

## Research Article

# Usage of the Fuzzy Adomian Decomposition Method for Solving Some Fuzzy Fractional Partial Differential Equations

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Received 25 February 2023; Revised 23 October 2023; Accepted 25 November 2023; Published 5 January 2024

Academic Editor: A. Ghareeb

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In this study, we examine the numerical solutions of nonlinear fuzzy fractional partial differential equations under the Caputo derivative utilizing the technique of fuzzy Adomian decomposition. This technique is used as an alternative method for obtaining approximate fuzzy solutions to various types of fractional differential equations and also investigated some new existence and uniqueness results of fuzzy solutions. Some examples are given to support the effectiveness of the proposed technique. We present the numerical results in graphical form for different values of fractional order and uncertainty  $\gamma \in [0, 1]$ .

#### 1. Introduction

During the last few decades, fractional calculus has been the focus of many studies due to its frequent appearance in many applications, such as viscoelasticity, physics, biology, signal processing, engineering, economics, and financial markets [1–5]. One of its most important applications is fractional partial differential equations (FPDEs), as most natural phenomena can be modeled by using such types of equations. The frequent use of FPDEs in engineering and scientific applications has led many researchers in this field to develop new results in theoretical and applied research methods [6–14]. Recently, several authors have solved linear and nonlinear FPDEs by using different methods, such as the homotopy perturbation method, Adomian decomposition method, variational iteration method, and homotopy analysis method, as mentioned in [15–22].

Physical models of real-world phenomena often contain uncertainty, which can come from a variety of sources. Fuzzy set theory, introduced by Zadeh [23] in 1965, is a suitable tool for modeling this uncertainty, as it can represent imprecise and vague concepts. Chang and Zadeh extended the concept of fuzzy sets by introducing the notions of fuzzy control and fuzzy mapping [24]. Many researchers have built

on the concept of fuzzy mapping and control to develop elementary fuzzy calculus [25-29]. This led to detailed studies of fuzzy fractional differential and integral equations in the field of physical science. Agarwal et al. [30] introduced the concept of solving fuzzy fractional differential equations (FFDEs). Authors in [31, 32] used this concept to prove the uniqueness and existence of solutions to initial value problems involving FFDEs. Long et al. [33] investigated the existence and uniqueness of fuzzy fractional partial differential equations (FFPDEs). Salahshour et al. [34] used Laplace transforms to find solutions for FFDEs. They converted the FFDEs into algebraic equations using Laplace transforms, which made them easier to solve. They also found a closed-form solution for one of the FFDEs. Allahviranloo et al. [35] presented an explicit solution for FFDEs. They found a solution for FFDE that can be written in simple and closed form. This is a significant achievement, as it makes it easier to use FFDEs in practical applications.

In recent years, numerous researchers have used different numerical methods to solve FFDEs analytically or numerically [36–39]. The Adomian decomposition method, introduced by mathematician Adomian [40] in 1984, is a simple and effective method in both linear and nonlinear differential equations. This method is a powerful tool for approximating the solution of fuzzy differential equations. It works by expressing the solution as an infinite series, which often converges to the exact solution. Although there are a few potential limitations or challenges associated with the Adomian decomposition method, such as it can be computationally expensive for complex problems, it is a good choice for problems that are difficult or impossible to solve using other methods. Recently, several researchers used this method for solving various linear and nonlinear systems in a fuzzy environment. Das and Roy [41] studied the numerical solution of linear fuzzy fractional differential equations by applying the fuzzy Adomian decomposition method. Osman et al. [42] presented a comparison of the fuzzy Adomian decomposition method with the fuzzy variational iteration method for solving fuzzy heat-like and wave-like equations with variable coefficients under  $g\mathcal{H}$ differentiability. Ullah et al. [43] used fuzzy Laplace transform along with Adomian decomposition to obtain general numerical results for the complex population dynamical model under the fuzzy Caputo fractional derivative [44–48].

Motivated by the above research, in this paper, we investigate some existence, uniqueness, and numerical results by using the fuzzy Adomian decomposition method of the following nonlinear fuzzy fractional partial differential equation (FFPDE):

$${}^{c}\mathfrak{D}_{t}^{\varphi}\nu(\xi,t,\gamma) = \aleph(\xi,t,\gamma,\nu,\nu_{\xi},\nu_{\xi\xi}) + \mathfrak{g}(\xi,t,\gamma),$$
  

$$\nu(\xi,0,\gamma) = E(\xi,\gamma),$$
(1)

where  ${}^{c}\mathfrak{D}_{t}^{\varphi}$  is the fuzzy Caputo derivative with respect to t,  $0 < \varphi < 1, \gamma \in [0, 1], (\xi, t) \in J = [0, c] \times [0, d], c, d \in \mathbb{R}^{+}, \mathbb{N}$  is a nonlinear fuzzy-valued function, and  $E, \mathfrak{g}$  are known fuzzy-valued functions.

#### 2. Preliminary Concepts

In this section, we present certain definitions and theorems that will be helpful in our further discussion.

Definition 1 (see [49]). A fuzzy number is a mapping  $\mathbf{m}: \mathbb{R} \longrightarrow [0, 1]$  satisfying the following features:

- (1) For  $\xi_0 \in \mathbb{R}$ , **m** is normal, it means  $\mathbf{m}(\xi_0) = 1$
- (2) For  $\xi_1, \xi_2, \in \mathbb{R}$  and  $t \in [0, 1]$ , **m** is convex such that

$$\mathfrak{m}(t\xi_1 + (1-t)\xi_2) \ge \min\{\mathfrak{m}(\xi_1), \mathfrak{m}(\xi_2)\}, \qquad (2)$$

- (3) m is semicontinuous
- (4)  $cl{\xi \in \mathbb{R}, \mathfrak{m}(\xi) > 0}$  is compact

*Definition 2* (see [50]). A fuzzy number  $\mathfrak{m}$  can be represented in parametric form as  $[\mathfrak{m}(\gamma), \overline{\mathfrak{m}}(\gamma)]$ , for  $0 \le \gamma \le 1$ , if and only if

- (i)  $\underline{\mathbf{m}}(\gamma)$  is increasing bounded function and left continuous over (0, 1]
- (ii) m
   (γ) is decreasing bounded function and right continuous over (0, 1]
- (iii)  $\underline{\mathfrak{m}}(\gamma) \leq \overline{\mathfrak{m}}(\gamma)$

Here, we employ the notations listed as follows:

- (i)  $\mathcal{F}_R$  is the set of all fuzzy numbers on  $\mathbb{R}$
- (ii)  $\mathfrak{C}[J, \mathscr{F}_R]$  is a space of all continuous fuzzy-valued functions which are on  $J \in \mathbb{R}^2$
- (iii)  $\mathfrak{Q}[J, \mathscr{F}_R]$  is the set of Lebesque integrable for fuzzyvalued functions on **B**, where  $\mathbf{B} \in \mathbb{R}^m, m \in \mathbb{N}$

