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Research Article

IR-Improved DGLAP-CS Theory

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We show that it is possible to improve the infrared aspects of the standard treatment of the DGLAP-CS evolution theory to take into account a large class of higher-order corrections that significantly improve the precision of the theory for any given level of fixed-order calculation of its respective kernels. We illustrate the size of the effects we resum using the moments of the parton distributions.

In the preparation of the physics for the precision QCD × EW (electroweak) [1–10] LHC physics studies, all aspects of the calculation of the cross sections and distributions for the would-be physical observables must be re-examined if precision tags such as that envisioned for the luminosity theoretical precision are to be realized, that is, 1% cross section predictions for single heavy gauge boson production in 14 TeV pp collisions when that heavy gauge boson decays into a light lepton pair. The QCD [11–21] evolution of the structure functions from the typical reference scale of data input, $\mu_0 \sim 1$–2 GeV, to the respective hard scale is one step that warrants further study, as it is well-known to many. Many authors [22–25] have provided excellent realizations of this evolution in the recent literature. Here, we will re-examine the infrared aspects of the basic evolution theory itself as it is represented via the approach of [17–21] to that theory to try to improve the treatment to a level consistent with the new era of precision QCD × EW physics needed for the LHC physics objectives.

Throughout the discussion, then, we work in the parton model; and we focus on the kernels of what in the literature are commonly referred to as the DGLAP [17–21] evolution equations for the respective parton distributions. These equations, under Mellin transformation, are entirely implied by those of the Callan-Symanzik-type [11–13] analyzed in [15, 16] in their classic analysis of the deep inelastic scattering processes. Thus, henceforward, we will refer to these equations as the DGLAP-CS equations.

Specifically, the motivation for the improvement which we develop can be seen already in the basic results in [17–21] for the kernels that determine the evolution of the structure functions by the attendant DGLAP-CS evolution of the corresponding parton

densities by the standard methodology. Consider the evolution of the non-singlet (NS) parton density function $q^{\mathrm{NS}}(x)$, where x can be identified as Bjorken's variable as usual. The basic starting point of our analysis is the infrared divergence in the kernel that determines this evolution:

$$\frac{dq^{\mathrm{NS}}(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi}\int_x^1 \frac{dy}{y} q^{\mathrm{NS}}(y,t)P_{qq}(x/y), \tag{1}$$

where the well-known result for the kernel $P_{qq}(z)$ is, for $z < 1$,

$$P_{qq}(z) = C_F \frac{1+z^2}{1-z} \tag{2}$$

when we set $t = \ln \mu^2/\mu_0^2$ for some reference scale μ_0 with which we study evolution to the scale of interest μ. (We will generally follow [26] and set $\mu_0 = \Lambda_{\mathrm{QCD}}$ without loss of content since $dt = dt'$ when $t = \ln \mu^2/\Lambda_{\mathrm{QCD}}^2$, $t' = \ln \mu^2/\mu_0^2$) for fixed values of Λ_{QCD}, μ_0.) Here, $C_F = (N_c^2 - 1)/(2N_c)$ is the quark color representation's quadratic Casimir invariant, where N_c is the number of colors and so that it is just 3. This kernel has an unintegrable IR singularity at $z = 1$, which is the point of zero energy gluon emission; and this is as it should be. The standard treatment of this very physical effect is to regularize it by the replacement

$$\frac{1}{(1-z)} \longrightarrow \frac{1}{(1-z)_+} \tag{3}$$

with the distribution $1/(1-z)_+$ defined so that for any suitable test function $f(z)$ we have

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{(1-z)}. \tag{4}$$

A possible representation of $1/(1-z)_+$ is seen to be

$$\frac{1}{(1-z)_+} = \frac{1}{(1-z)}\theta(1-\epsilon-z) + \ln \epsilon \delta(1-z) \tag{5}$$

with the understanding that $\epsilon \downarrow 0$. We use the notation $\theta(x)$ for the step function from 0 for $x < 0$ to 1 for $x \geq 0$ and $\delta(x)$ is Dirac's delta function. The final result for $P_{qq}(z)$ is then obtained by imposing the physical requirement [17–21] that

$$\int_0^1 dz P_{qq}(z) = 0, \tag{6}$$

which is satisfied by adding the effects of virtual corrections at $z = 1$ so that finally

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right). \tag{7}$$

Note that we can write the last result as

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+ \tag{8}$$

from which it follows that (6) holds identically.

The smooth behavior in the original real emission result from the Feynman rules, with a divergent $1/(1-z)$ behavior as $z \to 1$, has been replaced with a mathematical artifact: the regime $1-\epsilon < z < 1$ now has no probability at all; and at $z = 1$ we have a large negative integrable contribution so that we end-up finally with a finite (zero) value for the total integral of $P_{qq}(z)$. This mathematical artifact is what we wish to improve here; for, in the precision studies of Z physics [27–32] at LEP1, it has been found that such mathematical artifacts can indeed impair the precision tag which one can achieve with a given fixed order of perturbation theory. An analogous case is now well-known in the theory of QCD higher-order corrections, where the FNAL data on p_T spectra clearly show the need for improvement of fixed-order results by resumming large logs associated with soft gluons [33, 34]. For reference, note that at the LHC, 2 TeV partons are realistic so that $z \cong 0.001$ means ~2–3 GeV soft gluons, which are clearly above the LHC detector thresholds (here we intend the combined effect of such gluons), in complete analogy with the situation at LEP where $z \cong 0.001$ meant ~100 MeV photons which were also above the LEP detector thresholds—just as resummation was necessary to describe this view of the LEP data, so too we may argue it will be necessary to describe the LHC data on the corresponding view; more importantly, why should we have to set $P_{qq}(z)$ to 0 for $1-\epsilon < z < 1$ when it actually has its largest values in this very regime?

By mathematical artifact we do not mean that there is an error in the computations that lead to it; indeed, it is well-known that this +-function behavior is exactly what one gets at $\mathcal{O}(\alpha_s)$ for the bremsstrahlung process. The artifact is that the behavior of the differential spectrum of the process for $z \to 1$ in $\mathcal{O}(\alpha_s)$ is unintegrable and has to be cut-off; and thus this spectrum is only poorly represented by the $\mathcal{O}(\alpha_s)$ calculation; for, the resummation of the large soft higher-order effects as we present below changes the $z \to 1$ behavior nontrivially, as from our resummation we will find that the $1/(1-z)$-behavior is modified to $(1-z)^{\gamma-1}$, $\gamma > 0$. This is a testable effect, as we have seen in its QED analogs in Z physics at LEP1 [27–32]: if the experimentalist measures the cross section for bremsstrahlung for gluons (photons) down to energy fraction ϵ_0, $\epsilon_0 > 0$, in our new resummed theory presented below, the result will approach a finite value from below as $-\epsilon_0^{\gamma}$ whereas the $\mathcal{O}(\alpha_s)$ +-function prediction would increase without limit as $-\ln \epsilon_0$. The exponentiated result has been verified by the data at LEP1.

To illustrate the issue, consider the QED example of the Bonneau-Martin cross section formula for the process $e^+(p_1) + e^-(p_2) \to \mu^+(q_1) + \mu^-(q_2)(+\gamma(k))$:

$$\sigma = \int_0^1 dv \rho_{\mathrm{BM}}(v) \sigma^{\mathrm{Born}}\big(s(1-v)\big), \tag{9}$$

where

$$\begin{gathered}\rho_{\mathrm{BM}}(v) = \delta(v)\big(1 + \gamma \ln \epsilon + \Delta_{VS}^{(1)}\big) + \gamma\theta(v-\epsilon)\frac{1+(1-v)^2}{2v}, \\ \gamma = \frac{2\alpha}{\pi}(L-1), \quad L = \ln\frac{s}{m_e^2}, \qquad \Delta_{VS}^{(1)} = \left(\frac{\alpha}{\pi}\right)\left(\frac{3}{2}L - 2 + \frac{1}{3}\pi^2\right)\end{gathered} \tag{10}$$

and for the single photon emission $v = 2k^0/\sqrt{s}$ in the center of momentum system (cms), with $s = (p_1 + p_2)^2$ as usual. The parameter $\epsilon \downarrow 0$ then defines the +-distribution in the single photon emission

$$\gamma \frac{1+(1-v)^2}{2v_+} \tag{11}$$

just as we have indicated above for the single gluon emission in P_{qq}. The result (9) is inadequate for precision LEP physics and has to be replaced with an exponentiated formula such as that obtained from substituting [28–32] $\rho_{\mathrm{BM}}(v) \to \rho_{\mathrm{YFS}}(v)$ with

$$\begin{aligned}
\rho_{\mathrm{YFS}}(v) &= F_{\mathrm{YFS}}(\gamma) e^{\delta_{\mathrm{YFS}}} \gamma v^{\gamma-1}\left(1 + \Delta_S^{(1)} + \Delta_H^{(1)}(v)\right),\\
\Delta_S^{(1)} &= \frac{\alpha}{\pi}(L-1),\\
\Delta_H^{(1)}(v) &= v\left(-1 + \frac{1}{2}v\right),\\
\delta_{\mathrm{YFS}} &= \left(\frac{\alpha}{\pi}\right)\left(\frac{1}{2}L - 1 + \frac{1}{3}\pi^2\right),\\
F_{\mathrm{YFS}}(\gamma) &= \frac{e^{-C\gamma}}{\Gamma(1+\gamma)},\\
C &= 0.57721566\ldots,
\end{aligned} \tag{12}$$

where C is Euler's constant and $\Gamma(w)$ is Euler's gamma function. (See [27] for a complete discussion of all available variants of this substitution.) We see as advertised that the +-function has been replaced with an integrable function in v for $v \downarrow 0$. See [27] for more discussion of this phenomenology.

The important point is that the traditional resummations in N-moment space for the DGLAP-CS kernels address only the short-distance contributions to their higher-order corrections. The deep question we deal with in this paper concerns, then, how much of the complete soft limit of the DGLAP-CS kernels is contained in the anomalous dimensions of the leading twist operators in Wilson's expansion, an expansion which resides on the very tip of the light-cone? Are all of the effects of the very soft gluon emission, involving, as they most certainly do, arbitrarily long wavelength quanta, representable by the physics at the tip of the light-cone? The Heisenberg uncertainty principle surely tells us that answer cannot be affirmative. In this paper, we calculate these long-wavelength gluon effects on the DGLAP-CS kernels that are not included (see the discussion below) in the standard treatment of Wilson's expansion. We therefore do not contradict the results of the large N-moment space resummations such as that presented in [35] nor do we contradict the renormalon chain-type resummation as done in [36].

We employ the exact rearrangement of the Feynman series for QCD as it has been shown in [37–48]. For completeness, as this QCD exponentiation theory is not generally familiar, we reproduce its essential aspects in our appendix. The idea is to sum up the leading IR terms in the corrections to P_{qq} with the goal that they will render integrable the IR singularity that we have in its lowest-order form. This will remove the need for mathematical artifacts and exhibit more accurately the true predictions of the full QCD theory in terms of fully physical results.

