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Research Article

The Investigation of Some Essential Concepts of Extended Fuzzy-Valued Convex Functions and Their Applications

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In this paper, we are thus motivated to define and introduce the extended fuzzy-valued convex functions that can take the singleton fuzzy values $-\tilde{\infty}$ and $+\tilde{\infty}$ at some points. Such functions can be characterized using the notions of effective domain and epigraph. In this way, we study important concepts such as fuzzy indicator function and fuzzy infimal convolution for extended fuzzy-valued functions. Finally, we introduce the concept of directional generalized derivative for extended above functions and its properties. Eventually, we give a practical example that will illustrate well the directional g-derivative for the extended fuzzy-valued convex function.

1. Introduction

Since Zadeh [1] began to study the essential concepts and definitions of fuzzy theory, many studies have concentrated on the theoretical and practical aspects of fuzzy numbers. In this way, fuzzy numbers have been extensively researched by many researchers. For instance, Diamond and Kloeden [2], Puri and Ralescu [3], and many other researchers [4–8] brought up the concepts of Hukuhara differentiability (H-differentiability in short) and integrability for fuzzy mappings. The fuzzy convex analysis is one of the fundamental concepts in fuzzy optimal control and fuzzy optimization. Nanda and Kar [9] proposed the concept of convexity for fuzzy mapping in 1992. Accordingly, various studies on convexity for fuzzy mappings and application in fuzzy optimization have been introduced [10–13]. Yan and Xu [12] have explored the concepts of convexity and quasiconvexity of fuzzy-valued functions. Syau and Lee [14] have studied the concepts of quasiconvex and pseudoconvex multivariable fuzzy functions. Convexity and Lipschitz continuity of fuzzy-valued functions have been discussed by Furukawa [15]. Accordingly, some definitions for various types of convexity or generalized convexity of fuzzy mapping have been proposed, and their properties have been perused [10, 16]. Noor [17] has expressed the concept and properties of fuzzy preinvex functions in the $\mathbb{R}$ field. A generalization of the Hukuhara difference (H-difference in short), called the generalized Hukuhara difference (gH-difference in short), was proposed by Stefanini in 2010 [18] because the H-difference exists between two fuzzy numbers only under very restricted positions. Compared to the H-difference, the gH-difference exists in more cases but does not always exist. To solve this difficulty, Bede and Stefanini [19] introduced the generalized difference (g-difference in short), which always exists. It should be noted that this difference in some cases does not maintain the convexity condition of fuzzy numbers, therefore may not be a fuzzy number. So, this difficulty is resolved by considering the convex hull of the resulting set by Gomes and Barros [20]. Based on these two differences, the generalized Hukuhara differentiability (gH-differentiability in short), level-wise generalized Hukuhara differentiability (L_{gH}-differentiability in short), and generalized differentiability (g-differentiability in short) have been introduced [19]. For more recent interesting results related to Jensen's and related inequalities, we recommended [21, 22].

In this paper, we consider a generalization for a fuzzy-valued convex function whose range can be the extended fuzzy values. Also, we investigate some essential concepts of extended fuzzy-valued convex functions. We are thus motivated to introduce the extended fuzzy-valued convex functions that can take the singleton fuzzy values $-\tilde{\infty}$ and $+\tilde{\infty}$ at some points.

Hereupon, the theoretical aspect of extended fuzzy number-valued functions is described, and our aim is not to consider the real applications. It is clear that this research has many applications in dynamic systems of biomedical science, such as problems with cancer, problems with drug release, and so on.

In the following, we describe a comparative study between the convex functions with fuzzy values and the extended fuzzy-valued functions. In general, we prefer to work with fuzzy convex functions containing fuzzy numbers defined over the whole space $\mathbb{R}^n$ (and not only over a convex subset). However, in some situations, arising mainly in the context of fuzzy optimization and fuzzy conjugation or fuzzy duality, we will encounter operations with fuzzy number-valued functions that produce extended fuzzy-valued functions. An example is a fuzzy-valued function of the form.

$$f(x) = \sup_{i\in\Lambda} f_i(x), \tag{1}$$

where Λ is an infinite index set and can take the fuzzy value $+\tilde{\infty}$ even if the functions f_i are fuzzy number-valued. Furthermore, we will encounter functions f that are fuzzy-valued convex over convex subset and cannot be extended to functions that are fuzzy number-valued and convex over the entire space $\mathbb{R}^n$ (e.g., the fuzzy number-valued function $f: (0,+\infty) \longrightarrow \mathbb{R}_{\mathscr{F}}$ defined by $f(x) = < -1,1,2 > \odot 1/x$).

In such situations, it may be convenient, instead of restricting the domain of f to the subset where f takes fuzzy numbers values, to extend the domain to all of $\mathbb{R}^n$, but allow f to take fuzzy values $+\tilde{\infty}$. This process of extension enables us to treat fuzzy number-valued convex functions with different domains as fuzzy-valued convex functions with extended fuzzy values in $+\tilde{\infty}$ and defined throughout $\mathbb{R}^n$. A difficulty in defining extended fuzzy-valued convex functions φ that can take both fuzzy values $-\tilde{\infty}$ and $+\tilde{\infty}$ is that the term $\theta\odot\varphi(x)\oplus(1-\theta)\odot\varphi(y)$ arising in earlier papers for the fuzzy-valued convex case may involve the forbidden fuzzy sum $-\tilde{\infty}\oplus+\tilde{\infty}$ (this, of course, may happen only if φ is fuzzy improper but fuzzy improper function may arise on occasion in proofs or other analyses). So, the notions of effective domain and epigraph provide an effective way of dealing with this difficulty. Furthermore, we present some of the essential concepts such as the fuzzy indicator function, the epigraph, the fuzzy infimal convolution, the directional generalized derivatives, and their properties for extended fuzzy values.

This paper is divided into seven sections; in Section 2, several definitions besides the results about fuzzy numbers and the g-difference and g-differentiability are expressed at first. Moreover, in Section 3, we introduced the specific case of fuzzy Jensen's inequality for fuzzy-valued convex functions, and in Section 4, the g-differentiability for extended fuzzy-valued convex function is considered. Then, the concepts of fuzzy indicator function and the epigraph are discussed, and some outcomes are gained in Section 5. Furthermore, the fuzzy infimal convolution is considered in Section 6. At the end of this paper, in Section 7, the directional generalized derivatives with their properties for extended fuzzy-valued convex function are presented, and eventually, the above concepts are presented with several examples.

2. Preliminaries

In this section, the basic definitions and concepts which will be used throughout the paper will be presented and introduced. Also, we use $\mathbb{R}_{\mathscr{F}}$ to denote the fuzzy numbers set, that is normal, quasiconcave, upper semicontinuous, and compactly supported fuzzy sets that are defined on the real line. Suppose that $X\in\mathbb{R}_{\mathscr{F}}$ is a fuzzy number; for $r\in[0,1]$, the r-cuts of X are described by $[X]_r = \{x\in\mathbb{R}|X(x)\geq r\}$, and for $r=0$ by $[X]_0 = \overline{\{x\in\mathbb{R}|X(x)>0\}}$ is illustrated. Moreover, $[X]_r = [X_r^-, X_r^+]$ is explained, so the r-cut $[X]_r$ is a closed interval for all $r\in[0,1]$. If $X, Y\in\mathbb{R}_{\mathscr{F}}$, and $\theta\in\mathbb{R}$, the addition and scalar multiplication are described as having the $\theta\in\mathbb{R}$-cuts of $[X+Y]_r = [X]_r + [Y]_r$ and $[\theta\odot X]_r = \theta[X]_r$, relatively. By $X = \langle a,b,c,d\rangle$, a trapezoidal fuzzy number defined so that $a\leq b\leq c\leq d$, and has r-cuts $[X]_r = [a + r(b-a), d - r(d-c)]$ for $0\leq r\leq 1$; if $b=c$, we have the triangular fuzzy number. The support of fuzzy numbers X is specified as follows:

$$\operatorname{supp}(X) = \overline{\{x\in\mathbb{R}|X(x)>0\}}. \tag{2}$$

The standard Hukuhara difference (H-difference $\ominus_H$) is defined by $X\ominus_H Y = Z \Leftrightarrow X = Y + Z$, being $+$ the standard fuzzy addition; if $X\ominus_H Y$ exists, its r-cuts are $[X\ominus_H Y]_r = [X_r^- - Y_r^-, X_r^+ + Y_r^+]$. It is outstanding that

$X \ominus_H X = 0$ (here 0 signifies the singleton $\{0\}$) for any fuzzy number X but $X - X \neq 0$.whenever $\varphi'_g(x_0) \in \mathbb{R}_{\mathcal{F}}$ is uniquely determined by (19). It is called the g-derivative of φ at x_0.

Definition 1. The family of all closed and bounded intervals in $\mathbb{R}$ is demonstrated by $\mathbb{K}_C$, i.e.,

$$\mathbb{K}_C = \{[a^-, a^+] \in \mathbb{R} : a^- \leq a^+\}. \tag{3}$$

Definition 2 (see [9]). A singleton fuzzy number like $\tilde{a}$ can be defined for each $a \in \mathbb{R}$ as follows:

$$\tilde{a}(t) = \begin{cases} 1, & t = a, \\ 0, & t \neq a. \end{cases} \tag{4}$$

$\mathbb{R}$ can be embedded in $\mathbb{R}_{\mathcal{F}}$.