The set of a fuzzy number  $\mathfrak{m}(\xi) \in \mathscr{F}_R$  in the  $\gamma$ -level form is denoted by  $[\mathfrak{m}]^{\gamma}$  and defined as:  $[\mathfrak{m}]^{\gamma} = \begin{cases} \{\xi \in \mathbb{R} \mid \mathfrak{m}(\xi) \ge \gamma\}, if \ 0 < \gamma \le 1, \\ cl(supp \ \mathfrak{m}), if \ \gamma = 0. \end{cases}$ For any  $\mathfrak{p}, \mathfrak{q} \in \mathscr{F}_R$ . If there exists  $z \in \mathscr{F}_R$  such that

For any  $\mathfrak{p}, \mathfrak{q} \in \mathscr{F}_R$ . If there exists  $z \in \mathscr{F}_R$  such that  $\mathfrak{p} = \mathfrak{q} + z$ , then z is called the Hukuhara difference of  $\mathfrak{p}$  and  $\mathfrak{q}$  and it is denoted by  $\mathfrak{p} \ominus \mathfrak{q}$ .

*Definition 3* (see [49]). The generalized Hukuhara difference of two fuzzy numbers  $\mathfrak{p}, \mathfrak{q} \in \mathcal{F}_R$  (*g* $\mathcal{H}$  difference for short) is defined as the element  $\mathbf{z} \in \mathcal{F}_R$  such that

$$\mathfrak{p}_{\mathcal{G}}\mathfrak{g}_{\mathcal{H}}\mathfrak{q} = \mathbf{z} \Leftrightarrow (i) \mathfrak{p} = \mathfrak{q} + \mathbf{z} \text{ or } (ii) \mathfrak{q} = \mathfrak{p} + (-1)\mathbf{z}.$$
(3)

Note: if case (i) exists, then there is no need to consider case (ii), but if both cases exist, it means that both types of difference are the same and equal.

Allahviranloo [49] introduced the definition of the fuzzy partial derivative as follows:

Definition 4. Let  $\mathcal{G}: J \longrightarrow \mathcal{F}_R$ , then  $g\mathcal{H}$ -partial derivative of the first order at the point  $(\mathfrak{z}_0, \mathfrak{t}_0) \in J$  with respect to variables  $\mathfrak{z}, \mathfrak{t}$  is denoted by  $\partial \mathcal{G}(\mathfrak{z}_0, \mathfrak{t}_0)/\partial \mathfrak{z}, \partial \mathcal{G}(\mathfrak{z}_0, \mathfrak{t}_0)/\partial \mathfrak{t}$  and given by

$$\frac{\partial \mathscr{G}(\mathfrak{z}_{0}, \mathfrak{t}_{0})}{\partial \mathfrak{z}} = \lim_{h \longrightarrow 0} \frac{\mathscr{G}(\mathfrak{z}_{0} + h, \mathfrak{t}_{0}) \ominus_{g\mathscr{H}} \mathscr{G}(\mathfrak{z}_{0}, \mathfrak{t}_{0})}{h},$$

$$\frac{\partial \mathscr{G}(\mathfrak{z}_{0}, \mathfrak{t}_{0})}{\partial \mathfrak{t}} = \lim_{k \longrightarrow 0} \frac{\mathscr{G}(\mathfrak{z}_{0}, \mathfrak{t}_{0} + k) \ominus_{g\mathscr{H}} \mathscr{G}(\mathfrak{z}_{0}, \mathfrak{t}_{0})}{k},$$
(4)

provided that  $\partial \mathscr{G}(\mathfrak{z}_0, \mathfrak{t}_0)/\partial \mathfrak{z}$  and  $\partial \mathscr{G}(\mathfrak{z}_0, \mathfrak{t}_0)/\partial \mathfrak{t} \in \mathscr{F}_R$ .

Definition 5. Let  $\mathscr{G}: J \longrightarrow \mathscr{F}_R$  be  $g\mathscr{H}$ -partial differentiable with respect to  $\mathfrak{z}$  at  $(\mathfrak{z}_0, \mathfrak{t}_0) \in J$ . We say that

 
 (1) S is (i) gℋ-partial differentiable with respect to 3 at (30, t₀) ∈ J if

$$\left[\frac{\partial \mathscr{G}(\mathfrak{z}_{0},\mathfrak{t}_{0},\gamma)}{\partial\mathfrak{z}}\right] = \left[\frac{\partial \mathscr{G}(\mathfrak{z}_{0},\mathfrak{t}_{0},\gamma)}{\partial\mathfrak{z}}, \frac{\partial \overline{\mathscr{G}}(\mathfrak{z}_{0},\mathfrak{t}_{0},\gamma)}{\partial\mathfrak{z}}\right], \quad \forall \gamma \in [0,1].$$

$$(5)$$

(2) G is (ii) gℋ-partial differentiable with respect to 3 at (30, t0) ∈ J if

$$\frac{\partial \mathscr{G}(\mathfrak{z}_{0},\mathfrak{t}_{0},\gamma)}{\partial \mathfrak{z}} = \left[\frac{\partial \overline{\mathscr{G}}(\mathfrak{z}_{0},\mathfrak{t}_{0},\gamma)}{\partial \mathfrak{z}}, \frac{\partial \underline{\mathscr{G}}(\mathfrak{z}_{0},\mathfrak{t}_{0},\gamma)}{\partial \mathfrak{z}}\right], \quad \forall \gamma \in [0,1].$$

$$(6)$$

The following Newton-Leibniz formula is given in [33].

**Lemma 6** (Newton–Leibniz formula). Let  $\mathscr{P} \in \mathfrak{C}(\mathbb{R}^2, \mathscr{F}_R)$ .

 If 𝒫 is (i) gℋ-partial differentiable with respect to 𝔅 such that the type of gℋ-partial differentiability does not change on ℝ × [b, 𝔅], then

$$\int_{b}^{\mathfrak{y}} \frac{\partial \mathscr{P}(\mathbf{x}, \mathfrak{y})}{\partial \mathfrak{y}} dt = \mathscr{P}(\mathbf{x}, \mathfrak{y}) \ominus \mathscr{P}(\mathbf{x}, b).$$
(7)

(2) If P is (ii) gℋ-partial differentiable with respect to y such that the type of gℋ-partial differentiability does not change on R × [b, y], then

$$\int_{b}^{\mathfrak{y}} \frac{\partial \mathscr{P}(x,\mathfrak{y})}{\partial \mathfrak{y}} dt = (-1)\mathscr{P}(x,b) \ominus (-1)\mathscr{P}(x,\mathfrak{y}).$$
(8)

The authors in [34, 35] have defined the concepts of Riemann–Liouville integral and Caputo's  $g\mathcal{H}$ -derivative of fuzzy-valued functions as follows.

