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Research Article

Solving a System of Linear Equations Based on Z-Numbers to Determinate the Market Balance Value

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In this article, a general linear equations system with Z-number's data is introduced. Since the nature of Z-numbers has two parameters, namely, reliability and fuzziness, it is difficult to find the exact solution to these systems. Therefore, a numerical procedure for calculating the solution is designed. The proposed method is illustrated with some applied examples. Determining the value of the market balance is one of the examined examples.

1. Introduction

Many problems in various fields, such as engineering and economics, lead to the solution of linear equations systems. Since, in the real world, some parameters are presented inaccurately and colloquially, we must focus on solving linear equations systems with fuzzy data or Z-number data. In recent years, many authors have studied fuzzy linear equations systems (see [1–3]), but fuzzy numbers do not contain reliable information as Z-numbers. The concept of Z-numbers was introduced as a new topic by Professor Zadeh in 2011 [4]. The Z-number has a greater ability to describe human knowledge than the classical fuzzy numbers. This concept is able to deal with uncertain information effectively in a decision, and it is indicated by the symbol $Z = (A, B)$.

Here, a Z-number is defined as a pair fuzzy number in the form (A, B), where A denotes a fuzzy subset of the constraints including x values and the second component, B, is a measure of reliability (certainty) of the first component. In addition, the contract (x, A, B) can also be introduced as a Z-value. Semantically, using the value proposition developed in [5], the variable x can be interpreted in such a way that the probability that x takes a value of A is equal to B. It means as follows:

$$\text{Prob}\,(x \text{ is } A) \text{ is } B. \tag{1}$$

It should be mentioned that these numbers are directly related to quantity; it can be observed that they have more ability to describe the uncertainty environment than fuzzy numbers that do not have this feature. In this regard, several researchers have conducted many studies in various fields by utilizing Z-numbers [4–12].

In this article, we first introduce the linear equations system of Z-numbers and then propose a numerical method to solve it numerically. Finally, we suppose the balance value of the market as a linear equations system with Z-number's data and then determine the balance value of the market. The rest of the article is as follows.

Section 2 contains the basic concepts that are used in the continuation of the article. In Section 3, a linear equations system with Z-number's data is introduced, and a method is presented for solving it. Section 4 provides several examples, one of them shows the use of a linear equation system to determine the balance value of the market. Finally, Section 6 contains the conclusions and suggestions.

2. Preliminaries

In this section, the required explanations about fuzzy sets and Z-numbers are provided from [13].

Definition 1. Fuzzy sets A fuzzy Set $\widetilde{A}$ is defined on a universe X may be introduced as follows:

$$A = \left\{ < x, \mu_{\widetilde{A}}(x) > \mid x \,\varepsilon\, X \right\}, \tag{2}$$

where $\mu_{\widetilde{A}}: X \longrightarrow [0,1]$ is the membership function A. The membership value $\mu_{\widetilde{A}}(x)$ describes the degree of belongingness of $x \,\varepsilon\, X$ in $\widetilde{A}$.

Definition 2. A triangular fuzzy number $\widetilde{A}$ can be defined by a triplet (a_1, a_2, a_3), where the membership can be determined as follows: A triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$ can be considered as in the following equation (3).

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & x \in (-\infty, a_1), \\ \dfrac{x - a_1}{a_2 - a_1}, & x \in [a_1, a_2], \\ \dfrac{a_3 - x}{a_3 - a_2}, & x \in [a_2, a_3], \\ 0, & x \in (a_3, +\infty,). \end{cases} \tag{3}$$

Definition 3. The parametric form of a fuzzy number $\widetilde{A}$ is a pair of functions $(\underline{A}, \overline{A})$ in which the functions $\underline{A}(r)$ and $\overline{A}(r)$ apply to values $0 \le r \le 1$ under the following conditions:

(1) The function $\underline{A}$ is a nondecreasing continuous function bounded on [0, 1] from left

(2) The function $\overline{A}$ is a nonincreasing continuous function bounded on [0, 1] from left

(3) $\underline{A}(r) \le \overline{A}(r)$ for $0 \le r \le 1$

In the above definition, if $\underline{A}(r)$ and $\overline{A}(r)$ are linear functions, the parametric representation of a triangular fuzzy number is obtained.

Definition 4 (Definition of Z-number). Zadeh introduced the idea of Z-numbers with an unknown variable x. A Z-number is a pair of fuzzy numbers $(\widetilde{A}, \widetilde{B})$ [14]. Here, $\widetilde{A}$ is a fuzzy subset of constraints that x can have and $\widetilde{B}$ is a reliability of the $\widetilde{A}$ component. Zadeh also defined $(x, \widetilde{A}, \widetilde{B})$ as a Z-valuation and indicated that this value means that x equals $(\widetilde{A}, \widetilde{B})$. According to the suggestion of Zadeh, this valuation for Z was observed as a constraint in x and interpreted as follows:

$$\text{Prob}\,(x \text{ is } \widetilde{A}) \text{ is } \widetilde{B}. \tag{4}$$

Actually, this means that

$$R(x)\colon\ x \text{ is } \widetilde{A} \longrightarrow \text{Poss}\,(x = u) = \mu_{\widetilde{A}}(u), \tag{5}$$

$$P(x \text{ is } \widetilde{A}) = \int_R \mu_{\widetilde{A}}(u) P_x(u) \mathrm{d}u \text{ is } \widetilde{B}. \tag{6}$$

Here, μ_A is the membership function of the fuzzy set $\widetilde{A}$ and u is a value of x. $P_x(u)$ is the probability density function of x, and $P(x = u)$ is the probability function of x. When we do not know the basic probability distribution, it is clear from these data that the probability distribution is a fuzzy number.

Definition 5. Let a fuzzy set $\widetilde{A}$ be defined on a universe X and may be given as follows: $\widetilde{A} = \left\{ \langle x; \mu_{\widetilde{A}}(x) \rangle \mid x \in X \right\}$, where $\mu_{\widetilde{A}}: X \longrightarrow [0;1]$ is the membership function $\widetilde{A}$. The membership value $\mu_{\widetilde{A}}(x)$ describes the degree of belongingness of $x \in X$ in $\widetilde{A}$. The expected value of a fuzzy set is denoted as follows:

$$E_{\widetilde{A}}(x) = \int_X x \mu_{\widetilde{A}}(x) \mathrm{d}x, \tag{7}$$

which is not the same as the meaning of the expectation of probability space. It can be considered as the information strength supporting the fuzzy set $\widetilde{A}$. In the following section, we will introduce the proposed approach according to the same expectation of fuzzy numbers [1].

Definition 6. The weighted Z-number is denoted as follows:

$$\widetilde{Z}^{\alpha} = \left\{ \left(x, \mu_{\widetilde{A}^{\alpha}}(x) \right) \middle| \mu_{\widetilde{A}^{\alpha}} = \alpha \mu_{\widetilde{A}}(x), x \in [0,1] \right\}. \tag{8}$$

Note that, α represents the weight of the reliability component of Z-number.

Theorem 1 (Reference [4]).

$$E_{\widetilde{A}^{\alpha}}(x) = \alpha E_{\widetilde{A}}(x); \quad x \in X, \tag{9}$$

$$s.t.\mu_{\widetilde{A}^{\alpha}}(x) = \alpha \mu_{\widetilde{A}}(x); x \,\varepsilon\, X. \tag{10}$$

Proof

$$E_{\widetilde{A}^{\alpha}}(x) = \int_X x \mu_{\widetilde{A}^{\alpha}}(x) \mathrm{d}x = \int_X \alpha x \mu_{\widetilde{A}}(x) \mathrm{d}x = \alpha \int_X x \mu_{\widetilde{A}}(x) \mathrm{d}x = \alpha E_{\widetilde{A}}(x). \tag{11}$$

□

3. System of Linear Equations Based on Z-Numbers

Definition 7. The following $n \times n$system.of.linear.equations is called a Z-system of linear equations and is abbreviated as ZSLE.

$$\sum_{j=1}^{n} a_{ij}\left[x_j\right]^Z = \left[y_i\right]^Z, i = 1, \ldots, n, \tag{12}$$

where the matrix $A = [a_{ij}]$ is a real matrix, and also, $[.]^Z$ is the symbol for valuing Z.

In this article, the aim is to introduce a method to solve ZSLE (12); so using Definition 4, we can rewrite equations system (12) as follows:

$$\sum_{j=1}^{n} a_{ij}\left(\tilde{x}_j(r), B_j\right) = \left(\tilde{y}_i(r), R_i\right), i = 1, \ldots, n. \tag{13}$$

Here, B_j and R_i is reliability of the $\tilde{x}_j(r)$ and $\tilde{y}_i(r)$, respectively.

Now, we limit this problem to the following two cases:

Case 1. The reliability (probability) functions B_j and R_i are known and are defined as follows:

$$B_j = 1 - e^{-\lambda x_j}, \tag{14}$$

and also

$$R_i = 1 - e^{-\lambda y_i}. \tag{15}$$

Case 2. The criteria of reliability (probability) functions B_j and R_i are not known, and only their value is available. We propose a numerical method for both of the above cases, and in the numerical results section of an example, we compare the results obtained from both methods.

3.1. The Proposed Method for Solving Z-System of Linear Equations: Case 1

Definition 8. A fuzzy vector $[x_j]^Z = ([x_1]^Z, \ldots, [x_n]^Z)^T = ((\tilde{x}_1(r), B_1), \ldots, (\tilde{x}_n(r), B_n))^T$ is called a solution for a ZSLE (12) whenever it holds in the equations system. If $a_{ij} > 0$, we consider the ZSLE (12) as follows:

$$\sum_{j=1}^{n} a_{ij}\left[x_j\right]^Z = \sum_{j=1}^{n} a_{ij}\left(\tilde{x}_j(r), B_j\right). \tag{16}$$

Consider the parametric form $[x_j]^Z = (\tilde{x}_j(r), B_j)$. So,

$$\sum_{j=1}^{n} a_{ij}\left(\tilde{x}_j(r), B_j\right) = \left(\tilde{y}_i(r), R_i\right). \tag{17}$$

Therefore, to solve a ZSLE, a system of ordinary linear equations $2n \times 2n$ must be solved, in which the column of values in the right side of system is $[y_i]^Z = ([y_1]^Z, \ldots, [y_n]^Z)^T = ((\tilde{y}_1(r), R_1), \ldots, (\tilde{y}_n(r), R_n))^T$. By using Definition 6, we show the new $2n \times 2n$ system as follows:

$$\begin{cases} s_{11}\left(\underline{x}_1(r), B_{\underline{x}_1}\right) + \cdots + s_{1n}\left(\underline{x}_n(r), B_{\underline{x}_n}\right) + s_{1,n+1}\left(\overline{x}_1(r), B_{\overline{x}_1}\right) + \cdots + s_{1,2n}\left(-\left(\overline{x}_n(r), B_{\overline{x}_n}\right)\right) = \underline{y}_1^{\alpha}, \\ \vdots \\ s_{n1}\left(\underline{x}_1(r), B_{\underline{x}_1}\right) + \cdots + s_{nn}\left(\underline{x}_n(r), B_{\underline{x}_n}\right) + s_{n,n+1}\left(-\left(\overline{x}_1(r), B_{\overline{x}_1}\right)\right) + \cdots + s_{n,2n}\left(-\left(\overline{x}_n(r), B_{\overline{x}_n}\right)\right) = \underline{y}_n^{\alpha}, \\ s_{n+1,1}\left(\underline{x}_1(r), B_{\underline{x}_1}\right) + \cdots + s_{n+1,n}\left(\underline{x}_n(r), B_{\underline{x}_n}\right) + s_{n+1,n+1}\left(-\left(\overline{x}_1(r), B_{\overline{x}_1}\right)\right) + \cdots + s_{n+1,2n}\left(-\left(\overline{x}_n(r), B_{\overline{x}_n}\right)\right) = -\overline{y}_1^{\alpha}, \\ \vdots \\ s_{2n1}\left(\underline{x}_1(r), B_{\underline{x}_1}\right) + \cdots + s_{2n,n}\left(\underline{x}_n(r), B_{\underline{x}_n}\right) + s_{2n,n+1}\left(-\left(\overline{x}_1(r), B_{\overline{x}_1}\right)\right) + \cdots + s_{2n,2n}\left(-\left(\overline{x}_n(r), B_{\overline{x}_n}\right)\right) = -\overline{y}_n^{\alpha}, \end{cases} \tag{18}$$

where s_{ij} is defined as follows:

$$\begin{cases} a_{ij} \geq 0 \Rightarrow s_{ij} = s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \Rightarrow s_{i+n,j} = s_{i,j+n} = a_{ij}, \end{cases} \quad 1 \leq i \leq n, 1 \leq j \leq n. \tag{19}$$

The other components of the "S" that were not initialized above are set to zero. In this case, we will have

$$A = D + C. \tag{20}$$

Using the system matrix symbol, (14) is written as follows:

$$S\left(\underline{u}_j(r), B_{1j}\right) = \left(\underline{Y}_i(r), R_{1i}\right), \tag{21}$$

where $s_{ij} \geq 0$, for $1 \leq i, j \leq 2n$, and

$$(\tilde{x}(r), B) = \left((\underline{u}_1(r), B_{11}), \ldots, (\underline{u}_n(r), B_{1n}), (\overline{x}_1(r), B_{21}), \ldots, (\overline{x}_n(r), B_{2n})\right)^T, \tag{22}$$

where $B = \{B_{11}, \ldots, B_{1n}, B_{21}, \ldots, B_{2n}\}$ and

$$(\tilde{Y}(r), R) = \left((\overline{Y}_1(r), R_{11}), \ldots, (\overline{Y}_n(r), R_{1n}), (\overline{Y}_1(r), R_{21}), \ldots, (\overline{Y}_n(r), R_{2n})\right)^T, \tag{23}$$

where $R = \{R_{11}, \dots, R_{1n}, R_{21}, \dots, R_{2n}\}$.

Thus,

$$\begin{bmatrix} D & C \\ C & D \end{bmatrix} \begin{bmatrix} (\underline{u}(r), B_1) \\ (-\overline{x}(r), B_2) \end{bmatrix} = \begin{bmatrix} \left(\underline{Y}(r), R_1\right) \\ \left(-\overline{Y}(r), R_2\right) \end{bmatrix}, \tag{24}$$

where the matrices B and C contain the positive and negative elements of the matrix A, respectively.

It is clear in the system $S(\tilde{x}(r), B) = (\tilde{Y}(r), R)$ that whenever the matrix S is invertible, the response vector of the system is calculated as $(\tilde{x}(r), B) = S^{-1}(\tilde{Y}(r), R)$. The following theorem can be useful in this regard [1].

Theorem 2 (see [7]). *Matrix S is invertible if and only if $D + C$ and $D - C$ matrices are invertible.*

Definition 9. Suppose the answer $\tilde{x}^\alpha(r) = (\underline{u}_j^\alpha(r), -\overline{x}_j^\alpha(r)), 1 \le j \le n$, r is unique.

$u_j^\alpha(r) = (\underline{u}_j^\alpha(r), \overline{u}_j^\alpha(r)) 1 \le j \le n$ where

$$\begin{cases} \left(\overline{u}_j^\alpha(r)\right) = \min\left\{\overline{x}_j^\alpha(r), \overline{x}_j^\alpha(r), \overline{x}_j^\alpha(1)\right\}, \\ \overline{u}_j^\alpha(r) = \max\left\{\overline{x}_j^\alpha(r), \overline{x}_j^\alpha(r), \overline{x}_j^\alpha(1)\right\}, \end{cases} \tag{25}$$

is called a Z^α answer for the system $S\tilde{X}^\alpha = \tilde{Y}^\alpha$. If $\tilde{X}_j^\alpha = (\underline{x}_j^\alpha(r), \overline{x}_j^\alpha(r)) 1 \le j \le n$ and the $\tilde{Z}^\alpha$ are triangular, then $\underline{u}_j^\alpha(r) = \overline{x}_j^\alpha(r)$ and $\overline{u}_j^\alpha(r) = \overline{x}_j^\alpha(r)$, for all values $1 \le j \le n$, and $\overline{u}_j^\alpha(r)$ is called a strong Z^α answer for the system.

3.2. The Proposed Method for Solving Z-System of Linear Equations: Case 2. In this section, suppose that $\tilde{B}$, the probability distribution function of the confidence functions $\tilde{R}$, is specified in (3).

