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Research Article

Fixed-Point Results for Mappings Satisfying Implicit Relation in Orthogonal Fuzzy Metric Spaces

Sonam,[1] **Vandana Rathore**,[2] **Amita Pal**,[3] **Ramakant Bhardwaj**,[1] **and Satyendra Narayan**[4]

[1]*Department of Mathematics, Amity Institute of Applied Sciences, Amity University Kolkata, Kadampukur, West Bengal-700135, India*
[2]*School of Engineering and Technology, Jagran Lakecity University, Bhopal, Madhya Pradesh-462044, India*
[3]*Marathwada Mitramandal's Institute of Technology, Lohegaon, Pune-411047, India*
[4]*Department of Applied Computing, Sheridan Institute of Technology, Oakville, ON L6H 2L1, Canada*

Correspondence should be addressed to Ramakant Bhardwaj; rkbhardwaj100@gmail.com

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This research paper introduces a comprehensive study on fixed points in orthogonal fuzzy metric spaces. The primary objective is to establish the existence and uniqueness of fixed points for self-mappings satisfying implicit relation criteria in complete orthogonal fuzzy metric spaces. By doing so, our proven results extend and generalise well-known findings in the field of fixed-point theory. To demonstrate the significance of the established results, several related examples are provided, serving to support and validate the theoretical findings in orthogonal fuzzy metric spaces. The implications of these results are discussed, shedding light on their potential applications in various practical scenarios. In addition to theoretical advancements, the paper also demonstrates a practical application of our established results in solving integral equations. This application exemplifies the effectiveness and versatility of the proposed approach in real-world problem-solving scenarios.

1. Introduction

The study of nonlinear analysis is greatly benefited by the adoption of fixed-point theory. It is a multidisciplinary topic that has applications in numerous areas of mathematics and in domains such as biology, chemistry, engineering, physics, game theory, mathematical economics, optimization theory, approximation theory, and variational inequality. It became even more fascinating through its applicability to solve differential and integral equations. It is typical practice to use mathematical techniques to examine diverse structural aspects and the functioning of multiple departments. As a consequence, handling ambivalence and imprecise evidence in a diverse range of situations happens naturally. In 1965, Zadeh [1] presented the idea of fuzzy sets, and since then, it has become an important mechanism for resolving cases involving ambiguity and uncertainty. Kramosil and Michálek [2] presented the conception of a fuzzy metric space in 1975, which was updated by George and Veeramani [3] in an attempt to generate a Hausdorff topology for a particular category of fuzzy metric spaces. Furthermore, many researchers explored fuzzy metric spaces, their applications, and their related scopes in [4–17].

In 2017, Gordji et al. [18] developed the conception of orthogonal sets and introduced orthogonal metric spaces as a generalisation of metric spaces. Later, in [19–22] etc., the authors added some generalisations of orthogonal metric spaces along with several fixed-point results. In 2018, Hezarjaribi [19] presented the notion of orthogonal fuzzy metric spaces, and a little work has been performed in this generalisation of a metric space [23–25]. The research gap addressed in our work lies in the establishment of new fixed-point results in orthogonal fuzzy metric spaces, specifically focusing on the satisfaction of implicit relation by self-mappings. While existing literature provides valuable insights into fixed-point theory and fuzzy metric spaces, there is a need to explore the implications of considering self-mappings satisfying implicit relation in orthogonal fuzzy metric spaces.

By bridging this research gap, our paper contributes to the theoretical development of generalisation of fuzzy metric spaces and expands the understanding of fixed-point theory. The consideration of implicit relations and self-mappings enriches the existing framework and offers a versatile foundation that can be applied across various domains.

Furthermore, the results obtained in this study have direct applications in solving integral equations, as well as in addressing fractional differential equations. Integral equations arise in various fields, including physics, engineering, and economics, and often pose challenges due to their nonlinearity and complex nature. The fixed-point results established in our research can provide a valuable framework for developing efficient numerical methods to solve integral equations, leading to accurate solutions and improved computational efficiency. Moreover, the utilization of these results can be extended to fractional differential equations, which have gained significant attention in recent years due to their ability to model various phenomena in science and engineering. By incorporating the obtained fixed-point theorems, we can tackle the solution of fractional differential equations, contributing to the advancement of this important area of research.