Definition 7. Let  $v(\mathfrak{z}) \in \mathfrak{C}[\mathscr{I}, \mathscr{F}_R] \cap \mathfrak{L}[\mathscr{I}, \mathscr{F}_R], \mathscr{I} \in \mathbb{R}$ . The fuzzy fractional integral in the Riemann–Liouville sense of order  $\varphi > 0$  is defined as

$$\mathfrak{F}^{\varphi}\upsilon(\mathfrak{z},\gamma) = \big[\mathfrak{F}^{\varphi}\,\underline{\upsilon}(\mathfrak{z},\gamma),\mathfrak{F}^{\varphi}\overline{\upsilon}(\mathfrak{z},\gamma)\big], \quad \gamma \in [0,1], \qquad (9)$$

where

$$\mathfrak{T}^{\varphi} \, \underline{v}(\mathfrak{z}, \gamma) = \frac{1}{\Gamma(\varphi)} \int_{0}^{\mathfrak{z}} (\mathfrak{z} - \tau)^{\varphi - 1} \, \underline{v}(\tau, \gamma) \mathrm{d}\tau, \quad \mathfrak{z} > 0,$$

$$\mathfrak{T}^{\varphi} \overline{v}(\mathfrak{z}, \gamma) = \frac{1}{\Gamma(\varphi)} \int_{0}^{\mathfrak{z}} (\mathfrak{z} - \tau)^{\varphi - 1} \overline{v}(\tau, \gamma) \mathrm{d} \quad \tau, \mathfrak{z} > 0.$$

$$(10)$$

Definition 8. Let  $v(\mathfrak{z}) \in \mathfrak{C}[\mathcal{F}, \mathcal{F}_R] \cap \mathfrak{L}[\mathcal{F}, \mathcal{F}_R]$ . Then, the fuzzy fractional Caputo's  $g\mathcal{H}$ -derivative under (i)  $g\mathcal{H}$ -differentiability is given as follows:

$${}^{c}\mathfrak{D}_{\mathfrak{z}}^{\varphi}\upsilon(\mathfrak{z},\mathfrak{y}) = \Big[{}^{c}\mathfrak{D}_{\mathfrak{z}}^{\varphi}\underline{\upsilon}(\mathfrak{z},\mathfrak{y}), {}^{c}\mathfrak{D}_{\mathfrak{z}}^{\varphi}\overline{\upsilon}(\mathfrak{z},\mathfrak{y})\Big], \qquad (11)$$

and under (ii)  $g\mathcal{H}$ -differentiability is given as follows:

$${}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{z}}\upsilon(\mathfrak{z},\gamma) = \Big[{}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{z}}\overline{\upsilon}(\mathfrak{z},\gamma), {}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{z}}\underline{\upsilon}(\mathfrak{z},\gamma)\Big], \qquad (12)$$

where

$${}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{z}}\underline{\nu}(\mathfrak{z},\gamma)] = \frac{1}{\Gamma(\ell-\varphi)} \int_{0}^{\mathfrak{z}} (\mathfrak{z}-\tau)^{\ell-\varphi-1} \frac{\langle \ell \rangle}{\underline{\nu}}(\tau,\gamma) d\tau,$$

$${}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{z}}\overline{\nu}(\mathfrak{z},\gamma) = \frac{1}{\Gamma(\ell-\varphi)} \int_{0}^{\mathfrak{z}} (\mathfrak{z}-\tau)^{\ell-\varphi-1} \overline{\nu}^{\langle \ell \rangle}(\tau,\gamma) d\tau.$$

$$(13)$$

The Caputo derivative is a powerful tool for modeling and analyzing complex phenomena. It has several advantages over other fractional derivatives, such as its ability to use traditional initial and boundary conditions, its clear physical interpretation, and its mathematical tractability [51–53].

**Proposition 9** (see [49]). If  $v(\varkappa)$ :  $[0, a] \longrightarrow \mathcal{F}_R$  is an integrable fuzzy function and  $\varphi > 0$ ,  $\beta > 0$ , then

$$(\mathfrak{T}^{\varphi})(\mathfrak{T}^{\beta})v(\varkappa) = (\mathfrak{T}^{\varphi+\beta})v(\varkappa), \quad \varkappa \in [0,a].$$
(14)

**Theorem 10** (see [54]) (Banach contraction principle). Let (N, d) be a complete metric space, then each contraction mapping  $T: N \longrightarrow N$  has a unique fixed point z of T in N, that is, Tz = z.

#### 3. Existence and Uniqueness Results

Let  $0 < \varphi < 1$  and  $\nu \in \mathfrak{C}[J, \mathcal{F}_R]$ . The following equivalent formulations of equation (1) are satisfied. From Proposition 9 and Lemma 6, we have

Case (1): If v is (i)- $g\mathcal{H}$  differentiable, then

$$\nu(\xi, t, \gamma) = E(\xi, \gamma) + \mathfrak{F}_t^{\varphi} \Big[ \aleph \Big( \xi, t, \gamma, \nu, \nu_{\xi}, \nu_{\xi\xi} \Big) + \mathfrak{g}(\xi, t, \gamma) \Big].$$
(15)

(19)

Case (2): If  $\nu$  is (ii)- $g\mathcal{H}$  differentiable, then

$$\nu(\xi, t, \gamma) = E(\xi, \gamma) \ominus (-1) \mathfrak{I}_{t}^{\varphi} \Big[ \aleph \Big( \xi, t, \gamma, \nu, \nu_{\xi}, \nu_{\xi\xi} \Big) + \mathfrak{g}(\xi, t, \gamma) \Big].$$

$$(16)$$

Note: in this paper, we study our results for case (1) only. Now, we will demonstrate the existence and uniqueness of the fuzzy solution to the problem (1), by introducing the following assumptions. A1: For any  $v, w \in \mathfrak{C}(J, \mathcal{F}_R)$ , there exist constants  $\mathfrak{K}_{1,1} = 1, 2, 3$  such that

$$\left|\Re\left(\xi,t,\gamma,\nu,\nu_{\xi},\nu_{\xi\xi}\right)-\aleph\left(\xi,t,\gamma,w,w_{\xi},w_{\xi\xi}\right)\right| \leq \Re_{1}|\nu(\xi,t,\gamma)-w(\xi,t,\gamma)| + \Re_{2}|\nu_{\xi}(\xi,t,\gamma)-w_{\xi}(\xi,t,\gamma)| + \Re_{3}|\nu_{\xi\xi}(\xi,t,\gamma)-w_{\xi\xi}(\xi,t,\gamma)|.$$

$$(17)$$

A2: There exist constants  $\mathfrak{M}_1$ ,  $\mathfrak{M}_2 > 0$  such that

$$\begin{aligned} \left| \nu_{\xi}(\xi,t,\gamma) - w_{\xi}(\xi,t,\gamma) \right| &\leq \mathfrak{M}_{1} |\nu(\xi,t,\gamma) - w(\xi,t,\gamma)|, \\ \left| \nu_{\xi\xi}(\xi,t,\gamma) - w_{\xi\xi}(\xi,t,\gamma) \right| &\leq \mathfrak{M}_{2} |u(\xi,t,\gamma) - w(\xi,t,\gamma)|. \end{aligned}$$
(18)