Suppose $\tilde{R}_i = (\mathrm{r}_{1i}, \mathrm{r}_{2i}, \mathrm{r}_{3i})$. Now, using Definition 6, the right side of (3) can be rewritten as follows:

$$(\tilde{y}_i(r), \tilde{R}_i) = (\tilde{y}_i(r), (\mathrm{r}_{1i}, \mathrm{r}_{2i}, \mathrm{r}_{3i})), \tag{26}$$

where α is calculated as follows (Assuming $\tilde{y}_i(r) = (p_{1i}, p_{2i}, p_{3i})$):

Using Definition 6, we have

$$\tilde{y}_i^\alpha = ((p_{1i}, p_{2i}, p_{3i}), (\mathrm{r}_{1i}, \mathrm{r}_{2i}, \mathrm{r}_{3i})) = (\hat{\mathrm{r}}_{1i}, \hat{\mathrm{r}}_{2i}, \hat{\mathrm{r}}_{3i}), \tag{27}$$

where $(\hat{\mathrm{r}}_{1i}, \hat{\mathrm{r}}_{2i}, \hat{\mathrm{r}}_{3i}) = (\sqrt{\alpha}\, p_{1i}, \sqrt{\alpha}\, p_{2i}, \sqrt{\alpha}\, p_{3i})$. The α calculation algorithm is given in [10].

Using (13), (26), and (27), we can rewrite the system of (2) as follows:

$$\sum_{j=1}^{n} a_{ij}\left[x_j\right]^Z = (\hat{\mathrm{r}}_{1i}, \hat{\mathrm{r}}_{2i}, \hat{\mathrm{r}}_{3i}), i = 1, \dots, n, \tag{28}$$

where $(\hat{\mathrm{r}}_{1i}, \hat{\mathrm{r}}_{2i}, \hat{\mathrm{r}}_{3i}) \in \mathbb{R}_{\mathscr{F}}$, is given and determining the unknown Z-valuation vector $[x_j]^Z$ including $[x_j]^Z = (\tilde{x}_j(r), \tilde{B}_j) = (\tilde{x}_j(r), \beta\mu_{\tilde{x}_j})$ is the purpose. Assuming $\tilde{x}_j(r) = (q_{1j}, q_{2j}, q_{3j})$. Using the Definition 6, we have

$$\left(\tilde{x}_j(r), \beta\mu_{\tilde{x}_j}\right) = \left(\hat{q}_{1j}, \hat{q}_{2j}, \hat{q}_{3j}\right) = \tilde{x}_j^\beta, \tag{29}$$

where $(\hat{q}_{1j}, \hat{q}_{2j}, \hat{q}_{3j}) = (\sqrt{\beta}\, q_{1j}, \sqrt{\beta}\, q_{2j}, \sqrt{\beta}\, q_{3j})$, and β is calculated similarly to α.

Definition 10. A fuzzy vector $[x_j]^Z = ([x_1]^Z, \dots, [x_n]^Z)^t = ((\tilde{x}_1(r), \beta\mu_{\tilde{x}_1}), \dots, (\tilde{x}_n(r), \beta\mu_{\tilde{x}_n}))^t = (\tilde{x}_1^\beta, \dots, \tilde{x}_n^\beta)^t$ is called a solution for a ZSLE (12) whenever it holds in the equations system. If $a_{ij} > 0$, we consider the ZSLE (12) as follows:

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j^\beta(r) = \left(\hat{b}_{1i}, \hat{b}_{2i}, \hat{b}_{3i}\right). \tag{30}$$

Consider the parametric form $\tilde{x}_j^\beta(r) = (\overline{x}_j^\beta(r), \overline{x}_j(r))$; so,

$$\begin{cases} \sum_{j=1}^{n} a_{ij}\underline{x}_j^\beta(r) = \hat{y}_i(r), \sum_{j=1}^{n} a_{ij}\overline{x}_j^\beta(r) = \overline{y}_i(r). \end{cases} \tag{31}$$

Therefore, to solve a ZSLE, a system of ordinary linear equations $2n \times 2n$ must be solved, in which the column of values in the right side of the system is $(y^\beta, \dots, \overline{y}_n^\beta, \overline{y}_1^\beta, \dots, \overline{y}_n^\beta)^T$. We show the new $2n \times 2n$ system as follows:

$$\begin{cases} s_{11}\underline{x}_1^\beta + s_{12}\underline{x}_2^\beta + \cdots + s_{1n}\underline{x}_n^\beta + s_{1,n+1}\overline{x}_1^\beta + \cdots + s_{1,n}\overline{x}_n^\beta = \underline{y}_1^\alpha, \\ \vdots \\ s_{n1}\underline{x}_1^\beta + s_{n2}\underline{x}_2^\beta + \cdots + s_{nn}\underline{x}_n^\beta + s_{n,n+1}\left(-\overline{x}_1^\beta\right) + \cdots + s_{n,2n}\left(-\overline{x}_n^\beta\right) = \underline{y}_n^\alpha, \\ s_{n+1,1}\underline{x}_1^\beta + s_{n+1,2}\underline{x}_2^\beta + \cdots + s_{n+1,n}\underline{x}_n^\beta + s_{n+1,n+1}\left(-\overline{x}_1^\beta\right) + \cdots + s_{n+1,2n}\left(-\overline{x}_n^\beta\right) = -\overline{y}_1^\alpha, \\ \vdots \\ s_{2n1}\underline{x}_1^\beta + s_{2n2}\underline{x}_2^\beta + \cdots + s_{2n,n}\underline{x}_n^\beta + s_{2n,n+1}\left(-\overline{x}_1^\beta\right) + \cdots + s_{2n,2n}\left(-\overline{x}_n^\beta\right) = -\overline{y}_n^\alpha, \end{cases} \tag{32}$$

in which s_{ij} are defined as follows:

$$\begin{cases} a_{ij} \geq 0 \Rightarrow s_{ij} = s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \Rightarrow s_{i+n,j} = s_{i,j+n} = a_{ij}, \end{cases} \quad 1 \leq i \leq n, 1 \leq j \leq n, \tag{33}$$

and whenever s_{ij} are not specified above, then they are equal to zero. In this case, we will have

$$A = D + C. \tag{34}$$

Using the system matrix symbol, (30) is written as follows:

$$S\widetilde{X}^{\beta} = \widetilde{Y}^{\alpha}, \tag{35}$$

where $s_{ij} \geq 0$, for $1 \leq i, j \leq 2n$ and $\widetilde{X}^{\beta} = (\underline{x}_1^{\beta}, \ldots, \underline{x}_n^{\beta}, \overline{x}_1^{\beta}, \ldots, \overline{x}_n^{\beta})^T$ and $\widetilde{Y}^{\alpha} = (\underline{y}_1^{\alpha}, \ldots, \underline{y}_n^{\alpha}, \overline{y}_1^{\alpha}, \ldots, \overline{y}_n^{\alpha})^T$; thus,

$$\begin{bmatrix} D & C \\ C & D \end{bmatrix} \begin{bmatrix} X^{\beta} & -\overline{X}^{\beta} \end{bmatrix} = \begin{bmatrix} Y^{\alpha} & -\overline{Y}^{\alpha} \end{bmatrix}, \tag{36}$$

where the matrices B and C contain the positive and negative elements of the matrix A, respectively.

It is clear in the system $S\widetilde{X}^{\beta} = \widetilde{Y}^{\alpha}$ that whenever the matrix S is invertible, the response vector of the system is calculated as $\widetilde{X}^{\beta} = S^{-1}\widetilde{Y}^{\alpha}$. The following theorem can be useful in this regard.

Definition 11 (Reference [15]). If $X = (\underline{u}_1^{\beta}, \underline{x}_2^{\beta}, \ldots, \underline{x}_n^{\beta}, \overline{x}_1^{\beta}, \overline{x}_2^{\beta}, \ldots, \overline{x}_n^{\beta})^T$ satisfies in $SX = b$ and for each $1 \leq j \leq n$, the inequalities $\underline{x}_j^{\beta} \leq \overline{x}_j^{\beta}$ holds, then the solution $X = (\underline{x}_1^{\beta}, \underline{x}_2^{\beta}, \ldots, \underline{x}_n^{\beta}, \overline{x}_1^{\beta}, \overline{x}_2^{\beta}, \ldots, \overline{x}_n^{\beta})^T$ is called a strong solution of the $SX = b$.

3.3. Jacobi Method. Let $A[X]^Z = [b]^Z$ be a square system of n linear equations, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$[X]^Z = \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \\ \vdots \\ [x_n]^Z \end{bmatrix}, \tag{37}$$

$$[b]^Z = \begin{bmatrix} [b_1]^Z \\ [b_2]^Z \\ \vdots \\ [b_n]^Z \end{bmatrix}.$$

A can be decomposed into a diagonal component D, a lower triangular part L, and an upper triangular part U and given as follows: $A = D + L + U$, where $D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, and $L + U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$.

Then, the solution is obtained iteratively via

$$\left[X^{(k+1)}\right]^Z = D^{-1}\left(b - (L+U)\left[X^{(k)}\right]^Z\right), \tag{38}$$

where $[X^{(k)}]^Z$ is the kth approximation or iteration of $[X]^Z$ and $[X^{(k+1)}]^Z$ is the next or$(k+1)$th iteration of X. The element-based formula is written as follows:

$$\left[x_i^{(k+1)}\right]^Z = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij}\left[x_j^{(k)}\right]^Z\right), i = 1, \ldots, n. \tag{39}$$

The computation of $[x_i^{(k+1)}]^Z$ requires each element in $[x_j^{(k)}]^Z$ except where $i = j$.

4. Determining the Balance Value of the Market by the Z-System of Linear Equations

First, some of the concepts that are required for this section are stated. Then, the amount of market equilibrium in environment Z is determined by using a system of linear equations.

Definition 12. Demand Individual demand is the number of goods that the purchaser is willing to purchase because of its price and the stability of other factors in a certain period [16]. Of course, it should be noted that there is a demand for services such as passenger transport services (see [16] for more details). Need and demand are different, in which a lot of goods and services may be required but may not be demand for them. For example, a person may need a plane but does not demand it. Some of our needs become demand due to price and income (see [16] for more details).

The amount of demand for goods is affected by the following factors:

$$Q_x^d = F\left(P_x, I, P_y, T, A_x, E_d, \ldots,\right), \tag{40}$$

where P_x is the price of the good x, P_y is the price of other goods, I is the consumer income or budget, T is the taste and preference of the consumer that can be created due to his needs, in which the source of these needs can be due to social customs and habits or, above all, the product of his values and beliefs, A_x are the advertisements for goods, and E_d represents the factor of price expectations is demand so that consumer demand is influenced by his expectations of either availability or nonavailability of goods in the future, as along with his prediction of future price trends of this product.

Note that if the price of goods is not in the above equation, we keep other factors affecting demand constantly. Hence,

$$Q_x^d = f\left(P_x\right). \tag{41}$$

Table 1: The demands.

P_x	Q_x^d
0	100
10	80
20	60
30	40
40	20
50	0

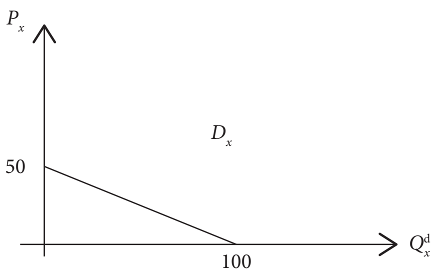

Figure 1: The demand curve.

This equation is called the demand function, which indicates the relationship between the price of a good and the amount of demand for the same good, assuming that other factors are constant. Table 1 reports the individual's demand for good x and the different quantities of commodity, which the consumer wants to purchase at various prices other than good x, assuming that other factors are constant. Moreover, Figure 1 illustrates the demand curve based on Table 1.

D_x is demand curve of x in Figure 1. According to Table 1, the equation of individual demand for good x is $Q_X = 100 - 2P_x$ (by assuming that other factors are constant).

Definition 13. Supply Individual supply is considered as the number of goods that the seller is willing to offer in the market for a certain period of time because of their prices and the stability of other factors [6]. The supply of a good is defined as the maximum quantity of that good, which the seller offers based on the price of that good in the market. Of course, it should be noted that there are service supplies in the economy, such as the supply of taxi services (see [16] for more details). Apart from the price factor P_x that is a substantial variable affecting supply, there are some important factors such as production total costs (TCs). TC includes the price of production institutions and so on, P_y is the price of related goods, T is the level of technology or technical knowledge, and E_s is the factor of supply price expectations, which are effective to supply a product. Overall, the supply is related to profit; it means if profit increases, supply also increases, and vice versa. Meanwhile, profit also depends on the above factors. Therefore, the general form of the individual supply function can be described as follows:

$$Q_x^s = F\left(P_x, I, TC, P_y, T, E_s, \ldots\right). \tag{42}$$

Table 2: The supply table.

P_x	Q_x^s
5	0
10	20
15	40
20	60
25	80

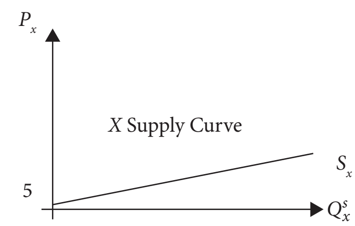

Figure 2: The supply curve.

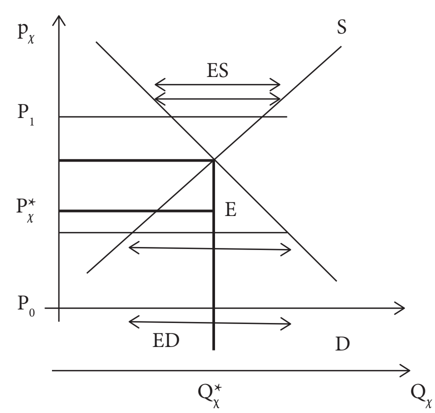

Figure 3: The balance quantity of the goods x.

If in the above equation (42), we assume other factors are constant except for the price of goods, we have the following equation (43):

$$Q_x^s = f\left(P_x\right). \tag{43}$$

This equation is called the supply function. As a result, it can be concluded that the supply function is a function that shows the relationship between the price of a good and the supply of the same good, assuming that other factors are constant. It is worth noting that this function can be interpreted in two ways:

(1) The minimum price that the supplier is willing to offer the goods

(2) It shows the maximum amount offered for each price

In the following, Table 2 lists the supply table of a person for a good x and different quantities of that commodity that he is willing to sell at various prices other than the good x, assuming that other factors are constant. Moreover, Figure 2 depicts the supply cure according to Table 2.

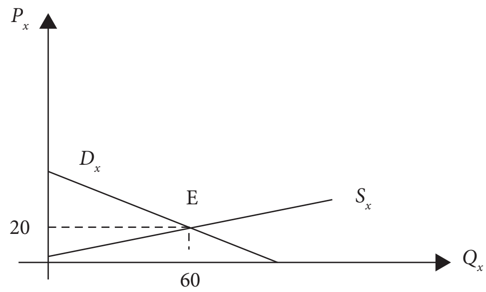

Figure 4: The generated balance condition.

According to Table 2, the supply equation of a person for a good x is equal to the equation of $Q_X = 100 - 2P_x$ (assuming that other factors are constant).

Definition 14. Balance A balance is defined as a state in which there is no motivation, stimulus, or force to change it. If we are not at balance, we tend to change [16]. The condition for market balance is that the supply equals demand; in other words, excess-demand or excess-supply would be equal to zero. Geometrically, as the balance occurs at the intersection of supply and demand curves, the balance price and value can be achieved [5]. In Figure 3, E is the balance point between P_x^* and Q_x^*, i.e., the price and the balance value in P_x^* is $Q_x^d = Q_x^s$. Note that P_1 is not a balance price because there is an excess-supply ($ES = Q_x^s - Q_x^d$) at this price. In other words, it is an incentive for suppliers to lower the price of their goods so that their goods can be sold. Besides, as there is excess-demand ($ED = Q_x^d - Q_x^s$) in P_0, there is an incentive to raise prices. In Figure 3, the balance quantity of the goods x can be seen.

According to the supply and demand functions, the balance price and quantity of the good X are obtained using the market balance condition as follows:

At the meantime, as the balance condition requires $S_x = D_x$, therefore, we have

$$-20 + 4P_x = 100 - 2P_x \Rightarrow P_x = 20, \tag{44}$$

$$Q_x = 100 - 2(20) = 60. \tag{45}$$

Thus, the balance condition is generated between the price and quantity of the good X, as illustrated in Figure 4.

