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Review Article

The Economical $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ Model

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The $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ gauge model with minimal scalar sector, two Higgs triplets, is presented in detail. One of the vacuum expectation values u is a source of lepton-number violations and a reason for mixing among charged gauge bosons—the standard model $W^{\pm}$ and the bilepton gauge bosons $Y^{\pm}$, as well as among the neutral non-Hermitian bilepton X^0 and neutral gauge bosons—the Z and the new Z'. An exact diagonalization of the neutral gauge boson sector is derived, and bilepton mass splitting is also given. Because of these mixings, the lepton-number violating interactions exist in both charged and neutral gauge boson sectors. Constraints on vacuum expectation values of the model are estimated and $u \simeq \mathcal{O}(1)$ GeV, $v \simeq v_{\mathrm{weak}} = 246$ GeV, and $\omega \simeq \mathcal{O}(1)$ TeV. In this model, there are three physical scalars, two neutral and one charged, and eight Goldstone bosons—the needed number for massive gauge bosons. The minimal scalar sector can provide all fermions including quarks and neutrinos consistent masses in which some of them require one-loop radiative corrections.

1. Introduction

In spite of all the successes of the standard model, it is unlikely to be the final theory. It leaves many striking features of the physics of our world unexplained. In the following, we list some of them which leads to the model's extensions. In particular, the models with $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ (3-3-1) gauge group are presented.

1.1. Generation problem and 3-3-1 models

In the standard model, the fundamental fermions come in generations. In writing down the theory, one may start by first introducing just one generation, then one may repeat the same procedure by introducing copies of the first generation. Why do quarks and leptons come in repetitive structures (generations)? How many generations are there? How to understand the interrelation between generations? These are the central issues of the weak interaction physics

known as the generation problem or the flavor question. Nowhere in physics this question is replied [1]. One of the most important experimental results in the past few years has been the determination of the number of these generations within the framework of the standard model. In the minimal electroweak model, the number of generations is given by the number of the neutrino species which are all massless, by definition. The number of generations is then computed from the invisible width of the Z^0:

$$\Gamma_{\text{inv}} \equiv \Gamma_{Z^0} - \left(\Gamma_h + \sum_l \Gamma_l\right), \tag{1.1}$$

where Γ_{Z^0} denotes the total width, the subscript h refers to hadrons, and Γ_l ($l = e, \mu, \tau$) is the width of the Z^0 decay into an $l\bar{l}$ pair. If Γ_ν is the theoretical width for just one massless neutrino, the number of generations is $N_{\text{gen}} = N_\nu = \Gamma_{\text{inv}}/\Gamma_\nu$, and recent results give a value very close to three $N_{\text{gen}} = 2.99 \pm 0.03$ [2, 3], but we do not understand why the number of standard model generations is three.

The answer to the generation problem may require a radical change in our approaches. It could be that the underlying objects are strings and all the low-energy phenomena will be determined by physics at the Planck scale. Grand unified theories (GUTs) have had a major impact on both cosmology and astrophysics; for cosmology they led to the inflationary scenario, while for astrophysics supernova, neutrinos were first observed in proton-decay detectors. It remains for GUTs to have impact directly on particle physics itself [4]. GUTs cannot explain the presence of fermion generations. On the other side, supersymmetry (SUSY) for the time being is an answer in search of question to be replied. It does not explain the existence of any known particle or symmetry. Some traditional approaches to the problem such as GUTs, monopoles, and higher dimensions introduce quite speculative pieces of new physics at high and experimentally inaccessible energies. Some years ago, there were hopes that the number of generations could be computed from first principles such as geometry of compactified manifolds, but these hopes did not materialize.

A very interesting alternative to explain the origin of generations comes from the cancelation of chiral anomalies of a gauge theory in all orders of perturbative expansion, which derives from the renormalizability condition. This constrains the fermion representation content. Three perturbative anomalies have been identified [5–10] for chiral gauge theories in four-dimensional space-time: (i) the triangle chiral gauge anomaly [11, 12] must be canceled to avoid violations of gauge invariance and the renormalizability of the theory; (ii) the global nonperturbative SU(2) chiral gauge anomaly, [13] which must be absent in order for the fermion integral to be defined in a gauge invariant way; and (iii) the mixed perturbative chiral gauge gravitational anomaly [14–16] which must be canceled in order to ensure general covariance. The general anomaly-free condition is

$$A^{ijk} \equiv \text{Tr}\left[\{T^i, T^j\}T^k\right] = \sum_{\text{representations}} \text{Tr}\left[\{T_L^i, T_L^j\}T_L^k - \{T_R^i, T_R^j\}T_R^k\right] = 0, \tag{1.2}$$

where T^i is the representation of the gauge algebra on the set of all left-handed fermion and antifermion fields put in a single column ψ, and "Tr" denotes a sum over these fermion and antifermion species; $T^i_{L,R}$ are the coupling matrices of fermions $\psi_{L,R}$ to the current $J^i_\mu = \overline{\psi}_L \gamma_\mu T^i_L \psi_L + \overline{\psi}_R \gamma_\mu T^i_R \psi_R$, respectively. The i index runs over the dimension of a simple SU(n) group, $i = 1, 2, \ldots, n^2 - 1$, with a rank $n - 1$, and $i = 0$ for the Abelian factor.