Let  $\mathfrak{C}(J, \mathcal{F}_R)$  be the Banach space of all continuous fuzzy valued functions with the norm

**Theorem 11.** Assume that the hypotheses A1 and A2 are fulfilled and

 $\|\nu\|_{\infty} = \sup\{|\nu(\xi, t)|; (\xi, t) \in J\}.$ 

$$\mathfrak{S} = \sup_{(\xi,t)\in I} \left| \varphi(\xi,t,\gamma) + \frac{1}{\Gamma(\varphi)} \int_{0}^{t} (t-\varrho)^{\varphi-1} \aleph(\xi,\varrho,\gamma,0,0,0) d\varrho \right| < +\infty,$$
(20)

where

If

$$\phi(\xi, t, \gamma) = E(\xi, \gamma) + \frac{1}{\Gamma(\varphi)} \int_0^t (t-\varrho)^{\varphi-1} \mathfrak{g}(\xi, \varrho, \gamma) d\varrho. \quad (21)$$

 $l^* = \frac{\left[\mathfrak{K}_1 + \mathfrak{M}_1\mathfrak{K}_2 + \mathfrak{M}_2\mathfrak{K}_3\right]\dot{\mathbf{d}}^{\varphi}}{\Gamma(\varphi + 1)} < 1, \qquad (22)$ 

then the problem (1) has a unique solution defined on J.

*Proof.* We define the operator  $S^*: \mathfrak{C}(J, \mathcal{F}_R) \longrightarrow \mathfrak{C}(J, \mathcal{F}_R)$  by

$$S^{*}(\nu(\xi,t,\gamma)) = E(\xi,\gamma) + \frac{1}{\Gamma(\varphi)} \int_{0}^{t} (t-\varrho)^{\varphi-1} \aleph(\xi,\varrho,\gamma,\nu,\nu_{\xi},\nu_{\xi\xi}) d\varrho + \frac{1}{\Gamma(\varphi)} \int_{0}^{t} (t-\varrho)^{\varphi-1} \mathfrak{g}(\xi,\varrho,\gamma) d\varrho,$$
(23)

for all  $(\xi, t) \in J$  and  $\gamma \in [0, 1]$ .

Assume that  $\nu \in \mathfrak{C}(J, \mathcal{F}_R)$ . Then, there exists  $\mathfrak{S}^* > 0$  such that  $\|\nu\| \leq \mathfrak{S}^*$ . Now, we prove that  $S^*$  is in the space  $\mathfrak{C}(J, \mathcal{F}_R)$ . We have

$$\begin{split} \left| S^* \nu(\xi, t, \gamma) \right| &\leq \left| \phi(\xi, t, \gamma) \right) + \frac{1}{\Gamma(\varphi)} \int_0^t \left( t - \varrho \right)^{\varphi - 1} \aleph(\xi, \varrho, \gamma, 0, 0, 0) d\varrho \right| \\ &+ \frac{1}{\Gamma(\varphi)} \int_0^t \left( t - \varrho \right)^{\varphi - 1} \left| \aleph \left( \xi, \varrho, \gamma, \nu, \nu_{\xi}, \nu_{\xi\xi} \right) - \aleph(\xi, \varrho, \gamma, 0, 0, 0) \right| d\varrho. \end{split}$$

$$(24)$$

From (20), we have

$$\begin{aligned} \left| S^{*} \left( \nu(\xi, t, \gamma) \right) \right| \\ &\leq \mathfrak{S} + \frac{1}{\Gamma(\varphi)} \int_{0}^{t} \left( t - \varrho \right)^{\varphi - 1} \left[ \mathfrak{K}_{1} | \nu(\xi, \varrho, \gamma)| + \mathfrak{K}_{2} | \nu_{\xi}(\xi, \varrho, \gamma)| + \mathfrak{K}_{3} | \nu_{\xi\xi}(\xi, \varrho, \gamma)| \right] d\varrho \\ &\leq \mathfrak{S} + \frac{1}{\Gamma(\varphi)} \int_{0}^{t} \left( t - \varrho \right)^{\varphi - 1} \left[ \mathfrak{K}_{1} + \mathfrak{M}_{1} \mathfrak{K}_{2} + \mathfrak{M}_{2} \mathfrak{K}_{3} \right] | \nu(\xi, \varrho, \gamma) | d\varrho \\ &\leq \mathfrak{S} + \frac{\left[ \mathfrak{K}_{1} + \mathfrak{M}_{1} \mathfrak{K}_{2} + \mathfrak{M}_{2} \mathfrak{K}_{3} \right]}{\Gamma(\varphi)} \int_{0}^{t} \left( t - \varrho \right)^{\varphi - 1} | \nu(\xi, \varrho, \gamma)| d\varrho. \end{aligned}$$

$$(25)$$

Applying supremum to both hands sides, we get

$$\left\|S^*\left(\nu\left(\xi,t,\gamma\right)\right)\right\| \le \mathfrak{S} + \frac{\|\nu\| \left[\mathfrak{K}_1 + \mathfrak{M}_1\mathfrak{K}_2 + \mathfrak{M}_2\mathfrak{K}_3\right]t^{\varphi}}{\Gamma(\varphi+1)}.$$
 (26)

For  $0 < t \le \dot{\mathbf{d}}$ , we get

$$\left\|S^{*}\left(\nu\left(\xi,t,\gamma\right)\right)\right\| \leq \mathfrak{S} + \frac{\mathfrak{S}^{*}\left[\mathfrak{K}_{1} + \mathfrak{M}_{1}\mathfrak{K}_{2} + \mathfrak{M}_{2}\mathfrak{K}_{3}\right]\dot{\mathbf{d}}^{\varphi}}{\Gamma\left(\varphi+1\right)} < +\infty,$$

$$(27)$$

it shows that  $S^*$  is in the space  $\mathfrak{C}(J, \mathcal{F}_R)$ . Thus,  $S^*$  maps  $\mathfrak{C}(J, \mathcal{F}_R)$  into itself.