Definition 3. Let us consider the singleton fuzzy values $-\tilde{\infty}$ and $+\tilde{\infty} \in \mathbb{R}_{\mathcal{F}}$ such that $-\tilde{\infty}(x) = 1$, if $x = -\infty$ and $-\tilde{\infty} = 0$, if $x \neq -\infty$, also, $+\tilde{\infty}(x) = 1$, if $x = +\infty$ and $+\tilde{\infty}(x) = 0$, if $x \neq +\infty$.

Remark 4. Throughout this paper, we use the extended fuzzy numbers, i.e., $\mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ by adjoining the singleton fuzzy elements $+\tilde{\infty}$ and $-\tilde{\infty}$.

Definition 5 (see [23]). Suppose that $X, Y \in \mathbb{R}_{\mathcal{F}}$, the partial order relations between two fuzzy numbers, i.e.,

$$X \preccurlyeq Y \Leftrightarrow [X]_r = [X_r^-, X_r^+] \leq [Y]_r = [Y_r^-, Y_r^+], \forall r \in [0,1]. \tag{5}$$

If $[X]_r \leq [Y]_r \Leftrightarrow X_r^- \leq Y_r^-$ and $X_r^+ \leq Y_r^+$. And $X \prec Y \Leftrightarrow X \preccurlyeq Y$ and $X \neq Y$.

Proposition 6 (see [19]). *X is a fuzzy number which is entirely distinguished by the pair of $X = (X^-, X^+)$ as functions $X^-, X^+ : [0,1] \longrightarrow \mathbb{R}$, denoting by the endpoints of the r-cuts, fulfilling the below situations:*

(1) As a function of r, $X^- : r \longrightarrow X_r^- \in \mathbb{R}$ is a bounded monotonic increasing left-continuous function for all $r \in (0, 1]$ and right-continuous at $r = 0$;

(2) As a function of r, $X^+ : r \longrightarrow X_r^+ \in \mathbb{R}$ is a bounded monotonic decreasing left-continuous function for all $r \in (0, 1]$ and right-continuous at $r = 0$;

(3) $X_1^- \leq X_1^+$ for $r = 1$, which supplies $X_r^- \leq X_r^+$, $\forall r \in [0,1]$.

The addendum outcome is well known [24]:

Proposition 7 (see [19]). *Suppose that an arbitrary real interval collection $\{C_r \mid r \in (0,1]\}$ that satisfied the below situations:*

(1) $C_r \in \mathbb{K}_C$ for every $r \in (0, 1]$;

(2) if $0 < r < \beta \leq 1$ then $C_\beta \subseteq C_r$;

(3) For any increasing sequence $r_n \in (0, 1]$ is given, such that $\lim_{n\longrightarrow\infty} r_n = r > 0$, then $C_r = \bigcup_{n=1}^{\infty} C_{r_n}$.

Therefore, there exists a unique LU-fuzzy quantity X (L for lower, U for upper) with $[X]_r = C_r$, $\forall r \in (0, 1]$ and $[X]_0 = (\overline{\cup_{r\in(0,1]} C_r})$.

Lemma 8 (see [19]). *Suppose that $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$, and $x_0 \in \mathbb{R}$. Then, if*

(1) $\lim_{x\longrightarrow x_0} [\varphi(x)]_r = C_r = [C_r^-, C_r^+]$ uniformly w.r.t. $r \in [0,1]$,

(2) The collections of C_r^-, C_r^+ satisfy the situations in Proposition 6 or equivalently C_r satisfy the situations in Proposition 7, therefore $\lim_{x\longrightarrow x_0} \varphi(x) = C$, with $[C]_r = C_r = [C_r^-, C_r^+]$.

Definition 9 (see [19]). For each $X, Y \in \mathbb{R}_{\mathcal{F}}$, the gH-difference is determined by the form

$$X \ominus_{gH} Y = Z \Leftrightarrow \begin{cases} (1), & X = Y \oplus Z, \\ \text{or } (2), & Y = X \oplus (-1)Z. \end{cases} \tag{6}$$

In terms of r-cuts,

$$[X \ominus_{gH} Y]_r = [\min\{X_r^- - Y_r^-, X_r^+ - Y_r^+\}, \max\{X_r^- - Y_r^-, X_r^+ - Y_r^+\}], \tag{7}$$

and conditions for the entity of $Z = X \ominus_{gH} Y \in \mathbb{R}_{\mathcal{F}}$ are as the form

$$\begin{aligned} &\text{case}(1)\begin{cases} Z_r^- = X_r^- - Y_r^- \text{ and } Z_r^+ = X_r^+ - Y_r^+ \quad \forall r \in [0,1], \\ \text{with } Z_r^- \text{ increasing}, Z_r^+ \text{ decreasing}, Z_r^- \leq Z_r^+. \end{cases} \\ &\text{case}(2)\begin{cases} Z_r^- = X_r^+ - Y_r^+ \text{ and } Z_r^+ = X_r^- - Y_r^- \quad \forall r \in [0,1], \\ \text{with } Z_r^- \text{ increasing}, Z_r^+ \text{ decreasing}, Z_r^- \leq Z_r^+. \end{cases} \end{aligned} \tag{8}$$

It is obvious that the conditions (9) and (10) are both satisfied if and only if Z is a crisp number.

Definition 10 (see [19]). Suppose that $x_0 \in (a,b)$ and h with $x_0 + h \in (a,b)$, then the function with gH-derivative $\varphi : (a,b) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at x_0 is described by the form

$$\varphi'_{gH}(x_0) = \lim_{h\longrightarrow 0} \frac{1}{h}[\varphi(x_0 + h) \ominus_{gH} \varphi(x_0)]. \tag{9}$$

If $\varphi'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ fulfilling (9) exists, it is said to be φ is generalized Hukuhara differentiable (gH-differentiable in short) at x_0.

Definition 11 (see [19]). Suppose that $\varphi: (a, b) \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$ in terms of r-cuts $\varphi_r^-(x)$ and $\varphi_r^+(x)$ are differentiable at x_0. Then,

(1) if φ is [(9) gH]-differentiable at x_0

$$\varphi'_{gH}(x_0)_r = [(\varphi_r^-)'(x_0), (\varphi_r^+)'(x_0)], \forall\, 0 \leq r \leq 1. \tag{10}$$

(2) if φ is [(10) gH]-differentiable at x_0

$$\varphi'_{gH}(x_0)_r = [(\varphi_r^+)'(x_0), (\varphi_r^-)'(x_0)], \forall\, 0 \leq r \leq 1. \tag{11}$$

Definition 12 (see [19]). The g-difference of $X, Y \in \mathbb{R}_{\mathcal{F}}$ in terms of r-cuts is as follows:

$$\left[X \ominus_g Y\right]_r = \overline{\mathbf{conv}}\left(\cup_{\beta \geq r}\left([X]_\beta \ominus_{gH} [Y]_\beta\right)\right), \quad \forall r \in [0, 1], \tag{12}$$

such that the gH-difference of intervals is denoted by $[X]_\beta \ominus_{gH} [Y]_\beta$.

Proposition 13 (see [19]). *The g-difference (4) in terms of r-cuts as the form*

$$\left[X \ominus_g Y\right]_r = \left[\inf_{\beta \geq r} \min\left\{X_\beta^- - Y_\beta^-, X_\beta^+ - Y_\beta^+\right\}, \sup_{\beta \geq r} \max\left\{X_\beta^- - Y_\beta^-, X_\beta^+ - Y_\beta^+\right\}\right], \quad \forall r \in [0, 1]. \tag{13}$$

Remark 14 (see [19]). Assume that $X \ominus_{gH} Y \in \mathbb{R}_{\mathcal{F}}$ as well as $X \ominus_g Y = X \ominus_{gH} Y$.

Proposition 15 (see [19]). *The g-difference of every $X, Y \in \mathbb{R}_{\mathcal{F}}$ is denoted by $X \ominus_g Y \in \mathbb{R}_{\mathcal{F}}$.*

Proposition 16 (see [19]). *Suppose that $X, Y \in \mathbb{R}_{\mathcal{F}}$, then*

(1) $X \ominus_g Y = X \ominus_{gH} Y$, if the right side exists; particularly $X \ominus_g X = 0$;
(2) $(X + Y) \ominus_g Y = X$;
(3) $0 \ominus_g (X \ominus_g Y) = Y \ominus_g X$;
(4) $X \ominus_g Y = Y \ominus_g X = Z \Leftrightarrow Z = -Z$; also, $Z = 0 \Leftrightarrow X = Y$.

Definition 17. Whenever $X, Y \in \mathbb{R}_{\mathcal{F}}$, then

(1) $X \succcurlyeq Y \Leftrightarrow X \ominus_g Y \succcurlyeq 0$;
(2) $X \Leftrightarrow Y \Leftrightarrow X \ominus_g Y \Leftrightarrow 0$;
(3) $X \succcurlyeq 0 \Leftrightarrow \ominus_g X \Leftrightarrow 0$.

Definition 18 (see [19]). For every $r \in (0, 1]$, suppose that $X \in \mathbb{R}_{\mathcal{F}}$ is a fuzzy number. We can describe the Hausdorff distance on $\mathbb{R}_{\mathcal{F}}$ as the form

$$\mathbb{D}(X, Y) = \sup_{r \in [0,1]} \left\{\| [X]_r \ominus_{gH} [Y]_r \|_*\right\}, \tag{14}$$

where, for an interval $[x, y]$, is the norm on $\mathbb{R}$,

$$\| [x, y] \|_* = \max\{|x|, |y|\}. \tag{15}$$

Note that the metric $\mathbb{D}$ is well defined because of the interval gH-difference, $[X]_r \ominus_{gH} [Y]_r$ evermore exists. Therefore, $\mathbb{R}_{\mathcal{F}}$ with the Hausdorff distance $\mathbb{D}$ becomes a complete metric space. This definition is equivalent to the usual definitions for metric fuzzy numbers spaces, e.g., [2, 25, 26].