First, let us consider the relationship between anomaly cancelation and flavor problem in the standard model. The individual generations have the following structure under the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ (3-2-1) gauge group:

$$\begin{gathered}(\nu_{aL}, l_{aL}) \sim (1,2,-1), \qquad l_{aR} \sim (1,1,-2), \\ (u_{aL}, d_{aL}) \sim \left(3,2,\frac{1}{3}\right), \qquad u_{aR} \sim \left(3,1,\frac{4}{3}\right), \qquad d_{aR} \sim \left(3,1,-\frac{2}{3}\right).\end{gathered} \tag{1.3}$$

The values in the parentheses denote quantum numbers based on the ($SU(3)_C$, $SU(2)_L$, $U(1)_Y$) symmetry, where the subscripts C, L, and Y, respectively, indicate to the color, left-handed, and hypercharge. The electric charge operator is defined as $Q = T^3 + (1/2)Y$, where $T^i = (1/2)\sigma^i$ $(i = 1,2,3)$ with σ^i are Pauli matrices. The weak isospin group $SU(2)_L$ is a safe group due to the fact that

$$\mathrm{Tr}\left[\{\sigma^i, \sigma^j\}\sigma^k\right] = 2\delta^{ij}\mathrm{Tr}\left[\sigma^k\right] = 0. \tag{1.4}$$

However, in the case where at least one of the generators is hypercharge we have

$$\mathrm{Tr}\left[\sigma^i Y Y\right] \propto \mathrm{Tr}\left[\sigma^i\right] = 0, \qquad \mathrm{Tr}\left[\{\sigma^i, \sigma^j\}Y\right] = 2\delta^{ij}\mathrm{Tr}[Y]. \tag{1.5}$$

The anomaly contribution in the last condition is proportional to the sum of all fermionic discrete hypercharge values on the color, flavor, and weak hypercharge degrees of freedom:

$$\mathrm{Tr}[Y] = \sum_{\text{lepton}} (Y_L + Y_R) + \sum_{\text{quark}} (Y_L + Y_R). \tag{1.6}$$

The Tr $[Y]$ vanishes for the fermion content in the ath generation because

$$\begin{gathered}\sum_{\text{lepton}} (Y_L + Y_R) = Y(\nu_{aL}) + Y(l_{aL}) + Y(l_{aR}) = -4, \\ \sum_{\text{quark}} (Y_L + Y_R) = 3\left[Y(u_{aL}) + Y(d_{aL}) + Y(u_{aR}) + Y(d_{aR})\right] = +4,\end{gathered} \tag{1.7}$$

where 3 factors take into account the number of quark colors. In the last case, all the generators are hypercharge:

$$\mathrm{Tr}\left[Y^3\right] \propto \mathrm{Tr}\left[Q^2 T_3 - Q T_3^2\right], \tag{1.8}$$

where we used the fact that the electromagnetic vector neutral current vertices do not have anomalies. For the ath generation, we have

$$\begin{gathered}\sum_{\text{lepton}} (Q^2 T_3 - Q T_3^2) = \left[(0)^2\left(\frac{1}{2}\right) - (0)\left(\frac{1}{2}\right)^2\right] + \left[(-1)^2\left(-\frac{1}{2}\right) - (-1)\left(-\frac{1}{2}\right)^2\right] = -\frac{1}{4}, \\ \sum_{\text{quark}} (Q^2 T_3 - Q T_3^2) = 3\left[\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^2\right] + 3\left[\left(-\frac{1}{3}\right)^2\left(-\frac{1}{2}\right) - \left(-\frac{1}{3}\right)\left(-\frac{1}{2}\right)^2\right] = +\frac{1}{4}.\end{gathered} \tag{1.9}$$

It yields that the anomaly in standard model cancels within each individual generation, but not by generations. Flavor question and anomaly-free conditions do not seem to have any connection in the standard model. This leads us to questions when going beyond this model. Are the anomalies always canceled automatically within each generation of quarks or leptons? Do the anomaly cancelation conditions have any connection with flavor puzzle?

We wish to show that some very fundamental aspects of the standard model, in particular the flavor problem, might be understood by embedding the three-generation version in a Yang-Mills theory with the $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ semisimple gauge group with a corresponding enlargement of the lepton and quark representations [17–25]. In particular, the number of generations will be related by anomaly cancelation to the number of quark colors, and one generation of quarks will be treated differently from the two others. In the 3-2-1 low-energy limit, all three generations appear similarly and cancel anomalies separately. Let us consider the following 3-3-1 fermion representation content:

$$\begin{gathered}
\psi_{aL} = \begin{pmatrix} \nu_{aL} \\ l_{aL} \\ \nu^c_{aR} \end{pmatrix} \sim \left(1, 3, -\frac{1}{3}\right), \quad l_{aR} \sim (1, 1, -1),\ a = 1, 2, 3, \\
Q_{1L} = \begin{pmatrix} u_{1L} \\ d_{1L} \\ U_L \end{pmatrix} \sim \left(3, 3, \frac{1}{3}\right), \qquad Q_{\alpha L} = \begin{pmatrix} d_{\alpha L} \\ -u_{\alpha L} \\ D_{\alpha L} \end{pmatrix}_L \sim (3, 3^*, 0), \quad \alpha = 2, 3, \\
u_{aR} \sim \left(3, 1, \frac{2}{3}\right), \qquad d_{aR} \sim \left(3, 1, -\frac{1}{3}\right), \qquad U_R \sim \left(3, 1, \frac{2}{3}\right), \qquad D_{\alpha R} \sim \left(3, 1, -\frac{1}{3}\right).
\end{gathered} \tag{1.10}$$

The quantum numbers in the parentheses are based on the $(SU(3)_C, SU(3)_L, U(1)_X)$ symmetry. The right-handed neutrinos ν_R and the exotic quarks U and D_α are composed along with that of the standard model. We call 3-3-1 model with right-handed neutrinos. The electric charge operator in this case takes a form $Q = T^3 - (1/\sqrt{3})T^8 + X$ with $T^i = 1/2\lambda^i$ $(i = 1, 2, \ldots, 8)$ and X standing for $SU(3)_L$ and $U(1)_X$ charges, respectively. Electric charges of the exotic quarks are the same as of the usual quarks, that is, $q_U = 2/3$ and $q_{D_\alpha} = -1/3$.