5. Determining the Balance Value of the Market in Z Environment Using the Linear Equations System

Assume that supply and demand are linear functions of price as follows:

$$\begin{cases} q_d = a * p + b, \\ q_s = c * p + d, \end{cases} \tag{46}$$

where q_s and q_d, respectively, are the demand and supply values and they are equal. Price parameter p and real numbers a and c are definite, and coefficients b and d must be determined.

Case 3. Suppose there are some inaccurate parameters in the relationship between supply and demand that are given as follows:

$$\begin{cases} [q_d]^Z = [p]^Z + ((3 + 2r, 7 - 2r), R_d), \\ [q_s]^Z = -3[p]^Z + ((9 + 4r, 17 - 4r), R_s), \end{cases} \tag{47}$$

where the coefficients b and d are given as Z-numbers. $[.]^Z$ is the symbol for valuing Z.

Also, according to the contents of the first case from Section 3, R_d and R_s are considered as follows:

$$\begin{aligned} R_d &= 1 - \left(e^{-\lambda(3+2r)} - e^{-\lambda(7-2r)}\right), \\ R_s &= 1 - \left(e^{-\lambda(9+4r)} - e^{-\lambda(17-4r)}\right). \end{aligned} \tag{48}$$

Now, if $[x_1]^Z$ and $[x_2]^Z$ are considered supply (demand) and price values, respectively, then by equating supply and demand values, a set of linear equations Z must be solved as follows:

$$\begin{cases} [x_1]^Z = [x_2]^Z + \left((3 + 2r, 7 - 2r), 1 - \left(e^{-\lambda(3+2r)} - e^{-\lambda(7-2r)}\right)\right), \\ [x_1]^Z = -3[x_2]^Z + \left((9 + 4r, 17 - 4r), 1 - \left(e^{-\lambda(9+4r)} - e^{-\lambda(17-4r)}\right)\right). \end{cases} \tag{49}$$

By rewriting the above equations, the following system is obtained as follows:

$$\begin{cases} [x_1]^Z - [x_2]^Z = \left((3 + 2r, 7 - 2r), 1 - \left(e^{-\lambda(3+2r)} - e^{-\lambda(7-2r)}\right)\right), \\ [x_1]^Z + 3[x_2]^Z = \left((9 + 4r, 17 - 4r), 1 - \left(e^{-\lambda(9+4r)} - e^{-\lambda(17-4r)}\right)\right). \end{cases} \tag{50}$$

So, we have

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \end{bmatrix} = \begin{bmatrix} \left((3 + 2r, 7 - 2r), 1 - \left(e^{-\lambda(3+2r)} - e^{-\lambda(7-2r)}\right)\right) \\ \left((9 + 4r, 17 - 4r), 1 - \left(e^{-\lambda(9+4r)} - e^{-\lambda(17-4r)}\right)\right) \end{bmatrix}. \tag{51}$$

Using relation (13), we can convert relation (51) as follows:

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} (\tilde{x}_1(r), \tilde{B}_1) \\ (\tilde{x}_2(r), \tilde{B}_2) \end{bmatrix} = \begin{bmatrix} \left((3+2r, 7-2r), 1-\left(e^{-\lambda(3+2r)} - e^{-\lambda(7-2r)}\right)\right) \\ \left((9+4r, 17-4r), 1-\left(e^{-\lambda(9+4r)} - e^{-\lambda(17-4r)}\right)\right) \end{bmatrix}. \tag{52}$$

Using Definition 8, we can convert relation (52) as follows:

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} ((\overline{x}_1(r), \overline{x}_1(r)), \tilde{B}_1) \\ ((\overline{x}_2(r), \overline{x}_2(r)), \tilde{B}_2) \end{bmatrix} = \begin{bmatrix} \left((3+2r, 7-2r), 1-\left(e^{-\lambda(3+2r)} - e^{-\lambda(7-2r)}\right)\right) \\ \left((9+4r, 17-4r), 1-\left(e^{-\lambda(9+4r)} - e^{-\lambda(17-4r)}\right)\right) \end{bmatrix}, \tag{53}$$

Theorem 3. *If X and Y are two independent random variables with probability density quasi functions, f_y, f_x, then the probability density quasi functions for random variable $Z = X + Y$ is*

$$f_z(t) = f_X(t) * f_Y(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\mathrm{d}x. \tag{54}$$

To solve the system (53), the S matrix of extended coefficients is first obtained as follows:

$$\begin{aligned}
1-\left(e^{-\lambda(\underline{x}_1(r))}.e^{-\lambda(-\overline{x}_2(r))}\right) &= 1-e^{-\lambda(3+2r)},\\
1-\left(e^{-\lambda(\underline{x}_1(r))}.e^{-\lambda(3\underline{u}_2(r))}\right) &= 1-e^{-\lambda(7-2r)},\\
1-\left(e^{-\lambda(\overline{x}_1(r))}.e^{-\lambda(-\underline{x}_2(r))}\right) &= 1-e^{-\lambda(9+4r)},\\
1-\left(e^{-\lambda(\overline{x}_1(r))}.e^{-\lambda(3\overline{x}_2(r))}\right) &= 1-e^{-\lambda(17-4r)}.
\end{aligned} \tag{55}$$

So,

$$\begin{aligned}
e^{-\lambda(\underline{x}_1(r)+(-\overline{x}_2(r)))} &= e^{-\lambda(3+2r)},\\
e^{-\lambda(\underline{x}_1(r)+3\underline{u}_2(r))} &= e^{-\lambda(7-2r)},\\
e^{-\lambda(\overline{x}_1(r)+(-\underline{x}_2(r)))} &= e^{-\lambda(9+4r)},\\
e^{-\lambda(\overline{x}_1(r)+3\overline{x}_2(r))} &= e^{-\lambda(17-4r)}.
\end{aligned} \tag{56}$$

So,

$$\begin{aligned}
\underline{u}_1(r) + (-\overline{x}_2(r)) &= 3+2r,\\
\underline{u}_1(r) + 3\underline{u}_2(r) &= 7-2r,\\
\overline{x}_1(r) + (-\underline{u}_2(r)) &= 9+4r,\\
\overline{x}_1(r) + 3\overline{x}_2(r) &= 17-4r.
\end{aligned} \tag{57}$$

So, using (57), we have

$$S = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}. \tag{58}$$

Also,

$$S^{-1} = \begin{bmatrix} 1.1250 & -0.1250 & -0.3750 & 0.3750 \\ -0.3750 & 0.3750 & 0.1250 & -0.1250 \\ -0.3750 & 0.3750 & 1.1250 & -0.1250 \\ 0.1250 & -0.1250 & -0.3750 & 0.3750 \end{bmatrix}. \tag{59}$$

Using the Jacobi method mentioned in Section 3 to solve the system of linear equations, the answer to the problem can be calculated as follows:

$$(\tilde{x}(r), \tilde{B}) = S^{-1}\tilde{b} = \begin{bmatrix} 6+r \\ 1+r \\ 8-r \\ 3-r \end{bmatrix}. \tag{60}$$

So,

$$\begin{aligned}
\underline{u}_1(r) &= \min\{6+r, 8-r, 7, 7\} = 6+r,\\
B_{\underline{u}_1} &= e^{-\lambda\underline{u}_1(r)} = e^{-\lambda(6+r)},\\
\overline{u}_1(r) &= \max\{6+r, 8-r, 7, 7\} = 8-r,\\
B_{\overline{u}_1} &= e^{-\lambda\overline{u}_1(r)} = e^{-\lambda(8+r)},\\
\underline{u}_2(r) &= \min\{1+r, 3-r, 2, 2\} = 1+r,\\
B_{\underline{u}_2} &= e^{-\lambda\underline{u}_2(r)} = e^{-\lambda(1+r)},\\
\overline{u}_2(r) &= \max\{1+r, 3-r, 2, 2\} = 3-r,\\
B_{\overline{u}_2} &= e^{-\lambda\overline{u}_2(r)} = e^{-\lambda(3-r)},\\
\tilde{U} &= \begin{bmatrix} 6+r \\ 1+r \\ 8-r \\ 3-r \end{bmatrix}.
\end{aligned} \tag{61}$$

So,

$$[U]^Z = \begin{bmatrix} ((6+r, 8-r), R_1) \\ ((1+r, 3-r), R_2) \end{bmatrix}, \tag{62}$$

TABLE 3: The different values of $[U]^Z$ for $\lambda = 1$ and $r = 0, 0.1, \ldots, 1$.

r	$(\widetilde{U}_1, R_1)$	$(\widetilde{U}_2, R_2)$
0	((6, 8), 0.99)	((1, 3), 0.68)
0.1	((6.1, 7.9), 0.998)	((1.1, 2.9), 0.72)
0.2	((6.2, 7.8), 0.9983)	((1.2, 2.8), 0.75)
0.3	((6.3, 7.7), 0.9986)	((1.3, 2.7), 0.79)
0.4	((6.4, 7.6), 0.9988)	((1.4, 2.6), 0.82)
0.5	((6.5, 7.5), 0.999)	((1.5, 2.5), 0.85)
0.6	((6.6, 7.4), 0.9992)	((1.6, 2.4), 0.88)
0.7	((6.7, 7.3), 0.9994)	((1.7, 2.3), 0.91)
0.8	((6.8, 7.2), 0.9996)	((1.8, 2.2), 0.94)
0.9	((6.9, 7.1), 0.9998)	((1.9, 2.1), 0.97)
1	((7, 7), 1)	((2, 2), 1)

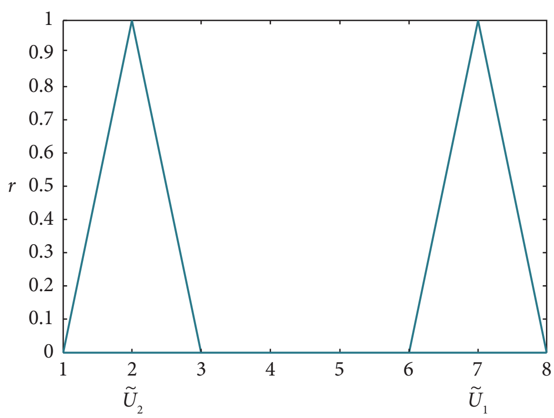

FIGURE 5: Chart of the restriction part of $\widetilde{U}_1$ and $\widetilde{U}_2$.

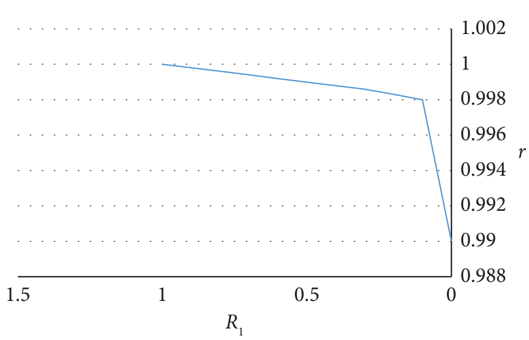

FIGURE 6: The confidence chart of $\widetilde{U}_1$.

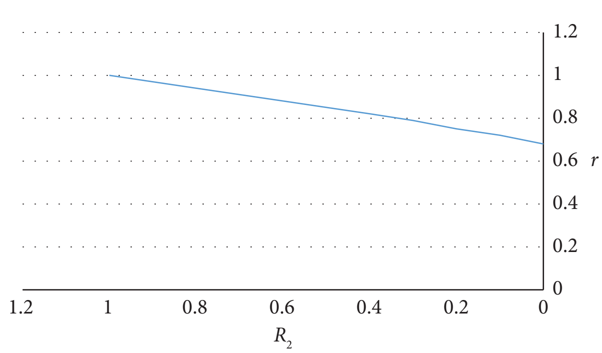

FIGURE 7: The confidence chart of $\widetilde{U}_2$.

where

$$\begin{aligned} R_1 &= 1 - \left(e^{-\lambda(6+r)} - e^{-\lambda(8+r)}\right), \\ R_2 &= 1 - \left(e^{-\lambda(1+r)} - e^{-\lambda(3-r)}\right). \end{aligned} \tag{63}$$

So,

$$[U]^Z = \begin{bmatrix} \left((6+r, 8-r), 1 - \left(e^{-\lambda(6+r)} - e^{-\lambda(8+r)}\right)\right) \\ \left((1+r, 3-r), 1 - \left(e^{-\lambda(1+r)} - e^{-\lambda(3-r)}\right)\right) \end{bmatrix}. \tag{64}$$

In Table 3, different values of $[U]^Z$ for $\lambda = 1$ and $r = 0, 0.1, \ldots 1$ can be seen.

In Figure 5, the constraint of $\widetilde{U}_1$ and $\widetilde{U}_2$ can be seen. Also, in Figures 6 and 7, the confidence diagram can be seen for $\widetilde{U}_1$ and $\widetilde{U}_2$, respectively.

Case 4. Suppose there are some inaccurate parameters in the relationship between supply and demand that are given as follows:

$$\begin{cases} [q_d]^Z = [p]^Z + ((3+2r, 7-2r), (0.677, 0.777, 0.877)), \\ [q_s]^Z = -3[p]^Z + ((9+4r, 17-4r), (0.677, 0.777, 0.877)), \end{cases} \tag{65}$$

where the coefficients b and d are given as Z-numbers, and also, $[.]^Z$ is the symbol for valuing Z. Now, if $[x_1]^Z$ and $[x_2]^Z$ are considered supply (demand) and price values, respectively, then by equating supply and demand values, we have a Z-system of linear equations that must be solved as follows:

$$\begin{cases} [x_1]^Z = [x_2]^Z + ((3+2r, 7-2r), (0.677, 0.777, 0.877)), \\ [x_1]^Z = -3[x_2]^Z + ((9+4r, 17-4r), (0.677, 0.777, 0.877)). \end{cases} \tag{66}$$

By rewriting the above equations, the following system is obtained as follows:

$$\begin{cases} [x_1]^Z - [x_2]^Z = ((3+2r, 7-2r), (0.677, 0.777, 0.877)), \\ [x_1]^Z + 3[x_2]^Z = ((9+4r, 17-4r), (0.677, 0.777, 0.877)). \end{cases} \tag{67}$$

So, we have

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \end{bmatrix} = \begin{bmatrix} ((3+2r, 7-2r), (0.677, 0.777, 0.877)) \\ ((9+4r, 17-4r), (0.677, 0.777, 0.877)) \end{bmatrix}. \tag{68}$$

Using relation (16) and (17), write the right side of relation (47) as follows:

$$\begin{aligned} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \end{bmatrix} &= \begin{bmatrix} (2.64 + 3.76r, 6.16 - 1.76r) \\ (7.92 + 3.52r, 14.96 - 3.52r) \end{bmatrix} \\ &= \begin{bmatrix} y_1^\alpha \\ y_2^\alpha \end{bmatrix}, \\ \alpha &= 0.777. \end{aligned} \tag{69}$$

Using relation (18), we can convert relation (52) as follows:

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1^\beta \\ x_2^\beta \end{bmatrix} = \begin{bmatrix} (2.64 + 3.76r, 6.16 - 1.76r) \\ (7.92 + 3.52r, 14.96 - 3.52r) \end{bmatrix}. \tag{70}$$

To solve the system (52), S, the matrix of extended coefficients is first obtained as follows:

$$\begin{aligned} \overline{x}_1^{\beta} + \left(-\overline{x}_2^{\beta}\right) &= 2.64 + 3.76r, \\ \overline{x}_1^{\beta} + 3\overline{x}_2^{\beta} &= 6.16 - 1.76r, \\ \overline{x}_1^{\beta} + \left(-\underline{u}_2^{\beta}\right) &= 7.92 + 3.52r, \\ \overline{x}_1^{\beta} + 3\overline{x}_2^{\beta} &= 14.96 - 3.52r. \end{aligned} \tag{71}$$

So,

$$S = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}. \tag{72}$$

Also,

$$S^{-1} = \begin{bmatrix} 1.1250 & -0.1250 & -0.3750 & 0.3750 \\ -0.3750 & 0.3750 & 0.1250 & -0.1250 \\ -0.3750 & 0.3750 & 1.1250 & -0.1250 \\ 0.1250 & -0.1250 & -0.3750 & 0.3750 \end{bmatrix}. \tag{73}$$

With suppose

$$\begin{bmatrix} (2.64 + 1.76r, 6.16 - 1.76r) \\ (7.92 + 3.52r, 14.96 - 3.52r) \end{bmatrix} = \begin{bmatrix} \tilde{b}_1^{0.777} \\ \tilde{b}_2^{0.777} \end{bmatrix}. \tag{74}$$

Using the Jacobi method mentioned in Subsection 3.3, to solve the system of linear equations, the answer to the problem can be calculated as follows:

$$\begin{aligned} \tilde{X}^{\beta} &= S^{-1}\tilde{b}^{0.777} \\ &= \begin{bmatrix} 0.88r + 5.28 \\ 0.88r + 0.88 \\ 7.04 - 0.88r \\ 2.64 - 0.88r \end{bmatrix}. \end{aligned} \tag{75}$$

So,

$$\begin{aligned} \underline{u}_1^{\gamma}(r) &= \min\{0.88r + 5.28, 7.04 - 0.88r, 6.16, 6.16\} = 0.88r + 5.28, \\ \overline{u}_1^{\gamma}(r) &= \max\{0.88r + 5.28, 7.04 - 0.88r, 6.16, 6.16\} = 7.04 - 0.88r, \\ \underline{u}_2^{\gamma}(r) &= \min\{0.88r + 0.88, 2.64 - 0.88r, 1.76, 1.76\} = 0.88r + 0.88, \\ \overline{u}_2^{\gamma}(r) &= \max\{0.88r + 0.88, 2.64 - 0.88r, 1.76, 1.76\} = 2.64 - 0.88r, \end{aligned} \tag{76}$$

where we suppose $\gamma = \beta$, then

$$\tilde{U}^{\gamma} = \begin{bmatrix} 0.88r + 5.28 \\ 0.88r + 0.88 \\ 7.04 - 0.88r \\ 2.64 - 0.88r \end{bmatrix}. \tag{77}$$

In the special case, suppose $\beta = \alpha$.