Proposition 19 (see [19]). *For all $X, Y \in \mathbb{R}_{\mathcal{F}}$*

$$\mathbb{D}(X, Y) = \sup_{r \in [0,1]} \| [X]_r \ominus_{gH} [Y]_r \|_* = \| X \ominus_g Y \|, \tag{16}$$

where $\|.\| = \mathbb{D}(., 0)$.

Remark 20 (see [19]). Note that since $\|.\| = \mathbb{D}(., 0)$ whenever the right expression exists, we also consummate $\mathbb{D}(X, Y) = \|X \ominus_g Y\| = \|X \ominus_{gH} Y\|$, whenever $X \ominus_{gH} Y$ exists.

Definition 21 (see [19]). Suppose that $x_0 \in (a, b)$ and h with $x_0 + h \in (a, b)$, then the level-wise gH-derivative (L_{gH}-derivative in short) of a function $\varphi: (a, b) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at x_0 is described as the interval-valued gH-derivatives set if they exist,

$$\varphi'_{L_{gH}}(x_0)_r = \lim_{h \longrightarrow 0} \frac{1}{h}\left([\varphi(x_0 + h)]_r \ominus_{gH} [\varphi(x_0)]_r\right), \tag{17}$$

If $\varphi'_{L_{gH}}(x_0)_r \in \mathbb{K}_C$, for all $r \in [0, 1]$, it is said to be φ is L_{gH}-differentiable at x_0, and the intervals' collection $\left\{\varphi'_{L_{gH}}(x_0)_r \mid r \in [0, 1]\right\}$ is the L_{gH}-derivative of φ at x_0 and indicated by $\varphi'_{L_{gH}}(x_0)$.

Definition 22. Suppose that $\varphi: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ is said to be g-continuous at $x_0 \in I$, if for every $h \in \mathbb{R}$ with $x_0 + h \in I$, then we have

$$\lim_{h \longrightarrow 0} \varphi(x_0 + h) \ominus_g \varphi(x_0) = 0. \tag{18}$$

Definition 23 (see [19]). Let x_0 be a point of (a, b) and h with $x_0 + h \in (a, b)$. Then, $\varphi: (a, b) \longrightarrow \mathbb{R}_{\mathcal{F}}$ is said to be g-differentiable at x_0 such that

$$\varphi'_g(x_0) = \lim_{h \longrightarrow 0} \frac{\varphi(x_0 + h) \ominus_g \varphi(x_0)}{h}, \tag{19}$$

Theorem 24 (see [19]). *Suppose that $\varphi: (a, b) \longrightarrow \mathbb{R}_{\mathcal{F}}$, the collection of interval $\left\{\varphi'_{LgH}(x_0)_r : r \in [0, 1]\right\}$ is uniformly L_{gH}-differentiable at x_0. Then, φ has a g-derivative at x_0 and*

$$\varphi'_g(x_0)_r = \overline{\mathbf{conv}}\left(\bigcup_{\beta \geq r} \varphi'_{L_{gH}}(x_0)_\beta\right), \quad \forall r \in [0, 1]. \tag{20}$$

Theorem 25 (see [19]). *Suppose that* $\varphi: [\mathrm{a}, \mathrm{b}] \longrightarrow \mathbb{R}_{\mathcal{F}}$ *with* $[\varphi(x)]_r = [\varphi_r^-(x), \varphi_r^+]$. *If the real-valued functions* $\varphi_r^-(x_0)$ *and* $\varphi_r^+(x_0)$ *are both differentiable w.r.t.* x_0, *uniformly w.r.t.* $r \in [0,1]$, *then* φ *has a* g*-derivative at* x_0,

$$\varphi_g'(x_0)_r = \left[\inf_{\beta\geq r} \min\left\{\left(\varphi_\beta^-\right)'(x_0), \left(\varphi_\beta^+\right)'(x_0)\right\}, \sup_{\beta\geq r} \max\left\{\left(\varphi_\beta^-\right)'(x_0), \left(\varphi_\beta^+\right)'(x_0)\right\}\right]. \tag{21}$$

3. The Fuzzy-Valued Convex Function in Sense of Jensen's Inequality

Therein-after, all of these below inequalities are now called the fuzzy-valued convex function Jensen's inequality. So, we shall designate by I a (closed, open, or half-open, finite or infinite) interval in $\mathbb{R}$. Also, we denoted the interior of I by $\mathrm{int}(\mathrm{I})$.

Definition 26 (see [23]). Suppose that $\varphi: \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$, then φ is said to be a fuzzy-valued convex function if

$$\varphi(\theta x + (1-\theta)y) \preccurlyeq \theta \odot \varphi(x) \oplus (1-\theta) \odot \varphi(y), \forall x, y \in \mathrm{I}, \quad \forall\, 0 \leq \theta \leq 1. \tag{22}$$

The basic fuzzy inequality equation (22) is sometimes called fuzzy Jensen's inequality.

Closely related to fuzzy convexity is the following concept.

Definition 27. Suppose that $\varphi: \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ is a midpoint fuzzy-valued convex function if

$$\varphi\left(\frac{x+y}{2}\right) \preccurlyeq \frac{1}{2} \odot [\varphi(x) \oplus \varphi(y)], \quad \forall x, y \in \mathrm{I}. \tag{23}$$

Note that if φ is a fuzzy-valued convex function, then φ is the midpoint fuzzy-valued convex function.

Theorem 28 (see [23]). *Suppose that* $\varphi: \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ *in terms of* $[\varphi(x)]_r = [\varphi_r^-(x), \varphi_r^+(x)]$, *then* φ *is a fuzzy-valued convex function if and only if for any fixed* $r \in [0,1]$, *the convex functions* $\varphi_r^-(x)$ *and* $\varphi_r^+(x)$ *are both real-valued of* x.

Definition 29. Suppose that $\varphi: \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ is a fuzzy-valued convex function. Let $x \in \mathrm{I}$ and h with $x+h, x-h \in \mathrm{I}$ then, we have

$$\varphi_{+g}'(x) = \lim_{h\longrightarrow 0^+} \frac{\varphi(x+h) \ominus_g \varphi(x)}{h}, \tag{24}$$

$$\varphi_{-g}'(x) = \lim_{h\longrightarrow 0^+} \frac{\varphi(x) \ominus_g \varphi(x-h)}{h}, \tag{25}$$

exists on $\mathrm{int}(\mathrm{I})$. If $\varphi_{+g}'(x)$ and $\varphi_{-g}'(x) \in \mathbb{R}_{\mathcal{F}}$ satisfying (24) and (25) exist, then φ is said to be right and left g-differentiable at x on $\mathrm{int}(\mathrm{I})$.

4. The Extended Fuzzy-Valued Convex Functions and g-Differentiability

In the previous section, we consider the fuzzy-valued convex function in sense of Jensen's inequality with fuzzy values in $\mathbb{R}_{\mathcal{F}}$. Now, in this part, we shall consider more general fuzzy-valued functions, with fuzzy values in $\mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$. In other words, we want to define the fuzzy-valued convex functions whose range of them be the extended fuzzy numbers in $\mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$. Throughout our paper, we consider for convenience extended fuzzy-valued functions, which take fuzzy values in $\mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$. The usual conventions of the fuzzy arithmetic are that $x \oplus + \tilde{\infty} = +\tilde{\infty}$ if $x \in \mathbb{R}$, $x \odot (-\tilde{\infty}) = -\tilde{\infty}$ if $x > 0$, $x \odot + \tilde{\infty} = -\tilde{\infty}$ if $x < 0$, but also the following less obvious one is as follows:

$$0 \odot (+\tilde{\infty}) = (+\tilde{\infty}) \odot 0 = 0 \odot (-\tilde{\infty}) = (-\tilde{\infty}) \odot 0 = 0. \tag{26}$$

Also, we discussed the g-differentiability and the basic facts of the g-differentiability for the extended fuzzy-valued convex functions that can be easily visualized. The expression $-\tilde{\infty} \oplus + \tilde{\infty}$ is undefined.

Definition 30. An extended fuzzy-valued function $\varphi: \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is called convex, if for all $x, y, \theta \in \mathbb{R}$ and $\mu, \nu \in \mathbb{R}_{\mathcal{F}}$ such that $\varphi(x) \prec \mu$, we have $\varphi(y) \prec \nu$, $0 < \theta < 1$

$$\varphi(\theta x + (1-\theta)y) \prec \theta \odot \mu \oplus (1-\theta) \odot \nu. \tag{27}$$

Lemma 31. *Suppose that* $\varphi: \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$, *then* φ *is the fuzzy-valued convex function in Definition 26 if and only if* φ *is the fuzzy-valued convex function in Definition 30.*

Proof. Suppose that $\varphi: \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ is convex fuzzy-valued function, and let $\varphi(x) \prec \mu$, $\varphi(y) \prec \nu$, $0 < \theta < 1$. Then,

$$\begin{aligned}\varphi(\theta x + (1-\theta)y) &\preccurlyeq \theta \odot \varphi(x) \oplus (1-\theta) \odot \varphi(y) \\ &\prec \theta \odot \mu \oplus (1-\theta) \odot \nu.\end{aligned} \tag{28}$$

Conversely, suppose that Definition 30 holds. Then, for any $\varepsilon > 0$, let $\mu := \varphi(x) \oplus \varepsilon$ and $\nu := \varphi(y) \oplus \varepsilon$, then by the hypothesis, we have

$$\begin{aligned}\varphi(\theta x+(1-\theta)y) &\prec \theta\odot\mu\oplus(1-\theta)\odot\nu\\ &=\theta\odot(\varphi(x)\oplus\varepsilon)\oplus(1-\theta)\odot(\varphi(y)\oplus\varepsilon)\\ &=\theta\odot\varphi(x)\oplus(1-\theta)\odot\varphi(y)\oplus\varepsilon,\quad\forall\varepsilon>0,\end{aligned}\tag{29}$$

as $\varepsilon\longrightarrow 0$, and so

$$\varphi(\theta x+(1-\theta)y)\preccurlyeq\theta\odot\varphi(x)\oplus(1-\theta)\odot\varphi(y).\tag{30}$$

Hence, φ is a fuzzy-valued convex function. It comes upon that Definition 30 is a generalization of Definition 26. □

Definition 32. The extended fuzzy-valued convex function effective domain of $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$, denoted by $dom(\varphi)$, is the set of $\{x\in\mathbb{R}:\varphi(x)\prec+\tilde{\infty}\}$.