The $SU(3)_L$ group is not safe in the sense of the standard model $SU(2)_L$ with the vanishing $\mathrm{Tr}[\{\sigma^i, \sigma^j\}, \sigma^k] = 0$. The $SU(3)_L$ generators proportional to the Gell-Mann matrices close among them the Lie algebra structure:

$$[\lambda^i, \lambda^j] = 2i f^{ijk} \lambda^k, \qquad \{\lambda^i, \lambda^j\} = \frac{4}{3}\delta^{ij} + 2d^{ijk}\lambda^k, \tag{1.11}$$

where the structure constant f^{ijk} is totally antisymmetric, and d^{ijk} is totally symmetric under exchange of the indices. We can normalize the λ-matrices such that $\mathrm{Tr}[\lambda^i \lambda^j] = 2\delta^{ij}$. Therefore, f^{ijk} and d^{ijk} are calculated by

$$f^{ijk} = \frac{1}{4i}\mathrm{Tr}\left[[\lambda^i, \lambda^j]\lambda^k\right], \qquad d^{ijk} = \frac{1}{4}\mathrm{Tr}\left[\{\lambda^i, \lambda^j\}\lambda^k\right]. \tag{1.12}$$

The anomaly is proportional to d^{ijk} in general, and of course such coefficients vanish in the case of the $SU(2)_L$ generators.

In the 3-3-1 model, there are six triangle anomalies which are potentially troublesome. In a self-explanatory notation, these are $(3_C)^3$, $(3_C)^2X$, $(3_L)^3$, $(3_L)^2X$, X^3, and $(\text{graviton})^2X$. The quantum chromodynamics anomaly $(3_C)^3$ is absent because the theory mentioned is

vectorlike (i.e., $T_L^i = U^{-1}T_R^iU$ with some unitary matrix U), and hence the condition $A^{ijk} = 0$ is automatically satisfied. For any D fermion representation, it satisfies the condition $A(D) = -A(D^*)$, where $A(D^*)$ is the anomaly of the conjugate representation of D [26]. The pure $\mathrm{SU}(3)_L$ anomaly $(3_L)^3$, therefore, vanishes because there is an equal number of triplets 3_L and antitriplets 3_L^* in the given fermion content. The remaining anomaly-free conditions are explicitly written as follows.

(1) $\mathrm{Tr}[\mathrm{SU}(3)_C]^2[\mathrm{U}(1)_X] = 0$:

$$3\sum_{\text{generation}} X_q^L - \sum_{\text{generation}}\sum_{\text{singlet}} X_q^R = 0; \tag{1.13}$$

(2) $\mathrm{Tr}[\mathrm{SU}(3)_L]^2[\mathrm{U}(1)_X] = 0$:

$$\sum_{\text{generation}} X_l^L + 3\sum_{\text{generation}} X_q^L = 0; \tag{1.14}$$

(3) $\mathrm{Tr}[\mathrm{U}(1)_X]^3 = 0$:

$$3\sum_{\text{generation}} \left(X_l^L\right)^3 + 9\sum_{\text{generation}} \left(X_q^L\right)^3 - 3\sum_{\text{generation}}\sum_{\text{singlet}} \left(X_q^R\right)^3 - \sum_{\text{generation}}\sum_{\text{singlet}} \left(X_l^R\right)^3 = 0; \tag{1.15}$$

(4) $\mathrm{Tr}[\text{graviton}]^2[\mathrm{U}(1)_X] = 0$:

$$3\sum_{\text{generation}} X_l^L + 9\sum_{\text{generation}} X_q^L - 3\sum_{\text{generation}}\sum_{\text{singlet}} X_q^R - \sum_{\text{generation}}\sum_{\text{singlet}} X_l^R = 0, \tag{1.16}$$

where X_l^L, X_q^L, X_l^R, and X_q^R indicate to the $\mathrm{U}(1)_X$ charges of the left-handed lepton, quark triplets or antitriplets, the right-handed lepton, and quark singlets, respectively. It is worth noting that some 3 factors in the conditions (2), (3), and (4) take into account the number of quark colors. With the fermion content as given, it is easily checked that all the above anomaly-free conditions are satisfied. For example, let us take condition (2). We first calculate the 3_L^2X anomaly for the first generation: $-1/3 + 3 \times (1/3) = 2/3$. The anomaly of the second or the third generation is $-1/3 + 3 \times 0 = -1/3$. It is especially interesting that this anomaly cancelation takes place between generations, unlike those of the standard model. Each individual generation possesses nonvanishing $(3_L)^3$, $(3_L)^2X$, X^3, and $(\text{gravion})^2X$ anomalies. Only with a matching of the number of generations with the number of quark colors does the overall anomaly vanish.