The paper is structured into six sections, each focusing on specific aspects of the study. In Section 1, the background and motivation for the research are introduced. Section 2 provides a comprehensive overview of basic concepts related to orthogonal fuzzy metric spaces, establishing the foundation for subsequent discussions. Section 3 is dedicated to the main theoretical contribution of the paper, where fixed-point results for functions satisfying implicit relation criteria in complete orthogonal fuzzy metric spaces are established. These results are supported by relevant examples, highlighting their significance and applicability. In Section 4, the consequences of the proven fixed-point results are explored, revealing their broader implications and generalisations. Furthermore, in Section 5, the practical relevance of established theory is demonstrated by applying it to the existence and uniqueness of solutions of nonlinear Volterra integral equations of the second kind. This application showcases the versatility and utility of the developed fixed-point results in real-world problem-solving. Finally, Section 6 provides the concluding remarks, summarizing the key findings and highlighting avenues for further research.

2. Preliminaries

The following are some preparatory considerations for the writing of the article.

Definition 1 (see [2, 3]). We consider a function $\star$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$. Then, $\star$ is purported to be a continuous t-norm if $\star$ agrees with the below listed conditions:

(i) $\star$ follows commutativity and associativity laws

(ii) Continuity of $\star$

(iii) $c\star 1 = c$ for all $c \in [0, 1]$

(iv) $c \star d \leq p \star q$ whenever $c \leq d$ and $p \leq q$, for $c, d, p, q \in [0, 1]$

Example 1. We define a binary operator $\star$ on $[0, 1]$ as $\lambda \star \mu = \min\{\lambda, \mu\}, \lambda \star \mu = \max\{0, \lambda + \mu - 1\}, or\ \lambda \star \mu = \lambda\mu$. Then, $\star$ is a continuous t-norm.

Definition 2 (see [2, 3]). Let X be a nonnull set, $\star$ be a continuous t-norm, and C: $X \times X \times (0, +\infty) \longrightarrow (0, 1]$ be a fuzzy set. Then, $(X, C, \star)$ is purported to be a fuzzy metric space under the following conditions:

For all $r, s, b \in X$ and $t, k > 0$

(FM1) $C(r, s, t) > 0$

(FM2) $C(r, s, t) = 1$ iff $r = s$

(FM3) $C(r, s, t) = C(s, r, t)$

(FM4) $C(r, b, t + k) \geq C(r, s, t)\star C(s, b, k)$

(FM5) $C(r, s, .)$: $(0, +\infty) \longrightarrow (0, 1]$ is a continuous function

Example 2. We consider $X = \mathbb{R}^6$. We define a continuous t-norm $\star$ as $\lambda \star \mu = \lambda\mu$, for all $\lambda, \mu \in [0, 1]$. Also, we define a mapping C: $X \times X \times (0, +\infty) \longrightarrow (0, 1]$ as follows.

For every $t > 0$,

$$C(r, s, t) = \frac{t}{t + \left(\Sigma_{j=1}^{6}\left(r_j - s_j\right)^4\right)^{1/4}}, \quad (1)$$

where, $r = \{r_j\}_{j=1}^{6}, s = \{s_j\}_{j=1}^{6} \in X$. Then, $(X, C, \star)$ is a fuzzy metric space.

Example 3. We consider $X = \mathscr{C}[0, 5/7](\mathbb{R})$. We define a mapping C: $X \times X \times (0, +\infty) \longrightarrow (0, 1]$ by

$$C(\mathbf{f}(l), \mathbf{g}(l), \gamma) = \sup_{l\in\left[0, \frac{5}{7}\right]} e^{-|\mathbf{f}(l) - \mathbf{g}(l)|/\gamma}, \gamma > 0, \quad (2)$$

and let $\star$ be a continuous t-norm defined as $\lambda \star \mu = \min\{\lambda, \mu\}$, for all $\lambda, \mu \in [0, 1]$. Then, $(X, C, \star)$ is a fuzzy metric space.

Lemma 3 (see [3]). *The function $C(r, s, .)$ is nondecreasing where $r, s \in X$.*

Definition 4 (see [18]). For a nonnull set X and a binary relation $\perp$ defined over X, the pair $(X, \perp)$ is purported to be an orthogonal set if there remains an element $r_0 \in X$ so that $r_0 \perp r$ or $r \perp r_0$ for each $r \in X$.

Definition 5 (see [19]). For a fuzzy metric space $(X, C, \star)$ and a binary relation $\perp$ defined over X, an ordered four-tuple $(X, C, \star, \perp)$ is purported to be an orthogonal fuzzy metric space if there remains an element $r_0 \in X$ so that $r_0 \perp r$ for each $r \in X$.