Lemma 33. *The extended fuzzy-valued convex function effective domain of $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ is a convex set.*

Proof. Suppose that $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ is a fuzzy-valued convex function, let $x,y\in dom(\varphi)$, $0<\theta<1$ we have

$$\varphi(\theta x+(1-\theta)y)\prec\theta\odot\mu\oplus(1-\theta)\odot\nu,\mu,\nu\in\mathbb{R}_{\mathscr{F}},\text{with }\varphi(\mathrm{x})\prec\mu,\quad\varphi(\mathrm{y})\prec\nu.\tag{31}$$

And so

$$\varphi(\theta x+(1-\theta)y)\prec+\tilde{\infty}\preccurlyeq\theta x+(1-\theta)y\in\mathrm{dom}(\varphi).\tag{32}$$

Hence, dom (φ) is a convex set. □

Definition 34. The extended fuzzy-valued function $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ is called fuzzy proper, if $\varphi(x)\neq-\tilde{\infty}$, $\forall x\in\mathbb{R}$ and $\varphi\equiv+\tilde{\infty}$.

Definition 35. The extended fuzzy-valued function $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ is called fuzzy improper, if φ is not fuzzy proper, i.e., $\varphi(x)\equiv+\tilde{\infty}$ or there exists $x\in\mathbb{R}$ such that $\varphi(x)=-\tilde{\infty}$.

The below theorem is the class of fuzzy-valued improper convex functions that is easy to describe.

Theorem 36. *Suppose that $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ be the fuzzy-valued improper convex function. Then, $\varphi(x)=-\tilde{\infty}$ whenever $x\in\mathrm{int}(dom(\varphi))$.*

Proof. The statement is trivially true if $\varphi\equiv+\tilde{\infty}$, i.e., $\varphi(x)=+\tilde{\infty}$ for all $x\in\mathbb{R}$, then $dom(\varphi)=\varnothing$ therefore $\mathrm{int}(\mathrm{dom}(\varphi))=\varnothing$. Henceforth, $\varphi\equiv+\tilde{\infty}$ on $\mathrm{int}(\mathrm{dom}(\varphi))$. Let $\varphi\equiv+\tilde{\infty}$, there exists $x_0\in\mathbb{R}$ such that $\varphi(x_0)=-\tilde{\infty}$, then $x_0\in\mathrm{dom}(\varphi)$. Let $x\in\mathrm{int}(\mathrm{dom}(\varphi))$, $x\neq x_0$ be arbitrary. There exists $y\in\mathrm{dom}(\mathrm{dom}(\varphi))$ and $0<\theta<1$ so that $x=\theta x_0+(1-\theta)y$. By Definition 30, for each $\mu\in\mathbb{R}_{\mathscr{F}}$ therefore $\varphi(y)\prec\mu\preccurlyeq+\tilde{\infty}$ and each $\nu\in\mathbb{R}_{\mathscr{F}}$

$$\varphi(x)=\varphi(\theta x_0+(1-\theta)y)\prec\theta\odot\nu\oplus(1-\theta)\odot\mu,\tag{33}$$

Since $\varphi(x_0)=-\tilde{\infty}\prec\nu$. Letting $\nu\longrightarrow-\infty$, we see that $\varphi(x)=-\tilde{\infty}$. The proof is complete. □

By the following lemma, it is often convenient to extend a fuzzy-valued convex function to all of $\mathbb{R}$ by defining its fuzzy value to be $\{+\tilde{\infty}\}$ outside its domain.

Lemma 37. *(The fuzzy-valued convex extension) Suppose that $\varphi:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}$ is a fuzzy number-valued convex function, where I is a convex set. We define its extended fuzzy-valued of $\hat{\varphi}:\mathbb{R}\longrightarrow\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}$, as follows:*

$$\hat{\varphi}(x):=\begin{cases}\varphi(x), & x\in\mathrm{dom}\,\varphi,\\ +\tilde{\infty}, & x\notin\mathrm{dom}\,\varphi.\end{cases}\tag{34}$$

Then, the extension $\hat{\varphi}$ is a fuzzy-valued convex function that defines on all $\mathbb{R}$ and takes the fuzzy values in $\mathbb{R}_{\mathscr{F}}\cup\{+\tilde{\infty}\}$.

Proof. Let $x,y\in\mathbb{R}$, $0\leq\theta\leq1$ be arbitrary. □

Case 1. If $x,y\in\mathrm{dom}\,\varphi$, then by (11), we have

$$\hat{\varphi}(x)=\varphi(x)\text{ and }\hat{\varphi}(\mathrm{y})=\varphi(\mathrm{y}).\tag{35}$$

Consider

$$\hat{\varphi}(\theta x+(1-\theta)y)=\varphi(\theta x+(1-\theta)y).\tag{36}$$

Note that since φ is the fuzzy-valued convex function on $dom\,\varphi$ and by (34), we have

$$\preccurlyeq\theta\odot\varphi(x)\oplus(1-\theta)\odot\varphi(y)\tag{37}$$

by (35),

$$=\theta\odot\hat{\varphi}(x)\oplus(1-\theta)\odot\hat{\varphi}(y).\tag{38}$$

Then,

$$\hat{\varphi}(\theta x+(1-\theta)y)\preccurlyeq\theta\odot\hat{\varphi}(x)\oplus(1-\theta)\odot\hat{\varphi}(y).\tag{39}$$

Case 2. If $x,y\notin\mathrm{dom}\varphi$, henceforth by (11),

$$\hat{\varphi}(x)=+\tilde{\infty},\hat{\varphi}(y)=+\tilde{\infty}.\tag{40}$$

Also, $\theta x+(1-\theta)y\notin\mathrm{dom}\varphi$, then by (34),

$$\hat{\varphi}(\theta x+(1-\theta)y)=+\tilde{\infty}.\tag{41}$$

Therefore.

$$\hat{\varphi}(\theta x + (1-\theta)y) = +\tilde{\infty} = (+\tilde{\infty}) \oplus (+\tilde{\infty}) \quad (42)$$

by (40),

$$= \theta \odot \hat{\varphi}(x) \oplus (1-\theta) \odot \hat{\varphi}(y). \quad (43)$$

Hence,

$$\hat{\varphi}(\theta x + (1-\theta)y) = \theta \odot \hat{\varphi}(x) \oplus (1-\theta) \odot \hat{\varphi}(y). \quad (44)$$

Case 3. If $x \in \mathrm{dom}\varphi$ and $y \notin \mathrm{dom}\varphi$, afterward by (11),

$$\hat{\varphi}(x) = \varphi(x) \text{ and } \hat{\varphi}(y) = +\tilde{\infty}. \quad (45)$$

Since $x \in \mathrm{dom}\varphi$ and $y \notin \mathrm{dom}\varphi$, then $\theta x + (1-\theta)y \notin I$, by (34), we have

$$\hat{\varphi}(\theta x + (1-\theta)y) = +\tilde{\infty}. \quad (46)$$

Therefore, we have

$$\hat{\varphi}(\theta x + (1-\theta)y) = +\tilde{\infty} = \theta \odot \varphi(x) \oplus (+\tilde{\infty}), \quad (47)$$

by (45),

$$= \theta \odot \hat{\varphi}(x) \oplus (1-\theta) \odot \hat{\varphi}(y). \quad (48)$$

So,

$$\hat{\varphi}(\theta x + (1-\theta)y) = \theta \odot \hat{\varphi}(x) \oplus (1-\theta) \odot \hat{\varphi}(y). \quad (49)$$

Thus, in all three cases, the definition of a fuzzy-valued convex function was established for $\hat{\varphi}$; hence, it is a fuzzy-valued convex function on $\mathbb{R}$.

Note that, by replacing the domain of a proper fuzzy-valued convex function with effective domain, we can convert it into a fuzzy-valued function.

In the below theorem, the sufficient conditions of left and right L_{gH}-differentiability for right and left g-differentiability for $\mathbb{R}_{\mathscr{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ proper fuzzy-valued convex functions in terms of r-cut are stated.

