})$ is an increasing function*

Then, the mapping $\widetilde{E}_{NS}(\widetilde{\alpha})$ defined by equation (A.10) satisfies $(\widetilde{E}_{NS}1) - (\widetilde{E}_{NS}4)$:

$$\widetilde{E}_{NS}(\widetilde{\alpha}) = \frac{2}{\langle l_{\widetilde{\alpha}} \rangle} \sum_{i=1}^{l_{\widetilde{\alpha}}} \sum_{j \geq i}^{l_{\widetilde{\alpha}}} \widetilde{g}\left(\widetilde{h}\left(\widetilde{\alpha}_{\sigma(i)} - \widetilde{\alpha}_{\sigma(j)}\right)\right). \tag{A.12}$$

Proof. Let $\widetilde{F}(\widetilde{x}, \widetilde{y}) = \widetilde{g}(\widetilde{h}(\widetilde{x} - \widetilde{y}))$. From Theorem 8, we have if $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}1) - (\widetilde{F}4)$, then $\widetilde{E}_{NS}(\widetilde{\alpha})$ defined by (A.12) satisfies $(\widetilde{E}_{NS}1) - (\widetilde{E}_{NS}4)$. Therefore, we just have to prove that $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}1) - (\widetilde{F}4)$.

(1) If $\widetilde{F}(\widetilde{x}, \widetilde{y}) = 0$, then $\widetilde{g}(\widetilde{h}(\widetilde{x} - \widetilde{y})) = 0$. From $(\widetilde{G}1)$, we obtain $\widetilde{h}(\widetilde{x} - \widetilde{y}) = [0,0]$. So, there exists $\alpha \in [0,1]$, such that $\widetilde{x} = \widetilde{y} = [a,a]$, thus the necessity holds. Conversely, if there exists $\alpha \in [0,1]$, such that $\widetilde{x} = \widetilde{y} = [a,a]$, then $\widetilde{h}(\widetilde{x} - \widetilde{y}) = [0,0]$. So, we have $\widetilde{F}(\widetilde{x}, \widetilde{y}) = \widetilde{g}(\widetilde{h}(\widetilde{x} - \widetilde{y})) = 0$. That is, $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}1)$, i.e., the sufficiency holds.

(2) If $\widetilde{F}(\widetilde{x}, \widetilde{y}) = 1$, from $(\widetilde{G}2)$, we have $\widetilde{h}(\widetilde{x} - \widetilde{y}) = [1,1]$, so $\widetilde{x} = [1,1]$, $\widetilde{y} = [0,0]$ or $\widetilde{x} = [0,0]$, $\widetilde{y} = [1,1]$ or $\widetilde{x} = \widetilde{y} = [0,1]$. That is to say that the necessity holds. Conversely, if $\widetilde{x} = [1,1]$, $\widetilde{y} = [0,0]$, or $\widetilde{x} = [0,0]$, $\widetilde{y} = [1,1]$ or $\widetilde{x} = \widetilde{y} = [0,1]$, we all obtain $\widetilde{h}(\widetilde{x} - \widetilde{y}) = [1,1]$, and from $(\widetilde{G}2)$, we have $\widetilde{F}(\widetilde{x}, \widetilde{y}) = \widetilde{g}(\widetilde{h}(\widetilde{x} - \widetilde{y})) = 1$. So, we proved the sufficiency. Then, we know that $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}2)$.

(3) $\widetilde{F}(\widetilde{x}, \widetilde{y}) = \widetilde{g}(\widetilde{h}(\widetilde{x} - \widetilde{y})) = \widetilde{g}(\widetilde{h}(\widetilde{y} - \widetilde{x})) = \widetilde{F}(\widetilde{y}, \widetilde{x})$ holds, for any $\widetilde{x}, \widetilde{y} \in \widetilde{I}$, i.e., $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}3)$.

(4) For any $\widetilde{x}, \widetilde{y} \in \widetilde{I}$, we observe that $\widetilde{F}([1,1] - \widetilde{y}, [1,1] - \widetilde{x}) = \widetilde{g}(\widetilde{h}(([1,1] - \widetilde{y}) - ([1,1] - \widetilde{x}))) = \widetilde{g}(\widetilde{h}(\widetilde{x}, \widetilde{y})) = \widetilde{F}(\widetilde{x}, \widetilde{y})$. That is, $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}4)$.