Next, let us introduce an alternative fermion content, where the three known left-handed lepton components for each generation are associated to three $\mathrm{SU}(3)_L$ triplets such that $(\nu_{aL}, l_{aL}, l_{aR}^c)^T \sim (1,3,0)$ (called minimal 3-3-1 model). Canceling the pure $\mathrm{SU}(3)_L$ anomaly requires that there are the same number of triplets and antitriplets, thus $Q_{1L} = (u_{1L}, d_{1L}, J_L)^T \sim (3,3,2/3)$, $Q_{\alpha L} = (d_{\alpha L}, -u_{\alpha L}, J_{\alpha L})^T \sim (3,3^*,-1/3)$. The respective right-handed fields are singlets: $u_{aR} \sim (3,1,2/3)$ and $d_{aR} \sim (3,1,-1/3)$ for the ordinary quarks; $J_R \sim (3,1,5/3)$ and $J_{\alpha R} \sim (3,1,-4/3)$ for the exotic quarks. Similarly, to the previous 3-3-1 model, the $(3_L)^3$, $(3_L)^2X$, X^3 anomalies vanish only if three generations of quarks and leptons take into account.

In a general case, we can verify that the number of generations must be multiple of the quark-color number in order to cancel the anomalies. On the other hand, if we suppose that the exotic quarks also contribute to the running of the coupling constants, the asymptotic-freedom principle requires that the number of quark generations is no more than five. It follows that the number of generations is *just* three. This provides a first step toward answering the flavor question. The asymmetric treatment of one generation of quarks breaks generation universality. This might provide an explanation of why the top quark is uncharacteristically heavy [27, 28]. An interesting alternative feature is that the electric charge quantization in nature might also be explained in this framework [23, 29–32]. Just enlarging $SU(2)_L$ to $SU(3)_L$, we have thus presented the simplest gauge extension of the standard model for the flavor question. The new models get five additional gauge bosons contained in a gauge adjoint octet: $8 = 3 + (2 + 2) + 1$ under $SU(2)_L$. The 1 is a neutral Z' and the two doublets are readily identifiable from the leptonic contents as non-Hermitian bilepton gauge bosons $(X, Y)^T$ and (X^*, Y^*). From the renormalization group analysis of the coupling constants [17, 33], the $SU(3)_L$ breaking scale is estimated to be lower than some TeV in the minimal 3-3-1 model. This is due to the fact that the squared sine of the Weinberg angle θ_W gets an upper bound, $\sin^2\theta_W < 1/4$. There is no "grand desert" in this model in comparison to GUTs. In contrast, the energy scale in the 3-3-1 model with right-handed neutrinos is very high, even larger than the Planck scale because of $\sin^2\theta_W < 3/4$. This version might allow the existence of a "desert." Anyway, the new physics in these models expected arise at not too high energies. The new particles such as the bilepton gauge bosons Z' and exotic quarks would be determinable in the next generation of collides.

1.2. Proposal of minimal Higgs sector

As mentioned above, there are two main versions of 3-3-1 models—the minimal model and the model with right-handed neutrinos, which have been subjects studied extensively over the last decade. In the minimal 3-3-1 model [17–19], the scalar sector is quite complicated and contains three scalar triplets and one scalar sextet. In the 3-3-1 model with right-handed neutrinos [20–22, 34, 35], the scalar sector requires three Higgs triplets. It is interesting to note that two Higgs triplets of this model have the same $U(1)_X$ charges with two neutral components at their top and bottom. Allowing these neutral components vacuum expectation values (VEVs), we can reduce number of Higgs triplets to be two. Note that the mentioned model contains very important advantage, namely, there is no new parameter, but it contains very simple Higgs sector, therefore, the significant number of free parameters is reduced. To mark the minimal content of the Higgs sector, this version that includes right-handed neutrinos is going to be called the *economical 3-3-1 model* [36–42]. The interested reader can find the supersymmetric version in [43–46].

This kind of model was proposed in [36] but has not got enough attention. In [37], phenomenology of this model was presented without mixing between charged gauge bosons as well as neutral ones. The mass spectrum of the mentioned scalar sector has also been presented in [36], and some couplings of the two neutral scalar fields with the charged W and the neutral Z gauge bosons in the standard model were presented. From explicit expression for the ZZH vertex, the authors concluded that two VEVs responsible for the second step of spontaneous symmetry breaking have to be in the same range $u \sim v$, or the theory needs an additional scalar triplet. As we will show in the following, this conclusion is incorrect.

It is well known that the electroweak symmetry breaking in the standard model is achieved via the Higgs mechanism. In the Weinberg-Salam model, there is a single complex scalar doublet, where the Higgs boson H is the physical neutral Higgs scalar which is the only remaining part of this doublet after spontaneous symmetry breaking. In the extended models, there are additional charged and neutral scalar Higgs particles. The prospects for Higgs coupling measurements at the CERN Large Hadron Collider (LHC) have recently been analyzed in detail in [47]. The experimental detection of the H will be great triumph of the standard model of electroweak interactions and will mark new stage in high-energy physics.

In extended Higgs models, which would be deduced in the low-energy effective theory of new physics models, additional Higgs bosons like charged and CP-odd scalar bosons are predicted. Phenomenology of these extra scalar bosons strongly depends on the characteristics of each new physics model. By measuring their properties like masses, widths, production rates, and decay branching ratios, the outline of physics beyond the electroweak scale can be experimentally determined.

The interesting feature compared with other 3-3-1 models is the Higgs physics. In the 3-3-1 models, the general Higgs sector is very complicated [48–51] and this prevents the models' predicability. The scalar sector of the considering model is one of subjects in the present work. As shown, by couplings of the scalar fields with the ordinary gauge bosons such as the photon, the W, and the neutral Z gauge bosons, we are able to identify full content of the Higgs sector in the standard model including the neutral H and the Goldstone bosons eaten by their associated massive gauge ones. All interactions among Higgs-gauge bosons in the standard model are recovered.