Theorem 41. *Suppose that $\varphi: \mathbb{R} \longrightarrow \mathbb{R}_{\mathscr{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is a proper fuzzy-valued convex function, and φ is right and left uniformly L_{gH}-differentiable at x. Then, φ has the right and left g-derivative throughout $dom(\varphi)$, provided the fuzzy values $+\tilde{\infty}$ and $-\tilde{\infty}$ are permitted.*

Proof. The proof is the same as Theorem 5.2 in [23] and Theorem 35 in [19] since the g-quotient $\varphi(x) \ominus_g \varphi(a)/x - a$ is nondecreasing and bounded from below on $dom(\varphi) = [a, b]$; therefore, there exists a subsequence $x_n > a$, in which the members of subsequence are $\varphi(x_n) \ominus_g \varphi(a)/x_n - a$ and as $\varphi(x_n) \ominus_g \varphi(a)/x_n - a$ converges to $\inf_{x>a}\varphi(x) \ominus_g \varphi(a)/x - a$ equals to $\varphi'_{+g}(a)$, i.e., there exists a subsequence $x_n > a$ such that

$$\varphi'_{+g}(a) = \lim_{n\longrightarrow\infty} \frac{\varphi(x_n) \ominus_g \varphi(a)}{x_n - a} = \inf_{x>a} \frac{\varphi(x) \ominus_g \varphi(a)}{x - a}. \quad (50)$$

Similarly, since the g-quotient $\varphi(x) \ominus_g \varphi(a)/x - a$ is nonincreasing and bounded from above on $\mathrm{dom}(\varphi) = [a, b]$, therefore there exists a subsequence $x_n < a$, in which the members of subsequence are $\varphi(x_n) \ominus_g \varphi(a)/x_n - a$ and as $n \longrightarrow \infty$ converges to $\sup_{x<a}\varphi(x) \ominus_g \varphi(a)/x - a$ equals to $\varphi'_{-g}(a)$, i.e., there exists a subsequence $x_n < a$ such that

$$\varphi'_{-g}(a) = \lim_{n\longrightarrow\infty} \frac{\varphi(x_n) \ominus_g \varphi(a)}{x_n - a} = \sup_{x<a} \frac{\varphi(x) \ominus_g \varphi(a)}{x - a}. \quad (51)$$

Thus, the left g-derivative φ'_{-g} for the case where $dom(\varphi) = [a, b]$ exists. Hence, by the above concepts, $\varphi'_{+g}(x)$ exists whenever $x \in (a, b]$, and $\varphi'_{-g}(x)$ exists whenever $x \in (a, b]$. But for any $x < a$, we have $\varphi(x) = +\tilde{\infty}$, so the quotient

$$\frac{\varphi(x) \ominus_g \varphi(a)}{x - a} = -\tilde{\infty}, \quad (52)$$

hence, $\varphi'_{-g}(a) = -\tilde{\infty}$, for any $x > b$, we have the quotient

$$\frac{\varphi(x) \ominus_g \varphi(b)}{x - b} = +\tilde{\infty}, \quad (53)$$

and so $\varphi'_{+g}(b) = +\tilde{\infty}$. □

5. The Fuzzy Concepts of Indicator Function and Epigraph

Now, we introduce the fuzzy indicator function and the epigraph for the extended fuzzy-valued convex function by the forms.

Definition 42. Suppose that $C \subseteq \mathbb{R}^n$ is a set. Define the fuzzy indicator function of C as follows:

$$\tilde{\mathrm{I}}_C: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathscr{F}} \cup \{+\tilde{\infty}\}, \quad (54)$$

is given by

$$\tilde{\mathrm{I}}_C(x) := \begin{cases} \tilde{0}, & \text{if } x \in C, \\ +\tilde{\infty}, & \text{if } x \notin C. \end{cases} \quad (55)$$

Theorem 43. *Let $C \subseteq \mathbb{R}^n$. Then, C is a convex set if and only if $\tilde{I}_C$ is a fuzzy-valued convex function.*

Proof. Suppose that C is a convex set. Let $x, y \in \mathbb{R}^n$, $0 < \theta < 1$. If $x, y \in C$, since C is a convex set, then $\theta x + (1-\theta) \in C$. Therefore,

$$\tilde{\mathrm{I}}_C(\theta x + (1-\theta)y) = \tilde{0} \preccurlyeq \theta \odot \tilde{\mathrm{I}}_C(x) \oplus (1-\theta) \odot \tilde{\mathrm{I}}_C(y) = \tilde{0}. \quad (56)$$

If $x, y \notin C$, then $\tilde{I}_C(x) = +\tilde{\infty}$, $\tilde{I}_C(y) = +\tilde{\infty}$. Since C is a convex set, then $(\theta x + (1-\theta)y) \notin C$, therefore

$$\theta \odot \tilde{I}_C(x) \oplus (1-\theta) \odot \tilde{I}_C(y) = +\tilde{\infty}. \tag{57}$$

Thus,

$$\begin{aligned}\tilde{I}_C(\theta x + (1-\theta)y) &= +\tilde{\infty} \\ &= \theta \odot \tilde{I}_C(x) \oplus (1-\theta) \odot \tilde{I}_C(y).\end{aligned} \tag{58}$$

Hence, $\tilde{I}_C$ is the fuzzy-valued convex function.

Conversely, suppose that $\tilde{I}_C$ is a fuzzy-valued convex function. Let $x, y \in C$, $0 < \theta < 1$, then $\tilde{I}_C(x) = \tilde{0}$, $\tilde{I}_C(y) = \tilde{0}$. Consider

$$\tilde{I}_C(\theta x + (1-\theta)y) \preccurlyeq \theta \odot \tilde{I}_C(x) \oplus (1-\theta) \odot \tilde{I}_C(y) = \tilde{0}, \tag{59}$$

then

$$\tilde{I}_C(\theta x + (1-\theta)y) = \tilde{0} \Longrightarrow \theta x + (1-\theta)y \in C. \tag{60}$$

Therefore, C is a convex set. The proof is complete. □

Definition 44. Let $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$. The epigraph of φ ($\mathrm{epi}(\varphi)$ in short) is a subset $\mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ by

$$epi(\varphi) := \left\{(x, \tilde{\theta}) \in \mathbb{R}^n \times \mathbb{R}_{\mathcal{F}} : \varphi(x) \preccurlyeq \tilde{\theta}\right\}. \tag{61}$$

Definition 45. Let $\varphi: \mathbb{R}^n\mathbb{R} \longrightarrow_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$. The fuzzy strictly epigraph of φ ($\mathrm{epi}_s(\varphi)$ in short) is a subset $\mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ by

$$\mathrm{epi}_s(\varphi) := \left\{(x, \tilde{\theta}) \in \mathbb{R}^n \times \mathbb{R}_{\mathcal{F}} : \varphi(x) \prec \tilde{\theta}\right\}. \tag{62}$$

Definition 46. An extended fuzzy-valued function $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is said to be convex, if for all $x, y \in \mathrm{dom}(\varphi)$, $\theta \in \mathbb{R}$, and $\mu, \nu \in \mathbb{R}_{\mathcal{F}}$ such that $\varphi(x) \prec \mu$, $\varphi(y) \prec \nu$, $0 < \theta < 1$

$$\varphi(\theta x + (1-\theta)y) \prec \theta \odot \mu \oplus (1-\theta) \odot \nu. \tag{63}$$

Theorem 47. *Let $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$, the following conditions are equivalent:*

(1) φ is a fuzzy-valued convex function.

(2) $epi(\varphi)$ is a convex set.

(3) $\mathrm{epi}_s(\varphi)$ is a convex set.

Proof. (1) ⇔ (3): Let $(x, \tilde{\alpha}), (y, \tilde{\beta}) \in F\mathrm{pei}_s(\varphi)$, $0 < \theta < 1$, $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}_{\mathcal{F}}$, then $\varphi(x) \prec \tilde{\alpha}$, $\varphi(y) \prec \tilde{\beta}$. Bring up

$$\theta \odot (x, \tilde{\alpha}) \oplus (1-\theta) \odot (y, \tilde{\beta}) = (\theta x + (1-\theta)y, \theta \odot \tilde{\alpha} \oplus (1-\theta) \odot \tilde{\beta}), \tag{64}$$

we have $\mathrm{epi}_s(\varphi)$ is convex set if and only if

$$\varphi(\theta x + (1-\theta)y) \prec \theta \odot \tilde{\alpha} \oplus (1-\theta) \odot \tilde{\beta}, \tag{65}$$

whenever $\varphi(x) \prec \tilde{\alpha}, \varphi(y) \prec \tilde{\beta}$, $0 < \theta < 1$.

Applying Definition 26, the equivalence of (9) and (11) is as follows. (2) ⇔ (3): Supposing that $\mathrm{epi}(\varphi)$ is a convex set. Let $(x, \tilde{\alpha}) \in \mathrm{epi}_s(\varphi)$, $(y, \tilde{\beta}) \in epi_s(\varphi)$, $0 < \theta < 1$. Because $\varphi(x) \prec \tilde{\alpha}, \varphi(y) \prec \tilde{\beta}$, we can select $\tilde{\alpha}_0, \tilde{\beta}_0 \in \mathbb{R}_{\mathcal{F}}$ so that $\varphi(x) \preccurlyeq \tilde{\alpha}_0 \prec \tilde{\alpha}, \varphi(y) \preccurlyeq \tilde{\beta}_0 \prec \tilde{\beta}$. We have $(x, \tilde{\alpha}_0) \in \mathrm{epi}(\varphi)$, $(y, \tilde{\beta}_0) \in \mathrm{epi}(\varphi)$. Since $\mathrm{epi}(\varphi)$ is a convex set, henceforth

$$\left(\theta x + (1-\theta)y, \theta \odot \tilde{\alpha}_0 + (1-\theta) \odot \tilde{\beta}_0\right) \in \mathrm{epi}(\varphi), \tag{66}$$

and so

$$\varphi(\theta x + (1-\theta)y) \preccurlyeq \theta \odot \tilde{\alpha}_0 \oplus (1-\theta) \odot \tilde{\beta}_0 \prec \theta \odot \tilde{\alpha} \oplus (1-\theta) \odot \tilde{\beta}. \tag{67}$$

It follows that

$$\begin{aligned}&(\theta x + (1-\theta)y, \theta \odot \tilde{\alpha} \oplus (1-\theta) \odot \tilde{\beta}) \in \mathrm{epi}_s(\varphi) \\ &\Longrightarrow \theta \odot (x, \tilde{\alpha}) \oplus (1-\theta) \odot (y, \tilde{\beta}) \in \mathrm{epi}_s(\varphi).\end{aligned} \tag{68}$$

We conclude that $\mathrm{epi}_s(\varphi)$ is a convex set.