(5) If $\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{w} \in \widetilde{I}$ and $\widetilde{h}(\widetilde{x} - \widetilde{y}) \succ \widetilde{h}(\widetilde{z} - \widetilde{w})$, then because $\widetilde{g}$ satisfies $(\widetilde{G}3)$, we have $\widetilde{F}(\widetilde{x}, \widetilde{y}) = \widetilde{g}(\widetilde{h}(\widetilde{x} - \widetilde{y})) \geq \widetilde{g}(\widetilde{h}(\widetilde{z} - \widetilde{w})) = \widetilde{F}(\widetilde{z}, \widetilde{w})$, that is, $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}5)$.

From (1)–(5), we obtain that $\widetilde{F}(\widetilde{x}, \widetilde{y})$ satisfies $(\widetilde{F}1) - (\widetilde{F}5)$, and from Theorem 8, we have $\widetilde{E}_{NS}(\widetilde{\alpha})$ defined by (A.10) satisfying $(\widetilde{E}_{NS}1) - (\widetilde{E}_{NS}4)$. Thus, we have proved Theorem 10. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] A. De Luca and S. Termini, "A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory," *Information and Control*, vol. 20, no. 4, pp. 301–312, 1972.

[3] L. Xuecheng, "Entropy, distance measure and similarity measure of fuzzy sets and their relations," *Fuzzy Sets and Systems*, vol. 52, no. 3, pp. 305–318, 1992.

[4] J. L. Fan and Y. L. Ma, "Some new fuzzy entropy formulas," *Fuzzy Sets and Systems*, vol. 128, no. 2, pp. 277–284, 2002.

[5] R. R. Yager, "Default knowledge and measures of specificity," *Information Sciences*, vol. 61, no. 1-2, pp. 1–44, 1992.

[6] L. Garmendia, R. R. Yager, E. Trillas, and A. Salvador, "On t-norms based measures of specificity," *Fuzzy Sets and Systems*, vol. 133, no. 2, pp. 237–248, 2003.

[7] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[8] M. M. Xia and Z. S. Xu, "Entropy/cross entropy-based group decision making under intuitionistic fuzzy environment," *Information Fusion*, vol. 13, no. 1, pp. 31–47, 2012.

[9] W. L. Hung, "A note on entropy of intuitiontistic fuzzy sets," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 11, no. 05, pp. 627–633, 2003.

[10] W. L. Hung and M. S. Yang, "Fuzzy entropy on intuitionistic fuzzy sets," *International Journal of Intelligent Systems*, vol. 21, no. 4, pp. 443–451, 2006.

[11] N. R. Pal, H. Bustince, M. Pagola, U. K. Mukherjee, D. P. Goswami, and G. Beliakov, "Uncertainties with Atanassov's intuitionistic fuzzy sets: fuzziness and lack of knowledge," *Information Sciences*, vol. 228, no. 228, pp. 61–74, 2013.

[12] V. Torra and Y. Narukawa, "On hesitant fuzzy sets and decision," in *Proceedings of the The 18th IEEE International Conference on Fuzzy Systems*, pp. 1378–1382, Korea, Jeju Island, January 2009.

[13] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. 529–539, 2010.

[14] Y. J. Chen, J. H. Xu, J. H. Shi, and Y. Liu, "Three-way decision models based on intuitionistic hesitant fuzzy sets and its applications," *Computer Science*, vol. 47, no. 8, pp. 144–150, 2020.

[15] Z. S. Xu and M. M. Xia, "Hesitant fuzzy entropy and cross-entropy and their use in multiattribute decision-making," *International Journal of Intelligent Systems*, vol. 27, no. 9, pp. 799–822, 2012.

[16] B. Farhadinia, "Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets," *Information Sciences*, vol. 240, no. 240, pp. 129–144, 2013.

[17] S. Singh and A. H. Ganie, "Generalized hesitant fuzzy knowledge measure with its application to multi-criteria decision-making," *Granular Computing*, vol. 2021, no. 4, pp. 1–14, 2021.

[18] N. Zhao, Z. Xu, and F. Liu, "Uncertainty measures for hesitant fuzzy information: uncertainty measures for hesitant fuzzy information," *International Journal of Intelligent Systems*, vol. 30, no. 7, pp. 818–836, 2015.