Therefore, the amount of fuzzy demand based on Z-numbers (also fuzzy supply based on Z-numbers) is $(5.28, 7.04)$, that is, $\tilde{Z}^{0.777}$-valuation (it is equivalent to $((6 + r, 8 - r), 0.777)$ as a Z-valuation); also, the amount of fuzzy price based on Z-numbers is $(0.88, 2.64)$, that is, $\tilde{Z}^{0.777}$-valuation, (it is equivalent to $((1 + r, 3 - r), 0.777)$ as a Z-valuation).

We now compare the results obtained from the two proposed methods.

Comparison of $\tilde{u}_1(r)$ and $\tilde{u}_2(r)$ result using the two proposed methods is given in Tables 4–7.

In many real issues, especially in economic issues, the decision maker has to make decisions in situations where the assumption of information in definite circumstances is out of the question. One of these issues in economics occurs when the objective is to determine the equilibrium value of supply and demand. In these cases, a linear equations system is created to determine the amount of supply (demand) and price. Here, due to uncertainty, a system of linear equations with Z-value was created and solved using two proposed methods. The results obtained from the two methods were compared with different landings. Since many real problems, especially economic problems, can be experienced in a system of linear equations involving Z-number with more than two variables. Next, two linear equation systems including Z-numbers 3×3 and 5×5 are simulated and solved.

Table 4: Comparison of $\tilde{u}_1(r)$ result using the two proposed methods $(\lambda = 0)$.

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((6, 8), 1)	((6, 8), 0.777)
0.2	((6.2, 7.8), 1)	((6.2, 7.8), 0.777)
0.4	((6.4, 7.6), 1)	((6.4, 7.6), 0.777)
0.6	((6.6, 7.4), 1)	((6.6, 7.4), 0.777)
0.8	((6.8, 7.2), 1)	((6.8, 7.2), 0.777)
1	((7, 7), 1)	((7, 7), 0.777)

Table 5: Comparison of $\tilde{u}_2(r)$ result using the two proposed methods $(\lambda = 0)$.

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((1, 3), 1)	((1, 3), 0.777)
0.2	((1.2, 2.8), 1)	((1.2, 2.8), 0.777)
0.4	((1.4, 2.6), 1)	((1.4, 2.6), 0.777)
0.6	((1.6, 2.4), 1)	((1.6, 2.4), 0.777)
0.8	((1.8, 2.2), 1)	((1.8, 2.2), 0.777)
1	((2, 2), 1)	((2, 2), 0.777)

Table 6: Comparison of $\tilde{u}_1(r)$ result using the two proposed methods $(\lambda = 0.05)$.

r	The first proposed method ($\lambda = 0.05$)	The second proposed method
0	((6, 8), 0.495)	((6, 8), 0.777)
0.2	((6.2, 7.8), 0.495)	((6.2, 7.8), 0.777)
0.4	((6.4, 7.6), 0.495)	((6.4, 7.6), 0.777)
0.6	((6.6, 7.4), 0.495)	((6.6, 7.4), 0.777)
0.8	((6.8, 7.2), 0.495)	((6.8, 7.2), 0.777)
1	((7, 7), 0.495)	((7, 7), 0.777)

Table 7: Comparison of $\tilde{u}_2(r)$ result using the two proposed methods $(\lambda = 0.05)$.

r	The first proposed method ($\lambda = 0.05$)	The second proposed method
0	((1, 3), 0.8187)	((1, 3), 0.777)
0.2	((1.2, 2.8), 0.8187)	((1.2, 2.8), 0.777)
0.4	((1.4, 2.6), 0.8187)	((1.4, 2.6), 0.777)
0.6	((1.6, 2.4), 0.8187)	((1.6, 2.4), 0.777)
0.8	((1.8, 2.2), 0.8187)	((1.8, 2.2), 0.777)
1	((2, 2), 0.8187)	((2, 2), 0.777)

Example 1. Consider the 3×3 Z-system.

$$\begin{cases} 2[x_1]^Z = -[x_2]^Z - 3[x_3]^Z + ((11 + 8r, 27 - 8r), R_1), \\ 4[x_1]^Z = -[x_2]^Z + [x_3]^Z + ((-23 + 10r, -5 - 8r), R_2), \\ -[x_1]^Z = -3[x_2]^Z - [x_3]^Z + ((10 + 5r, 27 - 12r), R_3). \end{cases} \tag{78}$$

Case 5. We solve the problem by the first proposed method.

According to the contents of Case 5 from Section 3, R_1, R_2, and R_3 are considered as follows:

$$\begin{aligned} R_1 &= e^{-\lambda(11+8r)} - e^{-\lambda(27-8r)}, \\ R_2 &= e^{-\lambda(-23+10r)} - e^{-\lambda(-5-8r)}, \\ R_3 &= e^{-\lambda(10+5r)} - e^{-\lambda(27-12r)}. \end{aligned} \tag{79}$$

So,

$$\begin{cases} 2[x_1]^Z = -[x_2]^Z - 3[x_3]^Z + \left((11+8r, 27-8r), \left(e^{-\lambda(11+8r)} - e^{-\lambda(27-8r)}\right)\right), \\ 4[x_1]^Z = -[x_2]^Z + [x_3]^Z + \left((-23+10r, -5-8r), \left(e^{-\lambda(-23+10r)} - e^{-\lambda(-5-8r)}\right)\right), \\ -[x_1]^Z = -3[x_2]^Z - [x_3]^Z + \left((10+5r, 27-12r), \left(e^{-\lambda(10+5r)} - e^{-\lambda(27-12r)}\right)\right). \end{cases} \quad (80)$$

By rewriting the above equations, the following system is obtained as follows:

$$\begin{cases} 2[x_1]^Z + [x_2]^Z + 3[x_3]^Z = \left((11+8r, 27-8r), \left(e^{-\lambda(11+8r)} - e^{-\lambda(27-8r)}\right)\right), \\ 4[x_1]^Z + [x_2]^Z - [x_3]^Z = \left((-23+10r, -5-8r), \left(e^{-\lambda(-23+10r)} - e^{-\lambda(-5-8r)}\right)\right), \\ -[x_1]^Z + 3[x_2]^Z + [x_3]^Z = \left((10+5r, 27-12r), \left(e^{-\lambda(10+5r)} - e^{-\lambda(27-12r)}\right)\right). \end{cases} \quad (81)$$

So, we have

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \\ [x_3]^Z \end{bmatrix} = \begin{bmatrix} \left((11+8r, 27-8r), \left(e^{-\lambda(11+8r)} - e^{-\lambda(27-8r)}\right)\right) \\ \left((-23+10r, -5-8r), \left(e^{-\lambda(-23+10r)} - e^{-\lambda(-5-8r)}\right)\right) \\ \left((10+5r, 27-12r), \left(e^{-\lambda(10+5r)} - e^{-\lambda(27-12r)}\right)\right) \end{bmatrix}. \quad (82)$$

Using relation (13), we can convert relation (82) as follows:

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} (\tilde{x}_1(r), \tilde{B}_1) \\ (\tilde{x}_2(r), \tilde{B}_2) \\ (\tilde{x}_3(r), \tilde{B}_3) \end{bmatrix} = \begin{bmatrix} \left((11+8r, 27-8r), \left(e^{-\lambda(11+8r)} - e^{-\lambda(27-8r)}\right)\right) \\ \left((-23+10r, -5-8r), \left(e^{-\lambda(-23+10r)} - e^{-\lambda(-5-8r)}\right)\right) \\ \left((10+5r, 27-12r), \left(e^{-\lambda(10+5r)} - e^{-\lambda(27-12r)}\right)\right) \end{bmatrix}. \quad (83)$$

Using Definition 8, we can convert relation (52) as follows:

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} ((\underline{x}_1(r), \overline{x}_1(r)), \tilde{B}_1) \\ ((\underline{x}_2(r), \overline{x}_2(r)), \tilde{B}_2) \\ ((\underline{x}_3(r), \overline{x}_3(r)), \tilde{B}_3) \end{bmatrix} = \begin{bmatrix} \left((11+8r, 27-8r), \left(e^{-\lambda(11+8r)} - e^{-\lambda(27-8r)}\right)\right) \\ \left((-23+10r, -5-8r), \left(e^{-\lambda(-23+10r)} - e^{-\lambda(-5-8r)}\right)\right) \\ \left((10+5r, 27-12r), \left(e^{-\lambda(10+5r)} - e^{-\lambda(27-12r)}\right)\right) \end{bmatrix}. \quad (84)$$

Consider $\widetilde{B}_1, \widetilde{B}_2$, and $\widetilde{B}_3$ as follows:

$$\begin{aligned}\widetilde{B}_1 &= e^{-\lambda \underline{x}_1(r)} - e^{-\lambda \overline{x}_1(r)},\\ \widetilde{B}_2 &= e^{-\lambda \underline{x}_2(r)} - e^{-\lambda \overline{x}_2(r)},\\ \widetilde{B}_3 &= e^{-\lambda \underline{x}_3(r)} - e^{-\lambda \overline{x}_3(r)}.\end{aligned} \tag{85}$$

To solve the system (84), S the matrix of extended coefficients is first obtained as follows:

$$\begin{aligned}2\underline{x}_1(r) + \underline{x}_2(r) + 3\underline{u}_3(r) &= 11 + 8r,\\ 4\underline{x}_1(r) + \underline{x}_2(r) + \left(-\overline{x}_3(r)\right) &= 27 - 8r,\\ -\overline{x}_1(r) + 3\underline{x}_2(r) + \underline{x}_3(r) &= -23 + 10r,\\ 2\overline{x}_1(r) + \overline{x}_2(r) + 3\overline{x}_3(r) &= -5 - 8r,\\ 4\overline{x}_1(r) + \overline{x}_2(r) + \left(-\underline{x}_3(r)\right) &= 10 + 5r,\\ -\underline{x}_1(r) + 3\overline{x}_2(r) + \overline{x}_3(r) &= 27 - 12r.\end{aligned} \tag{86}$$

So,

$$\begin{aligned}\left(e^{-\lambda\left(2\underline{x}_1(r)\right)}.e^{-\lambda\left(\underline{x}_2(r)\right)}.e^{-\lambda\left(3\underline{x}_3(r)\right)}\right) &= e^{-\lambda(11+8r)},\\ \left(e^{-\lambda\left(4\underline{u}_1(r)\right)}.e^{-\lambda\left(\underline{u}_2(r)\right)}.e^{-\lambda\left(-\overline{x}_3(r)\right)}\right) &= e^{-\lambda(27-8r)},\\ \left(e^{-\lambda\left(-\overline{x}_1(r)\right)}.e^{-\lambda\left(3\underline{u}_2(r)\right)}.e^{-\lambda\left(\underline{u}_3(r)\right)}\right) &= e^{-\lambda(-23+10r)},\\ \left(e^{-\lambda\left(2\overline{x}_1(r)\right)}.e^{-\lambda\left(\overline{x}_2(r)\right)}.e^{-\lambda\left(3\overline{x}_3(r)\right)}\right) &= e^{-\lambda(-5-8r)},\\ \left(e^{-\lambda\left(4\overline{x}_1(r)\right)}.e^{-\lambda\left(\overline{x}_2(r)\right)}.e^{-\lambda\left(-\underline{u}_3(r)\right)}\right) &= e^{-\lambda(10+5r)},\\ \left(e^{-\lambda\left(-\underline{u}_1(r)\right)}.e^{-\lambda\left(3\overline{x}_2(r)\right)}.e^{-\lambda\left(\overline{x}_3(r)\right)}\right) &= e^{-\lambda(27-12r)}.\end{aligned} \tag{87}$$

So,

$$\begin{aligned}\left(e^{-\lambda\left(2\underline{u}_1(r)+\underline{u}_2(r)3\underline{u}_3(r)\right)}\right) &= e^{-\lambda(11+8r)}\\ \left(e^{-\lambda\left(4\underline{u}_1(r)+\underline{u}_2(r)+\left(-\overline{x}_3(r)\right)\right)}\right) &= e^{-\lambda(27-8r)}\\ \left(e^{-\lambda\left(-\overline{x}_1(r)+3\underline{u}_2(r)+\underline{u}_3(r)\right)}\right) &= e^{-\lambda(-23+10r)}\\ \left(e^{-\lambda\left(2\overline{x}_1(r)+\overline{x}_2(r)+3\overline{x}_3(r)\right)}\right) &= e^{-\lambda(-5-8r)}\\ \left(e^{-\lambda\left(4\overline{x}_1(r)+\overline{x}_2(r)+\left(-\underline{u}_3(r)\right)\right)}\right) &= e^{-\lambda(10+5r)}\\ \left(e^{-\lambda\left(-\underline{u}_1(r)+3\overline{x}_2(r)+\overline{x}_3(r)\right)}\right) &= e^{-\lambda(27-12r)}.\end{aligned} \tag{88}$$

So, using (86), we have

$$S = \begin{bmatrix} 2 & 1 & 3 & 0 & 0 & 0\\ 4 & 1 & 0 & 0 & 0 & -1\\ 0 & 3 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 2 & 1 & 3\\ 0 & 0 & -1 & 4 & 1 & 0\\ -1 & 0 & 0 & 0 & 3 & 1\end{bmatrix}. \tag{89}$$

Also,

$$S^{-1} = \begin{bmatrix} 0.0070 & 0.2448 & -0.0839 & 0.0839 & -0.0629 & -0.00070\\ -0.0918 & 0.0376 & 0.3514 & 0.0236 & 0.0760 & -0.0332\\ 0.3593 & 0.1757 & -0.0612 & -0.0638 & 0.0166 & 0.0157\\ 0.0839 & -0.0629 & -0.0070 & -0.0070 & 0.2448 & -0.0839\\ 0.0236 & 0.0760 & -0.0332 & -0.0918 & 0.0376 & 0.3514\\ -0.0638 & 0.0166 & 0.0157 & 0.3593 & -0.01757 & -0.0612\end{bmatrix}. \tag{90}$$

Using the Jacobi method mentioned in Section 3 to solve the system of linear equations, the answer to the problem can be calculated as follows:

$$\begin{aligned}(\widetilde{x}(r), \widetilde{B}) &= S^{-1}\widetilde{b}\\ &= \begin{bmatrix} 2r-4\\ r+1\\ r+6\\ -1-r\\ 5-3r\\ 8-r\end{bmatrix}.\end{aligned} \tag{91}$$

So,

$$\begin{aligned}\underline{u}_1(r) &= \min\{2r-4, -1-r, -2, -2\} = 2r-4\\ B_{\underline{u}_1} &= e^{-\lambda\underline{u}_1(r)} = e^{-\lambda(2r-4)}\\ \overline{u}_1(r) &= \max\{2r-4, -1-r, -2, -2\} = 2r-4 = -1-r\\ B_{\overline{u}_1} &= e^{-\lambda\overline{u}_1(r)} = e^{-\lambda(-1-r)}\\ \underline{u}_2(r) &= \min\{1+r, 5-3r, 2, 2\} = 1+r\\ B_{\underline{u}_2} &= e^{-\lambda\underline{u}_2(r)} = e^{-\lambda(1+r)}\\ \overline{u}_2(r) &= \max\{1+r, 5-3r, 2, 2\} = 5-3r\end{aligned}$$

$$B_{\overline{u}_2} = e^{-\lambda \overline{u}_2(r)} = e^{-\lambda(5-3r)}$$
$$\underline{u}_3(r) = \min\{r+6, 8-r, 7, 7\} = 6+r$$
$$B_{\underline{u}_2} = e^{-\lambda \underline{u}_3(r)} = e^{-\lambda(6+r)}$$
$$\overline{u}_3(r) = \max\{r+6, 8-r, 7, 7\} = 8-r$$
$$B_{\overline{u}_2} = e^{-\lambda \overline{u}_3(r)} = e^{-\lambda(8-r)} \tag{92}$$
$$\widetilde{U} = \begin{bmatrix} 2r-4 \\ 1+r \\ 6+r \\ -r-1 \\ 5-3r \\ 8-r \end{bmatrix}.$$

So,

$$[U]^Z = \begin{bmatrix} ((2r-4, -r-1), R_1) \\ ((1+r, 5-3r), R_2) \\ ((6+r, 8-r), R_3) \end{bmatrix}, \tag{93}$$

where

$$R_1 = 1 - \left(e^{-\lambda(2r-4)} - e^{-\lambda(-r-1)}\right),$$
$$R_2 = 1 - \left(e^{-\lambda(1+r)} - e^{-\lambda(5-3r)}\right), \tag{94}$$
$$R_3 = 1 - \left(e^{-\lambda(6+r)} - e^{-\lambda(8-r)}\right).$$

So,

$$[U]^Z = \begin{bmatrix} \left((2r-4, -r-1), 1 - \left(e^{-\lambda(2r-4)} - e^{-\lambda(-r-1)}\right)\right) \\ \left((1+r, 5-3r), 1 - \left(e^{-\lambda(1+r)} - e^{-\lambda(5-3r)}\right)\right) \\ \left((6+r, 8-r), 1 - \left(e^{-\lambda(6+r)} - e^{-\lambda(8-r)}\right)\right) \end{bmatrix}. \tag{95}$$

In Table 8, different values of $[U]^Z$ for $\lambda = 1$ and $r = 0, 0.1, \ldots 1$ can be seen.