Next, we establish that  $S^*$  is a contraction mapping. For  $\nu, \omega \in \mathfrak{C}(J, \mathcal{F}_R)$  and  $(\xi, t) \in J$ , we obtain

$$\begin{split} \left| \left( S^* \nu(\xi, t, \gamma), S^* \bar{\omega}(\xi, t, \gamma) \right) \right| \\ &\leq \frac{1}{\Gamma(\varphi_1)} \int_0^t \left( t - \varrho \right)^{\varphi - 1} \left| \aleph(\xi, \varrho, \gamma, \nu, \nu_{\xi}, \nu_{\xi\xi}) - \aleph(\xi, \varrho, \gamma, \bar{\omega}, \bar{\omega}_{\xi}, \bar{\omega}_{\xi\xi}) \right| d\varrho \\ &\leq \frac{1}{\Gamma(\varphi)} \int_0^t \left( t - \varrho \right)^{\varphi - 1} \left[ \Re_1 | \nu(\xi, \varrho, \gamma) - \bar{\omega}(\xi, \varrho, \gamma)| + \Re_2 | \nu_{\xi}(\xi, \varrho, \gamma) - \bar{\omega}_{\xi}(\xi, \varrho, \gamma)| \right] \\ &+ \Re_3 | \nu_{\xi\xi}(\xi, \tau, \gamma) - \bar{\omega}_{\xi\xi}(\xi, \tau, \gamma)| \right] d\varrho \\ &\leq \frac{1}{\Gamma(\varphi)} \int_0^t \left( t - \varrho \right)^{\alpha - 1} \left[ \Re_1 + \mathfrak{M}_1 \Re_2 + \mathfrak{M}_2 \Re_3 \right] | \nu(\xi, \varrho, \gamma) - \bar{\omega}(\xi, \varrho, \gamma)| d\varrho \\ &\leq \frac{\left[ \Re_1 + \mathfrak{M}_1 \Re_2 + \mathfrak{M}_2 \Re_3 \right]}{\Gamma(\varphi)} \| \nu - \bar{\omega} \| \int_0^t \left( t - \varrho \right)^{\varphi - 1} d\varrho. \end{split}$$

This implies that

$$\left\| \left( S^* \nu(\xi, t, \gamma), S^* \bar{\omega}(\xi, t, \gamma) \right) \right\| \leq \frac{\left[ \widehat{\mathbf{x}}_1 + \widehat{\mathbf{m}}_1 \widehat{\mathbf{x}}_2 + \widehat{\mathbf{m}}_2 \widehat{\mathbf{x}}_3 \right] \dot{\mathbf{d}}^{\varphi}}{\Gamma(\varphi + 1)} \| \nu - \bar{\omega} \|$$

$$\leq l^* \| \nu - \bar{\omega} \|.$$
(29)

Since  $l^* < 1$ , the operator  $S^*$  is a contraction mapping. Thus, by Banach's fixed point theorem, problem (1) has a unique fuzzy solution  $\nu$  defined on J.

## 4. Analysis of the Fuzzy Adomian Decomposition Method (FADM)

Now, we employ the FADM to analyze the system (1) as follows.

The decomposition method requires writing the nonlinear fuzzy fractional differential equation (1) in terms of general operator form as

$${}^{c}\mathfrak{D}_{t}^{\alpha}\nu(\xi,t,\gamma)+\widehat{L}\nu(\xi,t,\gamma)+\widehat{N}\nu(\xi,t,\gamma)=\mathfrak{g}(\xi,t,\gamma),\quad(30)$$

where  $\hat{L}$  represents a linear operator and  $\hat{N}$  represents a nonlinear operator.

Now applying the operator  $\mathfrak{T}_t^{\varphi}$  to both sides of equation (30), we get

$$\nu(\xi, t, \gamma) = \nu(\xi, 0, \gamma) + \mathfrak{F}_t^{\varphi} \mathfrak{g}(\xi, t, \gamma) - \mathfrak{F}_t^{\varphi} [\widehat{L}\nu(\xi, t, \gamma) + \widehat{N}\nu(\xi, t, \gamma)].$$
(31)

$$\nu(\xi, t, \gamma) = [\underline{\nu}(\xi, t, \gamma), \overline{\nu}(\xi, t, \gamma)], \qquad (32)$$

Now, we consider the equation (31) in parametric form as follows:

where

$$\underline{\nu}(\xi,t,\gamma) = E\left(\xi,\gamma\right) + \mathfrak{F}_{t}^{\varphi} \underline{\mathfrak{g}}(\xi,t,\gamma) - \mathfrak{F}_{t}^{\varphi} \left[\widehat{L}\,\underline{\nu}(\xi,t,\gamma) + \widehat{N}\underline{\nu}(\xi,t,\gamma)\right],\tag{33}$$

$$\overline{\nu}(\xi,t,\gamma) = \overline{E}(\xi,\gamma) + \mathfrak{F}_{t}^{\varphi}\overline{\mathfrak{g}}(\xi,t,\gamma) - \mathfrak{F}_{t}^{\varphi}[\widehat{L}\overline{\nu}(\xi,t,\gamma) + \widehat{N}\overline{\nu}(\xi,t,\gamma)].$$
(34)

The standard Adomian method defines the solution  $v(\xi, t, \gamma)$  in the form of the series

$$\underline{\nu}(\xi, t, \gamma) = \sum_{k=0}^{\infty} \underline{\nu}_k(\xi, t, \gamma),$$
(35)

$$\overline{\nu}(\xi, t, \gamma) = \sum_{k=0}^{\infty} \overline{\nu}_k(\xi, t, \gamma), \qquad (36)$$

and the nonlinear term  $\hat{N}$  is decomposed as

$$\widehat{N} \underline{\nu} = \sum_{k=0}^{\infty} \underline{M}_k,$$

$$\widehat{N}\overline{\nu} = \sum_{k=0}^{\infty} \overline{M}_k,$$
(37)

where  $\underline{M}_k$  and  $\overline{M}_k$  are Adomian polynomials given by

$$\underline{M}_{k} = \frac{1}{k!} \frac{\partial^{k}}{\partial \mathbf{q}^{k}} \left[ \widehat{N} \left( \sum_{i=0}^{\infty} \mathbf{q}^{i} \underline{\nu}_{i} \right) \right]_{\mathbf{q}=0},$$

$$\overline{M}_{k} = \frac{1}{k!} \frac{\partial^{k}}{\partial \mathbf{p}^{k}} \left[ \widehat{N} \left( \sum_{i=0}^{\infty} \mathbf{p}^{i} \overline{\nu}_{i} \right) \right]_{\mathbf{p}=0}.$$
(38)

Moreover, we employ the following recurrence relations:

$$\underline{\underline{\nu}}_{0}(\xi, t, \gamma) = E(\xi, \gamma) + \mathfrak{F}_{t}^{\psi} \underline{\mathfrak{g}}(\xi, t, \gamma),$$

$$\underline{\underline{\nu}}_{k+1}(\xi, t, \gamma) = \mathfrak{F}_{t}^{\psi} \left( \widehat{L}\underline{\underline{\nu}}_{k} + \sum_{k=0}^{\infty} \underline{\underline{M}}_{k} \right),$$
(39)

and

$$\overline{\nu}_{0}\left(\xi,t,\gamma\right) = \overline{E}\left(\xi,\gamma\right) + \mathfrak{T}_{t}^{\varphi}\overline{\mathfrak{g}}\left(\xi,t,\gamma\right), \overline{\nu}_{k+1}\left(\xi,t,\gamma\right) = \mathfrak{T}_{t}^{\varphi}\left(\widehat{L}\overline{\nu}_{k} + \sum_{k=0}^{\infty}\overline{M}_{k}\right). \tag{40}$$

Finally, the series (35) and (36) provide the approximate solution to the problem (1).