[19] C. P. Wei, R. M. Rodríguez, and P. Li, "Note on entropies of hesitant fuzzy linguistic term sets and their applications," *Information Sciences*, vol. 512, no. 512, pp. 352–368, 2020.

[20] T. T. Xu, H. Zhang, and B. Q. Li, "Fuzzy entropy and hesitancy entropy in probabilistic hesitant fuzzy information and their applications," *Soft Computing*, vol. 26, no. 18, pp. 9101–9115, 2022.

[21] B. Fang, "Some uncertainly measures for probabilistic hesitant fuzzy information," *Information Sciences*, vol. 2023, no. 625, pp. 255–276, 2023.

[22] C. P. Wei, R. M. Rodríguez, and L. Martínez, "Uncertainty measures of extended hesitant fuzzy linguistic term sets," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 3, pp. 1763–1768, 2018.

[23] B. Fang, "Probabilistic linguistic decision-making based on the hybrid entropy and cross-entropy measures," *Fuzzy Optimization and Decision Making*, vol. 15, 2022.

[24] Z. Wang, F. Xiao, and Z. Cao, "Uncertainty measurements for Pythagorean fuzzy set and their applications in multiple-criteria decision making," *Soft Computing*, vol. 26, no. 19, pp. 9937–9952, 2022.

[25] T. T. Xu, H. Zhang, and B. Q. Li, "Axiomatic framework of fuzzy entropy and hesitancy entropy in fuzzy environment," *Soft Computing*, vol. 25, no. 2, pp. 1219–1238, 2021.

[26] J. D. Qin, T. T. Xu, and P. Zheng, "Axiomatic framework of entropy measure for type-2 fuzzy sets with new representation method and its application to product ranking through online reviews," *Applied Soft Computing*, vol. 2022, no. 130, pp. 1–11, 2022.

[27] N. Chen, Z. Xu, and M. Xia, "Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis," *Applied Mathematical Modelling*, vol. 37, no. 4, pp. 2197–2211, 2013.

[28] Z. S. Xu and Q. L. Da, "The uncertain OWA operator," *International Journal of Intelligent Systems*, vol. 17, no. 6, pp. 569–575, 2002.

[29] F. Y. Meng, C. Wang, X. H. Chen, and Q. Zhang, "Correlation coefficients of interval-valued hesitant fuzzy sets and their application based on the shapley function: hesitant fuzzy sets and their application," *International Journal of Intelligent Systems*, vol. 31, no. 1, pp. 17–43, 2016.

[30] R. M. Rodríguez, L. Martínez, V. Torra, Z. S. Xu, and F. Herrera, "Hesitant fuzzy sets: state of the art and future directions: hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 29, no. 6, pp. 495–524, 2014.

[31] C. Fu and J. Zhao, "Method of multi-attribute decision making based on hesitant intuitionistic fuzzy numbers," *Systems Engineering*, vol. 32, no. 4, pp. 131–136, 2014.

[32] B. Zhu, Z. S. Xu, and M. M. Xia, "Dual hesitant fuzzy sets," *Journal of Applied Mathematics*, vol. 2012, no. 5, Article ID 879629, 13 pages, 2012.

[33] Z. L. Ren, Z. S. Xu, and H. Wang, "Normal wiggly hesitant fuzzy sets and their application to environmental quality evaluation," *Knowledge-Based Systems*, vol. 159, no. 159, pp. 286–297, 2018.

[34] S. Singh and A. H. Ganie, "Two-parametric generalized fuzzy knowledge measure and accuracy measure with applications," *International Journal of Intelligent Systems*, vol. 37, no. 7, pp. 3836–3880, 2022.

[35] Z. L. Li, Z. Zhang, and W. Y. Yu, "Consensus reaching with consistency control in group decision making with incomplete hesitant fuzzy linguistic preference relations," *Computers & Industrial Engineering*, vol. 2022, no. 170, pp. 1–13, 2022.

[36] Q. X. Dong, Q. Sheng, L. Martínez, and Z. Zhang, "An adaptive group decision making framework: individual and local world opinion based opinion dynamics," *Information Fusion*, vol. 78, no. 78, pp. 218–231, 2022.

[37] Y. C. Dong, Q. B. Zha, H. J. Zhang et al., "Consensus reaching in social network group decision making: research paradigms and challenges," *Knowledge-Based Systems*, vol. 162, no. 162, pp. 3–13, 2018.