In Figure 8, the constraint of $\widetilde{U}_1$, $\widetilde{U}_2$, and $\widetilde{U}_3$ can be seen. Also, in Figures 9–11, the confidence diagram can be seen for $\widetilde{U}_1$, $\widetilde{U}_2$, and $\widetilde{U}_3$, respectively.

Case 6. We solve the problem by the second proposed method.

$$\begin{cases} 2[q_d]^Z = -[p]^Z - 3[e]^Z + ((11+8r, 27-8r), (0.6, 0.7, .08)), \\ 4[q_s]^Z = -[p]^Z + [e]^Z + ((-23+10r, -5-8r), (0.6, 0.7, .08)), \\ -[q_k]^Z = -3[p]^Z - [e]^Z + ((10+5r, 27-12r), (0.6, 0.7, .08)). \end{cases} \tag{96}$$

So,

$$\begin{cases} 2[x_1]^Z = -[x_2]^Z - 3[x_3]^Z + ((11+8r, 27-8r), (0.6, 0.7, .08)), \\ 4[x_1]^Z = -[x_2]^Z + [x_3]^Z + ((-23+10r, -5-8r), (0.6, 0.7, .08)), \\ -[x_1]^Z = -3[x_2]^Z - [x_3]^Z + ((10+5r, 27-12r), (0.6, 0.7, .08)). \end{cases} \tag{97}$$

By rewriting the above equations, the following system is obtained:

$$\begin{cases} 2[x_1]^Z + [x_2]^Z + 3[x_3]^Z = ((11+8r, 27-8r), (0.6, 0.7, .08)), \\ 4[x_1]^Z + [x_2]^Z - [x_3]^Z = ((-23+10r, -5-8r), (0.6, 0.7, .08)), \\ -[x_1]^Z + 3[x_2]^Z + [x_3]^Z = ((10+5r, 27-12r), (0.6, 0.7, .08)). \end{cases} \tag{98}$$

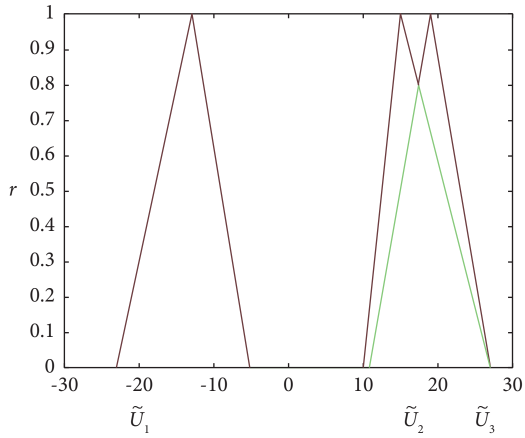

FIGURE 8: Chart of the restriction part of $\widetilde{U}_1$, $\widetilde{U}_2$, and $\widetilde{U}_3$.

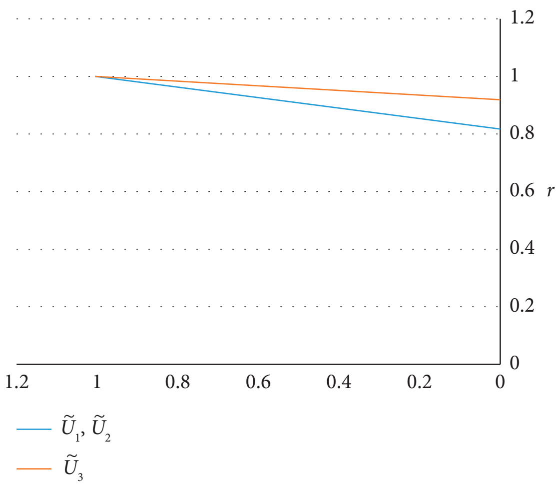

FIGURE 9: The confidence chart of $\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3$.

So, we have

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \\ [x_3]^Z \end{bmatrix} = \begin{bmatrix} ((11+8r, 27-8r), (0.6, 0.7, .08)) \\ ((-23+10r, -5-8r), (0.6, 0.7, .08)) \\ ((10+5r, 27-12r), (0.6, 0.7, .08)) \end{bmatrix}. \quad (99)$$

Using relation (16) and (17), write the right side of relation (52) as follows:

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \\ [x_3]^Z \end{bmatrix} = \begin{bmatrix} (9.13+6.64r, 22-6.64r) \\ (-19.09+8.3r, -4.15-6.64r) \\ (8.3+4.15r, 22.41-9.96r) \end{bmatrix} = \begin{bmatrix} y_1^\alpha \\ y_2^\alpha \end{bmatrix}, \quad \alpha = 0.7. \quad (100)$$

Using relation (18), we can convert relation (52) as follows:

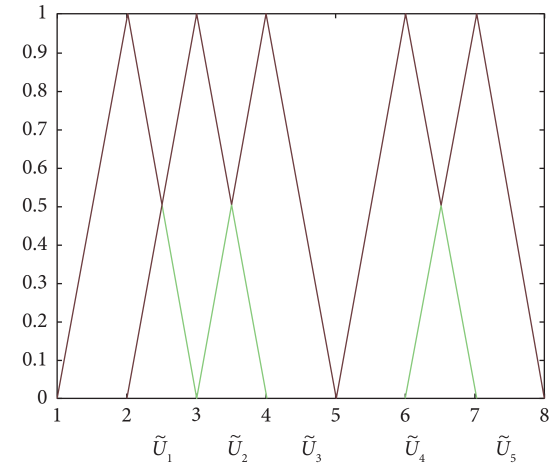

FIGURE 10: Chart of the restriction part of $\widetilde{U}_1$, $\widetilde{U}_2$, $\widetilde{U}_3$, $\widetilde{U}_4$, and $\widetilde{U}_5$.

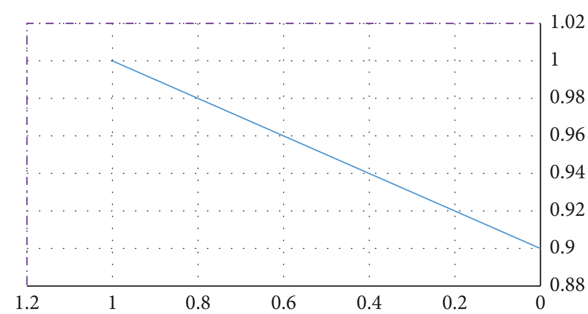

FIGURE 11: The confidence chart of $\widetilde{U}_1$, $\widetilde{U}_2$, $\widetilde{U}_3$, $\widetilde{U}_4$, and $\widetilde{U}_5$.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1^\beta \\ x_2^\beta \\ x_3^\beta \end{bmatrix} = \begin{bmatrix} (9.13+6.64r, 22.41-6.64r) \\ (-19.09+8.3r, -4.15-6.64r) \\ (8.3+4.15r, 22.41-9.96r) \end{bmatrix}. \quad (101)$$

To solve the system (96), S, the matrix of extended coefficients is first obtained as follows:

$$\begin{aligned} 2\underline{x}_1^\beta + \underline{x}_2^\beta + 3\underline{x}_3^\beta &= 9.13+6.64r \\ 4\underline{x}_1^\beta + \underline{x}_2^\beta + \left(-\overline{x}_3^\beta\right) &= 22-6.64r \\ -\overline{x}_1^\beta + 3\underline{x}_2^\beta + \underline{x}_3^\beta &= -19.09+8.3r \\ 2\overline{x}_1^\beta + \overline{x}_2^\beta + 3\overline{x}_3^\beta &= -4.15-6.64r \\ 4\overline{x}_1^\beta + \overline{x}_2^\beta + \left(-\underline{u}_3^\beta\right) &= 8.3+4.15r \\ -\underline{x}_1^\beta + 3\overline{x}_2^\beta + \overline{x}_3^\beta &= 22.41-9.96r. \end{aligned} \quad (102)$$

So,

$$S=\begin{bmatrix}2&1&3&0&0&0\\4&1&0&0&0&-1\\0&3&1&-1&0&0\\0&0&0&2&1&3\\0&0&-1&4&1&0\\-1&0&0&0&3&1\end{bmatrix}. \tag{103}$$

Also,

$$S^{-1}=\begin{bmatrix}0.0070&0.2448&-0.0839&0.0839&-0.0629&-0.00070\\-0.0918&0.0376&0.3514&0.0236&0.0760&-0.0332\\0.3593&0.1757&-0.0612&-0.0638&0.0166&0.0157\\0.0839&-0.0629&-0.0070&-0.0070&0.2448&-0.0839\\0.0236&0.0760&-0.0332&-0.0918&0.0376&0.3514\\-0.0638&0.0166&0.0157&0.3593&-0.01757&-0.0612\end{bmatrix}. \tag{104}$$

With suppose

$$\begin{bmatrix}(9.13+6.64r,22.41-6.64r)\\(-19.09+8.3r,-4.15-6.64r)\\(8.3+4.15r,22.41-9.96r)\end{bmatrix}=\begin{bmatrix}\widetilde{b}_1^{0.7}\\\widetilde{b}_2^{0.7}\\\widetilde{b}_3^{0.7}\end{bmatrix}. \tag{105}$$

Using the Jacobi method mentioned in Section 3 to solve the system of linear equations, the answer to the problem can be calculated as follows:

$$\widetilde{X}^{\beta}=S^{-1}\widetilde{b}^{0.7}=\begin{bmatrix}1.66r-3.32\\0.83r+0.83\\0.83r+4.98\\-0.83r-0.83\\4.15-2.49r\\6.64-0.83r\end{bmatrix}. \tag{106}$$

So,

$$\begin{aligned}\underline{u}_1^{\gamma}(r)&=\min\{1.66r-3.32,-0.83r-0.83,-1.66,-1.66\}=1.66r-3.32\\\overline{u}_1^{\gamma}(r)&=\max\{1.66r-3.32,-0.83r-0.83,-1.66,-1.66\}=-0.83r-0.83\\\underline{u}_2^{\gamma}(r)&=\min\{0.83r+0.83,4.15-2.49r,1.66,1.66\}=0.83r+0.83\\\overline{u}_2^{\gamma}(r)&=\max\{0.83r+0.83,4.15-2.49r,1.66,1.66\}=4.15-2.49r\\\underline{u}_3^{\gamma}(r)&=\min\{0.83r+4.98,6.64-0.83r,5.81,5.81\}=0.83r+4.98\\\overline{u}_3^{\gamma}(r)&=\max\{0.83r+4.98,6.64-0.83r,5.81,5.81\}=6.64-0.83r,\end{aligned} \tag{107}$$

where we suppose $\gamma=\beta$.

$$\widetilde{U}^{\gamma}=\begin{bmatrix}1.66r-3.32\\0.83r+0.83\\0.83r+4.98\\-0.83r-0.83\\4.15-2.49r\\6.64-0.83r\end{bmatrix}. \tag{108}$$

In the special case, suppose $\beta=\alpha$.

Therefore, the amount of fuzzy demand based on Z-numbers (also fuzzy supply based on Z-numbers) is $(-3.32,-0.83)$, that is, $\widetilde{Z}^{0.7}$-valuation (it is equivalent to $((-4+2r,-1-r),0.7)$ as a Z-valuation). Also, the amount of fuzzy price based on Z-numbers is $(0.83,4.15)$, that is, $\widetilde{Z}^{0.7}$-valuation (it is equivalent to $((1+r,5-3r),0.7)$ as a Z-valuation) and the amount of fuzzy price based on Z-numbers is $(-0.83,6.64)$, that is, $\widetilde{Z}^{0.7}$-valuation (it is equivalent to $((6+r,8-r),0.7)$ as a Z-valuation).

We now compare the results that are obtained from the two proposed methods. Comparison of $\widetilde{u}_1(r),\widetilde{u}_2(r)$, and $\widetilde{u}_3(r)$ result using the two proposed methods can be seen in Tables 9–11.