As we explain in detail in the appendix for the specific example of $\overline{Q}'Q \to \overline{Q}'''Q''+nG$, if $\mathcal{M}^{(n)}$ is the amplitude for any process $A \to B + nG$, the application of amplitude-based resummation as derived in [37–48] leads to the exact result

$$\mathcal{M}^{(n)} = e^{\alpha_s B_{\mathrm{QCD}}} \sum_{j=0}^{\infty} \mathsf{m}_j^{(n)}, \tag{13}$$

where we have defined

$$\alpha_s(Q)B_{\mathrm{QCD}} = \int \frac{d^4k}{\left(k^2-\lambda^2+i\epsilon\right)} S_{\mathrm{QCD}}(k), \tag{14}$$

where the amplitudes $\{\mathsf{m}_j^{(n)}\}$ are free of the IR singularities that are contained in the virtual IR function B_{QCD}. Here, j is the loop index and the virtual IR emission function $S_{\mathrm{QCD}}(k)$ is defined in the appendix. Upon squaring the amplitude in (13) and using the standard methods, we get the cross section representation, specializing to $A = \overline{Q}'(p_1)Q(q_1)$, $B = \overline{Q}'''(p_2)Q''(q_2)$ for definiteness:

$$\begin{aligned} d\widehat{\sigma}^n = \frac{e^{2\alpha_s \mathrm{Re}\, B_{\mathrm{QCD}}}}{n!} \int \prod_{m=1}^{n} \frac{d^3k_m}{\left(k_m^2+\lambda^2\right)^{1/2}} \delta\left(p_1+q_1-p_2-q_2-\sum_{i=1}^{n} k_i\right) \\ \times \overline{\rho}^{(n)}\left(p_1,q_1,p_2,q_2,k_1,\ldots,k_n\right) \frac{d^3p_2 d^3q_2}{p_2^0 q_2^0}, \end{aligned} \tag{15}$$

where we have defined

$$\overline{\rho}^{(n)}\left(p_1,q_1,p_2,q_2,k_1,\ldots,k_n\right) = \sum_{\text{color, spin}} \left\| \sum_{j=0}^{\infty} \mathsf{m}_j^{(n)} \right\|^2 \tag{16}$$

in the incoming $Q(q_1)\overline{Q}'(p_1)$ cms system and λ is an IR regulator mass only (it is not a parameter in the Lagrangian)—see the appendix for more details (Some kinematical factors are absorbed into the normalization of the amplitudes.) We show in the appendix that, upon summing over n, we can extract the dominant real emission contributions from the $\overline{\rho}^{(n)}$ to arrive at the master formula

$$\begin{aligned} d\widehat{\sigma}_{\mathrm{exp}} &= \sum_n d\widehat{\sigma}^n \\ &= e^{\Sigma_{\mathrm{IR}}(\mathrm{QCD})} \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{j=1}^{n} \frac{d^3k_j}{k_j} \int \frac{d^4y}{(2\pi)^4} e^{iy\cdot(p_1+q_1-p_2-q_2-\sum k_j)+D_{\mathrm{QCD}}} * \tilde{\overline{\beta}}_n\left(k_1,\ldots,k_n\right) \frac{d^3p_2}{p_2^0} \frac{d^3q_2}{q_2^0}, \end{aligned} \tag{17}$$

where now the hard gluon residuals $\tilde{\overline{\beta}}_n(k_1,\ldots,k_n)$ are defined in the appendix and are free of IR singularities to all orders in $\alpha_s(Q)$, Q is the relevant hard scale and

$$\begin{aligned} \Sigma_{\mathrm{IR}}(\mathrm{QCD}) &= 2\alpha_s \mathrm{Re}\, B_{\mathrm{QCD}} + 2\alpha_s \widetilde{B}_{\mathrm{QCD}}\left(K_{\max}\right), \\ 2\alpha_s \widetilde{B}_{\mathrm{QCD}}\left(K_{\max}\right) &= \int \frac{d^3k}{k^0} \widetilde{S}_{\mathrm{QCD}}(k)\theta\left(K_{\max}-k\right), \\ D_{\mathrm{QCD}} &= \int \frac{d^3k}{k} \widetilde{S}_{\mathrm{QCD}}(k)\left[e^{-iy\cdot k} - \theta\left(K_{\max}-k\right)\right], \end{aligned} \tag{18}$$

where the real IR function $\widetilde{S}_{\mathrm{QCD}}(k)$ is defined in the appendix. Note that (17) is independent of $K_{\max}$.

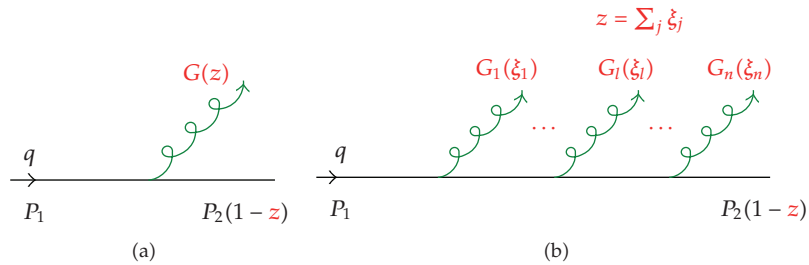

Figure 1: In (a), we show the usual process $q \to q(1-z) + G(z)$; in (b), we show its multiple gluon improvement $q \to q(1-z) + G_1(\xi_1) + \cdots + G_n(\xi_n)$, $z = \sum_j \xi_j$.

Here, we apply the QCD exponentiation master formula in (17) (see also [37–45]), following the analogous discussion then for QED in [28–32], to the gluon emission transition that corresponds to $P_{qq}(z)$, that is, to the squared amplitude for $q \to q(z) + G(1-z)$ so that in the appendix one replaces everywhere the squared amplitudes for the $\overline{Q}'Q \to \overline{Q}'''Q''$ processes with those for the former one plus its nG analoga with the attendant changes in the phase space and kinematics dictated by the standard methods; this implies that in [17, equation (53)] we have from (17) the replacement (see Figure 1)

$$P_{BA} = P^0_{BA} \equiv \frac{1}{2} z(1-z) \overline{\sum_{\text{spins}}} \frac{\left|V_{A\to B+C}\right|^2}{p_\perp^2}$$
$$\Longrightarrow \tag{19}$$
$$P_{BA} = \frac{1}{2} z(1-z) \overline{\sum_{\text{spins}}} \frac{\left|V_{A\to B+C}\right|^2}{p_\perp^2} z^{\gamma_q} F_{\text{YFS}}(\gamma_q) e^{(1/2)\delta_q},$$

where $A = q$, $B = G$, $C = q$, and $V_{A\to B+C}$ is the lowest-order amplitude for $q \to G(z) + q(1-z)$, so that we get the unnormalized exponentiated result

$$P_{qq}(z) = C_F F_{\text{YFS}}(\gamma_q) e^{(1/2)\delta_q} \frac{1+z^2}{1-z} (1-z)^{\gamma_q}, \tag{20}$$

where [28–32, 37–48] (note, $t = \ln(\mu^2/\Lambda^2_{\text{QCD}})$)

$$\gamma_q = C_F \frac{\alpha_s}{\pi} t = \frac{4C_F}{\beta_0},$$
$$\delta_q = \frac{\gamma_q}{2} + \frac{\alpha_s C_F}{\pi}\left(\frac{\pi^2}{3} - \frac{1}{2}\right), \tag{21}$$

and $F_{\text{YFS}}(z)$ was already defined. Here,

$$\beta_0 = 11 - \frac{2}{3} n_f, \tag{22}$$

where n_f is the number of active quark flavors. The function $F_{\text{YFS}}(z)$ was already introduced by Yennie et al. [49, 50] in their analysis of the IR behavior of QED. We see immediately,

as illustrated above for QED, that the exponentiation has removed the unintegrable IR divergence at $z = 1$. For reference, we note that we have in (20) resummed the terms $\mathcal{O}(\ln^k(1-z)t^\ell \alpha_s^n)$, $n \geq \ell \geq k$, which originate in the IR regime and which exponentiate. (Following the standard LEP Yellow Book [27] convention, we do not include the order of the first nonzero term in counting the order of its higher-order corrections.) The important point is that we have not dropped outright the terms that do not exponentiate but have organized them into the residuals $\tilde{\tilde{\beta}}_m$ in the analog of (17). The application of (17) to obtain (20) proceeds as follows. First, the exponent in the exponential factor in front of the expression on the RHS of (17) is readily seen to be from (A.16), using the well-known results for the respective real and virtual infrared functions from [37–48]:

$$\begin{aligned}\Sigma_{\mathrm{IR}}(\mathrm{QCD}) &= 2\alpha_s \mathrm{Re}\, B_{\mathrm{QCD}} + 2\alpha_s \tilde{B}_{\mathrm{QCD}}(K_{\max}) \\ &= \frac{1}{2}\left(2C_F \frac{\alpha_s}{\pi} t \ln \frac{K_{\max}}{E} + C_F \frac{\alpha_s}{2\pi} t + \frac{\alpha_s C_F}{\pi}\left(\frac{\pi^2}{3} - \frac{1}{2}\right)\right)'\end{aligned} \tag{23}$$

where on the RHS of the last result we have already applied the DGLAP-CS synthesization procedure in [39] to remove the collinear singularities, $\ln \Lambda_{\mathrm{QCD}}^2/m_q^2 - 1$, in accordance with the standard QCD factorization theorems [51–55]. This means that, identifying the LHS of (17) as the sum over final states and average over initial states of the respective process divided by the incident flux and replacing that incident flux by the respective initial state density according to the standard methods for the process $q \to q(1-z) + G(z)$, occurring in the context of a hard scattering at scale Q as it is for [17, equation (53)], the soft gluon effects for energy fraction $< z \equiv K_{\max}/E$ give the result, from (17), that, working through to the $\tilde{\tilde{\beta}}_1$-level and using q_2 to represent the momentum conservation via other degrees of freedom for the attendant hard process

$$\begin{aligned}&\int \frac{\alpha_s(t)}{2\pi} P_{BA}\, dt\, dz \\ &= e^{\Sigma_{\mathrm{IR}}(\mathrm{QCD})(z)} \int \left\{ \tilde{\tilde{\beta}}_0 \int \frac{d^4 y}{(2\pi)^4} e^{\{iy\cdot(p_1-p_2)+\int^{k<K_{\max}}(d^3k/k)\tilde{S}_{\mathrm{QCD}}(k)[e^{-iy\cdot k}-1]\}} \right. \\ &\qquad \left. + \int \frac{d^3k_1}{k_1} \tilde{\tilde{\beta}}_1(k_1) \int \frac{d^4 y}{(2\pi)^4} e^{\{iy\cdot(p_1-p_2-k_1)+\int^{k<K_{\max}}(d^3k/k)\tilde{S}_{\mathrm{QCD}}(k)[e^{-iy\cdot k}-1]\}} + \cdots \right\} \frac{d^3p_2}{p_2^0} \frac{d^3q_2}{q_2^0} \\ &= e^{\Sigma_{\mathrm{IR}}(\mathrm{QCD})(z)} \int \left\{ \tilde{\tilde{\beta}}_0 \int_{-\infty}^{\infty} \frac{dy}{(2\pi)} e^{\{iy\cdot(E_1-E_2)+\int^{k<K_{\max}}(d^3k/k)\tilde{S}_{\mathrm{QCD}}(k)[e^{-iy k}-1]\}} \right. \\ &\qquad \left. + \int \frac{d^3k_1}{k_1} \tilde{\tilde{\beta}}_1(k_1) \int_{-\infty}^{\infty} \frac{dy}{(2\pi)} e^{\{iy\cdot(E_1-E_2-k_1^0)+\int^{k<K_{\max}}(d^3k/k)\tilde{S}_{\mathrm{QCD}}(k)[e^{-iy\cdot k}-1]\}} + \cdots \right\} \frac{d^3p_2}{p_2^0 q_2^0},\end{aligned} \tag{24}$$