Definition 6 (see [19]). Let $(X, C, \star, \perp)$ be an orthogonal fuzzy metric space. Then,

(1) A sequence $\{r_m\}_{m\in\mathbb{N}}$ in X is said to be an O-sequence in X if $r_m \perp r_{m+1}$ for all $m \in \mathbb{N}$.

(2) An O-sequence $\{r_m\}_{m\in\mathbb{N}}$ is said to converge to a point $r \in X$ if for any $t > 0$,

$$\lim_{m\longrightarrow\infty} C(r_m, r, t) = 1. \tag{3}$$

(3) A self-mapping $\mathcal{T}$ on X is said to be $\perp-$ continuous at a point $r \in X$ if for any O-sequence $\{r_m\}_{m\in\mathbb{N}}$ of X $\lim_{m\longrightarrow\infty} C(r_m, r, t) = 1 \Rightarrow \lim_{m\longrightarrow\infty} C(\mathcal{T} r_m, \mathcal{T} r, t) = 1$, for all $t > 0$.

Moreover, $\mathcal{T}$ is said to be $\perp-$ continuous on X if $\mathcal{T}$ is $\perp-$ continuous at each point of X.

(4) A self-mapping $\mathcal{T}$ on X is said to be $\perp-$ preserving if for $r, s \in X$,

$$r \perp s \Rightarrow \mathcal{T} r \perp \mathcal{T} s. \tag{4}$$

(5) An O-sequence $\{r_m\}_{m\in\mathbb{N}}$ in X is said to be a Cauchy O-sequence if for any $0 < \epsilon < 1$ and $t > 0$, there exists a natural number m_0 so that $C(r_p, r_q, t) > 1 - \epsilon$, for all $p, q \geq m_0$.

(6) $(X, C, \star, \perp)$ is said to be an orthogonally complete (O-complete) fuzzy metric space if every Cauchy O-sequence in X converges to a point in X.

3. Main Results

We take $(X, C, \star, \perp)$ as an orthogonal fuzzy metric space and $Y (\neq \phi) \subseteq X$. For Y, we define $\eta_C(Y, t) = \inf\{C(r, s, t): r, s \in Y\}$ for all $t > 0$.

For $P_m = \{r_m, r_{m+1}, r_{m+2}, \ldots\}$ in the orthogonal fuzzy metric space $(X, C, \star, \perp)$, we consider $z_m(t) = \eta_C(P_m, t)$. Then, the following are some implications:

(1) $z_m(t)$ is a finite quantity for each $m \in \mathbb{N}$

(2) $\{z_m(t)\}_{m\in\mathbb{N}}$ is a nonincreasing sequence

(3) $\{z_m(t)\}_{m\in\mathbb{N}}$ converges to some $z(t) \in [0, 1]$

(4) $z_m(t) \leq C(r_p, r_q, t)$ for all $p, q \geq m$

We consider $\Omega = [0, 1]$ and $\mathcal{F}$ as the collection of all continuous functions $\mathfrak{F}$: $\Omega^3 \times \Omega \longrightarrow [-1, 1]$, which are all nondecreasing on Ω^3 and follow the below-stated conditions.

$[\mathfrak{F}_1]$: $\mathfrak{F}((x, x, x), y) \leq 0 \Rightarrow \mu(x) \leq y$ where μ: $\Omega \longrightarrow \Omega$ is a continuous nondecreasing function satisfying the condition $\mu(c) > c$ for $c \in \Omega/\{1\}$.

Example 4. We consider $\mathfrak{F}$: $\Omega^3 \times \Omega \longrightarrow [-1, 1]$ satisfying the following conditions:

(a) $\mathfrak{F}((x_1, x_2, x_3), x_4) = \mu(\min\{x_1, x_2, x_3\}) - x_4$

(b) $\mathfrak{F}((x_1, x_2, x_3), x_4) = \mu(\sum_{i=1}^{3} b_i x_i) - x_4$, for each $b_i \geq 0, \sum_{i=1}^{3} b_i = 1$, where $\mu(c) = c^k$ for some $k \in (0, 1)$

Theorem 7. *Let $(X, C, \star, \perp)$ be a complete orthogonal fuzzy metric space and $\mathcal{T}$: $X \longrightarrow X$ be a $\perp-$ preserving self-mapping satisfying the condition below: for $r, s \in X$ and $\mathfrak{F} \in \mathcal{F}$,*

$$\mathfrak{F}(C(r, s, t), C(\mathcal{T} r, r, t), C(\mathcal{T} r, s, t), C(\mathcal{T} r, \mathcal{T} s, t)) \leq 0. \tag{5}$$

Then, $\mathcal{T}$ admits a unique fixed point $L \in X$. Moreover, $\mathcal{T}$ is $\perp-$ continuous at point L.