Production of the Higgs boson in the 3-3-1 model with right-handed neutrinos at LHC has been considered in [52]. In scalar sector of the considered model, there exists the singly-charged boson $H_2^{\pm}$, which is a subject of intensive current studies [53, 54]. The trilinear coupling $ZW^{\pm}H^{\mp}$ which differs at the tree level, from zero only in the models with Higgs triplets plays a special role on study phenomenology of these exotic representations. We will pay particular interest on this boson.

At the tree level, the mass matrix for the upquarks has one massless state, and in the downquark sector there are two massless ones. This calls for radiative corrections. To solve this problem, the authors in [37] have introduced the third Higgs triplet. In this sense, the economical 3-3-1 model is not realistic. In the present work, we will show that this is a mistake! Without the third one, at the one loop level, the fermions in this model, with the given set of parameters, gain a consistent mass spectrum. A numerical evaluation leads us to conclusion that in the model under consideration, there are two scales for masses of the exotic quarks.

At the tree level, the neutrino spectrum is Dirac particles with one massless and two degenerate in mass $\sim h^{\nu}v$. This spectrum is not realistic under the data because there is only one squared-mass splitting. Since the observed neutrino masses are so small, the Dirac mass is unnatural. One must understand what physics gives $h^{\nu}v \ll h^{l}v$—the mass of charged leptons. In contrast to the seesaw cases [55–62] in which the problem can be solved, in this model the neutrinos including the right-handed ones get only small masses through radiative corrections [42, 49, 63–78]. We will obtain these radiative corrections and will provide a possible explanation of natural smallness of the neutrino masses. This is not the result of a seesaw, but it is due to a finite mass renormalization arising from a very different radiative mechanism. We will show that the neutrinos can get mass not only from the standard symmetry breakdown, but also from the electroweak $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ breaking associated with spontaneous lepton-number breaking (SLB), and even through the explicit

lepton-number violating processes due to a new physics. The total neutrino mass spectrum at the one-loop level is neat and can fit the data.

This report is organized as follows. In Section 2, we give a review of the model with stressing on the gauge bosons, currents, and constraints on the new physics. The Higgs-gauge interactions and scalar content are considered in Section 3. Section 4 is devoted to fermion masses. We summarize our results and make conclusions in the last section—Section 5.

2. The economical 3-3-1 model

We first recall the idea of constructing the model. An exact diagonalization of charged and neutral gauge boson sectors and their masses and mixings are presented. Because of the mixings, currents in this model have unusual features which are obtained then. Constraints on the parameters and some phenomena are sketched.

2.1. Particle content

The fermion content which is anomaly free is given by (1.10) like that of the 3-3-1 model with right-handed neutrinos. However, contrasting with the ordinary model in which the third generation of quarks should be discriminating [28], in the model under consideration the first generation has to be different from the two others. This results from the mass patterns for the quarks which will be derived in Section 4.

The 3-3-1 gauge group is broken spontaneously via two stages. In the first stage, it is embedded in that of the standard model via a Higgs scalar triplet:

$$\chi = \begin{pmatrix} \chi_1^0 \\ \chi_2^- \\ \chi_3^0 \end{pmatrix} \sim \left(1, 3, -\frac{1}{3}\right) \tag{2.1}$$

with the VEV given by

$$\langle \chi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ 0 \\ \omega \end{pmatrix}. \tag{2.2}$$

In the last stage, to embed the standard model gauge symmetry in $SU(3)_C \otimes U(1)_Q$, another Higgs scalar triplet is needed:

$$\phi = \begin{pmatrix} \phi_1^+ \\ \phi_2^0 \\ \phi_3^+ \end{pmatrix} \sim \left(1, 3, \frac{2}{3}\right) \tag{2.3}$$

with the VEV as follows:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}. \tag{2.4}$$

Table 1: Nonzero lepton number L of the model particles.

Field	ν_{aL}	$l_{aL,R}$	ν^c_{aR}	χ^0_1	χ^-_2	ϕ^+_3	$U_{L,R}$	$D_{\alpha L,R}$
L	1	1	−1	2	2	−2	−2	2

The Yukawa interactions which induce masses for the fermions can be written in the most general form as follows:

$$\mathcal{L}_{\mathrm{Y}} = \mathcal{L}_{\mathrm{LNC}} + \mathcal{L}_{\mathrm{LNV}}, \tag{2.5}$$

where LNC and LNV, respectively, indicate to the lepton number conserving and violating ones as shown below. Here, each part is defined by

$$\begin{aligned}\mathcal{L}_{\mathrm{LNC}} = {} & h^U \overline{Q}_{1L}\chi U_R + h^D_{\alpha\beta}\overline{Q}_{\alpha L}\chi^* D_{\beta R} + h^d_a \overline{Q}_{1L}\phi d_{aR} + h^u_{\alpha a}\overline{Q}_{\alpha L}\phi^* u_{aR} \\ & + h^l_{ab}\overline{\varphi}_{aL}\phi l_{bR} + h^\nu_{ab}\epsilon_{pmn}\left(\overline{\varphi}^c_{aL}\right)_p\left(\varphi_{bL}\right)_m(\phi)_n + \mathrm{H.c.},\end{aligned} \tag{2.6}$$

$$\mathcal{L}_{\mathrm{LNV}} = s^u_a \overline{Q}_{1L}\chi u_{aR} + s^d_{\alpha a}\overline{Q}_{\alpha L}\chi^* d_{aR} + s^D_\alpha \overline{Q}_{1L}\phi D_{\alpha R} + s^U_\alpha \overline{Q}_{\alpha L}\phi^* U_R + \mathrm{H.c.}, \tag{2.7}$$

where p, m, and n stand for $\mathrm{SU}(3)_L$ indices.