where we set $E_i = p_i^0$, $i = 1, 2$, and the real infrared function $\tilde{S}_{\mathrm{QCD}}(k)$ is well-known as well:

$$\tilde{S}_{\mathrm{QCD}}(k) = -\frac{\alpha_s C_F}{8\pi^2}\left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2}\right)^2 \Bigg|_{\text{DGLAP-CS synthesized}} \tag{25}$$

and we indicate as above that the DGLAP-CS synthesization procedure in [39] is to be applied to its evaluation to remove its collinear singularities; we are using the kinematics of [17] in their computation of $P_{BA}(z)$ in their (53), so that the relevant value of $k_\perp^2$ is indeed Q^2. It means that the computation can also be seen to correspond to computing the IR function for the standard t-channel kinematics and taking $1/2$ of the result to match the single line emission in P_{Gq}. The two important integrals needed in (24) were already studied in [49, 50]:

$$\begin{aligned} I_{\mathrm{YFS}}(zE,0) &= \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{[iy(zE)+\int^{k<zE}(d^3k/k)\tilde{S}_{\mathrm{QCD}}(k)(e^{-iyk}-1)]} \\ &= F_{\mathrm{YFS}}(\gamma_q)\frac{\gamma_q}{zE}, \\ I_{\mathrm{YFS}}(zE,k_1) &= \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{[iy(zE-k_1)+\int^{k<zE}(d^3k/k)\tilde{S}_{\mathrm{QCD}}(k)(e^{-iyk}-1)]} \\ &= \left(\frac{zE}{zE-k_1}\right)^{1-\gamma_q} I_{\mathrm{YFS}}(zE,0). \end{aligned} \tag{26}$$

When we introduce the results in (26) into (24), we can identify the factor

$$\int \left(\tilde{\tilde{\beta}}_0 \frac{\gamma_q}{zE} + \int dk_1 k_1 d\Omega_1 \tilde{\tilde{\beta}}_1(k_1)\left(\frac{zE}{zE-k_1}\right)^{1-\gamma_q}\frac{\gamma_q}{zE}\right)\frac{d^3p_2}{E_2 q_2^0} = \int dt \frac{\alpha_s(t)}{2\pi} P_{BA}^0 dz + \mathcal{O}(\alpha_s^2), \tag{27}$$

where P_{BA}^0 is the unexponentiated result in the first line of (19). This leads us finally to the exponentiated result in the second line of (19) by elementary differentiation:

$$P_{BA} = P_{BA}^0 z^{\gamma_q} F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q}. \tag{28}$$

The following observations are in order. First, unlike the light-cone gauge or light-like Wilson line singularity artifacts discussed in [56] for unintegrated definitions of parton density functions, the analyses just presented, both for the QED case and for the QCD case, show that the $1/(1-z)$ real emission singularity in P_{qq} (it would be in P_{ee} in the analog QED case) is a genuine property of soft radiation, it is gauge invariant. Second, from the explicit results for the exponent $\Sigma_{\mathrm{IR}}(\mathrm{QCD})$ in (23) and the results in (18), we see that the gluon mass regulator λ has completely canceled from our cross section, which is also then gauge invariant because we never introduced λ into the QCD Lagrangian—we only used λ to define IR singularities so that the Slavnov-Taylor, Ward-Takahashi identities were all the time maintained. Use of the n-dimensional regulator methods of [57, 58] gives the same results as our use of λ.

Here, we also may note how one can see that the terms we exponentiate are not included in the standard treatment of Wilson's expansion: from the standard methods [59, 60], the Nth moment of the invariants $T_{i,\ell}$, $i = L, 2, 3$, $\ell = q, G$, of the forward Compton amplitude in DIS, where we recall the structure functions F_i, $i = L, 2, 3$, satisfy $(1/2\pi)\Im T_i = (1/2x_{Bj})F_i$, $i = 2, L$, $(1/2\pi)\Im T_3 = F_3$, is projected by

$$p_N \equiv \left[\frac{q^{\{\mu_1}\cdots q^{\mu_N\}}}{N!}\frac{\partial^N}{\partial p^{\mu_1}\cdots\partial p^{\mu_N}}\right]\Bigg|_{p=0} \tag{29}$$

with $x_{Bj} = Q^2/(2qp)$ in the standard DIS notation; this projects the coefficient of $1/(2x_{Bj})^N$. For the dominant terms which we resum here, the characteristic behavior

would correspond formally to γ_q-dependent anomalous dimensions associated with the respective coefficient whereas by definition Wilson's expansion does not contain such. In more phenomenologically familiar language, it is well-known that the parton model used in this paper to calculate the large distance effects that improve the kernels contains such effects whereas Wilson's expansion does not; for example, the parton model can be used for Drell-Yan processes, whereas Wilson's expansion cannot. Similarly, any Wilson-expansion guided procedure used to infer the kernels via inverse Mellin transformation, by calculating the coefficient of $(1/z)^n$ in Wilson's expansion, will necessarily omit the dominant IR terms which we resum. Here, we stress that we refer to the properties of the expansion of the invariant functions T_i, *not to the expansion of the kernels themselves*, as the latter are related to the respective anomalous dimension matrix elements by inverse Mellin transformations.

The normalization condition in (6) then gives us the final expression

$$P_{qq}(z) = C_F F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q} \left[\frac{1+z^2}{1-z}(1-z)^{\gamma_q} - f_q(\gamma_q)\delta(1-z) \right], \tag{30}$$

where

$$f_q(\gamma_q) = \frac{2}{\gamma_q} - \frac{2}{\gamma_q+1} + \frac{1}{\gamma_q+2}. \tag{31}$$

The latter result is then our IR-improved kernel for NS DGLAP-CS evolution in QCD. We note that the appearance of the integrable function $(1-z)^{-1+\gamma_q}$ in the place of $1/(1-z)_+$ was already anticipated by Gribov and Lipatov in [18–21]. Here, we have calculated the value of γ_q in a systematic rearrangement of the QCD perturbation theory that allows one to work to any exact order in the theory without dropping any part of the theory's perturbation series.

The standard DGLAP-CS theory tells us that the kernel $P_{Gq}(z)$ is related to $P_{qq}(1-z)$ directly: for $z < 1$, we have

$$P_{Gq}(z) = P_{qq}(1-z) = C_F F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q} \frac{1+(1-z)^2}{z} z^{\gamma_q}. \tag{32}$$

This then brings us to our first nontrivial check of the new IR-improved theory; for, the conservation of momentum tells us that

$$\int_0^1 dz z \left(P_{Gq}(z) + P_{qq}(z) \right) = 0. \tag{33}$$

In view of new results in (30), (32), we note that, for any $P_{qq}(z)$ which satisfies the normalization condition (6) and which is related to $P_{Gq}(z)$ via the relation

$$P_{Gq}(z) = P_{qq}(1-z), \tag{34}$$

$z < 1$, we have the following result:

$$\begin{aligned} I &= \int_0^1 dz z \left(P_{qq}(z) + P_{Gq}(z) \right) \\ &= \int_0^1 dz \left(\left((z-1)+1 \right) P_{qq}(z) + z P_{qq}(1-z) \right) \\ &= \int_0^1 dz \left[-(1-z) P_{qq}(z) + z P_{qq}(1-z) + P_{qq}(z) \right]. \end{aligned} \tag{35}$$

The integral of the first term in square brackets on the RHS of this last equation is transformed to the negative of the integral of the second one by the change of variable $z \to 1 - z$ so that they exactly cancel while the third term integrates to zero by the normalization condition (6). Thus

$$I = 0 \tag{36}$$

and the quark momentum sum rule is satisfied. Since our new results for P_{qq}, P_{Gq} satisfy the conditions for $I = 0$, we conclude that the quark momentum sum rule holds for them as well.

Having improved the IR divergence properties of $P_{qq}(z)$ and $P_{Gq}(z)$, we now turn to $P_{GG}(z)$ and $P_{qG}(z)$. We first note that the standard formula for $P_{qG}(z)$,

$$P_{qG}(z) = \frac{1}{2}\left(z^2 + (1-z)^2\right), \tag{37}$$

is already well-behaved (integrable) in the IR regime. Thus, we do not need to improve it here to make it integrable; and we note that the singular contributions in the other kernels are expected to dominate the evolution effects in any case. We do not exclude improving it for the best precision [61] and we return to this point presently.

This brings us then to $P_{GG}(z)$. Its lowest-order form is

$$P_{GG}(z) = 2C_G\left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right) \tag{38}$$

which again exhibits unintegrable IR singularities at both $z = 1$ and $z = 0$. (Here, C_G is the gluon quadratic Casimir invariant, so that it is just $N_c = 3$.) If we repeat the QCD exponentiation calculation carried-out above by using the color representation for the gluon rather than that for the quarks, that is, if we apply the exponentiation analysis in the appendix to the squared amplitude for the process $G \to G(z) + G(1-z)$, we get the exponentiated unnormalized result

$$\begin{aligned} P_{GG}(z) &= 2C_G F_{\mathrm{YFS}}(\gamma_G) e^{(1/2)\delta_G} \\ &\quad \times \left(\frac{1-z}{z} z^{\gamma_G} + \frac{z}{1-z}(1-z)^{\gamma_G} + \frac{1}{2}\left(z^{1+\gamma_G}(1-z) + z(1-z)^{1+\gamma_G}\right)\right) \end{aligned} \tag{39}$$

wherein we obtain the γ_G and δ_G from the expressions for γ_q and δ_q by the substitution $C_F \to C_G$:

$$\begin{aligned} \gamma_G &= C_G \frac{\alpha_s}{\pi} t = \frac{4C_G}{\beta_0}, \\ \delta_G &= \frac{\gamma_G}{2} + \frac{\alpha_s C_G}{\pi}\left(\frac{\pi^2}{3} - \frac{1}{2}\right). \end{aligned} \tag{40}$$

We see again that exponentiation has again made the singularities at $z = 1$ and $z = 0$ integrable.