Proof. As $(X, C, \star, \perp)$ is an orthogonal fuzzy metric space, there exists $r_0 \in X$ satisfying $r_0 \perp s$ for all $s \in X$. This gives $r_0 \perp \mathcal{T} r_0$. Let $r_{m+1} = \mathcal{T} r_m$ for all $m \in \mathbb{N}$. As the mapping $\mathcal{T}$ is $\perp$ - preserving, the sequence $\{r_m\}_{m\in\mathbb{N}}$ is an O-sequence.

We consider $z_m(t) = \eta_C(P_m, t)$ where $P_m = \{r_m, r_{m+1}, r_{m+2}, \ldots\}, t > 0$. Then, $\lim_{m\longrightarrow\infty} z_m(t) = z(t)$ for $0 \leq z(t) \leq 1$. In the case $r_{m+1} = r_m$ for some $m \in \mathbb{N}$, $\mathcal{T}$ admits a fixed point $L \in X$. So we presume that $r_{m+1} \neq r_m$ for all $m \in \mathbb{N}$.

Now, for a fixed value $l \in \mathbb{N}$, take $r = r_{m-1}$ and $s = r_{m+p-1}$ in condition (5) where $m \geq l$ and $p \in \mathbb{N}$, we obtain

$$\begin{aligned}\mathfrak{F}\big(C(r_{m-1}, r_{m+p-1}, t), C(\mathcal{T} r_{m-1}, r_{m-1}, t), C(\mathcal{T} r_{m-1}, r_{m+p-1}, t), C(\mathcal{T} r_{m-1}, \mathcal{T} r_{m+p-1}, t)\big) \leq 0,\\ \mathfrak{F}\big(C(r_{m-1}, r_{m+p-1}, t), C(r_m, r_{m-1}, t), C(r_m, r_{m+p-1}, t), C(r_m, r_{m+p}, t)\big) \leq 0.\end{aligned} \tag{6}$$

By the nondecreasing nature of C, we have

$$\mathfrak{F}\big(z_{m-1}(t), z_{m-1}(t), z_m(t), C(r_m, r_{m+p}, t)\big) \leq 0. \tag{7}$$

By the nondecreasing nature of $\mathfrak{F}$ and the nonincreasing nature of $z_m(t)$,

$$\mathfrak{F}\big(z_{l-1}(t), z_{l-1}(t), z_{l-1}(t), C(r_m, r_{m+p}, t)\big) \leq 0. \tag{8}$$

This implies that

$$\mu(z_{m-1}(t)) \leq C(r_m, r_{m+p}, t). \tag{9}$$

So, for all $m \geq l$, $\inf_{m\geq l} C(r_m, r_{m+p}, t) = z_l(t) \geq \mu(z_{m-1}(t))$.

Taking $l \longrightarrow \infty$ gives $z(t) \geq \mu(z(t))$. This is a contradiction for the case $z(t) \neq 1$. Thus, $z(t) = 1$. So $\lim_{m\longrightarrow\infty} z_m(t) = 1$ i.e., for any given $\epsilon > 0$, there exists a natural number m_0 such that $z_m(t) > 1 - \epsilon$ for all $m \geq m_0$. Then, $C(r_m, r_{m+p}, t) > 1 - \epsilon$ for $p \in \mathbb{N}$ and $m \geq m_0$. Therefore, the sequence $\{r_m\}$ is an O-Cauchy sequence. By the O-completeness property of $(X, C, \star, \perp)$, this ensures that there exists a point $L \in X$ such that $\lim_{m\longrightarrow\infty} \mathcal{T} r_m = L$.

Now, by substituting $r = r_m$ and $s = L$ in condition (5), we obtain

$$\mathfrak{F}(C(r_m,L,t),C(\mathcal{T}r_m,L,t),C(\mathcal{T}r_m,r_m,t),C(\mathcal{T}r_m,\mathcal{T}L,t))\leq 0, \tag{10}$$

which implies

$$\mathfrak{F}(C(r_m,L,t),C(r_{m+1},L,t),C(r_{m+1},r_m,t),C(r_{m+1},\mathcal{T}L,t))\leq 0. \tag{11}$$

By leading $m\longrightarrow\infty$, we obtain

$$\mathfrak{F}(C(L,L,t),C(L,L,t),C(L,L,t),C(L,\mathcal{T}L,t))\leq 0, \tag{12}$$

which gives $C(L,\mathcal{T}L,t)\geq\mu(C(L,L,t))=\mu(1)=1.$

Thus, $\mathcal{T}L=L$, or L is a fixed point of $\mathcal{T}$.

