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Research Article

A Study of Some Properties of Fuzzy Laplace Transform with Their Applications in Solving the Second-Order Fuzzy Linear Partial Differential Equations

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In this paper, several results and theorems about the high-order strongly generalized Hukuhara differentiability of function defined via the fuzzy Riemann improper integral (in the sense of Wu) have been established. Then, some properties dealing with the partial derivatives of fuzzy Laplace transform for a fuzzy function of two real variables have been proved. Afterwards, an algorithm of fuzzy Laplace transform for solving second-order fuzzy partial differential equations has been proposed. Finally, two numerical examples, including the heat equation under fuzzy initial conditions, have been studied to justify the efficiency of the algorithm.

1. Introduction

Partial differential equations (PDEs) are extremely useful for the modeling of a variety of natural, physical, and biological phenomena. They have several engineering applications and intervene in many domains of science. Many researchers have investigated fuzzy differential equations (FDEs) and fuzzy partial differential equations (FPDEs) using the rigorous tool of the fuzzy Laplace transform. For example, Allahviranloo et al., Eljaoui et al., and Salahshour et al. have extended this method to solve different kinds of fuzzy differential problems: FDEs of first or second order, FPDEs, and fuzzy integral differential equations [1–5].

In this vein, we have studied the improper fuzzy Riemann integrals by establishing some important results about the continuity and differentiability of a fuzzy improper integral depending on a given crisp parameter in [6]. These results have been then employed to prove some fuzzy Laplace transform properties, which we have used to solve linear FPDEs of first order, under generalized Hukuhara differentiability.

For recent works about partial differential equations, their theory, and necessary materials, one can see [7–9] and the references therein.

The main purpose of this article is to present a fuzzy Laplace transform method for solving FPDEs of second order. To achieve this goal, we begin by developing some results about high-order Hukuhara differentiability of a function defined by a fuzzy improper integral.

The remainder of this work is organized as follows. Section 2 is reserved for preliminaries and recalls some important results about the continuity and differentiability of fuzzy improper integrals that we will need in the sequel. In Section 3, the main results about the high-order differentiability of fuzzy improper integral are studied and new properties of fuzzy Laplace transform are proved. Then, in Section 4, the procedure for solving FPDEs of second order by the fuzzy Laplace transform is proposed. Section 5 deals with numerical examples. Section 6 is reserved for the discussion of the obtained results. In the last section, we present conclusion and further research topic.

2. Preliminaries

Now we recall some basic results which are useful through the rest of this paper.

2.1. Fuzzy Numbers and Functions. A fuzzy number is a function $u: \mathbb{R} \longrightarrow [0,1]$ verifying the following four assumptions:

(1) u is normal, i.e., $\exists t_0 \in \mathbb{R}$ for which $u(t_0) = 1$
(2) u is fuzzy convex
(3) u is upper semicontinuous
(4) The closure of its support $\operatorname{supp}(u) = \{t \in \mathbb{R} | u(t) > 0\}$ is compact (see [10])

Denote E as the space of all fuzzy numbers.

For $0 < \alpha \leq 1$, let $[u]^\alpha = \{t \in \mathbb{R} | u(t) \geq \alpha\}$ be the α-level set of $u \in E$. Then, $[u]^\alpha$ is a nonempty compact interval of $\mathbb{R}$. For all $u, u_1, u_2 \in E, k \in \mathbb{R}$, $0 \leq \alpha \leq 1$, we have

$$\begin{aligned} [u_1 + u_2]^\alpha &= [u_1]^\alpha + [u_2]^\alpha, \\ [ku]^\alpha &= k[u]^\alpha. \end{aligned} \tag{1}$$

Define the Hausdorff distance on E by $D(u_1, u_2) = \sup_{0 \leq \alpha \leq 1} \max\{|\underline{u}_1^\alpha - \underline{u}_2^\alpha|, |\overline{u}_1^\alpha - \overline{u}_2^\alpha|\}$.

Then, (E, D) is a complete metric space (for more details, see [11]).

Definition 1 (see [1]). We define a fuzzy number u in parametric form as a couple $(\underline{u}, \overline{u})$ of mappings $\underline{u}(\alpha), \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$, verifying the following properties:

(1) $\underline{u}(\alpha)$ is bounded increasing, left continuous in $[0,1]$, and right continuous at 0
(2) $\overline{u}(\alpha)$ is bounded decreasing, left continuous in $[0,1]$, and right continuous at 0
(3) $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, for all $0 \leq \alpha \leq 1$

The length of $u \in E$ is level-wise given by $\operatorname{len}(u, \alpha) = \overline{u}(\alpha) - \underline{u}(\alpha) \geq 0$.

Theorem 1 (see [12]). *Let $f(x) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ be a fuzzy function defined on $[a, \infty$. Suppose that for all $\alpha \in [0,1]$, the maps $\underline{f}(x,\alpha), \overline{f}(x,\alpha)$ are integrable on $[a,b]$, and $\exists\, \underline{K}(\alpha), \overline{K}(\alpha) > 0$: $\int_a^b |\underline{f}(x,\alpha)| dx \leq \underline{K}(\alpha)$ and $\int_a^b |\overline{f}(x,\alpha)| dx \leq \overline{K}(\alpha)$, for every $b \geq a$. Then, f is fuzzy Riemann integrable on $[a, \infty$, $\int_a^\infty f(x) dx \in E$ and we have*

$$\int_a^\infty f(x) \mathrm{d}x = \left(\int_a^\infty \underline{f}(x,\alpha) \mathrm{d}x, \int_a^\infty \overline{f}(x,\alpha) \mathrm{d}x \right). \tag{2}$$

For $u_1, u_2 \in E$, if there exists an element u_3 in E such that $u_1 = u_2 + u_3$, then u_3 is called the Hukuhara difference of u_1 and u_2, which we denote by $u_1 \ominus u_2$.

Definition 2 (see [1]). A mapping $f: (a,b) \longrightarrow E$ is said to be strongly generalized differentiable at $x \in (a,b)$, if there exists $f'(x) \in E$ such that for all $h > 0$ very small, there exist the H-differences

(i) $f(x+h) \ominus f(x)$; $f(x) \ominus f(x-h)$ and the limits $\lim_{h \longrightarrow 0^+} f(x+h) \ominus f(x)/h = \lim_{h \longrightarrow 0^+} f(x) \ominus f(x-h)/h = f'(x)$
(ii) $f(x) \ominus f(x+h)$; $f(x-h) \ominus f(x)$ and the limits $\lim_{h \longrightarrow 0^+} f(x) \ominus f(x+h)/(-h) = \lim_{h \longrightarrow 0^+} f(x-h) \ominus f(x)/(-h) = f'(x)$
(iii) $f(x+h) \ominus f(x)$; $f(x-h) \ominus f(x)$ and the limits $\lim_{h \longrightarrow 0^+} f(x+h) \ominus f(x)/h = \lim_{h \longrightarrow 0^+} f(x-h) \ominus f(x)/(-h) = f'(x)$
(iv) $f(x) \ominus f(x+h)$; $f(x) \ominus f(x-h)$ and the limits $\lim_{h \longrightarrow 0^+} f(x) \ominus f(x+h)/(-h) = \lim_{h \longrightarrow 0^+} f(x) \ominus f(x-h)/h = f'(x)$

Theorem 2 (see [10]). *If $f: (a,b) \longrightarrow E$ is a strongly generalized differentiable function on (a,b) in the sense of Definition 2, (iii) or (iv), then $f'(x) \in \mathbb{R}$, for each $x \in (a,b)$.*

So, we can consider only Case (i) or (ii) of Definition 2 almost everywhere in (a,b).

Theorem 3 (see [13]). *Let $f(x) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ be a fuzzy strongly generalized differentiable function on (a,b); then, $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ are differentiable. Moreover,*

(1) If f is (i)-differentiable, then $f'(x) = (\underline{f}'(x,\alpha), \overline{f}'(x,\alpha))$.
(2) If f is (ii)-differentiable, then $f'(x) = (\overline{f}'(x,\alpha), \underline{f}'(x,\alpha))$.

Definition 3. We say that a mapping $f: (a,b) \longrightarrow E$ is strongly generalized differentiable of the n-th order at $x_0 \in (a,b)$ if there exists $f^{(k)}(x_0) \in E$, for all $k \in \{1, 2, \ldots, n\}$, such that for all $h > 0$ very small, there exist the H-differences

(i) $f^{(k-1)}(x_0 + h) \ominus f^{(k-1)}(x_0)$, $f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 - h)$ and the limits $\lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0 + h) \ominus f^{(k-1)}(x_0)/h = \lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 - h)/h = f^{(k)}(x_0)$
(ii) $f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 + h)$, $f^{(k-1)}(x_0 - h) \ominus f^{(k-1)}(x_0)$ and the limits $\lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 + h)/(-h) = \lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0 - h) \ominus f^{(k-1)}(x_0)/(-h) = f^{(k)}(x_0)$
(iii) $f^{(k-1)}(x_0 + h) \ominus f^{(k-1)}(x_0)$, $f^{(k-1)}(x_0 - h) \ominus f^{(k-1)}(x_0)$ and the limits $\lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0 + h) \ominus f^{(k-1)}(x_0)/h = \lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0 - h) \ominus f^{(k-1)}(x_0)/(-h) = f^{(k)}(x_0)$
(iv) $f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 + h)$, $f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 - h)$ and the limits $\lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 + h)/(-h) = \lim_{h \longrightarrow 0^+} f^{(k-1)}(x_0) \ominus f^{(k-1)}(x_0 - h)/h = f^{(k)}(x_0)$

Definition 4 (see [1]). If $f: [0, \infty[\longrightarrow E$ is a continuous mapping such that $e^{-sx} f(x)$ is fuzzy Riemann integrable on

$[0,\infty$, then $\mathbf{L}[f(x)] = \int_0^\infty e^{-sx} f(x)\mathrm{d}x$ is called the fuzzy Laplace transform of f. Notice that $\mathbf{L}[f(x)] = (\mathscr{L}(\underline{f}(x,\alpha)), \mathscr{L}(\overline{f}(x,\alpha)))$, where $\mathscr{L}(k(x))$ is the Laplace transform of a crisp function $k(x)$.

Theorem 4 (see [1, 2]). *Let f: $[0,\infty[\longrightarrow E$ be a fuzzy-valued function and f' and f'' be its derivatives on $[0,\infty$. If f is (i)-differentiable, then $\mathbf{L}[f'(x)] = s\mathbf{L}[f(x)] \ominus f(0)$, or if f is (ii)-differentiable, then $\mathbf{L}[f'(x)] = (-f(0)) \ominus (-s)\,\mathbf{L}[f(x)]$.*

Moreover, if f and f' are (i)-differentiable, then $\mathbf{L}[f''(x)] = \{p^2\mathbf{L}[f(x)] \ominus pf(0)\} \ominus f'(0)$. If f is (i)-differentiable and f' is (ii)-differentiable, then

$$\mathbf{L}[f''(x)] = (-f'(0)) \ominus \left\{-p^2\mathbf{L}[f(x)] \ominus (-pf(0))\right\}. \quad (3)$$

If f is (ii)-differentiable and f' is (i)-differentiable, then

$$\mathbf{L}\left[f''(x)\right] = \left\{(-pf(0)) \ominus -p^2\mathbf{L}[f(x)]\right\} \ominus f'(0). \quad (4)$$

If f and f' are (ii)-differentiable, then $\mathbf{L}[f''(x)] = (-f'(0)) \ominus \{pf(0) \ominus p^2 L[f(x)]\}$.