#### 5. Applications

In this section, we propose three examples of nonlinear FFPDEs to test the efficiency of the FADM.

*Example 1.* Consider the following nonlinear time fuzzy fractional advection equation:

$$\begin{cases} {}^{c}\mathfrak{D}_{t}^{\varphi}\nu(\xi,t,\gamma)+\nu(\xi,t,\gamma)\nu_{\xi}(\xi,t,\gamma)=0, & 0\leq\xi,t\leq1, \\ \nu(\xi,0,\gamma)=-[\gamma-1,1-\gamma]\xi, & 0\leq\gamma\leq1. \end{cases}$$

$$(41)$$

Applying the FADM step by step, we obtain the following recurrence relations:

$$\underline{\underline{\nu}}_{0}(\xi, t, \gamma) = -(\gamma - 1)\xi,$$

$$\underline{\underline{\nu}}_{k+1}(\xi, t, \gamma) = -\mathfrak{F}_{t}^{\varphi} \left(\sum_{k=0}^{\infty} \underline{\underline{M}}_{k}\right), k \ge 0.$$
(42)

and

$$\overline{\nu}_{0}\left(\xi,t,\gamma\right) = -(1-\gamma)\xi,$$

$$\overline{\nu}_{k+1}\left(\xi,t,\gamma\right) = -\mathfrak{F}_{t}^{\varphi}\left(\sum_{k=0}^{\infty}\overline{M}_{k}\right), k \ge 0,$$
(43)

where  $[\underline{M}_k, \overline{M}_k]$  represent the Adomian polynomials of the nonlinear function

$$\widehat{N}\nu = \left[\underline{\nu}(\xi, t, \gamma)\underline{\nu}_{\xi}(\xi, t, \gamma), \overline{\nu}(\xi, t, \gamma)\overline{\nu}_{\xi}(\xi, t, \gamma)\right].$$
(44)

Now, we calculate the first few iterations of the decomposition series as follows:

$$\underline{\nu}_{0}(\xi, t, \gamma) = -(\gamma - 1)\xi, 
\overline{\nu}_{0}(\xi, t, \gamma) = -(1 - \gamma)\xi, 
\underline{\nu}_{1}(\xi, t, \gamma) = -(\gamma - 1)^{2}\xi \left[\frac{t^{\varphi}}{\Gamma(\varphi + 1)}\right], 
\overline{\nu}_{1}(\xi, t, \gamma) = -(1 - \gamma)^{2}\xi \left[\frac{t^{\varphi}}{\Gamma(2\varphi + 1)}\right], 
\underline{\nu}_{2}(\xi, t, \gamma) = -(\gamma - 1)^{3}\xi \left[\frac{2t^{2\varphi}}{\Gamma(2\varphi + 1)}\right], 
\overline{\nu}_{2}(\xi, t, \gamma) = -(1 - \gamma)^{3}\xi \left[\frac{2t^{3\varphi}}{\Gamma(2\varphi + 1)}\right], 
\underline{\nu}_{3}(\xi, t, \gamma) = -(\gamma - 1)^{4}\xi \left[\frac{4}{\Gamma(3\varphi + 1)} + \frac{\Gamma(2\varphi + 1)}{\Gamma^{2}(\varphi + 1)\Gamma(3\varphi + 1)}\right] t^{3\varphi}, 
\overline{\nu}_{3}(\xi, t, \gamma) = -(1 - \gamma)^{4}\xi \left[\frac{4}{\Gamma(3\varphi + 1)} + \frac{\Gamma(2\varphi + 1)}{\Gamma^{2}(\varphi + 1)\Gamma(3\varphi + 1)}\right] t^{3\varphi}.$$

Similarly, we can find the other terms. Hence, the approximate solution of equation (41) is given by

$$\begin{cases} \underline{\nu}(\xi,t,\gamma) = -(\gamma-1)\xi \bigg[ 1 + (\gamma-1)\bigg[ \frac{t^{\varphi}}{\Gamma(\varphi+1)} \bigg] + (\gamma-1)^2 \bigg[ \frac{2t^{2\varphi}}{\Gamma(2\varphi+1)} \bigg] \\ + (\gamma-1)^3 \bigg[ \frac{4}{\Gamma(3\varphi+1)} + \frac{\Gamma(2\varphi+1)}{\Gamma^2(\varphi+1)\Gamma(3\varphi+1)} \bigg] t^{3\varphi} + \cdots \bigg], \end{cases}$$

$$(46)$$

$$\overline{\nu}(\xi,t,\gamma) = -(1-\gamma)\xi \bigg[ 1 + (1-\gamma)\bigg[ \frac{t^{\varphi}}{\Gamma(\varphi+1)} \bigg] + (1-\gamma)^2 \bigg[ \frac{2t^{2\varphi}}{\Gamma(2\varphi+1)} \bigg] \\ + (1-\gamma)^3 \bigg[ \frac{4}{\Gamma(3\varphi+1)} + \frac{\Gamma(2\varphi+1)}{\Gamma^2(\varphi+1)\Gamma(3\varphi+1)} \bigg] t^{3\varphi} + \cdots \bigg].$$

For  $\varphi = 1$ , the exact solution is given by

$$\begin{cases} \underline{\nu}(\xi, t, \gamma) = \frac{(\gamma - 1)\xi}{(\gamma - 1)t - 1}, \\ \overline{\nu}(\xi, t, \gamma) = \frac{(1 - \gamma)\xi}{(1 - \gamma)t - 1}. \end{cases}$$
(47)

*Remark 12.* When  $[\gamma - 1, 1 - \gamma] = 1$ , then the equation (46) recovers the results of the fractional order as in [55] and the solution (47) converges to the exact solution obtained in [56].

Figures 1 and 2 represent (a) the exact solutions and (b) the FADM solutions for the first three approximations of Example 1 with different fractional order and uncertainty  $\gamma = [0, 1]$ . We observe that the exact and derived results are in good contact, confirming the high accuracy of the proposed method for the fuzzy fractional problems in the sense of the Caputo operator.