We consider M as another fixed point of $\mathcal{T}$ such that $L\neq M$. By considering $r=L$ and $s=M$ in condition (5), we obtain

$$\mathfrak{F}(C(L,M,t),C(\mathcal{T}L,L,t),C(\mathcal{T}L,M,t),C(\mathcal{T}L,\mathcal{T}M,t))\leq 0, \tag{13}$$

or,

$$\mathfrak{F}(C(L,M,t),C(L,L,t),C(L,M,t),C(L,M,t))\leq 0. \tag{14}$$

Using the nondecreasing property of $\mathfrak{F}$ on Ω^3, we have

$$\mathfrak{F}(C(L,M,t),C(L,M,t),C(L,M,t),C(L,M,t))\leq 0. \tag{15}$$

which implies $\mu(C(L,M,t))\leq C(L,M,t)$. Also, $C(L,M,t)<\mu(C(L,M,t))$. This is a contradiction.

Hence, $L=M$.

We now show that the self-mapping $\mathcal{T}$ is $\perp$-continuous at the point L. We consider an O-sequence $\{s_m\}$ in X. Then, $\lim_{m\longrightarrow\infty}s_m=L$.

Putting $r=$ Land$s=s_m$ in condition (5), we get

$$\mathfrak{F}(C(L,s_m,t),C(\mathcal{T}L,L,t),C(\mathcal{T}L,s_m,t),C(\mathcal{T}L,\mathcal{T}s_m,t))\leq 0. \tag{16}$$

This implies

$$\begin{aligned}
&\mathfrak{F}(C(L,s_m,t),C(L,L,t),C(L,s_m,t),C(L,\mathcal{T}s_m,t))\leq 0,\\
&\Rightarrow C(L,\mathcal{T}s_m,t)\geq\mu(C(L,s_m,t))\\
&\Rightarrow \liminf_n C(L,\mathcal{T}s_m,t)\geq\liminf_n\ \mu\ (C(L,s_m,t))\\
&=\mu(1)\\
&=1.
\end{aligned} \tag{17}$$

Thus, $\lim_{m\longrightarrow\infty}\mathcal{T}s_m=L=\mathcal{T}L$. This gives $\liminf_m C(s_m,L,t)=1\Rightarrow\liminf_m C(\mathcal{T}s_m,\mathcal{T}L,t)=1$. Therefore, $\mathcal{T}$ is $\perp$- continuous at L. □

Example 5. We consider $X=\mathbb{R}^+\cup\{0\}$ and define a binary relation $\perp$ on X as $r\perp s$ if and only if $r\leq s$. Then, it is easy to verify the following.

$(X,\perp)$ is an orthogonal set.

We define the OFM by $C(r,s,t)=e^{-|r-s|/t}$ with a continuous t-norm $\star$ defined as, $\alpha\star\beta=\min\{\alpha,\beta\}$.

It is evident that $(X,C,\star,\perp)$ is an O-complete fuzzy metric space. We define a self-mapping $\mathcal{T}$ on X by $\mathcal{T}(r)=kr$, where $k\in(0,1)$. Then, $\mathcal{T}$ is $\perp-$ preserving. Also, we define $\Omega=[0,1]$ and $\mathfrak{F}((x_1,x_2,x_3),x_4)=[\min\{x_1,x_2,x_3\}]^k-x_4$, $k\in(0,1)$. Then, $\mathfrak{F}$ is continuous and nondecreasing on Ω^3.

Let $\mu(x)=x^k$. Then, $\mu(x)$ is a nondecreasing function, and $\mu(c)>c$ for all $c\in\Omega/\{1\}$. Also, $\mathfrak{F}((x,x,x),y)\leq 0$ implies $\mu(x)\leq y$. Thus, $\mathfrak{F}\in\mathcal{F}$.

Furthermore,

$$\begin{aligned} C(\mathcal{T}r,\mathcal{T}s,t) &= e^{-|kr-ks|/t} \\ &= \left[e^{-|r-s|/t}\right]^k \\ &= [C(r,s,t)]^k \\ &\geq [\min\{C(r,s,t),C(\mathcal{T}r,r,t),C(\mathcal{T}r,s,t)\}]^k. \end{aligned} \tag{18}$$

Therefore, all the hypotheses of Theorem 7 are satisfied, and hence, $\mathcal{T}$ admits a unique fixed point in $(X,C,\star,\perp)$. Moreover, $\mathcal{T}(0)=0$.