2.2. Continuity and Differentiability of Fuzzy Improper Integral. In the sequel, I denotes one of the intervals $-\infty, b]$ or $[b,\infty$ or $-\infty,\infty$, where $b \in \mathbb{R}$, J denotes another interval, and A is a nonempty subset of $\mathbb{R}$.

Let us recall the properties of continuity and differentiability of a function defined by a fuzzy improper integral that we had established and proved in [6].

Theorem 5 (see [6]). *Let $F(x,t)$: $A \times I \longrightarrow E$ satisfying the following conditions:*

(i) (H_1) For all $x \in A, t \longmapsto F(x,t)$ is continuous on I

(ii) (H_2) For each $t \in I, x \longmapsto F(x,t)$ is continuous on $A \subset \mathbb{R}$

(iii) (H_3) For all $\alpha \in [0,1]$, there exist a couple of nonnegative, continuous crisp functions $\varphi_\alpha(t)$ and $\psi_\alpha(t)$, which are integrable on I verifying, for all $x \in A, t \in I$:

$$|\underline{F}(x,t,\alpha)| \le \varphi_\alpha(t) \text{ and } |\overline{F}(x,t,\alpha)| \le \psi_\alpha(t). \quad (5)$$

Therefore, the fuzzy mapping $\phi(x) = \int_I F(x,t)\mathrm{d}t$ is continuous on A.

Theorem 6 (see [6]). *Let $F(x,t)$: $J \times I \longrightarrow E$ verifying the following assumptions:*

(i) (A_1) For all $x \in J, t \longmapsto F(x,t)$ is continuous and fuzzy Riemann integrable on I

(ii) (A_2) For all $t \in I, x \longmapsto F(x,t)$ is (i)-differentiable on the interval J

(iii) (A_3) For all $x \in J, t \longmapsto \partial F/\partial x(x,t)$ is continuous on I

(iv) (A_4) For all $t \in I, x \longmapsto \partial F/\partial x(x,t)$ is continuous on J

(v) (A_5) For all $\alpha \in [0,1]$, there exist a couple of continuous crisp functions $\varphi_\alpha(t)$ and $\psi_\alpha(t)$, which are integrable on I verifying, for all $x \in J, t \in I$:

$$\left|\frac{\partial \underline{F}}{\partial x}(x,t,\alpha)\right| \le \varphi_\alpha(t),$$
$$\left|\frac{\partial \overline{F}}{\partial x}(x,t,\alpha)\right| \le \psi_\alpha(t). \quad (6)$$

Therefore, the fuzzy mapping $\phi(x) = \int_I F(x,t)\mathrm{d}t$ is (i)-differentiable on J and

$$\phi'(x) = \int_I \frac{\partial F}{\partial x}(x,t)\mathrm{d}t, \quad \forall x \in J. \quad (7)$$

Moreover, if we replace the assumption (A_2) by the alternative condition

(i) (A_2') For all $t \in I, x \longmapsto F(x,t)$ is (ii)-differentiable on J

then the fuzzy function $\phi(x)$ is (ii)-differentiable on J and equation (7) remains true.

Theorem 7 (see [6]). *Let $u(\xi,\tau)$: $0,\infty \times 0,\infty \longrightarrow E$ be a fuzzy function such that $F(\xi,\tau) = e^{-s\tau}u(\xi,\tau)$ satisfies the assumptions $(A_1) - (A_5)$ above, for all $s \ge s_0 > 0$.*

Let $\mathbf{L}_\tau[u(\xi,\tau)]$ or $\mathbf{L}[u(\xi,\tau)]$ (for short) denote the fuzzy Laplace transform of $u(\xi,\tau)$ with respect to the time variable τ. Then,

$$\mathbf{L}_\tau\left[u_\xi(\xi,\tau)\right] = \frac{\partial}{\partial \xi}\left(\mathbf{L}_\tau[u(\xi,\tau)]\right). \quad (8)$$

Theorem 8 (see [6]). *Let $u(\xi,\tau)$ be a fuzzy-valued function on $0,\infty \times 0,\infty$ into E. Suppose that the mappings $\tau \longmapsto F(\xi,\tau) = e^{-s\tau}u(\xi,\tau)$ and $\tau \longmapsto G(\xi,\tau) = e^{-s\tau}u_\tau(\xi,\tau)$ are fuzzy Riemann integrable on $0,\infty$, for all $s \ge s_0$ for some $s_0 > 0$.*

(a) If $u(\xi,\tau)$ is (i)-differentiable with respect to τ, then

$$\mathbf{L}_\tau\left[u_\tau(\xi,\tau)\right] = s\mathbf{L}_\tau[u(\xi,\tau)] \ominus u(\xi,0). \quad (9)$$

(b) If $u(x,\tau)$ is (ii)-differentiable with respect to τ, then

$$\mathbf{L}_\tau[u_\tau(\xi,\tau)] = (-u(\xi,0)) \ominus (-s)\mathbf{L}_\tau[u(\xi,\tau)]. \quad (10)$$

3. High-Order Differentiability of Fuzzy Improper Integral

Theorem 9. *We consider a fuzzy-valued function $F(x,t)$: $J \times I \longrightarrow E$, verifying the following assumptions:*

(i) (B_1) For all $x \in J, t \longmapsto F(x,t)$ is continuous and fuzzy Riemann integrable on I

(ii) (B_2) *For all* $t \in I, x \longmapsto F(x,t)$ *is strongly generalized differentiable of the second order on the interval* J

(iii) (B_3) *For all* $x \in J, t \longmapsto \partial F/\partial x(x,t)$ *and* $t \longmapsto \partial^2 F/\partial x^2(x,t)$ *are continuous on* I

(iv) (B_4) *For all* $t \in I, x \longmapsto \partial F/\partial x(x,t)$ *and* $x \longmapsto \partial^2 F/\partial x^2(x,t)$ *are continuous on* J

(v) (B_5) *For all* $\alpha \in [0,1]$*, there exist four continuous crisp functions* $\varphi_\alpha(t), \psi_\alpha(t), \eta_\alpha(t)$*, and* $\theta_\alpha(t)$*, which are integrable on* I *verifying, for all* $x \in J, t \in I$*:*

$$\begin{aligned} \left|\frac{\partial \underline{F}}{\partial x}(x,t,\alpha)\right| &\le \varphi_\alpha(t), \\ \left|\frac{\partial \overline{F}}{\partial x}(x,t,\alpha)\right| &\le \psi_\alpha(t), \\ \left|\frac{\partial^2 \underline{F}}{\partial x^2}(x,t,\alpha)\right| &\le \eta_\alpha(t), \\ \left|\frac{\partial^2 \overline{F}}{\partial x^2}(x,t,\alpha)\right| &\le \theta_\alpha(t). \end{aligned} \tag{11}$$

Therefore, the fuzzy mapping $\phi(x) = \int_I F(x,t)\mathrm{d}t$ is strongly generalized differentiable of the second order on J and

$$\phi''(x) = \int_I \frac{\partial^2 F}{\partial x^2}(x,t)\mathrm{d}t, \quad \forall x \in J. \tag{12}$$

Proof. Using Theorem 6 and since F verifies assumptions $(A_1)-(A_5)$, then the fuzzy mapping $\phi(x)$ is strongly generalized differentiable on J and

$$\phi'(x) = \int_I \frac{\partial F}{\partial x}(x,t)\mathrm{d}t, \quad \forall x \in J. \tag{13}$$

Then, from the assumptions $(B_1)-(B_5)$, the function $\partial F/\partial x$ satisfies the conditions $(A_1)-(A_5)$. Hence, and by Theorem 6, $\phi'(x)$ is strongly differentiable on J and for all $x \in J$, we have

$$\phi''(x) = (\phi')'(x) = \int_I \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)(x,t)\mathrm{d}t = \int_I \frac{\partial^2 F}{\partial x^2}(x,t)\mathrm{d}t. \tag{14}$$

□

Theorem 10. *Let* u: $0,\infty \times 0,\infty \longrightarrow E$ *be a fuzzy function. Suppose that the mapping* $F(x,t) = e^{-st}u(x,t)$ *satisfies the assumptions* $(B_1)-(B_5)$ *above, for all* $s \ge s_0$ *for some* $s_0 > 0$*. Let* $\mathbf{L}_t[u(x,t)]$ *or* $\mathbf{L}[u(x,t)]$ *(for short) denote the fuzzy Laplace transform of* $u(x,t)$ *with respect to the time variable* t*. Then,*

$$\mathbf{L}_t\left[\frac{\partial^2 u}{\partial x^2}(x,t)\right] = \frac{\partial^2}{\partial x^2}\left(\mathbf{L}_t[u(x,t)]\right). \tag{15}$$

Proof. For fixed $s \ge s_0$, then using Theorem 9, we have

$$\begin{aligned} \mathbf{L}_t\left[\frac{\partial^2 u}{\partial x^2}(x,t)\right] &= \int_0^\infty e^{-st}\frac{\partial^2 u}{\partial x^2}(x,t)\mathrm{d}t \\ &= \int_0^\infty \frac{\partial^2 F}{\partial x^2}(x,t)\mathrm{d}t, \\ \mathbf{L}_t\left[\frac{\partial^2 u}{\partial x^2}(x,t)\right] &= \frac{\partial^2}{\partial x^2}\left(\int_0^\infty F(x,t)\mathrm{d}t\right) \\ &= \frac{\partial^2}{\partial x^2}\left(\mathbf{L}_t[u(x,t)]\right). \end{aligned} \tag{16}$$

□

Theorem 11. *Let* $u(x,t)$ *be a fuzzy-valued function on* $0,\infty\times 0,\infty$ *into* E*. Suppose that the mappings* $t \longmapsto F(x,t) = e^{-st}u(x,t), t \longmapsto G(x,t) = e^{-st}u_t(x,t)$*, and* $t \longmapsto H(x,t) = e^{-st}u_{tt}(x,t)$ *are fuzzy Riemann integrable on* $0,\infty$*, for all* $s \ge s_0$ *for some* $s_0 > 0$*.*

(a) *If* $u(x,t)$ *and* $u_t(x,t)$ *are (i)-differentiable with respect to* t*, then*

$\mathbf{L}_t[\partial^2 u(x,t)/\partial t^2] = \{s^2\mathbf{L}_t[u(x,t)] \ominus su(x,0)\} \ominus u_t(x,0)$.