Example 2. Consider the $5\times5Z$-system:

$$\begin{cases}6[q_d]^Z=-[p]^Z-3[e]^Z+[f]^Z-6[g]^Z+((1+r,3-r),R_1),\\5[q_s]^Z=-9[p]^Z-[e]^Z-2[f]^Z-3[g]^Z+((6+r,8-r),R_2),\\2[q_k]^Z=-3[p]^Z-9[e]^Z-2[f]^Z-3[g]^Z+((5+r,7-r),R_3),\\-[q_h]^Z=-[p]^Z-3[e]^Z-8[f]^Z-3[g]^Z+((3+r,5-r),R_4),\\[q_s]^Z=-2[p]^Z-2[e]^Z-2[f]^Z-9[g]^Z+((2+r,4-r),R_5).\end{cases} \tag{109}$$

Case 7. We solve the problem by the first proposed method. According to the contents of the first case from Section 3, R_1, R_2, R_3, R_4, R_5 are considered as follows:

$$\begin{cases} R_1 = e^{-\lambda(1+r)} - e^{-\lambda(3-r)}, \\ R_2 = e^{-\lambda(6+r)} - e^{-\lambda(8-r)}, \\ R_3 = e^{-\lambda(5+r)} - e^{-\lambda(7-r)}, \\ R_4 = e^{-\lambda(3+r)} - e^{-\lambda(5-r)}, \\ R_5 = e^{-\lambda(2+r)} - e^{-\lambda(4-r)}. \end{cases} \tag{110}$$

The set of linear equations Z must be solved as follows:

$$\begin{cases} 6[x_1]^Z = -[x_2]^Z - 3[x_3]^Z + [x_4]^Z - 6[x_5]^Z + \left((1+r, 3-r), e^{-\lambda(1+r)} - e^{-\lambda(3-r)}\right), \\ 5[x_1]^Z = -9[x_2]^Z - [x_3]^Z - 2[x_4]^Z - 3[x_5]^Z + \left((6+r, 8-r), e^{-\lambda(6+r)} - e^{-\lambda(8-r)}\right), \\ 2[x_1]^Z = -3[x_2]^Z - 9[x_3]^Z - 2[x_4]^Z - 3[x_5]^Z + \left((5+r, 7-r), e^{-\lambda(5+r)} - e^{-\lambda(7-r)}\right), \\ -[x_1]^Z = -[x_2]^Z - 3[x_3]^Z - 8[x_4]^Z - 3[x_5]^Z + \left((3+r, 5-r), e^{-\lambda(3+r)} - e^{-\lambda(5-r)}\right), \\ [x_1]^Z = -2[x_2]^Z - 2[x_3]^Z - 2[x_4]^Z - 9[gx_5]^Z + \left((2+r, 4-r), e^{-\lambda(2+r)} - e^{-\lambda(4-r)}\right). \end{cases} \tag{111}$$

By rewriting the above equations, the following system is obtained as follows:

$$\begin{cases} 6[x_1]^Z + [x_2]^Z + 3[x_3]^Z - [x_4]^Z + 6[x_5]^Z = \left((1+r, 3-r), e^{-\lambda(1+r)} - e^{-\lambda(3-r)}\right), \\ 5[x_1]^Z + 9[x_2]^Z + [x_3]^Z + 2[x_4]^Z + 3[x_5]^Z = \left((6+r, 8-r), e^{-\lambda(6+r)} - e^{-\lambda(8-r)}\right), \\ 2[x_1]^Z + 3[x_2]^Z + 9[x_3]^Z + 2[x_4]^Z + 3[x_5]^Z = \left((5+r, 7-r), e^{-\lambda(5+r)} - e^{-\lambda(7-r)}\right), \\ -[x_1]^Z + [x_2]^Z + 3[x_3]^Z + 8[x_4]^Z + 3[x_5]^Z = \left((3+r, 5-r), e^{-\lambda(3+r)} - e^{-\lambda(5-r)}\right), \\ [x_1]^Z + 2[px_2]^Z + 2[x_3]^Z + 2[x_4]^Z + 9[x_5]^Z = \left((2+r, 4-r), e^{-\lambda(2+r)} - e^{-\lambda(4-r)}\right). \end{cases} \tag{112}$$

So, we have

$$\begin{bmatrix} 6 & 1 & 3 & -1 & 6 \\ 5 & 9 & 1 & 2 & 3 \\ 2 & 3 & 9 & 2 & 3 \\ -1 & 1 & 3 & 8 & 3 \\ 1 & 2 & 2 & 1 & 9 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \\ [x_3]^Z \\ [x_4]^Z \\ [x_5]^Z \end{bmatrix} = \begin{bmatrix} \left((1+r, 3-r), e^{-\lambda(1+r)} - e^{-\lambda(3-r)}\right) \\ \left((6+r, 8-r), e^{-\lambda(6+r)} - e^{-\lambda(8-r)}\right) \\ \left((5+r, 7-r), e^{-\lambda(5+r)} - e^{-\lambda(7-r)}\right) \\ \left((3+r, 5-r), e^{-\lambda(3+r)} - e^{-\lambda(5-r)}\right) \\ \left((2+r, 4-r), e^{-\lambda(2+r)} - e^{-\lambda(4-r)}\right) \end{bmatrix}. \tag{113}$$

Using relation (13), we can convert relation (113) as follows:

$$\begin{bmatrix} 6 & 1 & 3 & -1 & 6 \\ 5 & 9 & 1 & 2 & 3 \\ 2 & 3 & 9 & 2 & 3 \\ -1 & 1 & 3 & 8 & 3 \\ 1 & 2 & 2 & 1 & 9 \end{bmatrix} \begin{bmatrix} (\tilde{x}_1(r), \widetilde{B}_1) \\ (\tilde{x}_2(r), \widetilde{B}_2) \\ (\tilde{x}_3(r), \widetilde{B}_3) \\ (\tilde{x}_4(r), \widetilde{B}_4) \\ (\tilde{x}_5(r), \widetilde{B}_5) \end{bmatrix} = \begin{bmatrix} \left((1+r, 3-r), e^{-\lambda(1+r)} - e^{-\lambda(3-r)}\right) \\ \left((6+r, 8-r), e^{-\lambda(6+r)} - e^{-\lambda(8-r)}\right) \\ \left((5+r, 7-r), e^{-\lambda(5+r)} - e^{-\lambda(7-r)}\right) \\ \left((3+r, 5-r), e^{-\lambda(3+r)} - e^{-\lambda(5-r)}\right) \\ \left((2+r, 4-r), e^{-\lambda(2+r)} - e^{-\lambda(4-r)}\right) \end{bmatrix}. \tag{114}$$

Using Definition 8, we can convert relation (114) as follows:

$$\begin{bmatrix} 6 & 1 & 3 & -1 & 6 \\ 5 & 9 & 1 & 2 & 3 \\ 2 & 3 & 9 & 2 & 3 \\ -1 & 1 & 3 & 8 & 3 \\ 1 & 2 & 2 & 1 & 9 \end{bmatrix} \begin{bmatrix} ((\underline{x}_1(r), \overline{x}_1(r)), \widetilde{B}_1) \\ ((\underline{x}_2(r), \overline{x}_2(r)), \widetilde{B}_2) \\ ((\underline{x}_3(r), \overline{x}_3(r)), \widetilde{B}_3) \\ ((\underline{x}_4(r), \overline{x}_4(r)), \widetilde{B}_4) \\ ((\underline{x}_5(r), \overline{x}_5(r)), \widetilde{B}_5) \end{bmatrix} = \begin{bmatrix} \left((1+r, 3-r), e^{-\lambda(1+r)} - e^{-\lambda(3-r)}\right) \\ \left((6+r, 8-r), e^{-\lambda(6+r)} - e^{-\lambda(8-r)}\right) \\ \left((5+r, 7-r), e^{-\lambda(5+r)} - e^{-\lambda(7-r)}\right) \\ \left((3+r, 5-r), e^{-\lambda(3+r)} - e^{-\lambda(5-r)}\right) \\ \left((2+r, 4-r), e^{-\lambda(2+r)} - e^{-\lambda(4-r)}\right) \end{bmatrix}. \tag{115}$$

Consider $\widetilde{B}_1$, $\widetilde{B}_2, \widetilde{B}_3, \widetilde{B}_4$, and $\widetilde{B}_5$ as follows:

$$\begin{aligned} \widetilde{B}_1 &= e^{-\lambda \underline{x}_1(r)} - e^{-\lambda \overline{x}_1(r)} \\ \widetilde{B}_2 &= e^{-\lambda \underline{x}_2(r)} - e^{-\lambda \overline{x}_2(r)} \\ \widetilde{B}_3 &= e^{-\lambda \underline{x}_3(r)} - e^{-\lambda \overline{x}_3(r)} \\ \widetilde{B}_4 &= e^{-\lambda \underline{x}_4(r)} - e^{-\lambda \overline{x}_4(r)} \\ \widetilde{B}_5 &= e^{-\lambda \underline{x}_5(r)} - e^{-\lambda \overline{x}_5(r)}. \end{aligned} \tag{116}$$

To solve the system (115), S, the matrix of extended coefficients is first obtained as follows:

$$\begin{aligned} 6\underline{x}_1(r) + \underline{x}_2(r) + 3\underline{x}_3(r) + \left(-\overline{x}_4(r)\right) + 6\underline{x}_5(r) &= 1 + r, \\ 5\underline{x}_1(r) + 9\underline{x}_2(r) + \underline{x}_3(r) + 2\underline{x}_4(r) + 3\underline{x}_5(r) &= 3 - r, \\ 2\underline{x}_1(r) + 3\underline{x}_2(r) + 9\underline{x}_3(r) + 2\underline{x}_4(r) + 3\underline{x}_5(r) &= 6 + r, \\ -\overline{x}_1(r) + \underline{x}_2(r) + 3\underline{x}_3(r) + 8\underline{x}_4(r) + 3\underline{x}_5(r) &= 8 - r, \\ \underline{x}_1(r) + 2\underline{x}_2(r) + 2\underline{x}_3(r) + \underline{x}_4(r) + 9\underline{x}_5(r) &= 5 + r, \\ 6\overline{x}_1(r) + \overline{x}_2(r) + 3\overline{x}_3(r) + \left(-\underline{x}_4(r)\right) + 6\overline{x}_5(r) &= 7 - r, \\ 5\overline{x}_1(r) + 9\overline{x}_2(r) + \overline{x}_3(r) + 2\overline{x}_4(r) + 3\overline{x}_5(r) &= 3 + r, \\ 2\overline{x}_1(r) + 3\overline{x}_2(r) + 9\overline{x}_3(r) + 2\overline{x}_4(r) + 3\overline{x}_5(r) &= 5 - r, \\ -\underline{x}_1(r) + \overline{x}_2(r) + 3\overline{x}_3(r) + 8\overline{x}_4(r) + 3\overline{x}_5(r) &= 2 + r, \\ \overline{x}_1(r) + 2\overline{x}_2(r) + 2\overline{x}_3(r) + \overline{x}_4(r) + 9\overline{x}_5(r) &= 4 - r. \end{aligned} \tag{117}$$

So,

$$\begin{aligned}
&\left(e^{-\lambda\left(6\underline{x}_1(r)\right)}\cdot e^{-\lambda\left(\underline{x}_2(r)\right)}\cdot e^{-\lambda\left(3x_3(r)\right)}\cdot e^{-\lambda\left(-\overline{x}_4(r)\right)}\cdot e^{-\lambda\left(6\underline{x}_5(r)\right)}\right)=e^{-\lambda(1+r)},\\
&\left(e^{-\lambda\left(5\underline{x}_1(r)\right)}\cdot e^{-\lambda\left(9\underline{x}_2(r)\right)}\cdot e^{-\lambda\left(\underline{x}_3(r)\right)}\cdot e^{-\lambda\left(2\underline{x}_4(r)\right)}\cdot e^{-\lambda\left(3\underline{x}_5(r)\right)}\right)=e^{-\lambda(3-r)},\\
&\left(e^{-\lambda\left(2\underline{x}_1(r)\right)}\cdot e^{-\lambda\left(3\underline{x}_2(r)\right)}\cdot e^{-\lambda\left(9\underline{x}_3(r)\right)}\cdot e^{-\lambda\left(2\underline{x}_4(r)\right)}\cdot e^{-\lambda\left(3\underline{x}_5(r)\right)}\right)=e^{-\lambda(6+r)},\\
&\left(e^{-\lambda\left(-\overline{x}_1(r)\right)}\cdot e^{-\lambda\left(\underline{x}_2(r)\right)}\cdot e^{-\lambda\left(3\underline{x}_3(r)\right)}\cdot e^{-\lambda\left(8\underline{x}_4(r)\right)}\cdot e^{-\lambda\left(3\underline{x}_5(r)\right)}\right)=e^{-\lambda(8-r)},\\
&\left(e^{-\lambda\left(\underline{x}_1(r)\right)}\cdot e^{-\lambda\left(2\underline{x}_2(r)\right)}\cdot e^{-\lambda\left(2\underline{x}_3(r)\right)}\cdot e^{-\lambda\left(\underline{x}_4(r)\right)}\cdot e^{-\lambda\left(9\underline{x}_5(r)\right)}\right)=e^{-\lambda(5+r)},\\
&\left(e^{-\lambda\left(6\underline{x}_1(r)\right)}\cdot e^{-\lambda\left(\overline{x}_2(r)\right)}\cdot e^{-\lambda\left(3\overline{x}_3(r)\right)}\cdot e^{-\lambda\left(\underline{x}_4(r)\right)}\cdot e^{-\lambda\left(6\overline{x}_5(r)\right)}\right)=e^{-\lambda(7-r)},\\
&\left(e^{-\lambda\left(5\overline{x}_1(r)\right)}\cdot e^{-\lambda\left(9\overline{x}_2(r)\right)}\cdot e^{-\lambda\left(\overline{x}_3(r)\right)}\cdot e^{-\lambda\left(2\overline{x}_4(r)\right)}\cdot e^{-\lambda\left(3\overline{x}_5(r)\right)}\right)=e^{-\lambda(3+r)},\\
&\left(e^{-\lambda\left(2\overline{x}_1(r)\right)}\cdot e^{-\lambda\left(3\overline{x}_2(r)\right)}\cdot e^{-\lambda\left(9\overline{x}_3(r)\right)}\cdot e^{-\lambda\left(2\overline{x}_4(r)\right)}\cdot e^{-\lambda\left(3\overline{x}_5(r)\right)}\right)=e^{-\lambda(5-r)},\\
&\left(e^{-\lambda\left(-\underline{x}_1(r)\right)}\cdot e^{-\lambda\left(\overline{x}_2(r)\right)}\cdot e^{-\lambda\left(3\overline{x}_3(r)\right)}\cdot e^{-\lambda\left(8\overline{x}_4(r)\right)}\cdot e^{-\lambda\left(3\overline{x}_5(r)\right)}\right)=e^{-\lambda(2+r)},\\
&\left(e^{-\lambda\left(\overline{x}_1(r)\right)}\cdot e^{-\lambda\left(\overline{x}_2(r)\right)}\cdot e^{-\lambda\left(2\overline{x}_3(r)\right)}\cdot e^{-\lambda\left(\overline{x}_4(r)\right)}\cdot e^{-\lambda\left(9\overline{x}_5(r)\right)}\right)=e^{-\lambda(4-r)}.
\end{aligned}\tag{118}$$

So,

$$\begin{aligned}
&\left(e^{-\lambda\left(6\underline{x}_1(r)+\underline{x}_2(r)+3\underline{x}_3(r)+\left(-\overline{x}_4(r)\right)+6\underline{x}_5(r)\right)}\right)=e^{-\lambda(1+r)},\\
&\left(e^{-\lambda\left(5\underline{x}_1(r)+9\underline{x}_2(r)+\underline{x}_3(r)+2\underline{x}_4(r)+3\underline{x}_5(r)\right)}\right)=e^{-\lambda(3-r)},\\
&\left(e^{-\lambda\left(2\underline{x}_1(r)+3\underline{x}_2(r)+9\underline{x}_3(r)+2\underline{x}_4(r)+3\underline{x}_5(r)\right)}\right)=e^{-\lambda(6+r)},\\
&\left(e^{-\lambda\left(-\overline{x}_1(r)+\underline{x}_2(r)+3\underline{x}_3(r)+8\underline{x}_4(r)+3\underline{x}_5(r)\right)}\right)=e^{-\lambda(8-r)},\\
&\left(e^{-\lambda\left(\underline{x}_1(r)+2\underline{x}_2(r)+2\underline{x}_3(r)+\underline{x}_4(r)+9\underline{x}_5(r)\right)}\right)=e^{-\lambda(5+r)},\\
&\left(e^{-\lambda\left(6\overline{x}_1(r)+\overline{x}_2(r)+3\overline{x}_3(r)+\left(-\underline{x}_4(r)\right)+6\overline{x}_5(r)\right)}\right)=e^{-\lambda(7-r)},\\
&\left(e^{-\lambda\left(5\overline{x}_1(r)+9\overline{x}_2(r)+\overline{x}_3(r)+2\overline{x}_4(r)+3\overline{x}_5(r)\right)}\right)=e^{-\lambda(3+r)},\\
&\left(e^{-\lambda\left(2\overline{x}_1(r)+3\overline{x}_2(r)+9\overline{x}_3(r)+2\overline{x}_4(r)+3\overline{x}_5(r)\right)}\right)=e^{-\lambda(5-r)},\\
&\left(e^{-\lambda\left(-\underline{x}_1(r)+\overline{x}_2(r)+3\overline{x}_3(r)+8\overline{x}_4(r)+3\overline{x}_5(r)\right)}\right)=e^{-\lambda(2+r)},\\
&\left(e^{-\lambda\left(\overline{x}_1(r)+2\overline{x}_2(r)+2\overline{x}_3(r)+\overline{x}_4(r)+9\overline{x}_5(r)\right)}\right)=e^{-\lambda(4-r)}.
\end{aligned}\tag{119}$$

So, using (117), we have

$$S = \begin{matrix} 6 & 1 & 3 & 0 & 6 & 0 & 0 & 0 & -1 & 0 \\ 5 & 9 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 9 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 8 & 3 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 6 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 5 & 9 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 9 & 2 & 3 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 8 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & 9 \end{matrix}. \quad (120)$$