Example 6. We consider $Z=\mathbb{R}^+\cup\{0\}$ and define a binary relation $\perp$ on Z as $r\perp s$ if and only if $rs\leq r\vee s$, where $r\vee s = r$ or s.

$(Z,\perp)$ is an orthogonal set. We define the OFM, C (where in $\star$ is defined as, $\alpha\star\beta=\alpha\beta$) by $C(r,s,t) = (t/1+t)^{|r-s|/10}$. It is evident that $(Z,C,\star,\perp)$ is an O-complete fuzzy metric space. Define $\mathcal{T}\colon Z\longrightarrow Z$ as $\mathcal{T}(r)=r/3$. Then, $\mathcal{T}$ is $\perp-$ preserving. Also, we define $\mathfrak{F}((z_1,z_2,z_3),z_4) = \sqrt[3]{\min\{z_1,z_2,z_3\}}-z_4$. Then, $\mathfrak{F}$ is continuous and nondecreasing on Ω^3. Also, $\mathfrak{F}((z,z,z),\zeta)\leq 0 \Rightarrow \mu(z)\leq\zeta$, where $\mu(z)=\sqrt[3]{z}$, which is nondecreasing, satisfies the condition $\mu(\eta)>\eta$ for all $\eta\in\Omega/\{1\}$. Thus, $\mathfrak{F}\in\mathcal{F}$.

Furthermore,

$$\begin{aligned} C(\mathcal{T}r,\mathcal{T}s,t) &= \left(\frac{t}{1+t}\right)^{|r/3-s/3|/10} \\ &= \left(\frac{t}{1+t}\right)^{|r-s|/30} \\ &= \sqrt[3]{C(r,s,t)} \\ &\geq \sqrt[3]{\min\{C(r,s,t),C(\mathcal{T}r,r,t),C(\mathcal{T}r,s,t)\}}. \end{aligned} \tag{19}$$

Thus, all the hypotheses of Theorem 7 are satisfied, and hence, $\mathcal{T}$ admits a unique fixed point in $(Z,C,\star,\perp)$. Moreover, the unique fixed point of $\mathcal{T}$ is 0.

4. Consequences

Corollary 8. *Let $(X,C,\star,\perp)$ be a complete orthogonal fuzzy metric space and $\mathcal{T}\colon X\longrightarrow X$ be a $\perp-$ preserving self-mapping with the following conditions:*

$$\mathfrak{F}\left(C(r,s,t),C\left(\mathcal{T}^k r,r,t\right),C\left(\mathcal{T}^k r,s,t\right),C\left(\mathcal{T}^k r,\mathcal{T}^k s,t\right)\right)\leq 0,\quad \forall r,s\in X \text{ and for } k\in\mathbb{N}, \tag{20}$$

where $\mathfrak{F}\in\mathcal{F}$.

Then, $\mathcal{T}$ admits a unique fixed point $L\in X$, and at the point L, $\mathcal{T}^k$ is $\perp-$ continuous.

Proof. By Theorem 7, it can be easily proved that $\mathcal{T}$ admits a unique fixed point $L\in X$ and $\mathcal{T}^k$ is $\perp-$ continuous at point L. Also, since $\mathcal{T}L=\mathcal{T}\mathcal{T}^k L=\mathcal{T}^k\mathcal{T}L, \mathcal{T}L$ also becomes a fixed point of $\mathcal{T}$, $\mathcal{T}L=L$ by the uniqueness of the fixed point of $\mathcal{T}$. □

Corollary 9. *Let $(X,C,\star,\perp)$ be a complete orthogonal fuzzy metric space and $\mathcal{T}\colon X\longrightarrow X$ be a $\perp-$ preserving self-mapping with the following conditions:*

$$C(\mathcal{T}r,\mathcal{T}s,t)\geq\mu(\min\{C(r,s,t),C(\mathcal{T}r,r,t),C(\mathcal{T}r,s,t)\}),\forall r,s\in X. \tag{21}$$

Then, $\mathcal{T}$ admits a unique fixed point $L\in X$, and $\mathcal{T}$ is $\perp-$ continuous at point L.

Corollary 10. *Let $(X,C,\star,\perp)$ be a complete orthogonal fuzzy metric space and $\mathcal{T}\colon X\longrightarrow X$ be a $\perp-$ preserving self-mapping with the following conditions:*

$$C\left(\mathcal{T}^k r,\mathcal{T}^k s,t\right)\geq\mu\left(\min\left\{C(r,s,t),C\left(\mathcal{T}^k r,r,t\right),C\left(\mathcal{T}^k r,s,t\right)\right\}\right),\forall r,s\in X \text{ and for } k\in\mathbb{N}. \tag{22}$$

Then, $\mathcal{T}$ admits a unique fixed point $L \in X$. Moreover, at point L, $\mathcal{T}^k$ is $\perp-$ continuous.