(b) *If* $u(x,t)$ *is (i)-differentiable and* $u_t(x,t)$ *is (ii)-differentiable with respect to* t*, then*
$\mathbf{L}_t[\partial^2 u(x,t)/\partial t^2] =$
$-u_t(x,0) \ominus \{-s^2\mathbf{L}_t[u(x,t)] \ominus (-su\ (x,0))\}$.

(c) *If* $u(x,t)$ *is (ii)-differentiable and* $u_t(x,t)$ *is (i)-differentiable with respect to* t*, then*
$\mathbf{L}_t[\partial^2 u(x,t)/\partial t^2] = \{-su(x,0) \ominus (-s^2\mathbf{L}_t[u(x,t)])\} = \ominus u_t(x,0)$.

(d) *If* $u(x,t)$ *and* $u_t(x,t)$ *are (ii)-differentiable with respect to* t*, then*
$\mathbf{L}_t[\partial^2 u/\partial t^2(x,t)] =$
$-u_t(x,0) \ominus \{su(x,0) \ominus s^2\mathbf{L}_t[u(x,t)]\}$.

Proof. This obviously results from Theorem 4, by fixing $x \ge 0$ and taking the Laplace transforms and derivations with respect to the variable t. □

Theorem 12. *We consider a fuzzy-valued function* $F(x,t)$: $J \times I \longrightarrow E$*, verifying the following assumptions:*

(i) (H_1) *For all* $x \in J, t \longmapsto F(x,t)$ *is continuous and fuzzy Riemann integrable on* I

(ii) (H_2) *For all* $t \in I, x \longmapsto F(x,t)$ *is strongly generalized differentiable of the n-th order on* J

(iii) (H_3) *For all* $k \in \{0,1,\ldots,n\}$ *and for each* $x \in J, t \longmapsto \partial^k F/\partial x^k(x,t)$ *is continuous on* I

(iv) (H_4) *For all* $k \in \{0,1,\ldots,n\}$ *and for each* $t \in I, x \longmapsto \partial^k F/\partial x^k(x,t)$ *is continuous on* J

(v) (H_5) *For all* $\alpha \in [0,1]$ *and for all* $k \in \{0,1,\ldots,n\}$*, there exist a couple of continuous and integrable crisp* *functions* $\varphi_{\alpha,k}(t), \psi_{\alpha,k}(t)$ *on* I *verifying, for all* $x \in J, t \in I$:

$$\left|\frac{\partial^k \underline{F}}{\partial x^k}(x,t,\alpha)\right| \le \varphi_{\alpha,k}(t),$$
$$\left|\frac{\partial^k \overline{F}}{\partial x^k}(x,t,\alpha)\right| \le \psi_{\alpha,k}(t). \tag{17}$$

Therefore, the fuzzy mapping $\phi(x) = \int_I F(x,t)\mathrm{d}t$ is strongly generalized differentiable of the n-th order on J and we have

$$\phi^{(n)}(x) = \int_I \frac{\partial^n F}{\partial x^n}(x,t)\mathrm{d}t, \quad \forall x \in J. \tag{18}$$

Proof. According to Theorem 6, the result holds true for $n = 1$. Also, by induction, assume that the result is true to the $(n-1)$-th order. In addition, let a function $F(x,t)\colon J \times I \longrightarrow E$, satisfying the conditions $(H_1)-(H_5)$. Then, $\phi(x) = \int_I F(x,t)\mathrm{d}t$ is strongly generalized differentiable of the n-th order on J and

$$\phi^{(n-1)}(x) = \int_I \frac{\partial^{n-1} F}{\partial x^{n-1}}(x,t)\mathrm{d}t, \quad \forall x \in J. \tag{19}$$

From the assumptions $(H_1)-(H_5)$ and using Theorem 6 and since $\partial^{n-1}F/\partial x^{n-1}$ verifies assumptions $(A_1)-(A_5)$, then the fuzzy mapping $\phi^{(n-1)}(x)$ is strongly generalized differentiable on J, that is, $\phi(x)$ is strongly generalized differentiable to the n-th order on J and we have

$$\phi^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\phi^{(n-1)}\right)(x) = \int_I \frac{\partial}{\partial x}\left(\frac{\partial^{n-1}F}{\partial x^{n-1}}\right)(x,t)\mathrm{d}t$$
$$= \int_I \frac{\partial^n F}{\partial x^n}(x,t)\mathrm{d}t, \quad \forall x \in J. \tag{20}$$

□

4. Fuzzy Laplace Transform for Second-Order Fuzzy Linear Partial Differential Equations

Our aim now is to solve the following second-order linear FPDE using the fuzzy Laplace transform method:

$$\begin{cases} u_{xx}(x,t) + a(x)u_{xt}(x,t) + b(x)u_{tt}(x,t) + c_1(x)u_x(x,t), \\ +c_2(x)u_t(x,t) + d(x)u(x,t) = f(x,t), x \ge 0, t \ge 0, \\ u(x,0) = g(x), u(0,t) = h(t), \text{ and } u_t(x,0) = k(x), \end{cases} \tag{21}$$

where $u(x,t)$ is a fuzzy strongly differentiable function of second order, with continuous partial derivatives. a, b, c_1, c_2, and d are real continuous functions, and $f(x,t), g(x), h(t)$, and $k(x)$ are continuous fuzzy functions. Without loss of generality, assume that the mappings a, b, c_1, c_2 are all positive.

4.1. Resolution of Equation (21) by Fuzzy Laplace Transform Method. By using fuzzy Laplace transform with respect to t, we get

$$\begin{aligned} &\frac{\partial^2 \mathbf{L}_t}{\partial x^2}[u(x,t)] + a(x)s\frac{\partial \mathbf{L}_t}{\partial x}[u(x,t)] - a(x)g'(x) \\ &+ b(x)s^2\mathbf{L}_t[u(x,t)] - b(x)sg(x) - b(x)k(x) \\ &+ c_1(x)\frac{\partial \mathbf{L}_t}{\partial x}[u_x(x,t)] + c_2(x)s\mathbf{L}_t[u_t(x,t)] - c_2(x)g(x) \\ &+ \mathrm{d}(x)\mathbf{L}_t[u(x,t)] = \mathbf{L}_t[f(x,t)]. \end{aligned} \tag{22}$$

Therefore, we have to distinguish the following 32 cases for solving this last equation.

(a) If u is (i)-differentiable with respect to x and t, u_x is (i)-differentiable with respect to x and t, and u_t is (i)-differentiable with respect to t, then using Theorems 10 and 11, we get

$$\begin{cases} \dfrac{\partial^2 \mathscr{L}_t}{\partial x^2}[\underline{u}(x,t,\alpha)] + a(x)s\dfrac{\partial \mathscr{L}_t}{\partial x}[\underline{u}(x,t,\alpha)] - a(x)\overline{g}'(x,\alpha) \\ + b(x)s^2\mathscr{L}_t[\underline{u}(x,t,\alpha)] - b(x)s\overline{g}(x,\alpha) \\ + c_1(x)\dfrac{\partial \mathscr{L}_t}{\partial x}[\underline{u}(x,t,\alpha)] + c_2(x)s\mathscr{L}_t[\underline{u}_t(x,t,\alpha)] \\ - c_2(x)\overline{g}(x,\alpha) + d(x)\mathscr{L}_t[\underline{u}(x,t,\alpha)] = \mathscr{L}_t[\underline{f}(x,t,\alpha)], \\ \dfrac{\partial^2 \mathscr{L}_t}{\partial x^2}[\underline{u}(x,t,\alpha)] + a(x)s\dfrac{\partial \mathscr{L}_t}{\partial x}[\underline{u}(x,t,\alpha)] - a(x)\underline{g}'(x,\alpha) \\ + b(x)s^2\mathscr{L}_t[\underline{u}(x,t,\alpha)] - b(x)s\underline{g}(x,\alpha) - b(x)\underline{k}(x,\alpha) \\ + c_1(x)\dfrac{\partial \mathscr{L}_t}{\partial x}[\underline{u}(x,t,\alpha)] + c_2(x)s\mathscr{L}_t[\underline{u}_t(x,t,\alpha)] \\ - c_2(x)\underline{g}(x,\alpha) + d(x)\mathscr{L}_t[\underline{u}(x,t,\alpha)] = \mathscr{L}_t[\underline{f}(x,t,\alpha)]. \end{cases} \tag{23}$$

Denote $U(x,s) = \mathbf{L}_t[u(x,t)]$ and $F(x,s) = \mathbf{L}_t[f(x,t)]$. Then,

$$\begin{cases} \dfrac{\partial^2 \underline{U}}{\partial x^2}(x,s,\alpha) + (a(x)s + c_1(x))\dfrac{\partial \underline{U}}{\partial x}(x,s,\alpha) \\ +\left(b(x)s^2 + c_2(x)s + d(x)\right)\underline{U}(x,s,\alpha) = a(x)\overline{g}'(x,\alpha) \\ +\left(b(x)\overline{k}(x,\alpha) + b(x)s + c_2(x)\right)\overline{g}(x,\alpha) + \underline{F}(x,s,\alpha), \\ \dfrac{\partial^2 \overline{U}}{\partial x^2}(x,s,\alpha) + (a(x)s + c_1(x))\dfrac{\partial \overline{U}}{\partial x}(x,s,\alpha) \\ +\left(b(x)s^2 + c_2(x)s + d(x)\right)\overline{U}(x,s,\alpha) = a(x)\,\underline{g}'(x,\alpha) \\ +b(x)\,\underline{k}(x,\alpha) + (b(x)s + c_2(x))\,\underline{g}(x,\alpha) + \overline{F}(x,s,\alpha), \end{cases} \tag{24}$$

satisfying the following initial conditions:

$$\begin{cases} \underline{U}(0,s,\alpha) = \mathscr{L}[\underline{u}(0,t,\alpha)] = \mathscr{L}[\underline{h}(t,\alpha)], \\ \overline{U}(0,s,\alpha) = \mathscr{L}[\overline{u}(0,t,\alpha)] = \mathscr{L}[\overline{h}(t,\alpha)]. \end{cases} \tag{25}$$

Assume that this leads to

$$\begin{cases} \mathscr{L}[\underline{u}(x,t,\alpha)] = H_1(x,s,\alpha), \\ \mathscr{L}[\overline{u}(x,t,\alpha)] = K_1(x,s,\alpha), \end{cases} \tag{26}$$

where $(H_1(x,s,\alpha), K_1(x,s,\alpha))$ is the solution of system (24) under (25).