To normalize P_{GG}, we take into account the virtual corrections such that the gluon momentum sum rule

$$\int_0^1 dz z\left(2n_f P_{qG}(z) + P_{GG}(z)\right) = 0 \tag{41}$$

is satisfied. This gives us finally the IR-improved result

$$P_{GG}(z) = 2C_G F_{\mathrm{YFS}}(\gamma_G) e^{(1/2)\delta_G} \times \left\{ \frac{1-z}{z} z^{\gamma_G} + \frac{z}{1-z}(1-z)^{\gamma_G} + \frac{1}{2}\left(z^{1+\gamma_G}(1-z) + z(1-z)^{1+\gamma_G}\right) - f_G(\gamma_G)\delta(1-z) \right\}, \tag{42}$$

where for $f_G(\gamma_G)$ we get

$$f_G(\gamma_G) = \frac{n_f}{6C_G F_{\mathrm{YFS}}(\gamma_G)} e^{-(1/2)\delta_G} + \frac{2}{\gamma_G(1+\gamma_G)(2+\gamma_G)} + \frac{1}{(1+\gamma_G)(2+\gamma_G)} + \frac{1}{2(3+\gamma_G)(4+\gamma_G)} + \frac{1}{(2+\gamma_G)(3+\gamma_G)(4+\gamma_G)}. \tag{43}$$

It is these improved results in (30), (32), (42) for $P_{qq}(z)$, $P_{Gq}(z)$, and $P_{GG}(z)$ that we use together with the standard result in (38) for $P_{qG}(z)$ as the IR-improved DGLAP-CS theory.

For clarity we summarize at this point the new IR-improved kernel set as follows:

$$P_{qq}^{\exp}(z) = C_F F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q} \left[\frac{1+z^2}{1-z}(1-z)^{\gamma_q} - f_q(\gamma_q)\delta(1-z) \right], \tag{44}$$

$$P_{Gq}^{\exp}(z) = C_F F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q} \frac{1+(1-z)^2}{z} z^{\gamma_q}, \tag{45}$$

$$P_{GG}^{\exp}(z) = 2C_G F_{\mathrm{YFS}}(\gamma_G) e^{(1/2)\delta_G} \times \left\{ \frac{1-z}{z} z^{\gamma_G} + \frac{z}{1-z}(1-z)^{\gamma_G} + \frac{1}{2}\left(z^{1+\gamma_G}(1-z) + z(1-z)^{1+\gamma_G}\right) - f_G(\gamma_G)\delta(1-z) \right\}, \tag{46}$$

$$P_{qG}(z) = \frac{1}{2}\left(z^2 + (1-z)^2\right), \tag{47}$$

where we have introduced the superscript exp to denote the exponentiated results henceforth.

Returning now to the improvement of $P_{qG}(z)$, let us record it as well for the sake of completeness and of providing better precision. Applying (17) to the process $G \to q + \overline{q}$, we get the exponentiated result

$$P_{qG}^{\exp}(z) = F_{\mathrm{YFS}}(\gamma_G) e^{(1/2)\delta_G} \frac{1}{2}\left\{ z^2(1-z)^{\gamma_G} + (1-z)^2 z^{\gamma_G} \right\}. \tag{48}$$

The gluon momentum sum rule then gives the new normalization constant for the $P_{GG}^{\exp}$ via the result

$$\overline{f}_G(\gamma_G) = \frac{n_f}{C_G} \frac{1}{(1+\gamma_G)(2+\gamma_G)(3+\gamma_G)} + \frac{2}{\gamma_G(1+\gamma_G)(2+\gamma_G)} + \frac{1}{(1+\gamma_G)(2+\gamma_G)} + \frac{1}{2(3+\gamma_G)(4+\gamma_G)} + \frac{1}{(2+\gamma_G)(3+\gamma_G)(4+\gamma_G)}. \tag{49}$$

The constant $\overline{f}_G$ should be substituted for f_G in $P_{GG}^{\exp}$ whenever the exponentiated result in (48) is used. These results, (47); (48); and (49), are our new improved DGLAP-CS kernel set, with the option exponentiating P_{qG} as well. Let us now look into their effects on the moments of the structure functions by discussing the corresponding effects on the moments of the parton distributions.

We know that moments of the kernels determine the exponents in the logarithmic variation [15–21] of the moments of the quark distributions and, thereby, of the moments of the structure functions themselves. To wit, in the nonsinglet case, we have

$$\frac{dM_n^{\mathrm{NS}}(t)}{dt} = \frac{\alpha_s(t)}{2\pi} A_n^{\mathrm{NS}} M_n^{\mathrm{NS}}(t), \tag{50}$$

where

$$M_n^{\mathrm{NS}}(t) = \int_0^1 dz z^{n-1} q^{\mathrm{NS}}(z,t) \tag{51}$$

and the quantity A_n^{NS} is given by

$$\begin{aligned} A_n^{\mathrm{NS}} &= \int_0^1 dz z^{n-1} P_{qq}^{\exp}(z) \\ &= C_F F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q} \left[B(n,\gamma_q) + B(n+2,\gamma_q) - f_q(\gamma_q) \right] \end{aligned} \tag{52}$$

where $B(x,y)$ is the beta function given by

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y). \tag{53}$$

This should be compared to the un-IR-improved result [15–21]:

$$A_n^{\mathrm{NS}^o} \equiv C_F \left[-\frac{1}{2} + \frac{1}{n(n+1)} - 2\sum_{j=2}^{n} \frac{1}{j} \right]. \tag{54}$$

The asymptotic behavior for large n is now very different, as the IR-improved exponent approaches a constant, a multiple of $-f_q$, as we would expect as $n \to \infty$ because $\lim_{n\to\infty} z^{n-1} = 0$ for $0 \le z < 1$ whereas, as it is well-known, the un-IR-improved result in (54) diverges as $-2C_F \ln n$ as $n \to \infty$. The two results are also different at finite n: for $n = 2$ we get, for example, for $\alpha_s \cong .118$ [62],

$$A_2^{\mathrm{NS}} = \begin{cases} C_F(-1.33), & \text{un-IR-improved}, \\ C_F(-0.966), & \text{IR-improved}, \end{cases} \tag{55}$$

so that the effects we have calculated are important for all n in general. For completeness, we note that the solution to (50) is given by the standard methods as

$$\begin{aligned} M_n^{\mathrm{NS}}(t) &= M_n^{\mathrm{NS}}(t_0) e^{\int_{t_0}^{t} dt' (\alpha_s(t')/2\pi) A_n^{\mathrm{NS}}(t')} \\ &= M_n^{\mathrm{NS}}(t_0) e^{\overline{a}_n [Ei((1/2)\delta_1 \alpha_s(t_0)) - Ei((1/2)\delta_1 \alpha_s(t))]} \underset{t,t_0 \text{ large with } t \gg t_0}{\Longrightarrow} M_n^{\mathrm{NS}}(t_0) \left(\frac{\alpha_s(t_0)}{\alpha_s(t)} \right)^{\overline{a}'_n}, \end{aligned} \tag{56}$$

where $Ei(x) = \int_{-\infty}^{x} dr e^{r}/r$ is the exponential integral function,

$$\begin{aligned} \overline{a}_n &= \frac{2C_F}{\beta_0} F_{\mathrm{YFS}}(\gamma_q) e^{\gamma_q/4} \left[B(n,\gamma_q) + B(n+2,\gamma_q) - f_q(\gamma_q) \right], \\ \overline{a}'_n &= \overline{a}_n \left(1 + \frac{\delta_1}{2} \frac{(\alpha_s(t_0) - \alpha_s(t))}{\ln(\alpha_s(t_0)/\alpha_s(t))} \right) \end{aligned} \tag{57}$$

with

$$\delta_1 = \frac{C_F}{\pi} \left(\frac{\pi^2}{3} - \frac{1}{2} \right). \tag{58}$$

We can compare with the un-IR-improved result in which the last line in (56) holds exactly with $\overline{a}'_n = 2A_n^{\mathrm{NS}^{\circ}}/\beta_0$. Phenomenologically, for $n = 2$, taking $Q_0 = 2\,\mathrm{GeV}$ and evolving to $Q = 100\,\mathrm{GeV}$, if we set $\Lambda_{\mathrm{QCD}} \cong .2\,\mathrm{GeV}$ and use $n_f = 5$ for definiteness of illustration, we see from (56), (57) that we get a shift of the respective evolved NS moment by ~5%, which is of some interest in view of the expected HERA precision [63]. (Although HERA is shutdown, HERA data analysis continues as the H1 and ZEUS combine their data to improve their results accordingly.)

We give now the remaining elements of the anomalous dimension matrix in its "best" IR-improved form for completeness:

$$\begin{aligned} A_n^{Gq} &= \int_0^1 dz z^{n-1} P_{Gq}^{\exp}(z) \\ &= C_F F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q} \left[\frac{1}{n+\gamma_q-1} + B(3, n+\gamma_q-1) \right], \end{aligned} \tag{59}$$

$$\begin{aligned} A_n^{GG} &= \int_0^1 dz z^{n-1} P_{GG}^{\exp}(z) \\ &= 2C_G F_{\mathrm{YFS}}(\gamma_G) e^{(1/2)\delta_G} \Big\{ B(n+1,\gamma_G) + B(n+\gamma_G-1,2) \\ &\qquad + \frac{1}{2}\left(B(n+1,\gamma_G+2) + B(n+\gamma_G+1,2) \right) - \overline{f}(\gamma_G) \Big\}, \end{aligned} \tag{60}$$

$$\begin{aligned} 2n_f A_n^{qG} &= 2n_f \int_0^1 dz z^{n-1} P_{qG}^{\exp}(z) \\ &= 2T(F) F_{\mathrm{YFS}}(\gamma_G) e^{(1/2)\delta_G} \left(B(n+2, 1+\gamma_G) + B(n+\gamma_G, 3) \right), \end{aligned} \tag{61}$$

where $T(F) = (1/2)n_f$. We note that the unexponentiated value of the last result in (61) is a well-known one [15–21], $2T(F)(2+n+n^2)/n(n+1)(n+2)$, and it would be used whenever we do not choose to exponentiate P_{qG}. We will investigate the further implications of these IR-improved results for LHC physics elsewhere [61].

In the discussion so far, we have used the lowest-order DGLAP-CS kernel set to illustrate how important the resummation which we present here can be. In the literature [64–74], there are now exact results up to $\mathcal{O}(\alpha_s^3)$ for the DGLAP-CS kernels. The question naturally arises as to the relationship of our work to these fixed-order exact results. We stress

first that we are presenting an improvement of the fixed-order results such that the singular pieces of the any exact fixed-order result, that is, the $1/(1-z)_+$ parts, are exponentiated so that they are replaced with integrable functions proportional to $(1-z)^{\gamma-1}$ with γ positive as we have illustrated above. Since the series of logs which we resum to accomplish this has the structure $\alpha_s^{\ell} t^{\ell} \ln^n(1-z)$, $\ell \geq n$, these terms are not already present in the results in [64–74]. As we use the formula in (17), there will be no double counting if we implement our IR-improvement of the exact fixed-order results in [64–74]. The detailed discussion of the application of our theory to the results in [64–74] will appear elsewhere [61]. For reference, we note that the higher-order kernel corrections in [64–74] are perturbatively related to the leading-order kernels, so one can expect that the size of the exponentiation effects illustrated above will only be perturbatively modified by the higher-order kernel corrections, leaving the same qualitative behavior in general.