Corollary 11. *Let $(X, C, \star, \perp)$ be a complete orthogonal fuzzy metric space and $\mathcal{T}: X \longrightarrow X$ be a $\perp-$ preserving self-mapping with the following conditions:*

$$C(\mathcal{T}r, \mathcal{T}s, t) \geq \mu\left(b_1 C(r, s, t) + b_2 C(\mathcal{T}r, r, t) + b_3 C(\mathcal{T}r, s, t)\right), \forall r, s \in X, \tag{23}$$

where each $b_i \geq 0, \sum_{i=1}^{3} b_i = 1$ Then, $\mathcal{T}$ admits a unique fixed point $L \in X$, and $\mathcal{T}$ is $\perp-$ continuous at point L.

Corollary 12. *Let $(X, C, \star, \perp)$ be a complete orthogonal fuzzy metric space and $\mathcal{T}: X \longrightarrow X$ be a $\perp-$ preserving self-mapping with the following conditions:*

$$C\left(\mathcal{T}^k r, \mathcal{T}^k s, t\right) \geq \mu\left(b_1 C(r, s, t) + b_2 C\left(\mathcal{T}^k r, r, t\right) + b_3 C\left(\mathcal{T}^k r, s, t\right)\right), \forall r, s \in X \text{ and } for\, k \in \mathbb{N}, \tag{24}$$

where each $b_i \geq 0, \sum_{i=1}^{3} b_i = 1$.

Then, $\mathcal{T}$ admits a unique fixed point, $L \in X$. Moreover, at point L, $\mathcal{T}^k$ is $\perp-$ continuous.

5. Application

The fixed-point theorems obtained in this work can be utilised to develop effective methods for solving integral equations, enabling the analysis and computation of solutions in scenarios where traditional methods may be limited. Furthermore, the applicability of these results extends to fractional differential equations, as many fractional calculus problems can be reduced to integral equations.

In this section, we apply Theorem 7 to demonstrate the existence and uniqueness of a solution of nonlinear integral equations.

We consider $X = \{\mathbf{u} \in \mathscr{C}([0, P], \mathbb{R}) \mid \mathbf{u}(\eta) \geq 0 \text{ and } P \in \mathbb{R}\}$ and an integral equation of the form:

$$\mathbf{u}(\eta) = \mathbb{A}(\eta) + \lambda \int_0^\eta \mathcal{I}_K(\eta, \zeta)\mathbf{u}(\zeta) d\zeta, \tag{25}$$

where $\lambda > 0$, $\mathbb{A}(\eta)$ is a fuzzy function of η where $\eta \in [0, P]$ and $\mathcal{I}_K: [0, P] \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is an integral kernel.

We aim to show the existence and uniqueness of the solution of (25) by applying Theorem 7.

We define a binary relation $\perp$ on X as $\mathbf{u}\perp\mathbf{v} \Leftrightarrow \mathbf{u}(\eta)\mathbf{v}(\eta) \leq \mathbf{u}(\eta) \vee \mathbf{v}(\eta)$ for all $\eta \in [0, P]$ where $\mathbf{u}(\eta) \vee \mathbf{v}(\eta) = \mathbf{u}(\eta)$ or $\mathbf{v}(\eta)$. Since $\mathbf{u} \equiv 0 \perp \mathbf{v}, \forall \mathbf{v} \in X$, $(X, \perp)$ is an orthogonal set.

We consider a function $C: X \times X \times (0, +\infty) \longrightarrow (0, 1]$ defined by

$$C(\mathbf{u}(\eta), \mathbf{v}(\eta), \gamma) = \sup_{\eta \in [0, P]} e^{-|\mathbf{u}(\eta) - \mathbf{v}(\eta)|^{8/5}/\gamma}, \gamma > 0, \tag{26}$$

and let $\star$ be a continuous t-norm defined as $\alpha \star \beta = \min\{\alpha, \beta\}$. Then, $(X, C, \star, \perp)$ is an O-complete fuzzy metric space, and integral equation (25) possesses a unique solution in X.