By the inverse Laplace transform, we obtain

$$\begin{cases} \underline{u}(x,t,\alpha) = \mathscr{L}^{-1}[H_1(x,s,\alpha)], \\ \overline{u}(x,t,\alpha) = \mathscr{L}^{-1}[K_1(x,s,\alpha)]. \end{cases} \tag{27}$$

(b) If u is (i)-differentiable with respect to x and t, u_x is (i)-differentiable with respect to t and (ii)-differentiable with respect to x, and u_t is (i)-differentiable with respect to t, then we obtain

$$\begin{cases} \dfrac{\partial^2 \mathscr{L}_t}{\partial x^2}[\overline{u}(x,t,\alpha)] + a(x)s\dfrac{\partial \mathscr{L}_t}{\partial x}\mathscr{L}_t[\overline{u}(x,t,\alpha)] - a(x)\overline{g}'(x,\alpha) \\ +b(x)s^2\mathscr{L}_t[\overline{u}(x,t,\alpha)] - b(x)\overline{k}(x,\alpha) - b(x)s\overline{g}(x,\alpha) \\ +c_1(x)\dfrac{\partial \mathscr{L}_t}{\partial x}[\overline{u}(x,t,\alpha)] + c_2(x)s\mathscr{L}_t[\underline{u}_t(x,t,\alpha)] - c_2(x)\overline{g}(x,\alpha) \\ +d(x)\mathscr{L}_t[\overline{u}(x,t,\alpha)] = \mathscr{L}_t\left[\underline{f}(x,t,\alpha)\right], \\ \dfrac{\partial^2 \mathscr{L}_t}{\partial x^2}[\overline{u}(x,t,\alpha)] + a(x)s\dfrac{\partial \mathscr{L}_t}{\partial x}[\overline{u}(x,t,\alpha)] - a(x)\,\underline{g}'(x,\alpha) \\ +b(x)s^2\mathscr{L}_t[\overline{u}(x,t,\alpha)] - b(x)\,\underline{k}(x,\alpha) - b(x)s\,\underline{g}(x,\alpha) \\ +c_1(x)\dfrac{\partial \mathscr{L}_t}{\partial x}[\overline{u}(x,t,\alpha)] + c_2(x)s\mathscr{L}_t[\underline{u}_t(x,t,\alpha)] - c_2(x)\,\underline{g}(x,\alpha) \\ +d(x)\mathscr{L}_t[\overline{u}(x,t,\alpha)] = \mathscr{L}_t\left[\underline{f}(x,t,\alpha)\right]. \end{cases} \tag{28}$$

Thus,

$$\begin{cases} \dfrac{\partial^2 \overline{U}(x,s,\alpha)}{\partial x^2} + (a(x)s + c_1(x)) \dfrac{\partial \underline{U}(x,s,\alpha)}{\partial x} + \left(b(x)s^2 + c_2(x)s + d(x)\right) \underline{U}(x,s,\alpha) = a(x)\overline{g}'(x,\alpha) + b(x)\overline{k}(x,\alpha) \\ + (b(x)s + c_2(x))\overline{g}(x,\alpha) + \underline{F}(x,s,\alpha), \\ \dfrac{\partial^2 \underline{U}(x,s,\alpha)}{\partial x^2} + (a(x)s + c_1(x)) \dfrac{\partial \overline{U}(x,s,\alpha)}{\partial x} + \left(b(x)s^2 + c_2(x)s + d(x)\right) \overline{U}(x,s,\alpha) = a(x)\,\underline{g}'(x,\alpha) + b(x)\,\underline{k}(x,\alpha) \\ + (b(x)s + c_2(x))\,\underline{g}(x,\alpha) + \overline{F}(x,s,\alpha). \end{cases} \quad (29)$$

Assume that this leads to

$$\begin{cases} \mathcal{L}[\underline{u}(x,t,\alpha)] = H_2(x,s,\alpha), \\ \mathcal{L}[\overline{u}(x,t,\alpha)] = K_2(x,t,\alpha), \end{cases} \quad (30)$$

where $(H_2(x,s,\alpha), K_2(x,s,\alpha))$ is the solution of system (29) under (25).

By the inverse Laplace transform, we get

$$\begin{cases} \underline{u}(x,s,\alpha) = \mathcal{L}^{-1}[H_2(x,s,\alpha)], \\ \overline{u}(x,s,\alpha) = \mathcal{L}^{-1}[K_2(x,s,\alpha)]. \end{cases} \quad (31)$$

(c) If u is (i)-differentiable with respect to x and (ii)-differentiable with respect to t, u_x is (i)-differentiable with respect to x and t, and u_t is (i)-differentiable with respect to t, then

$$\begin{cases} \dfrac{\partial^2 \underline{U}(x,s,\alpha)}{\partial x^2} + a(x)s \dfrac{\partial \overline{U}(x,s,\alpha)}{\partial x} + c_1(x) \dfrac{\partial \underline{U}(x,s,\alpha)}{\partial x} \\ + \left(b(x)s^2 + c_2(x)s + d(x)\right)\overline{U}(x,s,\alpha) = a(x)\overline{g}'(x,\alpha) \\ + b(x)\overline{k}(x,\alpha) + (b(x)s + c_2(x))\overline{g}(x,\alpha) + \underline{F}(x,s,\alpha), \\ \dfrac{\partial^2 \overline{U}(x,s,\alpha)}{\partial x^2} + a(x)s \dfrac{\partial \underline{U}(x,s,\alpha)}{\partial x} + c_1(x) \dfrac{\partial \overline{U}(x,s,\alpha)}{\partial x} \\ + \left(b(x)s^2 + c_2(x)s + d(x)\right)\underline{U}(x,s,\alpha) = a(x)\,\underline{g}'(x,\alpha) \\ + b(x)\,\underline{k}(x,\alpha) + (b(x)s + c_2(x))\,\underline{g}(x,\alpha) + \overline{F}(x,s,\alpha). \end{cases} \quad (32)$$

Assume that this leads to

$$\begin{cases} \mathcal{L}[\overline{u}(x,t,\alpha)] = H_3(x,s,\alpha), \\ \mathcal{L}[\overline{u}(x,t,\alpha)] = aK_3(x,s,\alpha), \end{cases} \quad (33)$$

where $(H_3(x,s,\alpha), K_3(x,s,\alpha))$ is the solution of system (32) under (25).

By the inverse Laplace transform, we obtain

$$\begin{cases} \underline{u}(x,t,\alpha) = \mathcal{L}^{-1}[H_3(x,s,\alpha)], \\ \overline{u}(x,t,\alpha) = \mathcal{L}^{-1}[K_3(x,s,\alpha)]. \end{cases} \quad (34)$$

(d) For the p-th case from the 29 remaining cases, with $p \in \{4, 5, \ldots, 32\}$, we get a differential system similar to one of the previous systems (24) and (29).

Assume that this leads to

$$\begin{cases} \mathcal{L}[\underline{u}(x,t,\alpha)] = H_p(x,s,\alpha), \\ \mathcal{L}[\overline{u}(x,t,\alpha)] = K_p(x,s,\alpha), \end{cases} \quad (35)$$

where $(H_p(x,s,\alpha), K_p(x,s,\alpha))$ is the solution of the latter system under initial condition (25).

By the inverse Laplace transform, we have

$$\begin{cases} \underline{u}(x,t,\alpha) = \mathcal{L}^{-1}\left[H_p(x,s,\alpha)\right], \\ \overline{u}(x,t,\alpha) = \mathcal{L}^{-1}\left[K_p(x,s,\alpha)\right]. \end{cases} \quad (36)$$

4.2. Algorithm of Fuzzy Laplace Transform. The steps of the proposed algorithm are as follows:

(i) Choose a case from the 32 possible ones according to the differentiability's type of each from the functions u, u_x, u_x, and u_t with respect to x and t, respectively.

(ii) Replace these functions by their parametric forms to transform equation (21) into an equivalent classical differential system of two linear equations with unknown $\underline{u}(x,t,\alpha), \overline{u}(x,t,\alpha)$, for $0 \le \alpha \le 1$.

(iii) Solve this differential system under the given initial and boundary conditions.

(iv) Calculate the length of the mappings $u, u_x, u_x, u_t, u_{xx}, u_{xt}$, and u_{tt}.

(v) Deduce the domain of definition for the solution $u(x,t,\alpha)$ using the nonnegativity of calculated lengths.

5. Numerical Examples

Example 1. We consider the heat equation with fuzzy initial and boundary conditions:

$$\begin{cases} u_t(x,t) = a u_{xx}(x,t), \\ u(0,t,\alpha) = u(\pi,t,\alpha) = (0,0), \alpha \in [0,1], \\ u(x,0,\alpha) = \sin x.(\alpha, 2-\alpha), \quad x \ge 0, t \ge 0, \end{cases} \quad (37)$$

where a is a positive real number. Here, we have to distinguish only 8 cases.

(a) If u is (i)-differentiable with respect to x and t and u_x is (i)-differentiable with respect to x, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] - \underline{u}(x,0,\alpha) = a\dfrac{\partial^2(\mathscr{L}_t[\underline{u}(x,t,\alpha)])}{\partial x^2}, \\ s\mathscr{L}_t[\underline{u}(x,t,\alpha)] - \overline{u}(x,0,\alpha) = a\dfrac{\partial^2(\mathscr{L}_t[\underline{u}(x,t,\alpha)])}{\partial x^2}. \end{cases} \quad (38)$$

Using the conditions $\underline{u}(x,0,\alpha) = \alpha \sin x, \overline{u}(x,0,\alpha) = (2-\alpha)\sin x$ leads to

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] = a\dfrac{\partial^2}{\partial x^2}(\mathscr{L}_t[\underline{u}(x,t,\alpha)]) + \alpha \sin x, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] = a\dfrac{\partial^2}{\partial x^2}(\mathscr{L}_t[\overline{u}(x,t,\alpha)]) + (2-\alpha)\sin x. \end{cases} \quad (39)$$

Solving (39), we get

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_1(s)e^{-\sqrt{s/a}\,x} + B_1(s)e^{\sqrt{s/a}\,x} + \dfrac{\alpha \sin x}{s+a}, \\ \mathscr{L}_t[\underline{u}(x,t,\alpha)] = C_1(s)e^{-\sqrt{s/a}\,x} + D_1(s)e^{\sqrt{s/a}\,x} + \dfrac{(2-\alpha)\sin x}{s+a}. \end{cases} \quad (40)$$

Since $\mathscr{L}_t[\underline{u}(0,t,\alpha)] = \mathscr{L}_t[\overline{u}(\pi,t,\alpha)] = \mathscr{L}_t[\underline{u}(\pi,t,\alpha)] = \mathscr{L}_t[\overline{u}(0,t,\alpha)] = 0$, then $A_1(s) = B_1(s) = C_1(s) = D_1(s) = 0$. Thus,

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = \dfrac{\alpha \sin x}{s+a}, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = \dfrac{(2-\alpha)\sin x}{s+a}. \end{cases} \quad (41)$$

By the inverse Laplace transform, we deduce

$$\begin{cases} \underline{u}(x,t,\alpha) = \alpha e^{-at}\sin x, \\ \overline{u}(x,t,\alpha) = (2-\alpha)e^{-at}\sin x. \end{cases} \quad (42)$$

The lengths of u, u_x, u_t, and u_{xx} are, respectively, given by

$$\begin{cases} \operatorname{len}(u(x,t,\alpha)) = 2(1-\alpha)e^{-at}\sin x, \\ \operatorname{len}(u_x(x,t,\alpha)) = 2(1-\alpha)e^{-at}\cos x, \\ \operatorname{len}(u_t(x,t,\alpha)) = -2a(1-\alpha)e^{-at}\sin x, \\ \operatorname{len}(u_{xx}(x,t,\alpha)) = -2(1-\alpha)e^{-at}\sin x. \end{cases} \quad (43)$$

Hence, this solution is invalid because $\operatorname{len}(u(x,t,\alpha)) \times \operatorname{len}(u_{xx}(x,t,\alpha)) \le 0$.