In the interest of specificness, let us illustrate the IR-improvement of P_{qq} when calculated to three loops using the results in [64–74]. Considering the nonsinglet case for definiteness (a similar analysis holds for the singlet case) we write in the notation of the latter references:

$$P_{\mathrm{ns}}^{+} = P_{qq}^{v} + P_{q\bar{q}}^{v} \equiv \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} P_{\mathrm{ns}}^{(n)+}, \tag{62}$$

where at order $\mathcal{O}(\alpha_s)$, we have

$$P_{\mathrm{ns}}^{(0)+}(z) = 2C_F \left\{ \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right\} \tag{63}$$

which shows that $P_{\mathrm{ns}}^{(0)+}(z)$ agrees with the unexponentiated result in (7) for P_{qq} except for an overall factor of 2. We use this latter identification to connect our work with that in [64–74] in the standard methodology. In [64–74], exact results are given for $P_{\mathrm{ns}}^{(1)+}(z)$, and in [73, 74] exact results are given for $P_{\mathrm{ns}}^{(2)+}(z)$. When we apply the result in (17) to the squared amplitudes for the processes $q \to q + X$, $\bar{q} \to q + X'$, we get the exponentiated result

$$\begin{aligned} P_{\mathrm{ns}}^{+,\exp}(z) = {} & \left(\frac{\alpha_s}{4\pi}\right) 2P_{qq}^{\exp}(z) + F_{\mathrm{YFS}}(\gamma_q) e^{(1/2)\delta_q} \\ & \times \left[\left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ (1-z)^{\gamma_q} \overline{P}_{\mathrm{ns}}^{(1)+}(z) + \overline{B}_2 \delta(1-z) \right\} \right. \\ & \left. + \left(\frac{\alpha_s}{4\pi}\right)^3 \left\{ (1-z)^{\gamma_q} \overline{P}_{\mathrm{ns}}^{(2)+}(z) + \overline{B}_3 \delta(1-z) \right\} \right], \end{aligned} \tag{64}$$

where $P_{qq}^{\exp}(z)$ is given in (47) and the resummed residuals $\overline{P}_{\mathrm{ns}}^{(i)+}$, $i = 1, 2$, are related to the exact results for $P_{\mathrm{ns}}^{(i)+}$, $i = 1, 2$, as follows:

$$\overline{P}_{\mathrm{ns}}^{(i)+}(z) = P_{\mathrm{ns}}^{(i)+}(z) - B_{1+i}\delta(1-z) + \Delta_{\mathrm{ns}}^{(i)+}(z), \tag{65}$$

where

$$\Delta_{\rm ns}^{(1)+}(z) = -4C_F\pi\delta_1\left\{\frac{1+z^2}{1-z} - f_q\delta(1-z)\right\},$$
$$\Delta_{\rm ns}^{(2)+}(z) = -4C_F(\pi\delta_1)^2\left\{\frac{1+z^2}{1-z} - f_q\delta(1-z)\right\} - 2\pi\delta_1\overline{P}_{\rm ns}^{(1)+}(z), \tag{66}$$

$$\overline{B}_2 = B_2 + 4C_F\pi\delta_1 f_q,$$
$$\overline{B}_3 = B_3 + 4C_F(\pi\delta_1)^2 f_q - 2\pi\delta_1\overline{B}_2. \tag{67}$$

Here, the constants B_i, $i = 2, 3$, are given by the results in [64–74] as

$$\begin{aligned}
B_2 &= 4C_GC_F\left(\frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3\right) - 4C_Fn_f\left(\frac{1}{12} + \frac{2}{3}\zeta_2\right) + 4C_F^2\left(\frac{3}{8} - 3\zeta_2 + 6\zeta_3\right),\\
B_3 &= 16C_GC_Fn_f\left(\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2^2 + \frac{25}{18}\zeta_3\right)\\
&\quad + 16C_GC_F^2\left(\frac{151}{64} + \zeta_2\zeta_3 - \frac{205}{24}\zeta_2 - \frac{247}{60}\zeta_2^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5\right)\\
&\quad + 16C_G^2C_F\left(-\frac{1657}{576} + \frac{281}{27}\zeta_2 - \frac{1}{8}\zeta_2^2 - \frac{97}{9}\zeta_3 + \frac{5}{2}\zeta_5\right)\\
&\quad + 16C_Fn_F^2\left(-\frac{17}{144} + \frac{5}{27}\zeta_2 - \frac{1}{9}\zeta_3\right)\\
&\quad + 16C_F^2n_F\left(-\frac{23}{16} + \frac{5}{12}\zeta_2 + \frac{29}{30}\zeta_2^2 - \frac{17}{6}\zeta_3\right)\\
&\quad + 16C_F^3\left(\frac{29}{32} - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}\zeta_2^2 + \frac{17}{4}\zeta_3 - 15\zeta_5\right),
\end{aligned} \tag{68}$$

where ζ_n is the Riemann zeta function evaluated at argument n. In arriving at the result in (64), we use the fact that the $\overline{MS}$ results for the higher-order kernels do not contain any of the powers of γ_q that we have resummed, so that the only issue for their improvement is the factor $e^{\delta_q/2}$, which then has to have the coefficients in the results for the higher-order kernels adjusted so that there is no double counting. It is in this way that we have derived the results in (65)–(67). The detailed phenomenological consequences of the fully exponentiated 2- and 3-loop DGLAP-CS kernel set will appear elsewhere [61].

In summary, we have used exact rearrangement of the QCD Feynman series to isolate and resum the leading IR contributions to the physical processes that generate the evolution kernels in DGLAP-CS theory. In this way, we have obviated the need to employ artificial mathematical regularization of the attendant IR singularities as the theory's higher-order corrections naturally tame these singularities. The resulting IR-improved anomalous dimension matrix behaves more physically for large n and receives significant effects at finite n from the exponentiation.

We in principle can make contact with the moment-space resummation results in [75] but we stress that these results have necessarily been obtained after making a Mellin

transform of the mathematical artifact which we address in this paper. Thus, the results in [75] do not in any way contradict the analysis in this paper.

We note that the program of improvement of the hadron cross section calculations for LHC physics advanced herein should be distinguished from the results in [76–78]. Indeed, recalling the standard hadron cross section formula

$$\sigma = \sum_{i,j} \int dx_1 dx_2 F_i(x_1) F_j(x_2) \widehat{\sigma}(x_1 x_2 s), \tag{69}$$

where $\{F_\ell(x)\}$ are the respective parton densities and $\widehat{\sigma}(x_1 x_2 s)$ is the respective reduced hard parton cross section, the resummation results in [76–78] address, by summing the large logs in Mellin transform space, the $x_1 x_2 \to 1$ limit of $\widehat{\sigma}(x_1 x_2 s)$ whereas the results above address the improvement, by resummation in x-space, of the calculation of the parton densities $\{F_i(x)\}$ for all values of x.Thus, the program of improvement presented above is entirely complementary to that in [76–78] and both programs of improvement are needed for precision LHC physics. The situation can be illustrated by comparing the results in [79] with our results herein. The key observation can already be made from (2.1) in the latter paper, wherein it is made manifest that the resummation carried out therein, as an application of the methods in [76–78], is a resummation for the large N-Mellin space limit of the Mellin transform of the hard scattering coefficient function so that all of the IR effects in the parton densities are not included in this resummation. What we deal with here is however resummation of the IR effects in the kernels which generate exactly these IR effects in these parton densities directly in configuration space so that we work on a complementary aspect the formula (69) and this we do directly in x-space rather than in N-Mellin space. There is then no contradiction or repetition between our results and those in [79].

The usual factorization theorems for mass singularities in QCD are fully consistent with our results: we act on the Feynman series for the hadron-hadron scattering in (69) after the mass singularities have been factorized into the parton densities, as our resummation is multiplicative in character. What one has to note is that, since the methods of [76–78], which are also consistent with the QCD factorization theorems, apply to the hard scattering coefficients, there is always the possibility to use them to improve any hard scattering effect where soft gluons are important. In particular, it is possible to use these methods to resum the soft gluon effects on the hard scattering contribution which one assigns in one's scheme to the kernels for example, as one can see in [79]. The resummation of the effects which we address, involving as they do terms of the form $\alpha_s^n t^n \ln^n(1-z)$, is genuinely associated with the external line initial-state parton density evolution aspects of the kernels, and is not addressed by the methods in [76–78]. Both resummations obtain because of the exclusive limit $1 - z \to 0$. One [76–78] is focused on the effects which remain after those associated with initial-state collinear singularities are removed so that they can be addressed by analyzing the respective hard coefficient function; and the other (that presented herein) is inclusive and allows one to focus on the effects associated with the initial-state collinear singularities as well as effects associated with the hard scattering coefficient, as we show now in the appendix by analyzing the result of [80] in our framework. From the discussion in the appendix, we see manifestly that there is no double counting of effects between the two approaches when they are used properly.

Finally, we address the issue of the relationship between the rearrangement that we have made of the exact leading-logs in the QCD perturbation theory and the usual treatment

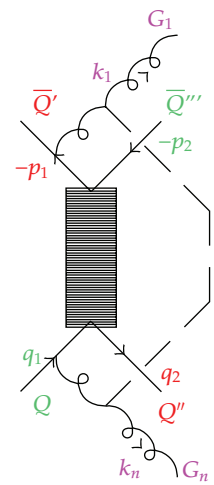

Figure 2: The process $\overline{Q}'Q \to \overline{Q}''' + Q'' + n(G)$. The four-momenta are indicated in the standard manner: q_1 is the four-momentum of the incoming Q, q_2 is the four-momentum of the outgoing Q'', and so forth, and $Q = u, d, s, c, b, G$.

in the nonexponentiated DGLAP-CS theory. If one expands out the exponentiated kernels, using the distribution identity

$$(1-z)^{a-1} = \frac{1}{a}\delta(1-z) + \frac{1}{(1-z)_+} + \sum_{j=1}^{\infty}\frac{a^j}{j!}\left[\frac{\ln^j(1-z)}{1-z}\right]_+, \tag{70}$$

one can see that for example P_{qq} and $P_{qq}^{\exp}$ agree to leading order, so that the leading log series which they generate for the respective NS parton distributions also agree through leading order in $(\alpha_s/\pi)L$, where L is the respective big log in momentum-space. At higher orders then, we have a different result for the $\{F_i\}$, let us denote them by $\{F'_i\}$, and a different result for the reduced cross section, let us denote it by $\widehat{\sigma}'$, such that we get the same perturbative QCD cross section

$$\sigma = \sum_{i,j}\int dx_1 dx_2 F_i(x_1)F_j(x_2)\widehat{\sigma}(x_1x_2s) = \sum_{i,j}\int dx_1 dx_2 F'_i(x_1)F'_j(x_2)\widehat{\sigma}'(x_1x_2s) \tag{71}$$

order by order in perturbation theory. The exponentiated kernels are used to factorize the mass singularities from the unfactorized reduced cross section and this generates $\widehat{\sigma}'$ instead of the usual $\widehat{\sigma}$ whose factorized form is generated using the usual DGLAP-CS kernels. We thus have the same leading log series for σ as does the usual calculation with unexponentiated DGLAP-CS kernels. We have an important advantage: the lack of +-functions in the generation of the configuration space functions $\{F'_i, \widehat{\sigma}'\}$ means that these functions lend themselves more readily to Monte Carlo realization to arbitrarily soft radiative effects, both for the generation of the parton shower associated to the $\{F'_i\}$ and for the attendant remaining radiative effects in $\widehat{\sigma}'$. Further consequences of our results for LHC physics will be presented elsewhere[61].