*Example 2.* Consider the following nonlinear fuzzy fractional gas dynamic equation:

$$\begin{cases} {}^{c}\mathfrak{D}_{t}^{\varphi}\nu(\xi,t,\gamma) + \nu(\xi,t,\gamma)\nu_{\xi}(\xi,t,\gamma) - \nu(\xi,t,\gamma) + \nu^{2}(\xi,t,\gamma) = 0, \quad 0 \le \xi, t \le 1, \\ \nu(\xi,0,\gamma) = [\gamma, 3 - 2\gamma]e^{-\xi}. \end{cases}$$
(48)

Applying the FADM step by step, we obtain the following recurrence relations:

$$\underline{\nu}_{0}\left(\xi,t,\gamma\right) = \gamma e^{-\xi},$$

$$\underline{\nu}_{k+1}\left(\xi,t,\gamma\right) = \mathfrak{T}_{t}^{\varphi}\left(\underline{\nu}_{k}\left(\xi,t,\gamma\right) - \sum_{k=0}^{\infty}\underline{M}_{k}\right), k \ge 0,$$
(49)

$$\overline{\nu}_{0}\left(\xi,t,\gamma\right) = (3-2\gamma)e^{-\xi},$$

$$\overline{\nu}_{k+1}\left(\xi,t,\gamma\right) = \mathfrak{F}_{t}^{\varphi}\left(\overline{\nu}_{k}\left(\xi,t,\gamma\right) - \sum_{k=0}^{\infty}\overline{M}_{k}\right), k \ge 0,$$
(50)

and

where  $[\underline{M}_k, \overline{M}_k]$  represent the Adomian polynomials of the nonlinear function

$$\widehat{N}\nu = \left[\underline{\nu}(\xi, t, \gamma)\underline{\nu}_{\xi}(\xi, t, \gamma) + \frac{2}{\nu}(\xi, t, \gamma), \overline{\nu}(\xi, t, \gamma)\overline{\nu}_{\xi}(\xi, t, \gamma) + \overline{\nu}^{2}(\xi, t, \gamma)\right].$$
(51)

Now, we calculate the first few iterations of the decomposition series as follows:

$$\underline{\nu}_{0}(\xi,t,\gamma) = \gamma e^{-\xi},$$

$$\overline{\nu}_{0}(\xi,t,\gamma) = (3-2\gamma)e^{-\xi},$$

$$\underline{\nu}_{1}(\xi,t,\gamma) = \gamma e^{-\xi} \left[ \frac{t^{\varphi}}{\Gamma(\varphi+1)} \right],$$

$$\overline{\nu}_{1}(\xi,t,\gamma) = (3-2\gamma)e^{-\xi} \left[ \frac{t^{\varphi}}{\Gamma(2\varphi+1)} \right],$$

$$\underline{\nu}_{2}(\xi,t,\gamma) = \gamma e^{-\xi} \left[ \frac{t^{2\varphi}}{\Gamma(2\varphi+1)} \right],$$

$$\underline{\nu}_{3}(\xi,t,\gamma) = (3-2\gamma)e^{-\xi} \left[ \frac{t^{3\varphi}}{\Gamma(3\varphi+1)} \right],$$

$$\overline{\nu}_{3}(\xi,t,\gamma) = (3-2\gamma)e^{-\xi} \left[ \frac{t^{3\varphi}}{\Gamma(3\varphi+1)} \right],$$

$$\overline{\nu}_{3}(\xi,t,\gamma) = (3-2\gamma)e^{-\xi} \left[ \frac{t^{3\varphi}}{\Gamma(3\varphi+1)} \right],$$

$$(52)$$



FIGURE 1: 2D simulation of the exact and approximate solutions of fuzzy upper and lower portions of Example 1 at  $\xi = 0.2$  and t = 1.



FIGURE 2: 3D simulation of the exact and approximate solutions of fuzzy upper and lower portions of Example 1 at  $\xi = 0.5$ .

and so on. Thus, the approximate solution is given by

$$\begin{cases} \underline{\nu}(\xi, t, \gamma) = \gamma e^{-\xi} \left[ 1 + \frac{t^{\varphi}}{\Gamma(\varphi+1)} + \frac{t^{2\varphi}}{\Gamma(2\varphi+1)} + \frac{t^{3\varphi}}{\Gamma(3\varphi+1)} + \cdots \right], \\ \overline{\nu}(\xi, t, \gamma) = (3 - 2\gamma) e^{-\xi} \left[ 1 + \frac{t^{\varphi}}{\Gamma(\varphi+1)} + \frac{t^{2\varphi}}{\Gamma(2\varphi+1)} + \frac{t^{3\varphi}}{\Gamma(3\varphi+1)} + \cdots \right]. \end{cases}$$
(53)

For  $\varphi = 1$ , the FADM solution (53) converges to the following exact solution:

$$\begin{cases} \underline{\nu}(\xi, t, \gamma) = \gamma e^{t-\xi}, \\ \overline{\nu}(\xi, t, \gamma) = (3 - 2\gamma)e^{t-\xi}. \end{cases}$$
(54)

*Remark 13.* When  $[\gamma, 3 - 2\gamma] = 1$ , then the equation (53) converts to the fractional order solution as in [57] and the solution (54) converges to the exact solution as in [58].

Figures 3 and 4 represent (a) the exact solutions and (b) the FADM solutions for the first three approximations of Example 2 with different fractional order and uncertainty  $\gamma = [0, 1]$ . We observe that the exact and derived results are in good contact, confirming the high accuracy of the proposed method for the fuzzy fractional problems in the sense of the Caputo operator.

*Example 3.* Consider the following nonlinear fuzzy fractional partial differential equation:

$$\begin{cases} {}^{c}\mathfrak{D}_{t}^{\varphi}\nu(\xi,t,\gamma) + \frac{1}{36}\xi\nu_{\xi\xi}^{2}(\xi,t,\gamma) = \xi^{3}, \quad \xi \in [0,1], t \in [0,2], \\ \nu(\xi,0,\gamma) = [\gamma, 2-\gamma]. \end{cases}$$
(55)

Applying the FADM step by step, we obtain the following recurrence relations:

$$\underline{\nu}_{0}\left(\xi, t, \gamma\right) = \gamma + \mathfrak{F}_{t}^{\varphi}\left(\xi^{3}\right),$$

$$\underline{\nu}_{k+1}\left(\xi, t, \gamma\right) = -\mathfrak{F}_{t}^{\varphi}\left(\sum_{k=0}^{\infty} \underline{M}_{k}\right), k \ge 0.$$
(56)

and

$$\overline{\nu}_{0}\left(\xi,t,\gamma\right) = (2-\gamma) + \mathfrak{F}_{t}^{\varphi}\left(\xi^{3}\right),$$

$$\overline{\nu}_{k+1}\left(\xi,t,\gamma\right) = -\mathfrak{F}_{t}^{\varphi}\left(\sum_{k=0}^{\infty}\overline{M}_{k}\right), k \ge 0,$$
(57)

where  $[\underline{M}_k, \overline{M}_k]$  represent the Adomian polynomials of the nonlinear function

$$\widehat{N}\nu = \left[-\frac{1}{36}\xi \underline{\nu}_{\xi\xi}^{2}(\xi,t,\gamma), -\frac{1}{36}\xi \overline{\nu}_{\xi\xi}^{2}(\xi,t,\gamma)\right].$$
(58)