(b) If u is (i)-differentiable with respect to x and t and u_x is (ii)-differentiable with respect to x, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] - \alpha \sin x = a\dfrac{\partial^2(\mathscr{L}_t[\overline{u}(x,t,\alpha)])}{\partial x^2}, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] - (2-\alpha)\sin x = a\dfrac{\partial^2(\mathscr{L}_t[\underline{u}(x,t,\alpha)])}{\partial x^2}. \end{cases} \quad (44)$$

Solving (44), we get

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_2(s)e^{-\sqrt{\frac{s}{a}}x} + B_2(s)e^{\sqrt{\frac{s}{a}}x} \\ \quad + C_2(s)\cos(\sqrt{s/a}\,x) + D_2(s)\sin(\sqrt{s/a}\,x) \\ \quad + \dfrac{(2s - s\alpha - a\alpha)}{s^2-a^2}\sin x, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = A_2(s)e^{-\sqrt{s/a}\,x} + B_2(s)e^{\sqrt{s/a}\,x} \\ \quad - C_2(s)\cos(\sqrt{s/a}\,x) - D_2(s)\sin(\sqrt{s/a}\,x) \\ \quad + \dfrac{(s\alpha - 2a + a\alpha)}{s^2-a^2}\sin x. \end{cases} \quad (45)$$

Since $\mathscr{L}_t[\underline{u}(0,t,\alpha)] = \mathscr{L}_t[\overline{u}(0,t,\alpha)] = \mathscr{L}_t[\underline{u}(\pi,t,\alpha)] = \mathscr{L}_t[\overline{u}(\pi,t,\alpha)] = 0$, then $A_2(s) = B_2(s) = C_2(s) = D_2(s) = 0$. Hence,

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = \dfrac{(s\alpha - 2a + a\alpha)}{s^2-a^2}\sin x, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = \dfrac{(2s - s\alpha - a\alpha)}{s^2-a^2}\sin x. \end{cases} \quad (46)$$

By the inverse Laplace transform, we obtain

$$\begin{cases} \underline{u}(x,t,\alpha) = \left[e^{-at} + (\alpha-1)e^{at}\right]\sin x, \\ \overline{u}(x,t,\alpha) = \left[e^{-at} + (1-\alpha)e^{at}\right]\sin x. \end{cases} \quad (47)$$

The lengths of u, u_x, u_t, and u_{xx} are, respectively, given by

$$\begin{cases} \operatorname{len}(u(x,t,\alpha)) = 2(1-\alpha)e^{at}\sin x, \\ \operatorname{len}(u_x(x,t,\alpha)) = 2(1-\alpha)e^{at}\cos x, \\ \operatorname{len}(u_t(x,t,\alpha)) = 2a(1-\alpha)e^{at}\sin x, \\ \operatorname{len}(u_{xx}(x,t,\alpha)) = 2(1-\alpha)e^{at}\sin x. \end{cases} \quad (48)$$

So, this solution (called solution 1) is valid over $[2k\pi, \pi/2 + 2k\pi] \times \mathbb{R}^+$, where $k \in \mathbb{Z}$ (see Figures 1–4).

(c) If u and u_x are (i)-differentiable with respect to x and u is (ii)-differentiable with respect to t, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] = a\dfrac{\partial^2(\mathscr{L}_t[\overline{u}(x,t,\alpha)])}{\partial x^2} + \alpha \sin x, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] = \dfrac{x^2}{2}\dfrac{\partial^2(\mathscr{L}_t[\underline{u}(x,t,\alpha)])}{\partial x^2} + (2-\alpha)\sin x. \end{cases} \tag{49}$$

Solving (49), we obtain

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_3(s)e^{-\sqrt{s/a}\,x} + B_3(s)e^{\sqrt{s/a}\,x} \\ + C_3(s)\cos(\sqrt{s/a}\,x) + D_3(s)\sin(\sqrt{s/a}\,x) \\ + \dfrac{(s\alpha - 2a + a\alpha)}{s^2 - a^2}\sin x, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = A_3(s)e^{-\sqrt{s/a}\,x} + B_3(s)e^{\sqrt{s/a}\,x} \\ - C_3(s)\cos(\sqrt{s/a}\,x) - D_3(s)\sin(\sqrt{s/a}\,x) \\ + \dfrac{(2s - s\alpha - a\alpha)}{s^2 - a^2}\sin x. \end{cases} \tag{50}$$

Similarly and as in Case 2, we obtain $A_3(s) = B_3(s) = C_3(s) = D_3(s) = 0$. So,

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = \dfrac{(s\alpha - 2a + a\alpha)}{s^2 - a^2}\sin x, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = \dfrac{(2s - s\alpha + \alpha)}{s^2 - a^2}\sin x. \end{cases} \tag{51}$$

By the inverse Laplace transform, we deduce

$$\begin{cases} \underline{u}(x,t,\alpha) = \left[e^{-at} + (\alpha-1)e^{at}\right]\sin x, \\ \overline{u}(x,t,\alpha) = \left[e^{-at} + (1-\alpha)e^{at}\right]\sin x. \end{cases} \tag{52}$$

Hence, this solution is invalid because $\operatorname{len}(u(x,t,\alpha)) \times \operatorname{len}(u_{xx}(x,t,\alpha)) \le 0$.

(d) If u is (i)-differentiable and u_x is (ii)-differentiable with respect to x and u is (ii)-differentiable with respect to t, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] = \dfrac{a\partial^2(\mathscr{L}_t[\underline{u}(x,t,\alpha)])}{\partial x^2} + \alpha \sin x, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] = \dfrac{a\partial^2(\mathscr{L}_t[\overline{u}(x,t,\alpha)])}{\partial x^2} + (2-\alpha)\sin x. \end{cases} \tag{53}$$

Solving (53), we get

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_4(s)e^{-\sqrt{s/a}\,x} + B_4(s)e^{\sqrt{s/a}\,x} + \dfrac{\alpha \sin x}{s+a}, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = C_4(s)e^{-\sqrt{s/a}\,x} + D_4(s)e^{\sqrt{s/a}\,x} + \dfrac{(2-\alpha)\sin x}{s+a}. \end{cases} \tag{54}$$

As in Case 1, we get $A_4(s) = B_4(s) = C_4(s) = D_4(s) = 0$.

Also, by the inverse Laplace transform,

$$\begin{cases} \underline{u}(x,t,\alpha) = \alpha e^{-at}\sin x, \\ \overline{u}(x,t,\alpha) = (2-\alpha)e^{-at}\sin x. \end{cases} \tag{55}$$

Thus, this solution (called solution 2) is valid over $[2k\pi, \pi/2 + 2k\pi]$, where $k \in \mathbb{Z}$ (see Figures 5–8).

(e) If u is (ii)-differentiable and u_x is (i)-differentiable with respect to x and is u (i)-differentiable with respect to t, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] = a\dfrac{\partial^2}{\partial x^2}(\mathscr{L}_t[\overline{u}(x,t,\alpha)]) + \alpha \sin x, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] = a\dfrac{\partial^2}{\partial x^2}(\mathscr{L}_t[\underline{u}(x,t,\alpha)]) + (2-\alpha)\sin x. \end{cases} \tag{56}$$

Solving (56), we get

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_5(s)e^{-\sqrt{s/a}\,x} + B_5(s)e^{\sqrt{s/a}\,x} \\ + C_5(s)\cos(\sqrt{s/a}\,x) + D_5(s)\sin(\sqrt{s/a}\,x) \\ + \dfrac{(s\alpha - 2a + a\alpha)}{s^2 - a^2}\sin x, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = A_5(s)e^{-\sqrt{s/a}\,x} + B_5(s)e^{\sqrt{s/a}\,x} \\ - C_5(s)\cos(\sqrt{s/a}\,x) - D_5(s)\sin(\sqrt{s/a}\,x) \\ + \dfrac{(2s - s\alpha - a\alpha)}{s^2 - a^2}\sin x. \end{cases} \tag{57}$$

As in Case 2, we get $A_5(s) = B_5(s) = C_5(s) = D_5(s) = 0$.

By the inverse Laplace transform,

$$\begin{cases} \underline{u}(x,t,\alpha) = \left[e^{-at} + (\alpha-1)e^{at}\right]\sin x, \\ \overline{u}(x,t,\alpha) = \left[e^{-at} + (1-\alpha)e^{at}\right]\sin x. \end{cases} \tag{58}$$

Thus, this solution (called solution 2) is valid over $[\pi/2 + 2k\pi, \pi + 2k\pi]$, where $k \in \mathbb{Z}$ (see Figures 5–8).

(f) If u and u_x are (ii)-differentiable with respect to x and u is (i)-differentiable with respect to t, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] = a\dfrac{\partial^2}{\partial x^2}(\mathscr{L}_t[\underline{u}(x,t,\alpha)]) + \alpha \sin x, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] = a\dfrac{\partial^2}{\partial x^2}(\mathscr{L}_t[\overline{u}(x,t,\alpha)]) + (2-\alpha)\sin x. \end{cases} \tag{59}$$

Solving (59), we get

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_6(s)e^{-\sqrt{s/a}\,x} + B_6(s)e^{\sqrt{s/a}\,x} + \dfrac{\alpha \sin x}{s+a}, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = C_6(s)e^{-\sqrt{s/a}\,x} + D_6(s)e^{\sqrt{s/a}\,x}. \\ +\dfrac{(2-\alpha)\sin x}{s+a} \end{cases} \tag{60}$$

As in Case 1, we have $A_6(s) = B_6(s) = C_6(s) = D_6(s) = 0$. Also, by the inverse Laplace transform,

$$\begin{cases} \underline{u}(x,t,\alpha) = \alpha e^{-at} \sin x, \\ \overline{u}(x,t,\alpha) = (2-\alpha)e^{-at} \sin x. \end{cases} \tag{61}$$

Therefore, this solution is invalid because $\operatorname{len}(u(x,t,\alpha)) \times \operatorname{len}(u_{xx}(x,t,\alpha)) \le 0$.