Note-added

The application of exact, amplitude-based YFS-style resummation to non-Abelian gauge theories is done for the first time in [37–48]. In [81, 82], cancellation of IR singularities for

QCD is approached from the KLN theorem perspective. As far as QED itself is concerned, the treatment in [81] is just the case of a singlet form-factor in which the exponentiated virtual IR function that is finally exhibited is not gauge invariant. The exponentiation of the soft real emission processes which cancel these virtual IR singularities is then done as an approximate treatment of the real emission processes in which momentum conservation for the soft real emission is ignored. In [82], the exponentiation and cancellation of IR singularities are demonstrated for any number of external electron lines as an approximate representation of the respective amplitudes in which the IR divergent terms are retained—finite terms are dropped. Thus, in neither case is the exact YFS theory for QED presented for the entire theory. Finally, we note that the discussion in [83] is a complete version of that in [82] but it still treats soft real photon emission in same soft photon approximation, so that it is not an exact rearrangement of the theory such as we have in the YFS formulation.

Appendix

In this appendix, we present the new QCD exponentiation theory which has been developed in [37–48] as it is not generally familiar. The goal is to make the current paper self-contained.

For definiteness, we will use the process in Figure 2, $\overline{Q}'(p_1)Q(q_1) \to \overline{Q}'''(p_2)Q''(q_2) + G_1(k_1)\cdots G_n(k_n)$, as the prototypical process, where we have written the kinematics as it is illustrated in the figure. This process, which dominates processes such as $t\bar{t}$ production at FNAL, contains all of the theoretical issues that we must face at the parton level to establish, as an extension of the original ideas of Yennie-Frautschi-Suura (YFS) [49, 50], QCD soft exponentiation by MC methods—applicability to other related processes will be immediate. For reference, let us also note that, in what follows, we use the GPS conventions of [84] for spinors $\{u, v, \mathfrak{u}\}$ and the attendant photon and gluon polarization vectors that follow therefrom:

$$\begin{aligned}
\left(\epsilon^{\mu}_{\sigma}(\beta)\right)^{*} &= \frac{\overline{u}_{\sigma}(k)\gamma^{\mu}u_{\sigma}(\beta)}{\sqrt{2}\,\overline{u}_{-\sigma}(k)u_{\sigma}(\beta)}, \\
\left(\epsilon^{\mu}_{\sigma}(\zeta)\right)^{*} &= \frac{\overline{u}_{\sigma}(k)\gamma^{\mu}\mathfrak{u}_{\sigma}(\zeta)}{\sqrt{2}\,\overline{u}_{-\sigma}(k)\mathfrak{u}_{\sigma}(\zeta)},
\end{aligned} \tag{A.1}$$

with $\beta^2 = 0$ and ζ defined in [84], so that all phase information is strictly known in our amplitudes. This means that, although we will use the older EEX realization of YFS MC exponentiation as defined in [85], the realization of our results via the the newer CEEX realization of YFS exponentiation in [85] is also possible and is in progress [61].

Specifically, the authors in [46–48] have analyzed how in the special case of Born level color exchange one applies the YFS theory to QCD by extending the respective YFS IR singularity analysis to QCD to all orders in α_s. Here, unlike what was emphasized in [46–48], we focus on the YFS theory as a general rearrangement of renormalized perturbation theory based on its IR behavior, just as the renormalization group is a general property of renormalized perturbation theory based on its ultra-violet (UV) behavior. We will thus keep our arguments entirely general from the outset, so that it will be immediate that our result applies to any renormalized perturbation theory in which the cross section under study is finite.

Let the amplitude for the emission of n real gluons in our prototypical subprocess, $Q^{\alpha} + \overline{Q}^{\prime\overline{\alpha}} \to Q^{\prime\prime\gamma}\overline{Q}^{\prime\prime\overline{\gamma}} + n(G)$, where α, $\overline{\alpha}$, γ, and $\overline{\gamma}$ are color indices, be represented by

$$\mathcal{M}^{(n)\alpha\overline{\alpha}}_{\gamma\overline{\gamma}} = \sum_{\ell} M^{(n)\alpha\overline{\alpha}}_{\gamma\overline{\gamma}\ell}, \tag{A.2}$$

$M^{(n)}_{\ell}$ is the contribution to $\mathcal{M}^{(n)}$ from Feynman diagrams with ℓ virtual loops. Symmetrization yields

$$M^{(n)}_{\ell} = \frac{1}{\ell!}\int \prod_{j=1}^{\ell} \frac{d^4k_j}{(k_j^2 - \lambda^2 + i\epsilon)} \rho^{(n)}_{\ell}(k_1, \ldots, k_{\ell}), \tag{A.3}$$

where this last equation defines $\rho^{(n)}_{\ell}$ as a symmetric function of its arguments $k_1, \ldots, k_{\ell}$. λ will be our infrared gluon regulator mass for IR singularities; n-dimensional regularization of the $'t$ Hooft-Veltman [57, 58] type is also possible as we will see. We now define the virtual IR emission factor $S_{\mathrm{QCD}}(k)$ for a gluon of 4-momentum k, for the $k \to 0$ regime of the respective 4-dimensional loop integration as in (A.3), such that

$$\lim_{k\to 0} k^2 \left(\rho^{(n)\alpha\overline{\alpha}}_{\gamma\overline{\gamma}1}(k)|_{\text{leading Casimir contribution}} - S_{\mathrm{QCD}}(k)\rho^{(n)\alpha\overline{\alpha}}_{\gamma\overline{\gamma}0}\right) = 0, \tag{A.4}$$

where we have now introduced the restriction to the leading color Casimir terms at one-loop (these correspond with maximally non-Abelian terms in [86] but computed exactly rather than in the eikonal approximation) so that in the expression for the respective one-loop correction $\rho^{(n)}_1$ and in that for $S_{\mathrm{QCD}}(k)$ given in [46–48], only the terms proportional to C_F should be retained here as we focus on the $f\overline{f} \to f\overline{f}$ case, where f denotes a fermion. (Henceforth, when we refer to $k \to 0$ gluons we are always referring for virtual gluons to the corresponding regime of the 4D loop integration in the computation of $M^{(n)}_{\ell}$.)

In [46–48], the respective authors have calculated $S_{\mathrm{QCD}}(k)$ using the running quark masses to regulate its collinear mass singularities, for example; n-dimensional regularization of the $'t$ Hooft-Veltman type is also possible for these mass singularities and we will also illustrate this presently.

We stress that $S_{\mathrm{QCD}}(k)$ has a freedom in it corresponding to the fact that any function $\Delta S_{\mathrm{QCD}}(k)$ which has the property that $\lim_{k\to 0} k^2 \Delta S_{\mathrm{QCD}}(k)\rho^{(n)}_0 = 0$ may be added to it.

Since the virtual gluons in $\rho^{(n)}_{\ell}$ are all on equal footing by the symmetry of this function, if we look at gluon ℓ, for example, we may write, for $k_{\ell} \to (0,0,0,0) \equiv O$ while the remaining k_i are fixed away from O, the representation

$$\rho^{(n)}_{\ell} = S_{\mathrm{QCD}}(k_{\ell}) * \rho^{(n)}_{\ell-1}(k_1, \ldots, k_{\ell-1}) + \beta^1_{\ell}(k_1, \ldots, k_{\ell-1}; k_{\ell}), \tag{A.5}$$

where the residual amplitude $\beta^1_{\ell}(k_1, \ldots, k_{\ell-1}; k_{\ell})$ will now be taken as defined by this last equation. It has two nice properties listed as follows:

(i) it is symmetric in its first $\ell - 1$ arguments;

(ii) the IR singularities for gluon ℓ *that are contained in* $S_{\mathrm{QCD}}(k_{\ell})$ are no longer contained in it.

We do not at this point discuss the extent to which there are any further remaining IR singularities for gluon ℓ in $\beta_\ell^1(k_1,\ldots,k_{\ell-1};k_\ell)$. In an Abelian gauge theory like QED, as has been shown by Yennie et al. [49, 50], there would not be any further such singularities; for a non-Abelian gauge theory like QCD, this point requires further discussion and we will come back to this point presently.

We rather now stress that if we apply the representation (A.5) again we may write

$$\begin{aligned}\rho_\ell^{(n)} = {} & S_{\mathrm{QCD}}(k_\ell)S_{\mathrm{QCD}}(k_{\ell-1})*\rho_{\ell-2}^{(n)}(k_1,\ldots,k_{\ell-2}) \\ & + S_{\mathrm{QCD}}(k_\ell)\beta_{\ell-1}^1(k_1,\ldots,k_{\ell-2};k_{\ell-1}) \\ & + S_{\mathrm{QCD}}(k_{\ell-1})\beta_{\ell-1}^1(k_1,\ldots,k_{\ell-2};k_\ell) \\ & + \beta_\ell^2(k_1,\ldots,k_{\ell-2};k_{\ell-1},k_\ell),\end{aligned} \tag{A.6}$$

where this last equation serves to define the function $\beta_\ell^2(k_1,\ldots,k_{\ell-2};k_{\ell-1},k_\ell)$. It has two nice properties listed below:

(i) it is symmetric in its first $\ell-2$ arguments and in its last two arguments $k_{\ell-1}$, k_ℓ;

(ii) the infrared singularities for gluons $\ell-1$ and ℓ that are contained in $S_{\mathrm{QCD}}(k_{\ell-1})$ and $S_{\mathrm{QCD}}(k_\ell)$ are no longer contained in it.

Continuing in this way, with repeated application of (A.5), we get finally the rigorous, exact rearrangement of the contributions to $\rho_\ell^{(n)}$ as

$$\rho_\ell^{(n)} = S_{\mathrm{QCD}}(k_1)\cdots S_{\mathrm{QCD}}(k_\ell)\beta_0^0 + \sum_{i=1}^{\ell}\prod_{j\neq i} S_{\mathrm{QCD}}(k_j)\beta_1^1(k_i) + \cdots + \beta_\ell^\ell(k_1,\ldots,k_\ell), \tag{A.7}$$

where the virtual gluon residuals $\beta_i^i(k_1',\ldots,k_i')$ have two nice properties:

(i) they are symmetric functions of their arguments

(ii) they do not contain any of the IR singularities which are contained in the product $S_{\mathrm{QCD}}(k_1')\cdots S_{\mathrm{QCD}}(k_i')$.