Proof. We define an operator $\mathcal{T}: X \longrightarrow X$ as

$$\mathcal{T}\mathbf{u}(\eta) = \mathbb{A}(\eta) + \lambda \int_0^\eta \mathcal{I}_K(\eta, \zeta)\mathbf{u}(\zeta) d\zeta. \tag{27}$$

We observe that, for any $\mathbf{u}, \mathbf{v} \in X$, $\mathbf{u}\perp\mathbf{v} \Rightarrow \mathcal{T}\mathbf{u}\perp\mathcal{T}\mathbf{v}$. Hence, $\mathcal{T}$ is $\perp-$ preserving.

We denote $\Omega = [0, 1]$. We define $F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left\{(\min\{\alpha_1, \alpha_2, \alpha_3\})^{(1/7)}\right\} - \alpha_4$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Omega$. Then, $\mathfrak{F}$ is continuous and nondecreasing on Ω^3. Also, we define $\mu: \Omega \longrightarrow \Omega$ by $\mu(\beta) = \beta^{1/7}$; then, $\mu(\beta)$ is nondecreasing and $\mu(\beta) > \beta$ for all $\beta \in \Omega/\{1\}$. Then, $\mathfrak{F}((\alpha, \alpha, \alpha), \beta) \leq 0 \Rightarrow \mu(\alpha) \leq \beta$, and hence, $\mathfrak{F} \in \mathcal{F}$.

Moreover, for any $\eta \in [0, P]$,

$$\begin{aligned}
C(\mathcal{T}\mathbf{u}(\eta), \mathcal{T}\mathbf{v}(\eta), \gamma) &= \sup_{\eta \in [0, P]} e^{-\left|\mathbb{A}(\eta) + \lambda \int_0^\eta \mathcal{I}_K(\eta, \zeta)\mathbf{u}(\zeta) d\zeta - \left(\mathbb{A}(\eta) + \lambda \int_0^\eta \mathcal{I}_K(\eta, \zeta)\mathbf{v}(\zeta) d\zeta\right)\right|^{8/5}/\gamma} \\
&= \sup_{\eta \in [0, P]} e^{-\left|(\mathbf{u}(\eta) - \mathbf{v}(\eta))\lambda \int_0^\eta \mathcal{I}_K(\eta, \zeta) d\zeta\right|^{8/5}/\gamma} \\
&\geq \sup_{\eta \in [0, P]} e^{-|\mathbf{u}(\eta) - \mathbf{v}(\eta)|^{8/5}/\gamma} \\
&= C(\mathbf{u}(\eta), \mathbf{v}(\eta), \gamma) \\
&\geq \sqrt[7]{\min\{C(\mathbf{u}(\eta), \mathbf{v}(\eta), \gamma), C(\mathcal{T}\mathbf{u}(\eta), \mathbf{u}(\eta), \gamma), C(\mathcal{T}\mathbf{u}(\eta), \mathbf{v}(\eta), \gamma)\}}.
\end{aligned} \tag{28}$$

Thus, $\mathcal{T}$ satisfies the implicit relation:

$$\mathfrak{F}\left(C(\mathbf{u},\mathbf{v},\gamma), C(\mathcal{T}\mathbf{u},\mathbf{u},\gamma), C(\mathcal{T}\mathbf{u},\mathbf{v},\gamma), C(\mathcal{T}\mathbf{u},\mathcal{T}\mathbf{v},\gamma)\right) \leq 0\ \forall \mathbf{u},\mathbf{v} \in X. \tag{29}$$

By establishing the fulfilment of condition (29), Theorem 7 guarantees the existence of a unique fixed point in X for the operator $\mathcal{T}$. Therefore, the integral equations of form (25) has a unique solution in X. □

6. Conclusion

In this work, some fixed-point results are proved in orthogonal fuzzy metric spaces, with a particular focus on the satisfaction of implicit relations through self-mappings. Several illustrations are provided to validate the established results. Some consequences of the established fixed-point results in orthogonal fuzzy metric spaces are also shown. Our findings contribute to the theoretical understanding of extension of fuzzy metric spaces and expanding the applicability of fixed-point theory in diverse domains. The fixed-point results presented in this paper have practical implications, one of which is the demonstration of the existence and uniqueness of solutions for integral equations. By applying the established fixed-point results, we provide evidence for the viability of proven results in solving integral equations. While our findings have practical implications for solving integral equations and addressing fractional differential equations, further research is needed to extend the framework, enhance computational aspects, and explore additional applications in various domains. The established results can be extended to the best proximity point results in the future. The findings in this article would enable researchers to enhance the further exploration of orthogonal fuzzy metric spaces in terms of developing a general framework for implementation of the fixed-point results to obtain techniques to handle real-world problems.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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