The VEV ω gives mass for the exotic quarks U and D_α: u gives mass for u_1, d_α, while v gives mass for u_α, d_1, and all ordinary leptons. In Section 4, we will provide more details on analysis of fermion masses. As mentioned, ω is responsible for the first stage of symmetry breaking, while the second stage is due to u and v; therefore, the VEVs in this model satisfies the constraint:

$$u^2, v^2 \ll \omega^2. \tag{2.8}$$

The Yukawa couplings in (2.6) possess an extra global symmetry [49, 50] which is not broken by v, ω, but by u. From these couplings, one can find the following lepton symmetry L as in Table 1 (only the fields with nonzero L are listed; all other fields have vanishing L). Here, L is broken by u which is behind $L(\chi^0_1) = 2$, that is, *u is a kind of the SLB scale* [79–83]. It is interesting that the exotic quarks also carry the lepton number (so-called leptoquarks); therefore, this L obviously does not commute with the gauge symmetry. One can then construct a new conserved charge $\mathcal{L}$ through L by making a linear combination $L = xT_3 + yT_8 + \mathcal{L}I$. Applying L on a lepton triplet, the coefficients will be determined:

$$L = \frac{4}{\sqrt{3}}T_8 + \mathcal{L}I. \tag{2.9}$$

Another useful conserved charge $\mathcal{B}$ which is exactly not broken by u, v, and ω is usual baryon number: $B = \mathcal{B}I$. Both the charges $\mathcal{L}$ and $\mathcal{B}$ for the fermion and Higgs multiplets are listed in Table 2.

Let us note that the Yukawa couplings of (2.7) conserve $\mathcal{B}$, however, violate $\mathcal{L}$ with ± 2 units which implies that these interactions are much smaller than the first ones [41]:

$$s^u_a, s^d_{\alpha a}, s^D_\alpha, s^U_\alpha \ll h^U, h^D_{\alpha\beta}, h^d_a, h^u_{\alpha a}. \tag{2.10}$$

Table 2: $\mathcal{B}$ and $\mathcal{L}$ charges of the model multiplets.

Multiplet	χ	ϕ	Q_{1L}	$Q_{\alpha L}$	u_{aR}	d_{aR}	U_R	$D_{\alpha R}$	ψ_{aL}	l_{aR}
$\mathcal{B}$-charge	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0
$\mathcal{L}$-charge	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	-2	2	$\frac{1}{3}$	1

In previous studies [19, 37, 84–86], the LNV terms of this kind have often been excluded, commonly by the adoption of an appropriate discrete symmetry. There is no reason within the 3-3-1 models why such terms should not be present.

In this model, the most general Higgs potential has very simple form:

$$V(\chi,\phi) = \mu_1^2\chi^\dagger\chi + \mu_2^2\phi^\dagger\phi + \lambda_1(\chi^\dagger\chi)^2 + \lambda_2(\phi^\dagger\phi)^2 + \lambda_3(\chi^\dagger\chi)(\phi^\dagger\phi) + \lambda_4(\chi^\dagger\phi)(\phi^\dagger\chi). \tag{2.11}$$

It is noteworthy that $V(\chi,\phi)$ does not contain trilinear scalar couplings and conserves both the mentioned global symmetries; this makes the Higgs potential much simpler and discriminative from the previous ones of the 3-3-1 models [48–51]. This potential is closer to that of the standard model. In the next section, we will show that after spontaneous symmetry breaking, there are eight Goldstone bosons—the needed number for massive gauge ones and three physical scalar fields (one charged and two neutral). One of two physical neutral scalars is the standard model Higgs boson.

To break the gauge symmetry spontaneously, the Higgs vacuums are not $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ singlets. Hence, nonzero values of χ and ϕ at the minimum value of $V(\chi,\phi)$ can be easily obtained by (for details, see Section 3):

$$\chi^\dagger\chi \equiv \frac{u^2+\omega^2}{2} = \frac{\lambda_3\mu_2^2 - 2\lambda_2\mu_1^2}{4\lambda_1\lambda_2 - \lambda_3^2}, \tag{2.12}$$

$$\phi^\dagger\phi \equiv \frac{v^2}{2} = \frac{\lambda_3\mu_1^2 - 2\lambda_1\mu_2^2}{4\lambda_1\lambda_2 - \lambda_3^2}. \tag{2.13}$$

It is important noting that any other choice of u, ω for the vacuum value of χ satisfying (2.12) gives the same physics because it is related to (2.2) by an $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ transformation. It is worth noting that the assumed $u \neq 0$ is, therefore, given in a general case. This model, however, does not lead to the formation of Majoron [79–83, 87].