(g) If u is (ii)-differentiable and u_x is (i)-differentiable with respect to x and u is (ii)-differentiable with respect to t, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] = \dfrac{a\partial^2(\mathscr{L}_t[\underline{u}(x,t,\alpha)])}{\partial x^2} + \alpha \sin x, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] = \dfrac{a\partial^2(\mathscr{L}_t[\overline{u}(x,t,\alpha)])}{\partial x^2} + (2-\alpha)\sin x. \end{cases} \tag{62}$$

Solving (62), we get

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_7(s)e^{-\sqrt{s/a}\,x} + B_7(s)e^{\sqrt{s/a}\,x} \\ +\dfrac{\alpha \sin x}{s+a}, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = C_7(s)e^{-\sqrt{s/a}\,x} + D_7(s)e^{\sqrt{s/a}\,x} \\ +\dfrac{(2-\alpha)\sin x}{s+a}. \end{cases} \tag{63}$$

As in Case 1, we get $A_7(s) = B_7(s) = C_7(s) = D_7(s) = 0$.

Also, by the inverse Laplace transform,

$$\begin{cases} \underline{u}(x,t,\alpha) = \alpha e^{-at} \sin x, \\ \overline{u}(x,t,\alpha) = (2-\alpha)e^{-at} \sin x. \end{cases} \tag{64}$$

Hence, this solution (called solution 2) is valid over $[\pi/2 + 2k\pi, \pi + 2k\pi]$, where $k \in \mathbb{Z}$ (see Figures 5–8).

(h) If u and u_x are (ii)-differentiable with respect to x and u is (ii)-differentiable with respect to t, then

$$\begin{cases} s\mathscr{L}_t[\underline{u}(x,t,\alpha)] = \dfrac{a\partial^2(\mathscr{L}_t[\overline{u}(x,t,\alpha)])}{\partial x^2} + \alpha \sin x, \\ s\mathscr{L}_t[\overline{u}(x,t,\alpha)] = \dfrac{a\partial^2(\mathscr{L}_t[\underline{u}(x,t,\alpha)])}{\partial x^2} + (2-\alpha)\sin x. \end{cases} \tag{65}$$

Solving (65), we get

$$\begin{cases} \mathscr{L}_t[\underline{u}(x,t,\alpha)] = A_8(s)e^{-\sqrt{s/a}\,x} + B_8(s)e^{\sqrt{s/a}\,x} \\ +C_8(s)\cos(\sqrt{s/a}\,x) + D_8(s)\sin(\sqrt{s/a}\,x) \\ +\dfrac{(s\alpha - 2a + a\alpha)}{s^2 - a^2}\sin x, \\ \mathscr{L}_t[\overline{u}(x,t,\alpha)] = A_8(s)e^{-\sqrt{s/a}\,x} + B_8(s)e^{\sqrt{s/a}\,x} \\ -C_8(s)\cos(\sqrt{s/a}\,x) - D_8(s)\sin(\sqrt{s/a}\,x) \\ +\dfrac{(2s - s\alpha - a\alpha)}{s^2 - a^2}\sin x. \end{cases} \tag{66}$$

As in Case 2, we get $A_8(s) = B_8(s) = C_8(s) = D_8(s) = 0$.

Also, by the inverse Laplace transform,

$$\begin{cases} \underline{u}(x,t,\alpha) = \left[e^{-at} + (\alpha-1)e^{at}\right]\sin x, \\ \overline{u}(x,t,\alpha) = \left[e^{-at} + (1-\alpha)e^{at}\right]\sin x. \end{cases} \tag{67}$$

Therefore, this solution is invalid because $\operatorname{len}(u(x,t,\alpha)) \times \operatorname{len}(u_{xx}(x,t,\alpha)) \le 0$.

Remark 1. Notice that in all cases, if we take $\alpha = 1$, we find the crisp solution $u(x,t) = e^{-at}\sin x$ of the corresponding classical heat equation

$$\begin{cases} u_t(x,t) = au_{xx}(x,t), x \ge 0, t \ge 0, \\ u(0,t) = u(\pi,t) = 0, u(x,0) = \sin x. \end{cases} \tag{68}$$

For the graph of the crisp solution, see Figures9 and 10.

Example 2. We consider the following FPDE:

$$\begin{cases} u_{xt}(x,t) = \widetilde{0} = (\alpha-1, 1-\alpha), \\ u(0,t,\alpha) = t^2, \ \alpha \in [0,1], \\ u(x,0,\alpha) = x^2, \quad x \ge 0, t \ge 0. \end{cases} \tag{69}$$

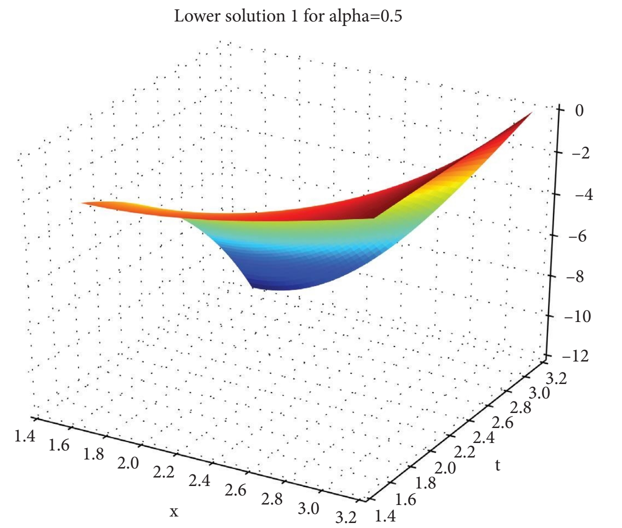

FIGURE 1: Lower part $\underline{u}(x,t,\alpha)$ of solution 1 for equation (37), with $\alpha = 0.5$.

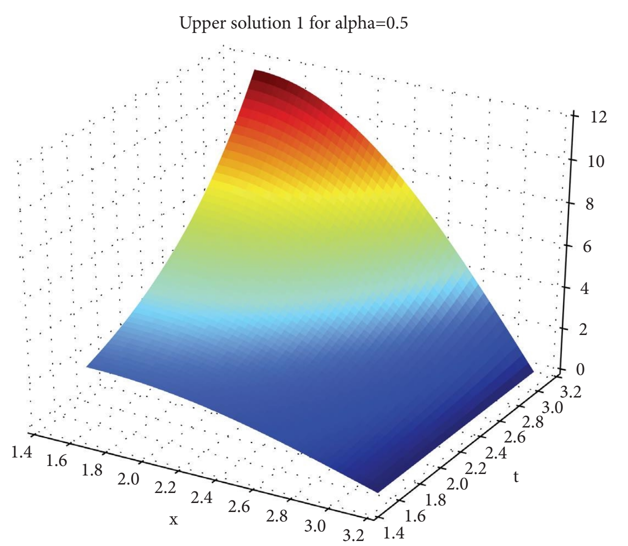

FIGURE 2: Upper part $\overline{u}(x,t,\alpha)$ of solution 1 for equation (37), with $\alpha = 0.5$.

Analogously, we obtain the solution's expression by distinguishing the following cases:

(a) If u (i)-differentiable with respect to t and u_t is (i)-differentiable with respect to x, then

$$\begin{cases} \underline{u}(x,t,\alpha) = (\alpha-1)xt + x^2 + t^2, \\ \overline{u}(x,t,\alpha) = (1-\alpha)xt + x^2 + t^2. \end{cases} \tag{70}$$

Therefore, this solution (called solution 3) is valid all over $(\mathbb{R}^+)^2$ (see Figures 11–14).

(b) If u (i)-differentiable with respect to t and u_t is (ii)-differentiable with respect to x, then

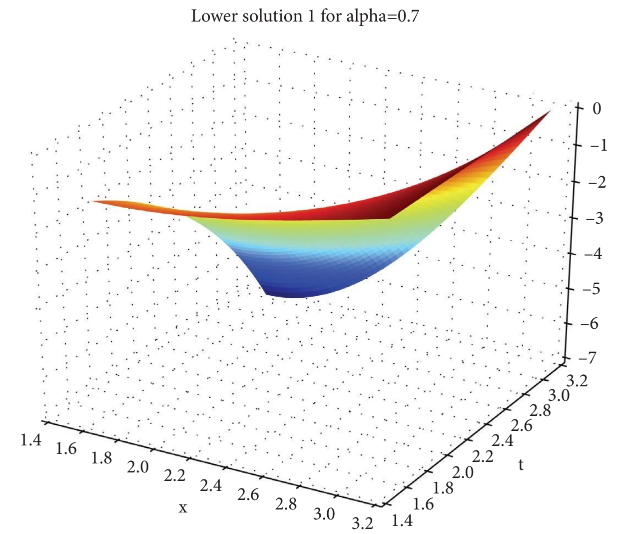

FIGURE 3: Lower part $\underline{u}(x,t,\alpha)$ of solution 1 for equation (37), with $\alpha = 0.7$.

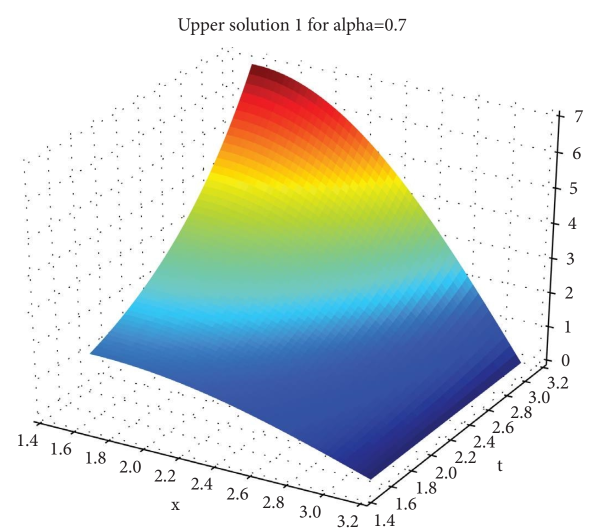

FIGURE 4: Upper part $\overline{u}(x,t,\alpha)$ of solution 1 for equation (37), with $\alpha = 0.7$.

$$\begin{cases} \underline{u}(x,t,\alpha) = (1-\alpha)xt + x^2 + t^2, \\ \overline{u}(x,t,\alpha) = (\alpha-1)xt + x^2 + t^2. \end{cases} \tag{71}$$

Thus, this solution is invalid because $len(u) \le 0$.

(c) If u (ii)-differentiable with respect to t and u_t is (i)-differentiable with respect to x, then

$$\begin{cases} \underline{u}(x,t,\alpha) = (1-\alpha)xt + x^2 + t^2, \\ \overline{u}(x,t,\alpha) = (\alpha-1)xt + x^2 + t^2. \end{cases} \tag{72}$$

Also, this solution is invalid because $\mathrm{len}(u) \le 0$.