Vice versa, suppose that $\mathrm{epi}_s(\varphi)$ is a convex set. Let $(x, \tilde{\alpha}), (y, \tilde{\beta}) \in \mathrm{epi}(\varphi)$, $0 < \theta < 1$, then $\varphi(x) \preccurlyeq \tilde{\alpha}$, $\varphi(y) \preccurlyeq \tilde{\beta}$, therefore for each $\varepsilon > 0$

$$\varphi(x) \prec \tilde{\alpha} \oplus \varepsilon, \varphi(y) \prec \tilde{\beta} \oplus \varepsilon, \tag{69}$$

thus

$$(x, \tilde{\alpha} \oplus \varepsilon) \in epi_s(\varphi), (y, \tilde{\beta} \oplus \varepsilon) \in \mathrm{epi}_s(\varphi). \tag{70}$$

Since $\mathrm{epi}_s(\varphi)$ is a convex set, then

$$\theta \odot (x, \tilde{\alpha} \oplus \varepsilon) \oplus (1-\theta) \odot (y, \tilde{\beta} \oplus \varepsilon) \in \mathrm{epi}_s(\varphi). \tag{71}$$

Hence,

$$\varphi(\theta x + (1-\theta)y) \prec \theta \odot (\tilde{\alpha} \oplus \varepsilon) \oplus (1-\theta) \odot (\tilde{\beta} \oplus \varepsilon) = \theta \odot \tilde{\alpha} \oplus (1-\theta) \odot \tilde{\beta} \oplus \varepsilon, \quad \forall \varepsilon > 0, \text{as } \varepsilon \longrightarrow 0^{+}. \tag{72}$$

It follows that

$$\varphi(\theta x + (1-\theta)y) \preccurlyeq \theta \odot \tilde{\alpha} \oplus (1-\theta) \odot \tilde{\beta}, \tag{73}$$

then

$$\begin{aligned} &(\theta x + (1-\theta)y, \theta \odot \tilde{\alpha} \oplus (1-\theta) \odot \tilde{\beta}) \in \operatorname{epi}(\varphi) \\ &\Longrightarrow \theta \odot (x, \tilde{\alpha}) \oplus (1-\theta) \odot (y, \tilde{\beta}) \in \operatorname{epi}(\varphi), \end{aligned} \tag{74}$$

and we consummate that $\operatorname{epi}(\varphi)$ is a convex set. □

Definition 48. Suppose that $A \subset \mathbb{R}^n \times \mathbb{R}_{\mathscr{F}}$ is a convex set. We define a fuzzy-valued function $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathscr{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ by

$$\varphi(x) \coloneqq \inf\left\{\tilde{\theta} \in \mathbb{R}_{\mathscr{F}} : (x, \tilde{\theta}) \in A\right\}, \quad \forall x \in \mathbb{R}^n, \tag{75}$$

i.e., $\varphi(x)$ is the greatest fuzzy-valued convex function on $\mathbb{R}^n$ which the epigraph contains A.

Theorem 49. *Suppose that $A \subset \mathbb{R}^n \times \mathbb{R}_{\mathscr{F}}$ is a convex set, and let*

$$\varphi(x) \coloneqq \inf\left\{\tilde{\theta} \in \mathbb{R}_{\mathscr{F}} : (x, \tilde{\theta}) \in A\right\}, \quad \forall x \in \mathbb{R}^n. \tag{76}$$

Then φ is a fuzzy-valued convex function.

Proof. We show that $\operatorname{epi}_s(\varphi)$ is a convex set. Let $(x, \tilde{\alpha}), (y, \tilde{\beta}) \in \operatorname{epi}_s(\varphi), 0 < \mu < 1$, then $\varphi(x) \prec \tilde{\alpha}$, $\varphi(y) \prec \tilde{\beta}$. By definition of infimum, there exists $\tilde{\theta}_1 \in \mathbb{R}_{\mathscr{F}}$ so that $(x, \tilde{\theta}_1) \in A$, $\tilde{\theta}_1 \prec \tilde{\alpha}$ and there exists $\tilde{\theta}_2 \in \mathbb{R}_{\mathscr{F}}$ so that $(y, \tilde{\theta}_2) \in A$, $\tilde{\theta}_2 \prec \tilde{\beta}$. Since A is a convex set, then

$$\begin{aligned} &\mu \odot \left(x, \tilde{\theta}_1\right) \oplus (1-\mu) \odot \left(y, \tilde{\theta}_2\right) \in A \\ &\Longrightarrow \left(\mu x + (1-\mu)y, \mu \odot \tilde{\theta}_1 \oplus (1-\mu) \odot \tilde{\theta}_2\right) \in A, \end{aligned} \tag{77}$$

by Definition 30, then

$$\varphi(\mu x + (1-\mu)y) \preccurlyeq \mu \odot \tilde{\theta}_1 \oplus (1-\mu)\tilde{\theta}_2 \prec \mu \odot \tilde{\alpha} \oplus (1-\mu) \odot \tilde{\beta}, \tag{78}$$

therefore

$$\begin{aligned} &\varphi(\mu x + (1-\mu)y) \prec \mu \odot \tilde{\alpha} \oplus (1-\mu) \odot \tilde{\beta} \\ &\qquad \Rightarrow (\mu x + (1-\mu)y, \mu \odot \tilde{\alpha} \oplus (1-\mu) \odot \tilde{\beta}) \in \operatorname{epi}_s(\varphi) \\ &\qquad \Rightarrow \mu \odot (x, \tilde{\alpha}) \oplus (1-\mu) \odot (y, \tilde{\beta}) \in \operatorname{epi}_s(\varphi), \end{aligned} \tag{79}$$

so $\operatorname{epi}_s(\varphi)$ is a convex set, hence by Theorem 47, φ is a fuzzy-valued convex function. □

6. The Fuzzy Infimal Convolution

Now, in the following, we introduce the fuzzy infimal convolution as a subset $\mathbb{R}^n \times \mathbb{R}_{\mathscr{F}}$ for extended fuzzy-valued convex functions φ and g that denote by $\varphi \Box g$.

Definition 50. Let $\varphi, g: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathscr{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ be the fuzzy-valued functions. Define the fuzzy infimal convolution φ and g as follows:

$$\varphi \Box g : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathscr{F}} \cup \left\{+\tilde{\infty}\right\} \cup \left\{-\tilde{\infty}\right\}, \tag{80}$$

by

$$(\varphi \Box g)(x) \coloneqq \inf\left\{\tilde{\theta} \in \mathbb{R}_{\mathscr{F}} : (x, \tilde{\theta}) \in epi(\varphi) \oplus \operatorname{epi}(g)\right\}. \tag{81}$$

Note that if φ and g are fuzzy-valued convex functions, then $epi(\varphi)$, $epi(g)$ in $\mathbb{R}^n \times \mathbb{R}_{\mathscr{F}}$ are convex sets, therefore $epi(\varphi) \oplus epi(g)$ in $\mathbb{R}^n \times \mathbb{R}_{\mathscr{F}}$ is a convex set, hence by Theorem 49, $\varphi \Box g$ is a fuzzy-valued convex function. The terminology is motivated by the case where φ and g are fuzzy-valued functions, $\varphi, g: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathscr{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$. Then, $\varphi \Box g$ can also be defined as

$$\begin{aligned} (\varphi \Box g)(x) &= \inf\left\{\tilde{\theta} \in \mathbb{R}_{\mathscr{F}} : (x, \tilde{\theta}) \in epi(\varphi) \oplus \operatorname{epi}(g)\right\} \\ &= \inf\left\{\tilde{\theta} \in \mathbb{R}_{\mathscr{F}} : \exists \left(x_1, \tilde{\theta}_1\right) \in \operatorname{epi}(\varphi), \exists \left(x_2, \tilde{\theta}_2\right) \in \operatorname{epi}(g) : (x, \tilde{\theta}) = \left(x_1, \tilde{\theta}_1\right) \oplus \left(x_2, \tilde{\theta}_2\right)\right\} \\ &= \inf\left\{\tilde{\theta} \in \mathbb{R}_{\mathscr{F}} : \tilde{\theta} = \tilde{\theta}_1 \oplus \tilde{\theta}_2, x = x_1 + x_2, \varphi(x_1) \preccurlyeq \tilde{\theta}_1, g(x_2) \preccurlyeq \tilde{\theta}_2\right)\right\} \\ &= \inf\left\{\tilde{\theta} \in \mathbb{R}_{\mathscr{F}} : \varphi(x_1) \oplus g(x_2) \preccurlyeq \tilde{\theta}, x = x_1 + x_2, x_1, x_2 \in \mathbb{R}^n\right\} \\ &= \inf\{\varphi(x_1) \oplus g(x_2) : x = x_1 + x_2, x_1, x_2 \in \mathbb{R}^n\} \\ &= \inf\{\varphi(y) \oplus g(x-y) : y \in \mathbb{R}^n\}. \end{aligned} \tag{82}$$

Hence.

$$(\varphi\square g)(x) = \inf\{\varphi(y)\oplus g(x-y):\ y\in\mathbb{R}^n\}, \tag{83}$$

which is analogous to the formula for fuzzy integral convolution

$$(\varphi * g)(x) = \int_{-\infty}^{+\infty}\varphi(y)\odot g(x-y)\,dy, \tag{84}$$

and $\varphi\square g$ is exact at $x\in\mathbb{R}^n$, if $(\varphi\square g)(x)=\min_{y\in\mathbb{R}^n}\varphi(y)\oplus g(x-y)$, i.e., there exists $y\in\mathbb{R}^n$ so that

$$(\varphi\square g)(x) = \varphi(y)\oplus g(x-y); \tag{85}$$

$\varphi\square g$ is exact if it is exact at every point of its domain, in which case it is denoted by $\varphi\boxdot g$.