Henceforth, we denote β_i^i as the function β_i for reasons of pedagogy. We cannot stress too much that (A.7) is an *exact* rearrangement of the contributions of the Feynman diagrams which contribute to $\rho_\ell^{(n)}$; it involves no approximations. Here also we note that the question of the absolute convergence of these Feynman diagrams from the standpoint of constructive field theory remains open as usual. Yennie et al. [49, 50] have already stressed that Feynman diagrammatic perturbation theory is nonrigorous from this standpoint. What we do claim is that the relationship between the YFS expansion and the usual perturbative Feynman diagrammatic expansion is itself rigorous even though neither of the two expansions themselves is rigorous.

Introducing (A.7) into (A.2) yields a representation similar to that of YFS, and we will call it a "YFS representation":

$$\mathcal{M}^{(n)} = e^{\alpha_s B_{\mathrm{QCD}}}\sum_{j=0}^{\infty}\mathsf{m}_j^{(n)}, \tag{A.8}$$

where we have defined

$$\alpha_s(Q)B_{\mathrm{QCD}} = \int \frac{d^4k}{(k^2-\lambda^2+i\epsilon)} S_{\mathrm{QCD}}(k), \tag{A.9}$$

$$\mathsf{m}_j^{(n)} = \frac{1}{j!}\int \prod_{i=1}^{j} \frac{d^4k_i}{k_i^2-\lambda^2+i\epsilon}\beta_j(k_1,\ldots,k_j). \tag{A.10}$$

We say that (A.8) is similar to the respective result of Yennie et al. in [49, 50] and is not identical to it because we have not proved that the functions $\beta_i(k_1,\ldots,k_i)$ are completely free of virtual IR singularities. What we have shown is that they do not contain the IR singularities in the product $S_{\mathrm{QCD}}(k_1)\cdots S_{\mathrm{QCD}}(k_i)$ so that $\mathsf{m}_j^{(n)}$ does not contain the virtual IR divergences generated by this product when it is integrated over the respective $4j$-dimensional j-virtual gluon phase space. In an Abelian gauge theory, there are no other possible virtual IR divergences; in the non-Abelian gauge theory that we treat here, such additional IR divergences are possible and are expected; but, the result (A.8) does have an improved IR divergence structure over (A.2) in that all of the IR singularities associated with $S_{\mathrm{QCD}}(k)$ are explicitly removed from the sum over the virtual IR improved loop contributions $\mathsf{m}_j^{(n)}$ to all orders in $\alpha_s(Q)$.

Turning now to the analogous rearrangement of the real IR singularities in the differential cross section associated with the $\mathcal{M}^{(n)}$, we first note that we may write this cross section as follows according to the standard methods:

$$\begin{aligned} d\hat{\sigma}^n = \frac{e^{2\alpha_s \mathrm{Re}\, B_{\mathrm{QCD}}}}{n!}\int \prod_{m=1}^{n} \frac{d^3k_m}{\left(k_m^2+\lambda^2\right)^{1/2}} \delta\left(p_1+q_1-p_2-q_2-\sum_{i=1}^{n}k_i\right) \\ \times \overline{\rho}^{(n)}(p_1,q_1,p_2,q_2,k_1,\ldots,k_n)\frac{d^3p_2 d^3q_2}{p_2^0 q_2^0}, \end{aligned} \tag{A.11}$$

where we have defined

$$\overline{\rho}^{(n)}(p_1,q_1,p_2,q_2,k_1,\ldots,k_n) = \sum_{\text{color, spin}} \left\| \sum_{j=0}^{\infty} \mathsf{m}_j^{(n)} \right\|^2 \tag{A.12}$$

in the incoming $Q\overline{Q}'$ cms system; and we have absorbed the remaining kinematical factors for the initial-state flux, spin, and color averages into the normalization of the amplitudes $\mathcal{M}^{(n)}$ for reasons of pedagogy so that the $\overline{\rho}^{(n)}$ are averaged over initial spins and colors and summed over final spins and colors. We now proceed in complete analogy with the discussion of $\rho_\ell^{(n)}$ above.

Specifically, for the functions $\overline{\rho}^{(n)}(p_1,q_1,p_2,q_2,k_1,\ldots,k_n) \equiv \overline{\rho}^{(n)}(k_1,\ldots,k_n)$ which are symmetric functions of their arguments $k_1,\ldots,k_n$, we define first, for $n=1$,

$$\lim_{|\vec{k}|\to 0} \vec{k}^2 \left(\overline{\rho}^{(1)}(k)\big|_{\text{leading Casimir contribution}} - \widetilde{S}_{\mathrm{QCD}}(k)\overline{\rho}^{(0)}\right) = 0, \tag{A.13}$$

where the real infrared function $\widetilde{S}_{\mathrm{QCD}}(k)$ is rigorously defined by this last equation and is explicitly computed in [46–48], wherein we retain here only the terms proportional to C_F

from the result in [46–48]; like its virtual counterpart $S_{\mathrm{QCD}}(k)$ it has a freedom in it as any function $\Delta\widetilde{S}_{\mathrm{QCD}}(k)$ with the property that $\lim_{|\vec{k}|\to 0}\vec{k}^{\,2}\,\Delta\widetilde{S}_{\mathrm{QCD}}(k) = 0$ may be added to it without affecting the defining relation (A.13).

We can again repeat the analogous arguments of [49, 50], following the corresponding steps in (A.5)–(A.10) above for S_{QCD} to get the "YFS-like" result

$$\begin{aligned} d\widehat{\sigma}_{\mathrm{exp}} &= \sum_n d\widehat{\sigma}^n \\ &= e^{\Sigma_{\mathrm{IR}}(\mathrm{QCD})}\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{j=1}^{n}\frac{d^3k_j}{k_j^0}\int\frac{d^4y}{(2\pi)^4}e^{iy\cdot(p_1+q_1-p_2-q_2-\sum k_j)+D_{\mathrm{QCD}}}*\overline{\beta}_n(k_1,\ldots,k_n)\frac{d^3p_2}{p_2^0}\frac{d^3q_2}{q_2^0} \end{aligned} \tag{A.14}$$

with

$$\begin{aligned} \Sigma_{\mathrm{IR}}(\mathrm{QCD}) &= 2\alpha_s\mathrm{Re}\,B_{\mathrm{QCD}} + 2\alpha_s\widetilde{B}_{\mathrm{QCD}}(K_{\max}), \\ 2\alpha_s\widetilde{B}_{\mathrm{QCD}}(K_{\max}) &= \int\frac{d^3k}{k^0}\widetilde{S}_{\mathrm{QCD}}(k)\theta(K_{\max}-k), \\ D_{\mathrm{QCD}} &= \int\frac{d^3k}{k}\widetilde{S}_{\mathrm{QCD}}(k)\left[e^{-iy\cdot k}-\theta(K_{\max}-k)\right], \end{aligned} \tag{A.15}$$

$$\begin{aligned} \frac{1}{2}\overline{\beta}_0 &= d\sigma^{(1\text{-loop})} - 2\alpha_s\mathrm{Re}\,B_{\mathrm{QCD}}d\sigma_B, \\ \frac{1}{2}\overline{\beta}_1 &= d\sigma^{B1} - \widetilde{S}_{\mathrm{QCD}}(k)d\sigma_B,\ldots, \end{aligned} \tag{A.16}$$

where the $\overline{\beta}_n$ are the QCD hard gluon residuals defined above; they are the non-Abelian analogs of the hard photon residuals defined by YFS. Here, for illustration, we have recorded the relationship between the $\overline{\beta}_n$, $n = 0, 1$ through $\mathcal{O}(\alpha_s)$ and the exact one-loop and single bremsstrahlung cross sections, $d\sigma^{(1\text{-loop})}$, $d\sigma^{B1}$, respectively, where the latter may be taken from [87]. We stress two things about the right-hand side of (A.14):

(i) it does not depend on the dummy parameter $K_{\max}$ which has been introduced for cancellation of the infrared divergences in $\Sigma_{\mathrm{IR}}(\mathrm{QCD})$ to all orders in $\alpha_s(Q)$, where Q is the hard scale in the parton scattering process under study here;

(ii) its analog can also be derived in our new CEEX [85] format.

We now return to the property of (A.14) that distinguishes it from the Abelian result derived by Yennie, Frautschi, and Suura—namely, the fact that, owing to its non-Abelian gauge theory origins, it is in general expected that there are infrared divergences in the $\overline{\beta}_n$ which were not removed into the S_{QCD}, $\widetilde{S}_{\mathrm{QCD}}$ when these infrared functions were isolated in our derivation of (A.14).

More precisely, the left-hand side of (A.14) is the fundamental reduced parton cross section and it should be infrared finite or else the entire QCD parton model has to be abandoned.

There is an observation in the literature [88–90] that unless we use the approximation of massless incoming quarks, the reduced parton cross section on the left-hand side of (A.14) diverges in the infrared regime at $\mathcal{O}(\alpha_s^2(Q))$. We do not go into this issue here but either use the quark masses strictly as collinear limit regulators so that they are set to zero in the

numerators of all Feynman diagrams in such a way that the limit $\lim_{m_q^2/E_q^2\to 0}$, where E_q is the quark energy, is taken everywhere that it is finite or, alternatively, we use n-dimensional methods to regulate such divergences while setting the quark masses to zero as that is an excellent approximation for the light quarks at FNAL and LHC energies—we take this issue up elsewhere.

From the infrared finiteness of the left-hand side of (A.14) and the infrared finiteness of $\Sigma_{\rm IR}(\rm QCD)$, it follows that the quantity

$$d\overline{\widehat{\sigma}}_{\rm exp} \equiv e^{-\Sigma_{\rm IR}(\rm QCD)} d\widehat{\sigma}_{\rm exp} \tag{A.17}$$

must also be infrared finite to all orders in α_s.

As we assume the QCD theory makes sense in some neighborhood of the origin for α_s, we conclude that each order in α_s must make an infrared finite contribution to $d\overline{\widehat{\sigma}}_{\rm exp}$. At $\mathcal{O}(\alpha_s^0(Q))$, the only contribution to $d\overline{\widehat{\sigma}}_{\rm exp}$ is the respective Born cross section given by $\overline{\beta}_0^{(0)}$ in (A.14) and it is obviously infrared finite, where we use henceforth the notation $\overline{\beta}_n^{(\ell)}$ to denote the $\mathcal{O}(\alpha_s^\ell(Q))$ part of $\overline{\beta}_n$. Thus, we conclude that the lowest hard gluon residual $\overline{\beta}_0^{(0)}$ is infrared finite.