2.2. Gauge bosons

The covariant derivative of a triplet is given by

$$D_\mu = \partial_\mu - igT_iW_{i\mu} - ig_XT_9XB_\mu \equiv \partial_\mu - i\mathcal{D}_\mu, \tag{2.14}$$

where the gauge fields W_i and B transform as the adjoint representations of $\mathrm{SU}(3)_L$ and $\mathrm{U}(1)_X$, respectively, and the corresponding gauge coupling constants g, g_X. Moreover, $T_9 = (1/\sqrt{6})\,\mathrm{diag}(1,1,1)$ is fixed so that the relation $\mathrm{Tr}(T_iT_j) = (1/2)\delta_{ij}$ $(i,j = 1,2,\ldots,9)$

is satisfied. The $\mathcal{P}_\mu$ matrix appeared in the above covariant derivative is rewritten in a convenient form:

$$\mathcal{P}_\mu = \frac{g}{2}\begin{pmatrix} W_{3\mu} + \frac{W_{8\mu}}{\sqrt{3}} + t\sqrt{\frac{2}{3}}XB_\mu & \sqrt{2}W'^{+}_\mu & \sqrt{2}X'^{0}_\mu \\ \sqrt{2}W'^{-}_\mu & -W_{3\mu} + \frac{W_{8\mu}}{\sqrt{3}} + t\sqrt{\frac{2}{3}}XB_\mu & \sqrt{2}Y'^{-}_\mu \\ \sqrt{2}X'^{0*}_\mu & \sqrt{2}Y'^{+}_\mu & -\frac{2W_{8\mu}}{\sqrt{3}} + t\sqrt{\frac{2}{3}}XB_\mu \end{pmatrix}, \tag{2.15}$$

where $t \equiv g_X/g$. Let us denote the following combinations:

$$W'^{\pm}_\mu \equiv \frac{W_{1\mu} \mp iW_{2\mu}}{\sqrt{2}}, \qquad Y'^{\mp}_\mu \equiv \frac{W_{6\mu} \mp iW_{7\mu}}{\sqrt{2}}, \qquad X'^{0}_\mu \equiv \frac{W_{4\mu} - iW_{5\mu}}{\sqrt{2}} \tag{2.16}$$

having defined charges under the generators of the $SU(3)_L$ group. For the sake of convenience in further reading, we note that W_4 and W_5 are pure real and imaginary parts of X'^{0}_μ and X'^{0*}_μ, respectively:

$$W_{4\mu} = \frac{1}{\sqrt{2}}(X'^{0}_\mu + X'^{0*}_\mu), \qquad W_{5\mu} = \frac{i}{\sqrt{2}}(X'^{0}_\mu - X'^{0*}_\mu). \tag{2.17}$$

The masses of the gauge bosons in this model are followed from

$$\begin{aligned}
&\mathcal{L}^{\mathrm{GB}}_{\mathrm{mass}} \\
&\quad = (D_\mu\langle\phi\rangle)^\dagger(D^\mu\langle\phi\rangle) + (D_\mu\langle\chi\rangle)^\dagger(D^\mu\langle\chi\rangle) \\
&\quad = \frac{g^2}{4}(u^2+v^2)W'^{-}_\mu W'^{+\mu} + \frac{g^2}{4}(\omega^2+v^2)Y'^{-}_\mu Y'^{+\mu} + \frac{g^2u\omega}{4}(W'^{-}_\mu Y'^{+\mu} + Y'^{-}_\mu W'^{+\mu}) \\
&\qquad + \frac{g^2v^2}{8}\left(-W_{3\mu} + \frac{1}{\sqrt{3}}W_{8\mu} + t\frac{2}{3}\sqrt{\frac{2}{3}}B_\mu\right)^2 + \frac{g^2u^2}{8}\left(W_{3\mu} + \frac{1}{\sqrt{3}}W_{8\mu} - t\frac{1}{3}\sqrt{\frac{2}{3}}B_\mu\right)^2 \\
&\qquad + \frac{g^2\omega^2}{8}\left(-\frac{2}{\sqrt{3}}W_{8\mu} - t\frac{1}{3}\sqrt{\frac{2}{3}}B_\mu\right)^2 + \frac{g^2u\omega}{4\sqrt{2}}\left(W_{3\mu} + \frac{1}{\sqrt{3}}W_{8\mu} - t\frac{1}{3}\sqrt{\frac{2}{3}}B_\mu\right)(X'^{0\mu} + X'^{0*\mu}) \\
&\qquad + \frac{g^2u\omega}{4\sqrt{2}}\left(-\frac{2}{\sqrt{3}}W_{8\mu} - t\frac{1}{3}\sqrt{\frac{2}{3}}B_\mu\right)(X'^{0\mu} + X'^{0*\mu}) \\
&\qquad + \frac{g^2}{16}(u^2+\omega^2)\left\{(X'^{0}_\mu + X'^{0*}_\mu)^2 + [i(X'^{0}_\mu - X'^{0*}_\mu)]^2\right\}.
\end{aligned} \tag{2.18}$$

The combinations W' and Y' are mixing via

$$\mathcal{L}^{\mathrm{CG}}_{\mathrm{mass}} = \frac{g^2}{4}(W'^{-}_\mu, Y'^{-}_\mu)\begin{pmatrix} u^2+v^2 & u\omega \\ u\omega & \omega^2+v^2 \end{pmatrix}\begin{pmatrix} W'^{+\mu} \\ Y'^{+\mu} \end{pmatrix}. \tag{2.19}$$

Diagonalizing this mass matrix, we get *physical* charged gauge bosons:

$$W_\mu = \cos\theta W'_\mu - \sin\theta\, Y'_\mu, \qquad Y_\mu = \sin\theta W'_\mu + \cos\theta\, Y'_\mu, \tag{2.20}$$

where the mixing angle is defined by

$$\tan\theta = \frac{u}{\omega}. \tag{2.21}$$

The mass eigenvalues are

$$M_W^2 = \frac{g^2 v^2}{4}, \tag{2.22}$$

$$M_Y^2 = \frac{g^2}{4}\left(u^2 + v^2 + \omega^2\right). \tag{2.23}$$

Because of the constraints in (2.8), the following remarks are in order:

(1) θ should be very small, and then $W_\mu \simeq W'_\mu,\ Y_\mu \simeq Y'_\mu$;

(2) $v \simeq v_{\text{weak}} = 246$ GeV due to identification of W as the W boson in the standard model.