(d) If u (ii)-differentiable with respect to t and u_t is (ii)-differentiable with respect to x, then

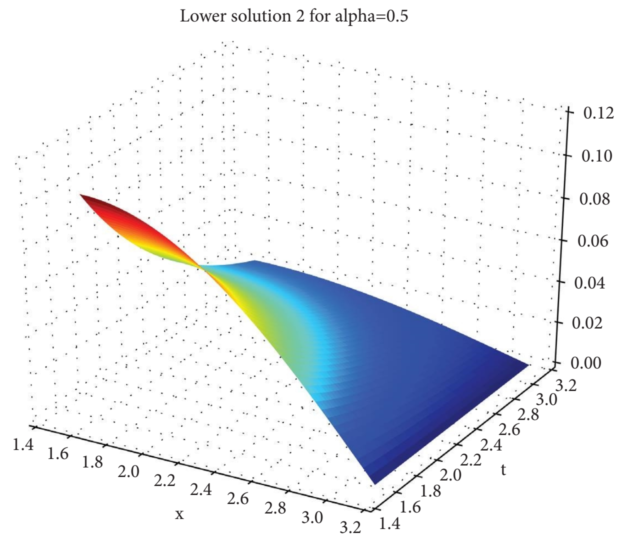

Figure 5: Lower part $\underline{u}(x,t,\alpha)$ of solution 2 for equation (37), with $\alpha = 0.5$.

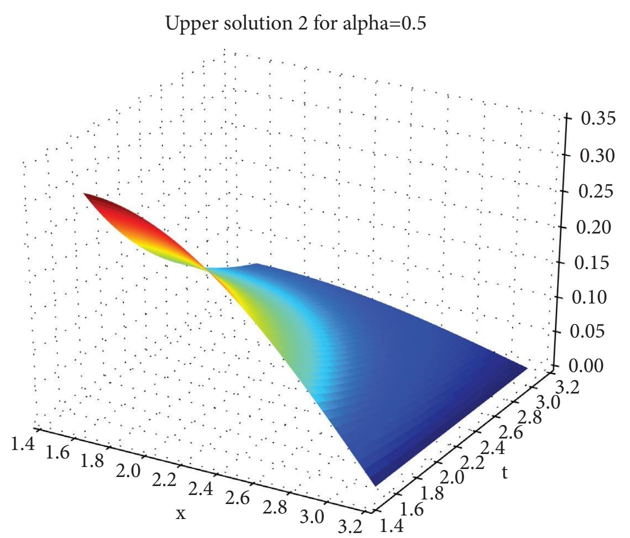

Figure 6: Upper part $\overline{u}(x,t,\alpha)$ of solution 2 for equation (37), with $\alpha = 0.5$.

$$\begin{cases} \underline{u}(x,t,\alpha) = (\alpha - 1)xt + x^2 + t^2, \\ \overline{u}(x,t,\alpha) = (1-\alpha)xt + x^2 + t^2. \end{cases} \quad (73)$$

So, this solution is invalid because u is not (ii)-differentiable with respect to t.

Remark 2. Notice that in all cases, if we take $\alpha = 1$, we find the crisp solution $u(x,t) = x^2 + t^2$ of the corresponding classical problem

$$\begin{cases} u_{xt}(x,t) = 0, \\ u(0,t) = t^2, u(x,0) = x^2, x \geq 0, t \geq 0. \end{cases} \quad (74)$$

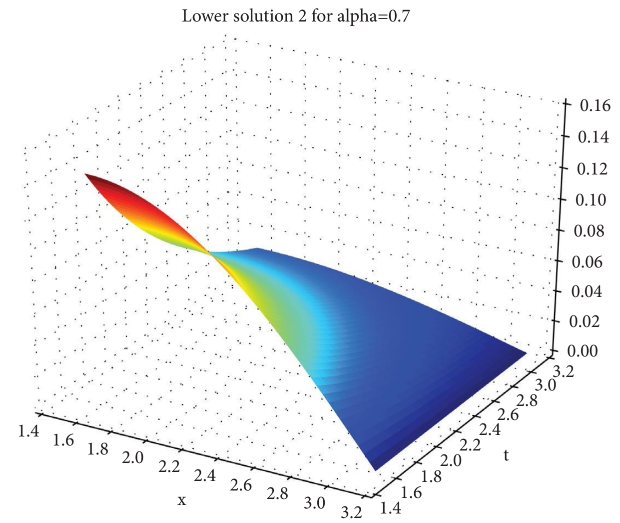

Figure 7: Lower part $\underline{u}(x,t,\alpha)$ of solution 2 for equation (37), with $\alpha = 0.7$.

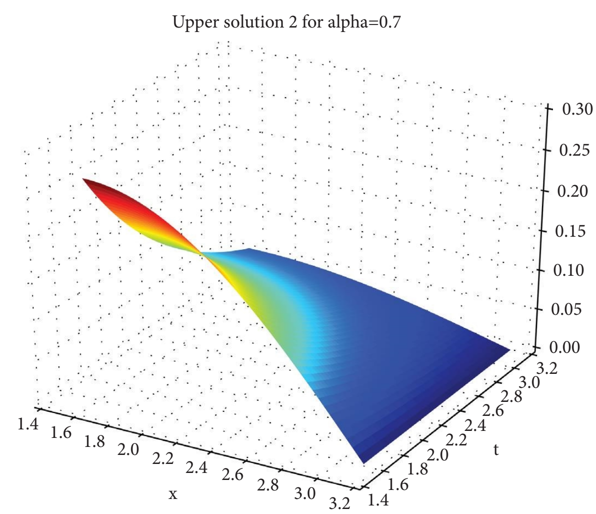

Figure 8: Upper part $\overline{u}(x,t,\alpha)$ of solution 2 for equation (37), with $\alpha = 0.7$.

6. Discussion

The Laplace transform method was used to compute the analytic solution $u(x,t,\alpha)$ of two linear FPDEs of second order. First, we decomposed each FPDE into a system of two crisp PDEs with the unknown $(\underline{u}(x,t,\alpha), \overline{u}(x,t,\alpha))$, for $0 \leq \alpha \leq 1$, for which we calculated the Laplace transforms using the properties proved in Section 3. Then, using the inverse Laplace transform, we obtained the lower and upper solution's parts $\underline{u}(x,t,\alpha), \overline{u}(x,t,\alpha)$, respectively. Finally, we determined the definition's domain of these fuzzy solutions by utilizing the positivity of the length for $u(x,t,\alpha)$ and its partial derivatives $u_x(x,t,\alpha), u_t(x,t,\alpha)$, $u_{xt}(x,t,\alpha), u_{xx}(x,t,\alpha)$, and $u_{tt}(x,t,\alpha)$.

In both numerical examples studied, the fuzzy solution $u(x,t,\alpha)$ of the second-order FPDE can be expressed as follows:

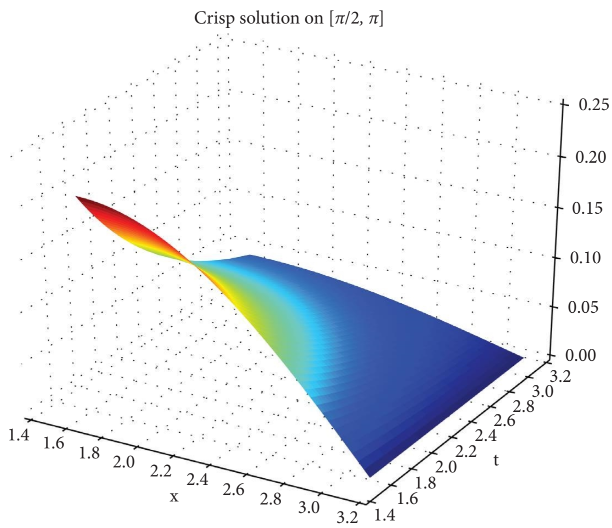

FIGURE 9: Crisp solution $u_c(x,t)$ over $[\pi/2, \pi]$ for equation (37).

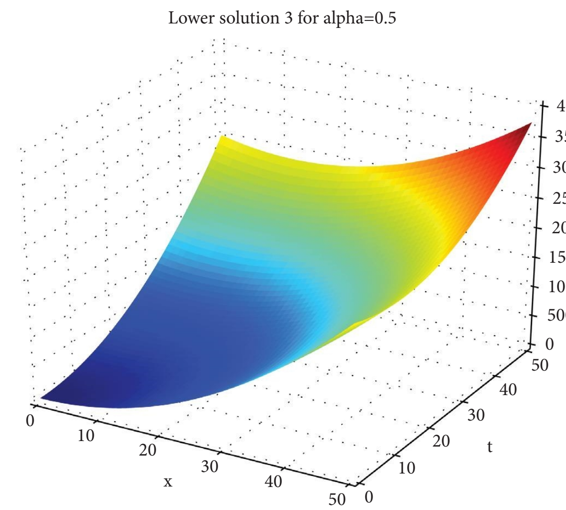

FIGURE 11: Lower part $\underline{u}(x,t,\alpha)$ of solution 3 for equation (69), with $\alpha = 0.5$.

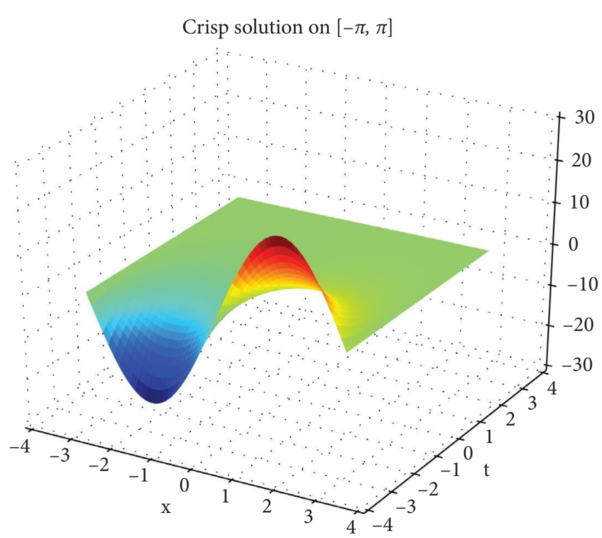

FIGURE 10: Crisp solution $u_c(x,t)$ over $[-\pi, \pi]$ for equation (37).

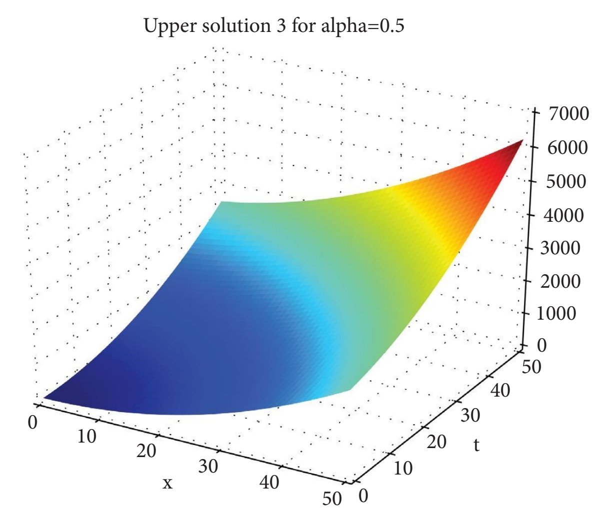

FIGURE 12: Upper part $\overline{u}(x,t,\alpha)$ of solution 3 for equation (69), with $\alpha = 0.5$.

$$u(x,t,\alpha) = u_c(x,t) + u_F(x,t,\alpha), \qquad (75)$$

where $u_c(x,t)$ is the crisp solution of the corresponding classical second-order PDE, obtained by letting $\alpha = 1$, and $u_F(x,t,\alpha)$ is an undesirable term, which represents the fuzzy pure part of the fuzzy solution $u(x,t,\alpha)$.