The first few iterations of the decomposition series are as follows:

$$\begin{split} \underline{\nu}_{0}\left(\xi,t,\gamma\right) &= \gamma + \xi^{3} \bigg[ \frac{t^{\varphi}}{\Gamma(\varphi+1)} \bigg], \\ \bar{\nu}_{0}\left(\xi,t,\gamma\right) &= (2-\gamma) + \xi^{3} \bigg[ \frac{t^{\varphi}}{\Gamma(\varphi+1)} \bigg], \\ \underline{\nu}_{1}\left(\xi,t,\gamma\right) &= -\xi^{3} \bigg[ \frac{\Gamma(2\varphi+1)t^{3\varphi}}{\Gamma^{2}(\varphi+1)\Gamma(3\varphi+1)} \bigg], \\ \bar{\nu}_{1}\left(\xi,t,\gamma\right) &= -\xi^{3} \bigg[ \frac{\Gamma(2\varphi+1)t^{3\varphi}}{\Gamma^{2}(\varphi+1)\Gamma(3\varphi+1)} \bigg], \\ \underline{\nu}_{2}\left(\xi,t,\gamma\right) &= 2\xi^{3} \bigg[ \frac{\Gamma(2\varphi+1)\Gamma(4\varphi+1)t^{5\varphi}}{\Gamma^{3}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} \bigg], \\ \bar{\nu}_{2}\left(\xi,t,\gamma\right) &= 2\xi^{3} \bigg[ \frac{\Gamma(2\varphi+1)\Gamma(4\varphi+1)t^{5\varphi}}{\Gamma^{3}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} \bigg], \\ \bar{\nu}_{3}\left(\xi,t,\gamma\right) &= -\xi^{3} \bigg[ \bigg( \frac{4\Gamma(2\varphi+1)\Gamma(4\varphi+1)\Gamma(6\varphi+1)}{\Gamma^{4}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} \bigg) t^{7\varphi} \bigg], \\ \bar{\nu}_{3}\left(\xi,t,\gamma\right) &= -\xi^{3} \bigg[ \bigg( \frac{4\Gamma(2\varphi+1)\Gamma(6\varphi+1)}{\Gamma^{4}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} \bigg) t^{7\varphi} \bigg], \\ \bar{\nu}_{3}\left(\xi,t,\gamma\right) &= -\xi^{3} \bigg[ \bigg( \frac{4\Gamma(2\varphi+1)\Gamma(4\varphi+1)\Gamma(6\varphi+1)}{\Gamma^{4}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} \bigg) t^{7\varphi} \bigg]. \end{split}$$

Similarly, we can find the other terms. Hence the approximate solution is given by

(59)



FIGURE 3: 2D simulation of the exact and approximate solutions of fuzzy upper and lower portions of Example 2 at  $\xi = 0.50$  and t = 0.5.



FIGURE 4: 3D simulation of the exact and approximate solutions of fuzzy upper and lower portions of Example 2 at  $\xi = 0.5$ .



FIGURE 5: 2D simulation of the exact and approximate solutions of fuzzy upper and lower portions of Example 3 at  $\xi = 1$  and t = 0.5.

$$\begin{split} \underline{\nu}(\xi,t,\gamma) &= \gamma + \xi^{3} \Biggl[ \frac{t^{\varphi}}{\Gamma(\varphi+1)} - \frac{\Gamma(2\varphi+1)t^{3\varphi}}{\Gamma^{2}(\varphi+1)\Gamma(3\varphi+1)} + \frac{\Gamma(2\varphi+1)\Gamma(4\varphi+1)t^{5\varphi}}{\Gamma^{3}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} \\ &- \Biggl( \frac{4\Gamma(2\varphi+1)\Gamma(4\varphi+1)\Gamma(6\varphi+1)}{\Gamma^{4}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} + \frac{\Gamma^{2}(2\varphi+1)\Gamma(6\varphi+1)}{\Gamma^{4}(\varphi+1)\Gamma^{2}(3\varphi+1)\Gamma(7\varphi+1)} \Biggr) \\ &\times t^{7\varphi} + \cdots \Biggr], \end{split}$$
(60)  
$$\overline{\nu}(\xi,t,\gamma) &= (2-\gamma) + \xi^{3} \Biggl[ \frac{t^{\varphi}}{\Gamma(\varphi+1)} - \frac{\Gamma(2\varphi+1)t^{3\varphi}}{\Gamma^{2}(\varphi+1)\Gamma(3\varphi+1)} + \frac{\Gamma(2\varphi+1)\Gamma(4\varphi+1)t^{5\varphi}}{\Gamma^{3}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} \\ &- \Biggl( \frac{4\Gamma(2\varphi+1)\Gamma(4\varphi+1)\Gamma(6\varphi+1)}{\Gamma^{4}(\varphi+1)\Gamma(3\varphi+1)\Gamma(5\varphi+1)} + \frac{\Gamma^{2}(2\varphi+1)\Gamma(6\varphi+1)}{\Gamma^{4}(\varphi+1)\Gamma^{2}(3\varphi+1)\Gamma(7\varphi+1)} \Biggr) \\ &\times t^{7\varphi} + \cdots \Biggr]. \end{split}$$

For  $\varphi = 1$ , the exact solution obtained as follows:

$$\begin{cases} \underline{\nu}(\xi, t, \gamma) = \gamma + \xi^{3} \text{tanht,} \\ \overline{\nu}(\xi, t, \gamma) = (2 - \gamma) + \xi^{3} \text{tanht.} \end{cases}$$
(61)

*Remark 14.* When  $[\gamma, 2 - \gamma] = 0$ , then the equation (61) converges to the exact solution obtained in [56].

Figures 5 and 6 represent (a) the exact solutions and (b) the FADM solutions for the first three approximations of Example 3 with different fractional order and uncertainty



FIGURE 6: 3D simulation of the exact and approximate solutions of fuzzy upper and lower portions of Example 3 at  $\xi = 1$ .

 $\gamma = [0, 1]$ . We observe that the exact and derived results are in good contact, confirming the high accuracy of the proposed method for the fuzzy fractional problems in the sense of the Caputo operator.

#### 6. Conclusion

This work aimed to investigate certain sufficient conditions for the existence and uniqueness of a solution of the nonlinear fuzzy fractional partial differential equations. Furthermore, we used the FADM to obtain the approximate solutions to the given problem. The proposed method provides more believable series solutions whose continuity depends on the fuzzy fractional derivative. As the number of decomposed terms increases, the numerical solution begins to converge. The performance and reliability of the FADM are studied by implementing three numerical examples. We also generated graphs of the numerical solution at different fractional orders. As can be seen in the figures, the plots converge to the curve at  $\varphi = 1$  as the fractional order  $\varphi$ approaches its integer value. This suggests that fractional calculus can be used to identify the global nature of the dynamics of equations related to fuzzy concepts.

### **Data Availability**

No data were used to support the findings of this study.

## **Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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