Also,

$$S^{-1} = \begin{bmatrix} 0.1912 & 0.0195 & -0.0455 & 0.0233 & -0.1266 & 0.0074 & 0.0018 & -0.0092 & 0.0272 & -0.0115 \\ -0.1101 & 0.1091 & 0.0291 & -0.0408 & 0.0410 & -0.0094 & -0.0015 & 0.0065 & -0.0163 & 0.0100 \\ -0.0109 & -0.0359 & 0.1297 & -0.0219 & -0.0167 & -0.0042 & -0.0005 & 0.0014 & -0.0020 & 0.0032 \\ 0.0174 & 0.0073 & -0.0439 & 0.1434 & -0.0472 & 0.0270 & 0.0028 & -0.0071 & 0.0055 & -0.0184 \\ 0.0037 & -0.0192 & -0.0254 & -0.0046 & 0.1250 & -0.0008 & -0.0001 & 0.0000 & 0.0004 & 0.0004 \\ 0.0074 & 0.0018 & -0.0092 & 0.0272 & -0.0115 & 0.1912 & 0.1912 & -0.0455 & 0.0233 & -0.1266 \\ -0.0094 & -0.0015 & 0.0065 & -0.0163 & 0.0100 & -0.1101 & 0.1091 & 0.0291 & -0.0408 & 0.0410 \\ -0.0042 & -0.0005 & 0.0014 & -0.0020 & 0.0032 & -0.0109 & -0.0359 & 0.1297 & -0.0219 & -0.0167 \\ 0.0270 & 0.0028 & -0.0071 & 0.0055 & -0.0184 & 0.0174 & 0.0073 & -0.0439 & 0.1434 & -0.0472 \\ -0.0008 & -0.0001 & 0.0000 & 0.0004 & 0.0004 & 0.0037 & -0.0192 & -0.0254 & -0.0046 & 0.1250 \end{bmatrix}. \quad (121)$$

Using the Jacobi method mentioned in Section 3, to solve the system of linear equations, the answer to the problem can be calculated as follows:

$$(\tilde{x}(r), \tilde{B}) = S^{-1}\tilde{b} = \begin{bmatrix} 0.046214r - 0.040903 \\ 0.613045 + 0.038839r \\ 0.319445 + 0.04646r \\ 0.185413 + 0.0671091r \\ 0.001054 + 0.079564r \\ 0.051525 - 0.046214r \\ 0.69072 - 0.038839r \\ 0.412365 - 0.04646r \\ 0.3196317 - 0.067109r \\ 0.158073 - 0.079564r \end{bmatrix}. \quad (122)$$

So,

$$\begin{aligned} \underline{u}_1(r) &= \min\{0.046214r - 0.040903, 0.051525 - 0.046214r, 0.005311, 0.005311\} \\ &= 0.046214r - 0.040903, \\ B_{\underline{u}_1} &= e^{-\lambda \underline{u}_1(r)} \\ &= e^{-\lambda(0.046214r - 0.040903)}, \end{aligned}$$

$$
\begin{aligned}
\overline{u}_1(r) &= \max\{0.046214r - 0.040903, 0.051525 - 0.046214r, 0.005311, 0.005311\} \\
&= 0.051525 - 0.046214r, \\
B_{\overline{u}_1} &= e^{-\lambda \overline{u}_1(r)} \\
&= e^{-\lambda(0.051525 - 0.046214r)}, \\
\underline{u}_2(r) &= \min\{0.613045 + 0.038839r, 0.69072 - 0.038839r, 0.651884, 0.651884\} \\
&= 0.613045 + 0.038839r, \\
B_{\underline{u}_2} &= e^{-\lambda \underline{u}_2(r)} \\
&= e^{-\lambda(0.613045 + 0.038839r)}, \\
\overline{u}_2(r) &= \max\{0.613045 + 0.038839r, 0.69072 - 0.038839r, 0.651884, 0.651884\} \\
&= 0.69072 - 0.038839r, \\
B_{\overline{u}_2} &= e^{-\lambda \overline{u}_2(r)} \\
&= e^{-\lambda(0.69072 - 0.038839r)}, \\
\underline{u}_3(r) &= \min\{0.319445 + 0.04646r, 0.412365 - 0.04646r, 0.365905, 0.365905\} \\
&= 0.319445 + 0.04646r, \\
B_{\underline{u}_3} &= e^{-\lambda \underline{u}_3(r)} \\
&= e^{-\lambda(0.319445 + 0.04646r)}, \\
\overline{u}_3(r) &= \max\{0.319445 + 0.04646r, 0.412365 - 0.04646r, 0.365905, 0.365905\} \\
&= 0.412365 - 0.04646r, \\
B_{\overline{u}_3} &= e^{-\lambda \overline{u}_3(r)} \\
&= e^{-\lambda(0.412365 - 0.04646r)}, \\
\underline{u}_4(r) &= \min\{0.185413 + 0.0671091r, 0.3196317 - 0.067109r, 0.2525, 0.2525\} \\
&= 0.185413 + 0.0671091r, \\
B_{\underline{u}_4} &= e^{-\lambda \underline{u}_4(r)} \\
&= e^{-\lambda(0.185413 + 0.0671091r)}, \\
\overline{u}_4(r) &= \max\{0.185413 + 0.0671091r, 0.3196317 - 0.067109r, 0.2525, 0.2525\} \\
&= 0.3196317 - 0.067109r, \\
B_{\overline{u}_4} &= e^{-\lambda \overline{u}_4(r)} \\
&= e^{-\lambda(0.3196317 - 0.067109r)}, \\
\underline{u}_5(r) &= \min\{0.001054 + 0.079564r, 0.158073 - 0.079564r, 0.0785, 0.0875\} \\
&= 0.001054 + 0.079564r, \\
B_{\underline{u}_5} &= e^{-\lambda \underline{u}_5(r)} \\
&= e^{-\lambda(0.001054 + 0.079564r)}, \\
\overline{u}_5(r) &= \max\{0.001054 + 0.079564r, 0.158073 - 0.079564r, 0.0785, 0.0875\} \\
&= 0.158073 - 0.079564r, \\
B_{\overline{u}_5} &= e^{-\lambda \overline{u}_5(r)} \\
&= e^{-\lambda(0.158073 - 0.079564r)}.
\end{aligned}
\tag{123}
$$

TABLE 8: The different values of $[U]^Z$ for $\lambda = 0.05$ and $r = 0, 0.1, \ldots, 1$.

r	$(\widetilde{U}_1, R_1)$	$(\widetilde{U}_2, R_2)$	$(\widetilde{U}_3, R_3)$
0	$((-4,-1), 0.82)$	$((1,5), 0.82)$	$((6,8), 0.92)$
0.2	$((-3.6,-1.2), 0.86)$	$((1.2,4.4), 0.86)$	$((6.2.8), 0.93)$
0.4	$((-3.2,-1.4), 0.89)$	$((1.4,3.8), 0.89)$	$((6.4,7.6), 0.95)$
0.6	$((-2.8,-1.6), 0.93)$	$((1.6,3.2), 0.93)$	$((6.6,7.4), 0.97)$
0.8	$((-2.4,-1.8), 0.96)$	$((1.8,2.6), 0.96)$	$((6.8,7.2), 0.98)$
1	$((-2,-2), 1)$	$((2,2), 1)$	$((7,7), 1)$

TABLE 9: Comparison of $\tilde{u}_1(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	$((-4, -1), 1)$	$((-4, -1), 0.7)$
0.2	$((-3.6, -1.2), 1)$	$((-3.6, -1.2), 0.7)$
0.4	$((-3.2, -1.4), 1)$	$((-3.2, -1.4), 0.7)$
0.6	$((-2.8, -1.6), 1)$	$((-2.8, -1.6), 0.7)$
0.8	$((-2.4, -1.8), 1)$	$((-2.4, -1.8), 0.7)$
1	$((-2, -2), 1)$	$((-2, -2), 0.7)$

TABLE 10: Comparison of $\tilde{u}_2(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	$((1, 2), 1)$	$((1, 2), 0.7)$
0.2	$((1.2, 4.4), 1)$	$((1.2, 4.4), 0.7)$
0.4	$((1.4, 3.8), 1)$	$((1.4, 3.8), 0.7)$
0.6	$((1.6, 3.2), 1)$	$((1.6, 3.2), 0.7)$
0.8	$((1.8, 2.6), 1)$	$((1.8, 2.6), 0.7)$
1	$((2, 2), 1)$	$((2, 2), 0.7)$

So,

$$\widetilde{U} = \begin{bmatrix} 0.046214r - 0.040903 \\ 0.613045 + 0.038839r \\ 0.319445 + 0.04646r \\ 0.185413 + 0.0671091r \\ 0.001054 + 0.079564r \\ 0.051525 - 0.046214r \\ 0.69072 - 0.038839r \\ 0.412365 - 0.04646r \\ 0.3196317 - 0.067109r \\ 0.158073 - 0.079564r \end{bmatrix}. \quad (124)$$

So,

$$[U]^Z = \begin{bmatrix} ((0.046214r - 0.040903, 0.051525 - 0.046214r), R_1) \\ ((0.613045 + 0.038839r, 0.69072 - 0.038839r), R_2) \\ ((0.319445 + 0.04646r, 0.412365 - 0.04646r), R_3) \\ ((0.185413 + 0.0671091r, 0.3196317 - 0.067109r), R_4) \\ ((0.001054 + 0.079564r, 0.158073 - 0.079564r), R_5) \end{bmatrix}, \quad (125)$$

Table 11: Comparison of $\widetilde{u}_3(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((6, 8), 1)	((6, 8), 0.7)
0.2	((6.2, 7.8), 1)	((6.2, 7.8), 0.7)
0.4	((6.4, 7.6), 1)	((6.4, 7.6), 0.7)
0.6	((6.6, 7.4), 1)	((6.6, 7.4), 0.7)
0.8	((6.8, 7.2), 1)	((6.8, 7.2), 0.7)
1	((7, 7), 1)	((7, 7), 0.7)

where

$$\begin{aligned}
R_1 &= 1-\left(e^{-\lambda(0.046214r-0.040903)}-e^{-\lambda(0.051525-0.046214r)}\right),\\
R_2 &= 1-\left(e^{-\lambda(0.613045+0.038839r)}-e^{-\lambda(0.69072-0.038839r)}\right),\\
R_3 &= 1-\left(e^{-\lambda(0.319445+0.04646r)}-e^{-\lambda(0.412365-0.04646r)}\right),\\
R_4 &= 1-\left(e^{-\lambda(0.185413+0.0671091r)}-e^{-\lambda(0.3196317-0.067109r)}\right),\\
R_5 &= 1-\left(e^{-\lambda(0.001054+0.079564r)}-e^{-\lambda(0.158073-0.079564r)}\right).
\end{aligned} \tag{126}$$

In Table 12, different values of $[U]^Z$ for $\lambda = 1$ and $r = 0, 0.1, \ldots 1$ can be seen.

In Figure 10, the constraint of $\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3, \widetilde{U}_3$, and $\widetilde{U}_5$ can be seen. Also, in Figure 11, the confidence diagram can be seen for $\widetilde{U}_1$, $\widetilde{U}_2$, $\widetilde{U}_3$, $\widetilde{U}_3$, and $\widetilde{U}_5$, respectively.

Case 8. We solve the problem by the second proposed method.

$$\begin{cases}
6[q_d]^Z+[p]^Z+3[e]^Z-[f]^Z+6[g]^Z=((1+r,3-r),(0.5,0.6,0.7)),\\
5[q_s]^Z+9[p]^Z+[e]^Z+2[f]^Z+3[g]^Z=((6+r,8-r),(0.5,0.6,0.7)),\\
2[q_k]^Z+3[p]^Z+9[e]^Z+2[f]^Z+3[g]^Z=((5+r,7-r),(0.5,0.6,0.7)),\\
-[q_h]^Z+[p]^Z+3[e]^Z+8[f]^Z+3[g]^Z=((3+r,5-r),(0.5,0.6,0.7)),\\
[q_s]^Z+2[p]^Z+2[e]^Z+2[f]^Z+9[g]^Z=((2+r,4-r),(0.5,0.6,0.7)).
\end{cases} \tag{127}$$

So,

$$\begin{cases}
6[x_1]^Z=-[x_2]^Z-3[x_3]^Z+[x_4]^Z-6[x_5]^Z+((1+r,3-r),(0.5,0.6,0.7)),\\
5[x_1]^Z=-9[x_2]^Z-[x_3]^Z-2[x_4]^Z-3[x_5]^Z+((6+r,8-r),(0.5,0.6,0.7)),\\
2[x_1]^Z=-3[x_2]^Z-9[x_3]^Z-2[x_4]^Z-3[x_5]^Z+((5+r,7-r),(0.5,0.6,0.7)),\\
-[x_1]^Z=-[x_2]^Z-3[x_3]^Z-8[x_4]^Z-3[x_5]^Z+((3+r,5-r),(0.5,0.6,0.7)),\\
[x_1]^Z=-2[x_2]^Z-2[x_3]^Z-2[x_4]^Z-9[gx_5]^Z+((2+r,4-r),(0.5,0.6,0.7)).
\end{cases} \tag{128}$$

By rewriting the above equations, the following system is obtained:

$$\begin{cases}
6[x_1]^Z+[x_2]^Z+3[x_3]^Z-[x_4]^Z+6[x_5]^Z=((1+r,3-r),(0.5,0.6,0.7)),\\
5[x_1]^Z+9[x_2]^Z+[x_3]^Z+2[x_4]^Z+3[x_5]^Z=((6+r,8-r),(0.5,0.6,0.7)),\\
2[x_1]^Z+3[x_2]^Z+9[x_3]^Z+2[x_4]^Z+3[x_5]^Z=((5+r,7-r),(0.5,0.6,0.7)),\\
-[x_1]^Z+[x_2]^Z+3[x_3]^Z+8[x_4]^Z+3[x_5]^Z=((3+r,5-r),(0.5,0.6,0.7)),\\
[x_1]^Z+2[px_2]^Z+2[x_3]^Z+2[x_4]^Z+9[x_5]^Z=((2+r,4-r),(0.5,0.6,0.7)).
\end{cases} \tag{129}$$

Table 12: The different values of $[U]^Z$ for $\lambda = 0.05$ and $r = 0, 0.1, \ldots, 1$.

r	$(\widetilde{U}_1, R_1)$	$(\widetilde{U}_2, R_2)$	$(\widetilde{U}_3, R_3)$	$(\widetilde{U}_4, R_4)$	$(\widetilde{U}_5, R_5)$
0	$((-0.04, 0.05), 0.90)$	$((0.61, 0.69), 0.95)$	$((0.31, 0.41), 0.9)$	$((0.18, 0.31), 0.89)$	$((0.001, 0.15), 0.85)$
0.2	$((-0.03, 0.04), 0.92)$	$((0.62, 0.68), 0.96)$	$((0.32, 0.40), 0.94)$	$((0.19, 0.30), 0.91)$	$((0.016, 0.14), 0.88)$
0.4	$((-0.02, 0.03), 0.94)$	$((0.62, 0.67), 0.97)$	$((0.33, 0.39), 0.96)$	$((0.21, 0.29), 0.93)$	$((0.03, 0.12), 0.913)$
0.6	$((-0.01, 0.02), 0.96)$	$((0.63, 0.66), 0.98)$	$((0.34, 0.38), 0.97)$	$((0.22, 0.27), 0.95)$	$((0.04, 0.11), 0.94)$
0.8	$((-0.003, 0.01), 0.98)$	$((0.64, 0.65), 0.99)$	$((0.35, 0.37), 0.98)$	$((0.23, 0.26), 0.97)$	$((0.06, 0.09), 0.97)$
1	$((0.005, 0.005), 1)$	$((0.65, 0.65), 1)$	$((0.360.36), 1)$	$((0.25, 0.25), 1)$	$((0.08, 0.07), 1)$

So, we have

$$\begin{bmatrix} 6 & 1 & 3 & -1 & 6 \\ 5 & 9 & 1 & 2 & 3 \\ 2 & 3 & 9 & 2 & 3 \\ -1 & 1 & 3 & 8 & 3 \\ 1 & 2 & 2 & 1 & 9 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \\ [x_3]^Z \\ [x_4]^Z \\ [x_5]^Z \end{bmatrix} = \begin{bmatrix} ((1+r, 3-r), (0.5, 0.6, 0.7)) \\ ((6+r, 8-r), (0.5, 0.6, 0.7)) \\ ((5+r, 7-r), (0.5, 0.6, 0.7)) \\ ((3+r, 5-r), (0.5, 0.6, 0.7)) \\ ((2+r, 4-r), (0.5, 0.6, 0.7)) \end{bmatrix}. \tag{130}$$