This fuzzy pure component $u_F(x,t,\alpha)$ results from the modeling choices and steps using fuzzy tools and theory. It also measures the uncertainty and vagueness in the adopted model due to the imprecisions in the initial and boundary conditions or in the fuzzy (respectively, real) second member of the FPDE (respectively, PDE).

On the one hand, we get for Example 1:

$$\begin{cases} \underline{u}(x,t,\alpha) = e^{-at}\sin x + (\alpha - 1)e^{at}\sin x, \\ \overline{u}(x,t,\alpha) = e^{-at}\sin x + (1-\alpha)e^{at}\sin x, \end{cases} \qquad (76)$$

that is,

$$u(x,t,\alpha) = e^{-at}\sin x + e^{at}\sin x.(\alpha - 1, 1-\alpha), \qquad (77)$$

for solution 1 of (37), given in its parametric form. Hence, we have

$$\begin{cases} u_c(x,t) = e^{-at}\sin x, \\ u_F(x,t,\alpha) = e^{at}\sin x.(\alpha - 1, 1-\alpha). \end{cases} \qquad (78)$$

On the other hand, we obtain

$$\begin{cases} \underline{u}(x,t,\alpha) = e^{-at}\sin x + (\alpha - 1)e^{-at}\sin x, \\ \overline{u}(x,t,\alpha) = e^{-at}\sin x + (1-\alpha)e^{-at}\sin x, \end{cases} \qquad (79)$$

that is, $u(x,t,\alpha) = e^{-at}\sin x + e^{-at}\sin x.(\alpha - 1, 1-\alpha)$, for solution 2 of (37), written in its parametric form. So, we have

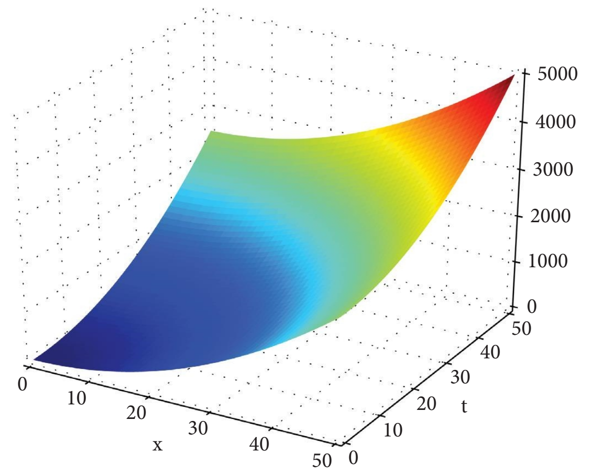

FIGURE 13: Lower part $\underline{u}(x,t,\alpha)$ of solution 3 for equation (69), with $\alpha = 0.999999$.

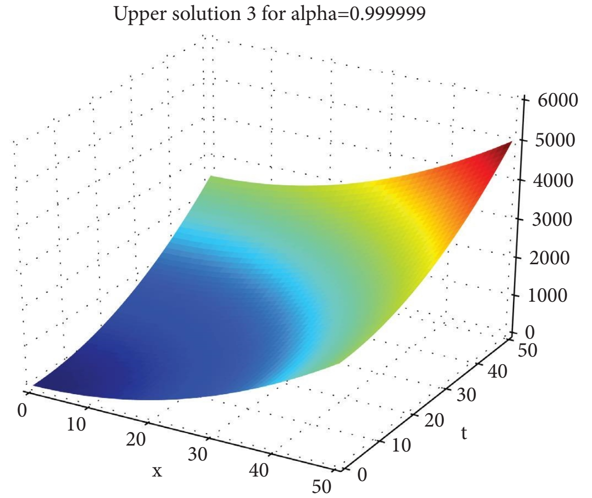

FIGURE 14: Upper part $\overline{u}(x,t,\alpha)$ of solution 3 for equation (69), with $\alpha = 0.999999$.

$$\begin{cases} u_c(x,t) = e^{-at}\sin x, \\ u_F(x,t,\alpha) = e^{-at}\sin x.(\alpha-1, 1-\alpha). \end{cases} \tag{80}$$

The presence of the pure fuzzy parts in these two solutions of the fuzzy heat equation can be explained by some uncontrollable parameters or omitted inputs in the modeling of the crisp problem. So, the fuzzy approach is better and more efficient than the ordinary classical way.

Furthermore, we have for Example 2:

$$\begin{cases} \underline{u}(x,t,\alpha) = x^2 + t^2 + (\alpha-1)xt, \\ \overline{u}(x,t,\alpha) = x^2 + t^2 + (1-\alpha)xt, \end{cases} \tag{81}$$

that is, $u(x,t,\alpha) = x^2 + t^2 + xt.(\alpha-1, 1-\alpha)$, for solution 3 of (69), given in its parametric form. Then, we have

$$\begin{cases} u_c(x,t) = x^2 + t^2, \\ u_F(x,t,\alpha) = (\alpha-1)xt. \end{cases} \tag{82}$$

In (37), the length of the fuzzy pure part $\text{len}(u_F(x,t,\alpha)) = 2(1-\alpha)e^{at}\sin x$ goes to infinity as $t \longrightarrow \infty$ for solution 2, while the length $\text{len}(u_F(x,t,\alpha)) = 2(1-\alpha)e^{-at}\sin x$ converges to 0 as $t \longrightarrow \infty$ for the fuzzy pure part of solution 1. Thus, the first solution is stable, whereas the second solution is unstable. In other words, solution 2 is most uncertain or vague than solution 1 for Example 1.

In Example 2, the crisp part of the solution 3 is $u_c(x,t) = x^2 + t^2$ and its fuzzy pure part is $u_F(x,t,\alpha) = xt.(\alpha-1, 1-\alpha)$, for which the length $\text{len}(u_F(x,t,\alpha)) = 2(1-\alpha)xt$ diverges to the infinity as $x \longrightarrow \infty$ (respectively, $t \longrightarrow \infty$). Hence, the uncertainty is increasing as the value of x or t increases, and this solution is unstable.

In general, as α approaches 1, the fuzziness and uncertainty become smaller and completely disappear, for $\alpha = 1$, yielding the crisp solution of the classic real problem.

Moreover, note that the existence, form, and asymptotic behavior of the solution depend on the choice of the kind of each used fuzzy partial derivative. Indeed, solution 2 of Example 1 is unstable, and it is obtained if we assume that u is (i)-differentiable with respect to x and it is (ii)-differentiable with respect to t and u_x is (ii)-differentiable with respect to x (see the fourth case (d)), while solution 1 is stable and is obtained provided that u is (i)-differentiable with respect to x and t and u_x is (ii)-differentiable with respect to x (see the second case (b)). But, in the other cases, there is no valid fuzzy solution.

In Example 2, we note that solution 3 exists and is unstable, and it is valid only if we assume that u is (i)-differentiable with respect to t and u_t is (ii)-differentiable with respect to x.

Furthermore, the uniqueness of the solution is lost in Example 1 (we have two (solutions 1 and 2)), although this unicity is preserved in Example 2.

Consequently, the study of these examples demonstrates that the type of fuzzy partial derivatives used influences the existence, uniqueness, and stability of the fuzzy solution(s) for a second-order FPDE under the strong generalized differentiability assumption.

Now, we proceed to the graphic interpretation of Figures 5–8. Firstly, in the solution 2 of (37), the lower and upper parts of this solution are proportional to the crisp solution of the corresponding classic equation, for all $\alpha \in [0,1]$. Indeed, we have

$$\underline{u}(x,t,\alpha) = (\alpha-1)u_c(x,t), \overline{u}(x,t,\alpha) = (1-\alpha)u_c(x,t). \tag{83}$$

So, all the obtained graphs are in fact images of the crisp solution's graph (see Figures 9 and 10) by the dilation of ratio equal to $\alpha-1$ for the graph of $\underline{u}(x,t,\alpha)$ and equal to $1-\alpha$ for the graph of $\overline{u}(x,t,\alpha)$.

Secondly, for each valid fuzzy solution (1, 2, and 3), in both examples, the maximum value of the upper solution $\overline{u}(x,t,\alpha)$ is greater than the maximum value of the lower solution $\underline{u}(x,t,\alpha)$. For instance, we get the following results for solution 2, $a = 1$, and $\alpha = 0.5$:

$$\max \overline{u}(x,t,\alpha) \simeq 0.35;\ \max\ \underline{u}(x,t,\alpha) \simeq 0.9, \tag{84}$$

and for solution 1 and $\alpha = 0.5$, we have

$$\max \overline{u}(x,t,\ \alpha) \simeq 11;\ \max\ \underline{u}(x,t,\alpha) = 0, \tag{85}$$

while we get for solution 3 and $\alpha = 0.5$:

$$\max \overline{u}(x,t,\alpha) \simeq 3500;\ \max\ \underline{u}(x,t,\alpha) \simeq 6000. \tag{86}$$

Finally, the graphs of the solutions $\underline{u}(x,t,\alpha)$ and $\overline{u}(x,t,\alpha)$ are almost equal and coincide with the crisp solution's graph for α close to 1, as it is shown in the graph of solution 3 for $\alpha = 0.999999$ (see Figures 13 and 14).

7. Conclusions

Theorems of high differentiability for a fuzzy function defined via a fuzzy improper integral have been investigated and proved, which have been employed to prove some results related to the partial derivatives of the fuzzy Laplace transform. Then, using the Laplace transform method, the solutions for linear FPDEs of second order have been given. For future research, one can apply this method to solve nonlinear FPDEs of first or high order. The influence of the choice of the kind of the fuzzy partial derivatives on the fuzzy solutions and their existence, uniqueness, and asymptotic behavior have been discussed.

Data Availability

The graph data used to support the findings of this study are included within the supplementary information file. This file contains programs developed by using the free Python software to plot each of the fourteen figures in the manuscript. For more information and documentation about Python, the readers can consult the website https://www.python.org.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Supplementary Materials

Here we give a concise description of the supplementary material used in this article. All the fourteen Python programs that we developed are similar and can be summarized as follows: Step 1: call the necessary Python libraries in the heading—numpy, matplotlib.pyplot, mpl_toolkits.mplot3d, and sklearn.datasets. Step 2: set up the axes using "plt.axes" and define the mapping $f(x, y)$ to plot using "def." Step 3: define the intervals for the values of each of the variables x, y, and z using "np.linspace." Step 4: plot the function $f(x, y)$ using "ax.plot_surface." Step 5: set up the figure's title using "ax.set_title" and the axis labels using "ax.set_xlabel" and "ax.set_ylabel." (*Supplementary Materials*)

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