Proposition 51. *Let φ, g, and h be the extended fuzzy-valued functions from $\mathbb{R}^n$ to $\mathbb{R}_{\mathcal{F}}\cup\{+\tilde{\infty}\}$. Then, the following cases hold:*

(1) $\varphi\square g = g\square\varphi$;

(2) $\varphi\square(g\square h) = (\varphi\square g)\square h$.

Proof. For the proof of (1), we can see that

$$(\varphi\square g)(x) = \inf\{\varphi(x_1)\oplus g(x_2):\ x = x_1+x_2,\ x_1,x_2\in\mathbb{R}^n\} = (g\square\varphi)(x). \tag{86}$$

Hence, $\varphi\square g = g\square\varphi$. Also, for the proof of (10), we have

$$\begin{aligned}((\varphi\square g)\square h)(x) &= \inf\{(\varphi\square g)(x_1)\oplus h(x_2):\ x = x_1+x_2, x_1,x_2\in\mathbb{R}^n\}\\ &= \inf\{\inf\{\varphi(t_1)\oplus g(t_2):\ x_1 = t_1+t_2, t_1,t_2\in\mathbb{R}^n\}\oplus h(x_2):\ x = x_1+x_2, x_1,x_2\in\mathbb{R}^n\}\\ &= \inf\{\varphi(t_1)\oplus g(t_2)\oplus h(x_2):\ x = t_1+t_2+x_2, t_1,t_2,x_2\in\mathbb{R}^n\}.\end{aligned} \tag{87}$$

On the other hand, we have

$$\begin{aligned}\varphi\square(g\square h)(x) &= \inf\{\varphi(x_1)\oplus(g\square h)(x_2):\ x = x_1+x_2, x_1,x_2\in\mathbb{R}^n\}\\ &= \inf\{\varphi(x_1)\oplus\inf\{g(t_1)\oplus h(t_2):\ x_2 = t_1+t_2, t_1,t_2\in\mathbb{R}^n\}:\ x = x_1+x_2, x_1,x_2\in\mathbb{R}^n\}\\ &= \inf\{\varphi(x_1)\oplus g(t_1)\oplus h(t_2):\ x = x_1+t_1+t_2, t_1,t_2\in\mathbb{R}^n\}.\end{aligned} \tag{88}$$

Hence, $(\varphi\square g)\square h = \varphi\square(g\square h)$. □

Example 1. Consider $C\subseteq\mathbb{R}^n$ to be a nonempty convex set and $x_0\in\mathbb{R}^n$. The fuzzy distance of x_0 from C is defined by

$$d(x_0,C) \coloneqq \langle -1,0,1\rangle\odot\inf_{y\in C}\|x_0-y\|. \tag{89}$$

We show that $d(x_0,C)$ is a fuzzy-valued convex function. Since C is a convex set, then by Theorem 43, $\tilde{\mathrm{I}}_C$ is a fuzzy-valued convex function. Let $\varphi(x) = \langle -1,0,1\rangle\odot\|x\|, \forall x\in\mathbb{R}^n$. Therefore, φ is a fuzzy-valued convex function. By Definition 50, hence $\varphi\square\tilde{\mathrm{I}}_C$ is a fuzzy-valued convex function. Consider

$$\begin{aligned}\left(\varphi\square\tilde{\mathrm{I}}_C\right)(x_0) &= \inf_{y\in\mathbb{R}^n}\left\{\tilde{\mathrm{I}}_C(x_0)\oplus\langle -1,0,1\rangle\odot\|x_0-y\|\right\}\\ &= \inf_{y\in\mathbb{R}^n}\left\{\tilde{\mathrm{I}}_C(x_0)\oplus\varphi(x_0-y)\right\}\\ &= \inf_{y\in C}\{\varphi(x_0-y)\} = \langle -1,0,1\rangle\odot\inf_{y\in C}\|x_0-y\| = d(x_0,C).\end{aligned} \tag{90}$$

So, $(\varphi \square \widetilde{\mathrm{I}}_C)(x_0) = d(x_0, C)$. According to Definition 50, $d(x_0, C)$ is a fuzzy-valued convex function.

Example 2. Consider $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\widetilde{\infty}\}$ is an extended fuzzy-valued function and let $y \in \mathbb{R}^n$. We show that $\widetilde{\mathrm{I}}_{\{y\}} \boxdot \varphi = \iota_y \varphi$, where

$$\left(\iota_y \varphi\right)(x) := \varphi(x - y), \quad \forall x \in \mathbb{R}^n, \tag{91}$$

is the translation of the extended fuzzy-valued function φ by $y \in \mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ be arbitrary. Then

$$\left(\widetilde{\mathrm{I}}_{\{y\}} \boxdot \varphi\right)(x) = \min_{z \in \mathbb{R}^n} \left\{\widetilde{\mathrm{I}}_{\{y\}}(z) \oplus \varphi(x - z)\right\} = \varphi(x - y) = \left(\iota_y \varphi\right)(x). \tag{92}$$

Hence, $\widetilde{\mathrm{I}}_{\{y\}} \boxdot \varphi = \iota_y \varphi$.

7. The Directional g-Derivative for Extended Fuzzy-Valued Convex Functions

Now, we introduce the directional g-derivative for extended fuzzy-valued convex functions and their properties are discussed.

Definition 52. Suppose that φ is an extended fuzzy-valued function, $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\widetilde{\infty}\} \cup \{-\widetilde{\infty}\}$ and $x_0, x \in \mathbb{R}^n$. The directional g-derivative of φ at x_0 in the direction x is as follows:

$$\varphi'_g(x_0, x) := \lim_{t \longrightarrow 0^+} \frac{\varphi(x_0 + tx) \ominus_g \varphi(x_0)}{t}. \tag{93}$$

If $\varphi'_g(x_0, x) \in \mathbb{R}_{\mathcal{F}} \cup \{+\widetilde{\infty}\} \cup \{-\widetilde{\infty}\}$ satisfying (93) exist. Note that if it exists ($+\widetilde{\infty}$ and $-\widetilde{\infty}$ being allowed as limits).

Theorem 53. *Suppose that $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\widetilde{\infty}\} \cup \{-\widetilde{\infty}\}$ is a proper fuzzy-valued convex function and $x_0 \in dom(\varphi)$. Then,*

(1) $\varphi'_g(x_0, x)$ exist, $\forall x \in \mathbb{R}^n$.

(2) $\varphi'_g(x_0, .)$ is fuzzy positively homogeneous and fuzzy-valued convex.

Proof. For proof of (1), let $x \in \mathbb{R}^n$ be arbitrary. Define $g_{x_0, x_0 + x}: \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\widetilde{\infty}\} \cup \{-\widetilde{\infty}\}$ by

$$\underset{x_0, x_0 + x}{g}(t) := \varphi((1 - t)x_0 + t(x_0 + x)) = \varphi(x_0 + tx), \quad \forall t \in \mathbb{R}. \tag{94}$$

Since by the hypothesis, φ is the proper fuzzy-valued convex function, then $g: \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\widetilde{\infty}\} \cup \{-\widetilde{\infty}\}$ is a proper fuzzy-valued convex function, therefore by Theorem 41, $g'_{+g}(0)$ exists where it is the right g-derivative of the proper fuzzy-valued convex function at $t = 0$, as follows:

$$\begin{aligned} g'_{+g}(0) &= \lim_{t \longrightarrow 0^+} \frac{g(t) \ominus_g g(0)}{t} \\ &= \lim_{t \longrightarrow 0^+} \frac{\varphi(x_0 + tx) \ominus_g \varphi(x_0)}{t} = \varphi'_g(x_0, x). \end{aligned} \tag{95}$$

Hence $\varphi'_g(x_0, x)$ exists, for all $x \in \mathbb{R}^n$. For proof of (10), let $\theta > 0, x \in \mathbb{R}^n$. Consider

$$\begin{aligned} \varphi'_g(x_0, \theta x) &= \lim_{t \longrightarrow 0^+} \frac{\varphi(x_0 + t(\theta x)) \ominus_g \varphi(x_0)}{t\theta} \\ &= \theta \odot \lim_{t\theta \longrightarrow 0^+} \frac{\varphi(x_0 + (t\theta)x) \ominus_g \varphi(x_0)}{t\theta} \end{aligned} \tag{96}$$

get $s = t\theta$, when $t \longrightarrow 0^+ \Rightarrow t\theta \longrightarrow 0^+, \forall \theta > 0 \Rightarrow s \longrightarrow 0^+$

$$= \theta \odot \lim_{s \longrightarrow 0^+} \frac{\varphi(x_0 + sx) \ominus_g \varphi(x_0)}{s} = \theta \odot \varphi'_g(x_0, x). \tag{97}$$

Hence, $\varphi'_g(x_0, \theta x) = \theta \odot \varphi'_g(x_0, x)$.