Using relation (27) and (28), write the right side of relation (130) as follows:

$$\begin{bmatrix} 6 & 1 & 3 & -1 & 6 \\ 5 & 9 & 1 & 2 & 3 \\ 2 & 3 & 9 & 2 & 3 \\ -1 & 1 & 3 & 8 & 3 \\ 1 & 2 & 2 & 1 & 9 \end{bmatrix} \begin{bmatrix} [x_1]^Z \\ [x_2]^Z \\ [x_3]^Z \\ [x_4]^Z \\ [x_5]^Z \end{bmatrix} = \begin{bmatrix} (0.77 + 0.77r, 2.31 - 0.77r) \\ (4.62 + 0.77, 6.16 - 0.77r) \\ (3.85 + 0.77r + 5.39 - 0.77r) \\ (2.31 + 0.77r, 3.85 - 0.77r) \\ (1.54 + 0.77r, 3.08 - 0.77r) \end{bmatrix} = \begin{bmatrix} y_1^\alpha \\ y_2^\alpha \end{bmatrix}, \alpha = 0.6. \tag{131}$$

Using relation (29), we can convert relation (131) as follows:

$$\begin{bmatrix} 6 & 1 & 3 & -1 & 6 \\ 5 & 9 & 1 & 2 & 3 \\ 2 & 3 & 9 & 2 & 3 \\ -1 & 1 & 3 & 8 & 3 \\ 1 & 2 & 2 & 1 & 9 \end{bmatrix} \begin{bmatrix} x_1^\beta \\ x_2^\beta \\ x_3^\beta \\ x_4^\beta \\ x_5^\beta \end{bmatrix} = \begin{bmatrix} (0.77 + 0.77r, 2.31 - 0.77r) \\ (4.62 + 0.77, 6.16 - 0.77r) \\ (3.85 + 0.77r + 5.39 - 0.77r) \\ (2.31 + 0.77r, 3.85 - 0.77r) \\ (1.54 + 0.77r, 3.08 - 0.77r) \end{bmatrix}. \tag{132}$$

To solve the system (127), S, the matrix of extended coefficients is first obtained as follows:

$$\begin{aligned}
6\underline{x}_1(r) + \underline{x}_2(r) + 3\underline{x}_3(r) + (-\overline{x}_4(r)) + 6\underline{x}_5(r) &= 0.77 + 0.77r, \\
5\underline{x}_1(r) + 9\underline{x}_2(r) + \underline{x}_3(r) + 2\underline{x}_4(r) + 3\underline{x}_5(r) &= 2.31 - 0.77r, \\
2\underline{x}_1(r) + 3\underline{x}_2(r) + 9\underline{x}_3(r) + 2\underline{x}_4(r) + 3\underline{x}_5(r) &= 4.62 + 0.77, \\
-\overline{x}_1(r) + \underline{x}_2(r) + 3\underline{x}_3(r) + 8\underline{x}_4(r) + 3\underline{x}_5(r) &= 6.16 - 0.77r, \\
\underline{x}_1(r) + 2\underline{x}_2(r) + 2\underline{x}_3(r) + \underline{x}_4(r) + 9\underline{x}_5(r) &= 3.85 + 0.77r, \\
6\overline{x}_1(r) + \overline{x}_2(r) + 3\overline{x}_3(r) + (-\underline{x}_4(r)) + 6\overline{x}_5(r) &= 5.39 - 0.77r, \\
5\overline{x}_1(r) + 9\overline{x}_2(r) + \overline{x}_3(r) + 2\overline{x}_4(r) + 3\overline{x}_5(r) &= 2.31 + 0.77r, \\
2\overline{x}_1(r) + 3\overline{x}_2(r) + 9\overline{x}_3(r) + 2\overline{x}_4(r) + 3\overline{x}_5(r) &= 3.85 - 0.77r, \\
-\underline{x}_1(r) + \overline{x}_2(r) + 3\overline{x}_3(r) + 8\overline{x}_4(r) + 3\overline{x}_5(r) &= 1.54 + 0.77r, \\
\overline{x}_1(r) + 2\overline{x}_2(r) + 2\overline{x}_3(r) + \overline{x}_4(r) + 9\overline{x}_5(r) &= 3.08 - 0.77r.
\end{aligned} \tag{133}$$

Table 13: Comparison of $\widetilde{u}_1(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((−0.0409, 0.908), 0.9)	((−0.0409, 0.0515), 0.6)
0.2	((−0.031, 0.926), 0.92)	((−0.03166, 0.04226), 0.6)
0.4	((−0.022, 0.944), 0.94)	((−0.02242, 0.03302), 0.6)
0.6	((−0.013, 0.963), 0.96)	((−0.01318, 0.02378), 0.6)
0.8	((−0.003, 0.981), 0.98)	((−0.00394, 0.01454), 0.6)
1	((0.005, 0.005), 1)	((0.0053, 0.0053), 0.6)

Table 14: Comparison of $\widetilde{u}_2(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((0.613, 0.69), 0.95)	((0.613, 0.6907), 0.6)
0.2	((0.620, 0.68), 0.96)	((0.62076, 0.68292), 0.6)
0.4	((0.628, 0.67), 0.97)	((0.62852, 0.67514), 0.6)
0.6	((0.636, 0.66), 0.98)	((0.63628, 0.66736), 0.6)
0.8	((0.644, 0.65), 0.99)	((0.64404, 0.65958), 0.6)
1	((0.6510.651), 1)	((0.6518, 0.6518), 0.6)

Table 15: Comparison of $\widetilde{u}_3(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((0.31, 0.41), 0.93)	((0.1854, 0.4123), 0.6)
0.2	((0.32, 0.40), 0.94)	((0.19468, 0.3762), 0.6)
0.4	((0.33, 0.39), 0.96)	((0.20396, 0.3401), 0.6)
0.6	((0.34, 0.38), 0.97)	((0.21324, 0.304), 0.6)
0.8	((0.35, 0.37), 0.98)	((0.22252, 0.2679), 0.6)
1	((0.36, 0.36), 1)	((0.2318, 0.2318), 0.6)

Table 16: Comparison of $\widetilde{u}_4(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((0.18, 0.31), 0.89)	((0.1854, 0.3196), 0.6)
0.2	((0.19, 0.30), 0.91)	((0.19882, 0.30618), 0.6)
0.4	((0.21, 0.29), 0.93)	((0.21224, 0.29276), 0.6)
0.6	((0.22, 0.27), 0.95)	((0.22566, 0.27934), 0.6)
0.8	((0.23, 0.26), 0.97)	((0.23908, 0.26592), 0.6)
1	((0.25, 0.25), 1)	((0.2525, 0.2525), 0.6)

Table 17: Comparison of $\widetilde{u}_5(r)$ result using the two proposed methods ($\lambda = 0$).

r	The first proposed method ($\lambda = 0$)	The second proposed method
0	((0.001, 0.158), 0.85)	((−0.001, 0.158), 0.6)
0.2	((0.016, 0.142), 0.88)	((0.158, 0.1421), 0.6)
0.4	((0.032, 0.126), 0.91)	((0.317, 0.1262), 0.6)
0.6	((0.048, 0.110), 0.94)	((0.476, 0.1103), 0.6)
0.8	((0.064, 0.094), 0.97)	((0.635, 0.0944), 0.6)
1	((0.080, 0.078), 1)	((0.794, 0.0785), 0.6)

So,

$$S=\begin{bmatrix} 6 & 1 & 3 & 0 & 6 & 0 & 0 & 0 & -1 & 0\\ 5 & 9 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0\\ 2 & 3 & 9 & 2 & 3 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 3 & 8 & 3 & -1 & 0 & 0 & 0 & 0\\ 1 & 2 & 2 & 1 & 9 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 6 & 1 & 3 & 0 & 6\\ 0 & 0 & 0 & 0 & 0 & 5 & 9 & 1 & 2 & 3\\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 9 & 2 & 3\\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 8 & 3\\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & 9 \end{bmatrix}. \tag{134}$$

Also, suppose

$$S^{-1}=\begin{bmatrix} 0.1912 & 0.0195 & -0.0455 & 0.0233 & -0.1266 & 0.0074 & 0.0018 & -0.0092 & 0.0272 & -0.0115\\ -0.1101 & 0.1091 & 0.0291 & -0.0408 & 0.0410 & -0.0094 & -0.0015 & 0.0065 & -0.0163 & 0.0100\\ -0.0109 & -0.0359 & 0.1297 & -0.0219 & -0.0167 & -0.0042 & -0.0005 & 0.0014 & -0.0020 & 0.0032\\ 0.0174 & 0.0073 & -0.0439 & 0.1434 & -0.0472 & 0.0270 & 0.0028 & -0.0071 & 0.0055 & -0.0184\\ 0.0037 & -0.0192 & -0.0254 & -0.0046 & 0.1250 & -0.0008 & -0.0001 & 0.0000 & 0.0004 & 0.0004\\ 0.0074 & 0.0018 & -0.0092 & 0.0272 & -0.0115 & 0.1912 & 0.1912 & -0.0455 & 0.0233 & -0.1266\\ -0.0094 & -0.0015 & 0.0065 & -0.0163 & 0.0100 & -0.1101 & 0.1091 & 0.0291 & -0.0408 & 0.0410\\ -0.0042 & -0.0005 & 0.0014 & -0.0020 & 0.0032 & -0.0109 & -0.0359 & 0.1297 & -0.0219 & -0.0167\\ 0.0270 & 0.0028 & -0.0071 & 0.0055 & -0.0184 & 0.0174 & 0.0073 & -0.0439 & 0.1434 & -0.0472\\ -0.0008 & -0.0001 & 0.0000 & 0.0004 & 0.0004 & 0.0037 & -0.0192 & -0.0254 & -0.0046 & 0.1250 \end{bmatrix},$$

$$\begin{bmatrix} (0.77+0.77r, 5.39-0.77r)\\ (2.31-0.77r, 2.31+0.77r)\\ (4.62+0.77, 3.85-0.77r)\\ (6.16-0.77r, 1.54+0.77r)\\ (3.85+0.77r, 3.08-0.77r) \end{bmatrix} = \begin{bmatrix} \tilde{b}_1^{0.6}\\ \tilde{b}_2^{0.6}\\ \tilde{b}_3^{0.6}\\ \tilde{b}_4^{0.6}\\ \tilde{b}_5^{0.6} \end{bmatrix}. \tag{135}$$

Using the Jacobi method mentioned in Section 3, to solve the system of linear equations, the answer to the problem can be calculated as follows:

$$\tilde{X}^{\beta} = S^{-1}\tilde{b}^{0.6} = \begin{bmatrix} (0.035585r-0.031496, 0.039674-0.035585r)\\ (0.029907r+0.472045, 0.531858-0.029907r)\\ (0.035774r+0.245973, 0.317522-0.035774r)\\ (0.051674r+0.142768, 0.246116-0.051674)\\ (0.061264r-0.000812, 0.121717-0.061264r) \end{bmatrix}. \tag{136}$$

So,

$$\begin{aligned}
\underline{u}_1^{\gamma}(r) &= \min\{0.035585r - 0.031496, 0.039674 - 0.035585r, 0.004, 0.004\} = 0.035585r - 0.031496,\\
\overline{u}_1^{\gamma}(r) &= \max\{0.035585r - 0.031496, 0.039674 - 0.035585r, 0.004, 0.004\} = 0.039674 - 0.035585r,\\
\underline{u}_2^{\gamma}(r) &= \min\{0.029907r + 0.472045, 0.531858 - 0.029907r, 0.501, 0.501\} = 0.029907r + 0.472045,\\
\overline{u}_2^{\gamma}(r) &= \max\{0.029907r + 0.472045, 0.531858 - 0.029907r, 0.501, 0.501\} = 0.531858 - 0.029907r,\\
\underline{u}_3^{\gamma}(r) &= \min\{0.035774r + 0.245973, 0.317522 - 0.035774r, 0.281, 0.281\} = 0.035774r + 0.245973,\\
\overline{u}_3^{\gamma}(r) &= \max\{0.035774r + 0.245973, 0.317522 - 0.035774r, 0.281, 0.281\} = 0.317522 - 0.035774r,\\
\underline{u}_4^{\gamma}(r) &= \min\{0.051674r + 0.142768, 0.246116 - 0.051674, 0.194, 0.194\} = 0.051674r + 0.142768,\\
\overline{u}_4^{\gamma}(r) &= \max\{0.051674r + 0.142768, 0.246116 - 0.051674r, 0.194, 0.194\} = 0.246116 - 0.051674r,\\
\underline{u}_5^{\gamma}(r) &= \min\{0.061264r - 0.000812, 0.121717 - 0.061264r, 0.06, 0.06\} = 0.061264r - 0.000812,\\
\overline{u}_5^{\gamma}(r) &= \max\{0.061264r - 0.000812, 0.121717 - 0.061264r, 0.06, 0.06\} = 0.121717 - 0.061264r.
\end{aligned} \tag{137}$$

Here, we suppose $\gamma = \beta$.

$$\widetilde{U}^{\gamma} = \begin{bmatrix} 0.035585r - 0.031496\\ 0.029907r + 0.472045\\ 0.035774r + 0.245973\\ 0.051674r + 0.142768\\ 0.061264r - 0.000812\\ 0.039674 - 0.035585r\\ 0.531858 - 0.029907r\\ 0.245973 - 0.035774r\\ 0.246116 - 0.051674r\\ 0.121717 - 0.061264r \end{bmatrix}. \tag{138}$$

In the special case, suppose $\beta = \alpha$.

Therefore, the amount of fuzzy demand based on Z-numbers (also fuzzy supply based on Z-numbers) is $(-0.031496, 0.039674)$, that is, $\widetilde{Z}_1^{0.6}$-valuation (it is equivalent to $((-0.0409 + 0.0462r, 0.0515 - 0.0462r), 0.6)$ as a Z -valuation).

Also, the amount of fuzzy price based on Z-numbers is $(0.472045, 0.531858)$, that is, $\widetilde{Z}_2^{0.6}$-valuation (it is equivalent to $((0.613 + 0.0388r, 0.6907 - 0.0389r), 0.6)$ as a Z -valuation).

$$\begin{aligned}
&(0.245973, 0.245973)\ \text{is}\ \widetilde{Z}_3^{0.6} - \text{valuation},\\
&(\text{it.is.equivalent.to}\ ((0.1854 + 0.0464r, 0.4123 - 0.1805r), 0.6)\ \text{as a}\ Z - \text{valuation}),\\
&(0.142768, 0.246116)\ \text{is}\ \widetilde{Z}_4^{0.6} - \text{valuation},\\
&(\text{it.is.equivalent.to}\ ((0.1854 + 0.0671r, 0.3196 - 0.0671r), 0.6)\ \text{as a}\ Z - \text{valuation}),\\
&(-0.000812, 0.121717)\ \text{is}\ \widetilde{Z}_5^{0.6} - \text{valuation},\\
&(\text{it.is.equivalent.to}\ ((-0.001 + 0.795r, 0.1580 - 0.0795r), 0.6)\ \text{as a}\ Z - \text{valuation}).
\end{aligned} \tag{139}$$

We now compare the results obtained from the two proposed methods. Comparison of $\widetilde{u}_1(r), \widetilde{u}_2(r), \widetilde{u}_3(r), \widetilde{u}_4(r)$, and $\widetilde{u}_5(r)$ result using the two proposed methods is given in Tables 13–17.

6. Conclusion and Recommendations

One of the most important uses of the Z-number arithmetic is to study linear equations systems based on Z-numbers. So far, no method has been provided to solve such systems, or if it has been provided, it is few.

In this article, we present a numerical method for solving a linear system based on Z-numbers. To show the capability of the proposed method for different systems, we simulated and solved several examples of linear equations including Z-numbers.

Also, we solved an economic problem (determining market balance values) that was modeled to use a linear equation system including Z-numbers. Determining the price and quantity in the market mechanism is the purpose of solving these systems. In the future, we can introduce the full Z-system of linear equations (FZSLE) and provide a solution to it that is not covered in this article.

In the future, the proposed method can be used in topics such as "Failure Mode and Effect Analysis" that Huang et al. using T-spherical fuzzy maximizing deviation and combined comparison solution methods [17]; also, fuzzy decision-making models in [14, 18] can be solved by the proposed method by assumption Z-numbers instead of fuzzy numbers and extending fuzzy linear equations systems to Z-system of linear equations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

M. Afshar Kermani, T. Allahviranloo, and S. Abbasbandy devised the project, the main conceptual ideas and proof outline, and verified the analytical methods. Z. Motamedi Pour developed the theory, performed the computations, and wrote the manuscript with support from M. Afshar Kermani, T. Allahviranloo, and S. Abbasbandy. M. Afshar Kermani encouraged Z. Motamedi Pour to investigate and supervise the findings of this work. All authors discussed the results and contributed to the final manuscript.

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