Let us now define the left-over non-Abelian infrared divergence part of each contribution $\overline{\beta}_n^{(\ell)}$ via

$$\overline{\beta}_n^{(\ell)} = \overline{\widetilde{\beta}}_n^{(\ell)} + D\overline{\beta}_n^{(\ell)}, \tag{A.18}$$

where the new function $\overline{\widetilde{\beta}}_n^{(\ell)}$ is now completely free of any infrared divergences and the function $D\overline{\beta}_n^{(\ell)}$ contains all left-over infrared divergences in $\overline{\beta}_n^{(\ell)}$, which are of non-Abelian origin, and is normalized to vanish in the Abelian limit $f_{abc} \to 0$ where f_{abc} are the group structure constants.

Further, we define $D\overline{\beta}_n^{(\ell)}$ by *a* minimal subtraction of the respective IR divergences in it so that it only contains the actual pole and transcendental constants, $1/\epsilon - C_E$ for $\epsilon = 2 - d/2$, where d is the dimension of space-time, in dimensional regularization or $\ln\lambda^2$ in the gluon mass regularization. Here, C_E is Euler's constant.

For definiteness, we write this out explicitly as follows:

$$\int d\Gamma\, D\overline{\beta}_n^{(\ell)} \equiv \sum_{i=1}^{n+\ell} d_i^{n,\ell} \ln^i(\lambda^2), \tag{A.19}$$

where the coefficient functions $d_i^{n,\ell}$ are independent of λ for $\lambda \to 0$ and $d\Gamma$ is the respective n-gluon Lorentz invariant phase space.

At $\mathcal{O}(\alpha_s^n(Q))$, the IR finiteness of the contribution to $d\overline{\widehat{\sigma}}_{\rm exp}$ then requires the contribution

$$\begin{aligned} d\overline{\widehat{\sigma}}_{\rm exp}^{(n)} \equiv \int \sum_{\ell=0}^{n} \frac{1}{\ell!} \prod_{j=1}^{\ell} \int_{k_j \geq K_{\max}} \frac{d^3k_j}{k_j} \widetilde{S}_{\rm QCD}(k_j) \sum_{i=0}^{n-\ell} \frac{1}{i!} \prod_{j=\ell+1}^{\ell+i} \\ \times \int \frac{d^3k_j}{k_j^0} \overline{\beta}_i^{(n-\ell-i)}(k_{\ell+1},\ldots,k_{\ell+i}) \frac{d^3p_2}{p_2^0} \frac{d^3q_2}{q_2^0} \end{aligned} \tag{A.20}$$

to be finite.

From this it follows that

$$D d\overline{\widehat{\sigma}}_{\exp}^{(n)} \equiv \int \sum_{\ell=0}^{n} \frac{1}{\ell!} \prod_{j=1}^{\ell} \int_{k_j \geq K_{\max}} \frac{d^3 k_j}{k_j} \widetilde{S}_{\mathrm{QCD}}(k_j) \sum_{i=0}^{n-\ell} \frac{1}{i!} \prod_{j=\ell+1}^{\ell+i} \times \int \frac{d^3 k_j}{k_j^0} D\overline{\beta}_i^{(n-\ell-i)}(k_{\ell+1}, \ldots, k_{\ell+i}) \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0} \tag{A.21}$$

is finite. Since the integration region for the final particles is arbitrary, the independent powers of the IR regulator $\ln(\lambda^2)$ in this last equation must give vanishing contributions. This means that we can drop the $D\overline{\beta}_n^{(\ell)}$ from our result (A.14) because they do not make a net contribution to the final parton cross section $\widehat{\sigma}_{\exp}$. We thus finally arrive at the new rigorous result

$$\begin{aligned} d\widehat{\sigma}_{\exp} &= \sum_n d\widehat{\sigma}^n \\ &= e^{\Sigma_{\mathrm{IR}}(\mathrm{QCD})} \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) \int \prod_{j=1}^{n} \frac{d^3 k_j}{k_j} \int \frac{d^4 y}{(2\pi)^4} e^{iy\cdot(p_1+q_1-p_2-q_2-\sum k_j)+D_{\mathrm{QCD}}} * \widetilde{\overline{\beta}}_n(k_1, \ldots, k_n) \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0}, \end{aligned} \tag{A.22}$$

where now the hard gluon residuals $\widetilde{\overline{\beta}}_n(k_1, \ldots, k_n)$ defined by

$$\widetilde{\overline{\beta}}_n(k_1, \ldots, k_n) = \sum_{\ell=0}^{\infty} \widetilde{\overline{\beta}}_n^{(\ell)}(k_1, \ldots, k_n) \tag{A.23}$$

are free of all infrared divergences to all orders in $\alpha_s(Q)$. This is a basic result of this appendix. It agrees with (17) in the text.

We note here that, contrary to what was claimed in the appendix of [46, 47] and consistent with what is explained in [47], the arguments in [46, 47] are not sufficient to derive the respective analog of (A.22); for, they did not really expose the compensation between the left over genuine non-Abelian IR virtual and real singularities between $\int d\Gamma \overline{\beta}_n$ and $\int d\Gamma \overline{\beta}_{n+1}$, respectively, that really distinguishes QCD from QED, where no such compensation occurs in the $\overline{\beta}_n$ residuals for QED.

We point-out that the general non-Abelian exponentiation of the eikonal cross sections in QCD has been proven formally in [86]. The contact between [86] and our result (A.22) is that, in the language of [86], our exponential factor corresponds to the $N = 1$ term in the exponent of (14) of the latter reference. One also sees immediately the fundamental difference between what we derive in (A.22) and the eikonal formula in [86]: our result (A.22) is an *exact* rearrangement of the complete cross section whereas the result in [86, equation (10)] is an *approximation* to the complete cross section in which all terms that could not be eikonalized and exponentiated have been dropped.

Finally, there is considerable confusion, apparently, in the literature regarding the various aspects of the IR limit in QCD, and the consequent use of the words soft gluon resummation. Let us try to clarify our work in this context in relation to the results in [76–79, 91], all of which are resumming soft gluons. The current paper is focused on the soft gluons emitted from the initial state lines that determine the IR behavior of the initial state parton densities via DGLAP-CS evolution. The latter references are focused on the soft gluons

in the hard scattering coefficients of a process and therefore do not address the resummation results in the current paper in the text. In fact, the authors in [76–79, 91] stress that they have canceled all initial line collinear IR (singular) effects from the coefficients which they resum—otherwise the coefficients would not be hard! It is exactly these canceled effects which we are treating in the text to get improved IR behavior of the DGLAP-CS kernels. To illustrate that there is thus no contradiction between our approach and that in [76–78], we visit with [80], which treats the $2 \to n$ parton process in the resummation theory of [76–78], working in the IR and collinear regime to exact two-loop order. The authors in [80] have arrived at the following representation for the amplitude for a general $2 \to n$ parton process $[f]$ at hard scale Q, $f_1(p_1, r_1) + f_2(p_2, r_2) \to f_3(p_3, r_3) + f_4(p_4, r_4) + \cdots + f_{n+2}(p_{n+2}, r_{n+2})$, where the p_i, r_i label 4-momenta and color indices, respectively, with all parton masses set to zero (so in our approach, we should have in mind that the masses of the quarks and the IR regulator mass of the gluon would all be taken to zero or, we could, as it is done [80], just set all masses to zero at the outset and use dimensional regularization to define both collinear and IR singular integrals):

$$\mathcal{M}^{[f]}_{\{r_i\}} = \sum_{L}^{C} \mathcal{M}^{[f]}_{L}(c_L)_{\{r_i\}} = J^{[f]} \sum_{L}^{C} S_{LI} H^{[f]}_{I}(c_L)_{\{r_i\}}, \tag{A.24}$$

where repeated indices are summed, and the functions $J^{[f]}$, S_{LI}, and $H^{[f]}_{I}$ are, respectively, the jet function, the soft function which describes the exchange of soft gluons between the external lines, and the hard coefficient function. The latter functions' infrared and collinear poles have been calculated to two-loop order in [80]. How do these results relate to (A.22)?

To make contact between (A.22), (A.24), identify in $\overline{Q}' Q \to \overline{Q}''' Q'' + m(G)$ in (A.22) $f_1 = Q, \overline{Q}'$, $f_2 = \overline{Q}'$, $f_3 = Q''$, $f_4 = \overline{Q}'''$, $\{f_5, \ldots, f_{n+2}\} = \{G_1, \ldots, G_m\}$, in (A.24), where we use the obvious notation for the gluons here. This means that $n = m + 2$. To use (A.24) in (A.22), one simply has to observe the following.

(1) By its definition in [80, equation (2.23)], the anomalous dimension of the matrix S_{LI} does not contain any of the diagonal effects described by our infrared functions $\Sigma_{\mathrm{IR}}(\mathrm{QCD})$ and D_{QCD}.

(2) By its definition in [80, equations (2.5) and (2.7)], the jet function $J^{[f]}$ contains the exponential of the virtual infrared function $\alpha_s \Re B_{\mathrm{QCD}}$, so that we have to take care that we do not double count when we use (A.24) in (A.22) and the equations that lead thereto.

When we observe these two latter points, we get the following realization of our approach using the results in [80]: in our result (A.11), we can identify the residual $\overline{\rho}^{(m)}$ as follows:

$$\begin{aligned}
&\overline{\rho}^{(m)}(p_1, q_1, p_2, q_2, k_1, \ldots, k_m) \\
&\quad = \overline{\sum_{\text{colors, spin}}} \left| \mathcal{M}^{[f]}_{\{r_i\}} \right|^2 \equiv \sum_{\text{spins},\{r_i\},\{r'_i\}} \mathfrak{h}^{cs}_{\{r_i\}\{r'_i\}} \left| \overline{J}^{[f]} \right|^2 \sum_{L=1}^{C} \sum_{L'=1}^{C} S^{[f]}_{LI} H^{[f]}_{I}(c_L)_{\{r_i\}} \left(S^{[f]}_{L'I'} H^{[f]}_{I'}(c_{L'})_{\{r'_i\}} \right),
\end{aligned} \tag{A.25}$$

where here we defined $\overline{J}^{[f]} = e^{-\alpha_s \Re B_{\mathrm{QCD}}} J^{[f]}$, and we introduced the color-spin density matrix for the initial state, $\mathfrak{h}^{cs}$, so that $\mathfrak{h}^{cs}_{\{r_i\}\{r'_i\}} = \mathfrak{h}^{cs}_{\{r_1, r_2\}\{r'_1, r'_2\}}$, suppressing the spin indices, that is,

$\mathfrak{h}^{cs}$ only depends on the initial-state colors and has the obvious normalization implied by (A.11). Proceeding then according to the steps leading from (A.11) to (A.22), we get the corresponding implementation of the results in [80] in our approach, without any double counting of effects.

The result in (A.22) for the case just considered would then require DGLAP-CS synthesization [39] to remove its collinear divergences to the respective parton densities as given by the factorization theorem. In this way, all of the results for hard coefficient soft gluon resummation in [76–78, 91] can then be included in our residuals $\tilde{\tilde{\beta}}_n$ without double counting, as these results are all free of both infrared and collinear divergences, so that they are naturally described by our $\tilde{\tilde{\beta}}_n$.

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