The convexity of this function is as follows:

Let $y, z \in \mathbb{R}^n, 0 < \theta < 1$. Consider

$$\varphi'_g(x_0, \theta y + (1 - \theta)z) = \lim_{t \longrightarrow 0^+} \frac{\varphi(x_0 + t(\theta y + (1 - \theta)z)) \ominus_g \varphi(x_0)}{t}. \tag{98}$$

Let $x_0 = \theta x_0 + (1-\theta)x_0$, then

$$\begin{aligned}
&= \lim_{t\longrightarrow 0^+} \frac{\varphi(\theta(x_0+ty)+(1-\theta)(x_0+tz))\ominus_g \varphi(x_0)}{t}\\
&\preccurlyeq \lim_{t\longrightarrow 0^+} \frac{\theta\odot\varphi(x_0+ty)\oplus(1-\theta)\odot\varphi(x_0+tz)\ominus_g \varphi(x_0)}{t}\\
&= \theta\odot \lim_{t\longrightarrow 0^+} \frac{\varphi(x_0+ty)\ominus_g \varphi(x_0)}{t}\oplus(1-\theta)\odot \lim_{t\longrightarrow 0^+} \frac{\varphi(x_0+tz)\ominus_g \varphi(x_0)}{t}\\
&= \theta\odot\varphi_g'(x_0,y)\oplus(1-\theta)\odot\varphi_g'(x_0,z).
\end{aligned} \tag{99}$$

Hence, $\varphi_g'(x_0,x)$ is an extended fuzzy-valued convex function. ☐

Proposition 54. *Let $\varphi:\mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ be a proper fuzzy-valued convex function and $x_0\in dom(\varphi)$ be so that $[\varphi(x)]_r = [\varphi_r^-(x),\varphi_r^+(x)]$. Assume that $\varphi_r^-(x)$ and $\varphi_r^+(x)$ of the proper real-valued convex functions are directional differentiable at x_0 in the direction of x, uniformly w.r.t. $r\in[0,1]$. Then, φ has a directional g-derivative at x_0 in the direction of x as follows:*

$$\varphi_g'(x_0,x)_r = \left[\inf_{\beta\geq r}\min\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\},\sup_{\beta\geq r}\max\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\}\right]. \tag{100}$$

Proof. According to Proposition 13, we get

$$\begin{aligned}
&\left[\frac{\varphi(x_0+tx)\ominus_g\varphi(x_0)}{t}\right]_r\\
&\quad=\left[\inf_{\beta\geq r}\min\left\{\frac{\varphi_\beta^-(x_0+tx)-\varphi_\beta^-(x_0)}{t},\frac{\varphi_\beta^+(x_0+tx)-\varphi_\beta^+(x_0)}{t}\right\},\right.\\
&\quad\left.\sup_{\beta\geq r}\max\left\{\frac{\varphi_\beta^-(x_0+tx)-\varphi_\beta^-(x_0)}{t},\frac{\varphi_\beta^+(x_0+tx)-\varphi_\beta^+(x_0)}{t}\right\}\right].
\end{aligned} \tag{101}$$

Since the proper real-valued convex functions $\varphi_r^-(x)$ and $\varphi_r^+(x)$ are directional differentiable at x_0 in the direction of x, we have

$$\begin{aligned}
&\lim_{t\longrightarrow 0^+}\left[\frac{\varphi(x_0+tx)\ominus_g\varphi(x_0)}{t}\right]_r\\
&\quad=\left[\inf_{\beta\geq r}\min\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\},\right.\\
&\quad\left.\sup_{\beta\geq r}\max\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\}\right],\quad \forall r\in[0,1].
\end{aligned} \tag{102}$$

Also, let us consider that if the functions $\varphi_r^-(x)$ and $\varphi_r^+(x)$ are left continuous w.r.t. $r\in(0,1]$ and right continuous at 0. From the definition of the directional derivative, for any $n\geq 1$, there exists a sequence $t_n>0$ such that $t_n\longrightarrow 0^+$ the quotients

$$\frac{\varphi_r^-(x_0+t_nx)-\varphi_r^-(x_0)}{t_n},\frac{\varphi_r^+(x_0+t_nx)-\varphi_r^+(x_0)}{t_n}, \tag{103}$$

as functions of $r\in[0,1]$ are left continuous at $r\in(0,1]$ and right continuous at 0. Also, for any $n\geq 1$, there exists a sequence $t_n>0$ such that $t_n\longrightarrow 0^+$, then the functions

$$\inf_{\beta\geq r}\min\left\{\frac{\varphi_\beta^-(x_0+t_nx)-\varphi_\beta^-(x_0)}{t_n},\frac{\varphi_\beta^+(x_0+t_nx)-\varphi_\beta^+(x_0)}{t_n}\right\}, \tag{104}$$

and

$$\sup_{\beta\geq r}\max\left\{\frac{\varphi_\beta^-(x_0+t_n x)-\varphi_\beta^-(x_0)}{t_n},\frac{\varphi_\beta^+(x_0+t_n x)-\varphi_\beta^+(x_0)}{t_n}\right\}, \tag{105}$$

satisfying the above properties. Thus, it follows that

$$\inf_{\beta\geq r}\min\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\}, \tag{106}$$

and

$$\sup_{\beta\geq r}\min\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\}, \tag{107}$$

as functions of $r\in[0,1]$ are left continuous at $r\in(0,1]$ and right continuous at 0. It is obvious to see that the function $\inf_{\beta\geq r}\min\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\}$ is increasing function w.r.t. $r\in[0,1]$ and the function $\sup_{\beta\geq r}$ min $\left\{\varphi_\beta^{'-}(x_0,x),\varphi_\beta^{'+}(x_0,x)\right\}$ is decreasing function w.r.t. $r\in[0,1]$, by Proposition 6, they define a fuzzy number. Consequently, the r-cuts $\varphi_g'(x_0,x)_r$ define a fuzzy number, by Lemma 8, the directional g-derivative with extended fuzzy-valued $\varphi_g'(x_0,x)$ exists at x_0 in the direction x. □

Below, we give a practical example that will illustrate well the directional g-derivative for the extended fuzzy-valued convex function.

Example 3. Let $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_{\mathcal{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ be an extended fuzzy-valued convex function defined by

$$\varphi(x_0)=\langle -1,0,1\rangle\odot|x_0|;\quad \forall x_0\in\operatorname{dom}(\varphi), \tag{108}$$

its r-cuts, $r\in[0,1]$, are defined by

$$[\varphi(x_0)]_r=[\varphi_r^-(x_0),\varphi_r^+(x_0)]=[(r-1)|x_0|,(1-r)|x_0|]. \tag{109}$$

For all $r\in[0,1]$, the functions $\varphi_r^-(x_0)$ and $\varphi_r^+(x_0)$ are extended real-valued differentiable at each point $0\neq x_0\in dom(\varphi)$, then $\varphi_r^{'-}(x_0)=(r-1)x_0/|x_0|$ and $\varphi_r^{'+}(x_0)=(1-r)x_0/|x_0|$. Now, for all $r\in[0,1]$, the two functions $\varphi_r^-(x_0)$ and $\varphi_r^+(x_0)$ are not differentiable at $x_0=0$. However, for all $r\in[0,1]$, the functions $\varphi_r^{'-}(x_0,x)=(r-1)|x|$ and $\varphi_r^{'+}(x_0,x)=(1-r)|x|$ are real-valued directional differentiable at $x_0=0$ in any direction $x\in\mathbb{R}$ and satisfy the conditions in Proposition 6, indeed for any direction $x\in\mathbb{R}$ and $r\in[0,1]$ we have

$$\begin{aligned}\varphi_g'(0,x)_r&=\left[\inf_{\beta\geq r}\min\left\{\varphi_\beta^{'-}(0,x),\varphi_\beta^{'+}(0,x)\right\},\sup_{\beta\geq r}\max\left\{\varphi_\beta^{'-}(0,x),\varphi_\beta^{'+}(0,x)\right\}\right]\\&=\left[\inf_{\beta\geq r}\min\{(\beta-1)|x|,(1-\beta)|x|\},\sup_{\beta\geq r}\max\{(\beta-1)|x|,(1-\beta)|x|\}\right]\\&=[(r-1)|x|,(1-r)|x|]=[r-1,1-r]|x|,\quad\forall x\in\mathbb{R}\end{aligned} \tag{110}$$

it follows that

$$\varphi_g'(0,x)_r=[r-1,1-r]|x|,\quad \forall x\in\mathbb{R}. \tag{111}$$

Hence, φ is directionally g-differentiable at $x_0=0$ in any direction $x\in\mathbb{R}$.

8. Conclusion

The concepts of g-difference and g-differentiability were introduced for fuzzy-valued functions in 2013 by Bede and Stefanini [19] which is the generalization concept of gH-difference and gH-differentiability. Here, we defined the fuzzy-valued convex functions whose range is extended fuzzy numbers, and some of their properties were expressed. Moreover, several important fuzzy concepts such as indicator function, epigraph, infimal convolution, and directional g-derivative with their properties for the extended fuzzy-valued convex functions have been stated and discussed. It is worth pursuing follow-up research by considering g-subgradient and g-subdifferential for the extended fuzzy-valued convex function. In this way, in the next studies and research, we propose the concepts of g-subgradient and g-subdifferential, which play an important role in extended fuzzy-valued optimization.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

TA is the supervisor of this study and was a major contributor to methodology, investigation, and validation. RS and MRBS worked on resources, investigation, and formal analysis of this study. OS, UFG, MRS, and SN worked on the software, writing reviews, and editing and validating the results. All authors have main contributions in writing the original draft preparation and in also writing a review and editing the paper. All authors read and approved the final manuscript.

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