Next, from (2.18), the W_5 gains mass as follows:

$$M_{W_5}^2 = \frac{g^2}{4}\left(\omega^2 + u^2\right). \tag{2.24}$$

Finally, there is a mixing among $W_3,\ W_8,\ B,\ W_4$ components. In the basis of these elements, the mass matrix is given by

$$M^2 = \frac{g^2}{4}\begin{pmatrix} u^2+v^2 & \dfrac{u^2-v^2}{\sqrt{3}} & -\dfrac{2t}{3\sqrt{6}}\left(u^2+2v^2\right) & 2u\omega \\ \dfrac{u^2-v^2}{\sqrt{3}} & \dfrac{1}{3}\left(4\omega^2+u^2+v^2\right) & \dfrac{\sqrt{2}t}{9}\left(2\omega^2-u^2+2v^2\right) & -\dfrac{2}{\sqrt{3}}u\omega \\ -\dfrac{2t}{3\sqrt{6}}\left(u^2+2v^2\right) & \dfrac{\sqrt{2}t}{9}\left(2\omega^2-u^2+2v^2\right) & \dfrac{2t^2}{27}\left(\omega^2+u^2+4v^2\right) & -\dfrac{8t}{3\sqrt{6}}u\omega \\ 2u\omega & -\dfrac{2}{\sqrt{3}}u\omega & -\dfrac{8t}{3\sqrt{6}}u\omega & u^2+\omega^2 \end{pmatrix}. \tag{2.25}$$

Note that the mass Lagrangian in this case has the form

$$\mathcal{L}_{\text{mass}}^{\text{NG}} = \frac{1}{2}V^T M^2 V, \quad V^T \equiv \left(W_3, W_8, B, W_4\right). \tag{2.26}$$

In the limit $u \to 0$, W_4 does not mix with $W_3,\ W_8,\ B$. In the general case $u \neq 0$, the mass matrix in (2.25) contains two *exact eigenvalues* such as

$$M_\gamma^2 = 0, \qquad M_{W'_4}^2 = \frac{g^2}{4}\left(\omega^2 + u^2\right). \tag{2.27}$$

Thus, the W'_4 and W_5 components have the same mass, and this conclusion *contradicts the previous analysis in* [36]. With this result, we should identify the combination of W'_4 and W_5:

$$\sqrt{2}X^0_\mu = W'_{4\mu} - iW_{5\mu} \tag{2.28}$$

as *physical* neutral *non-Hermitian* gauge boson. The subscript 0 denotes neutrality of gauge boson X. However, in the following, this subscript may be dropped. This boson caries the lepton number with two units. Hence, it is the bilepton like those in the usual 3-3-1 model with right-handed neutrinos. From (2.22), (2.23), and (2.27), it follows an interesting relation between the bilepton masses similar to the law of Pythagoras:

$$M^2_Y = M^2_X + M^2_W. \tag{2.29}$$

Thus, the charged bilepton Y is slightly heavier than the neutral one X. Remind that the similar relation in the 3-3-1 model with right-handed neutrinos is [88]: $|M^2_Y - M^2_X| \le m^2_W$.

Now, we turn to the eigenstate question. The eigenstates corresponding to the two values in (2.27) are determined as follows:

$$A_\mu = \frac{1}{\sqrt{18+4t^2}}\begin{pmatrix}\sqrt{3}t\\ -t\\ 3\sqrt{2}\\ 0\end{pmatrix}, \qquad W'_{4\mu} = \frac{1}{\sqrt{1+4\tan^2 2\theta}}\begin{pmatrix}\tan 2\theta\\ \sqrt{3}\tan 2\theta\\ 0\\ 1\end{pmatrix}. \tag{2.30}$$

To embed this model in the effective theory at the low energy, we follow an appropriate method in [89, 90], where the photon field couples with the lepton by strength:

$$\mathcal{L}^{\mathrm{EM}}_{\mathrm{int}} = -\frac{\sqrt{3}g_X}{\sqrt{18+4t^2}}\bar{l}\gamma^\mu l A_\mu. \tag{2.31}$$

Therefore, the coefficient of the electromagnetic coupling constant can be identified as

$$\frac{\sqrt{3}g_X}{\sqrt{18+4t^2}} = e. \tag{2.32}$$

Using continuation of the gauge coupling constant g of $\mathrm{SU}(3)_L$ at the spontaneous symmetry breaking point,

$$g = g\left[\mathrm{SU}(2)_L\right] = \frac{e}{s_W} \tag{2.33}$$

from which it follows

$$t = \frac{3\sqrt{2}s_W}{\sqrt{3-4s^2_W}}. \tag{2.34}$$

The eigenstates are now rewritten as follows:

$$\begin{aligned} A_\mu &= s_W W_{3\mu} + c_W\left(-\frac{t_W}{\sqrt{3}}W_{8\mu} + \sqrt{1-\frac{t^2_W}{3}}B_\mu\right),\\ W'_{4\mu} &= \frac{t_{2\theta}}{\sqrt{1+4t^2_{2\theta}}}W_{3\mu} + \frac{\sqrt{3}t_{2\theta}}{\sqrt{1+4t^2_{2\theta}}}W_{8\mu} + \frac{1}{\sqrt{1+4t^2_{2\theta}}}W_{4\mu}, \end{aligned} \tag{2.35}$$

where we have denoted $s_W \equiv \sin\theta_W$, $t_{2\theta} \equiv \tan 2\theta$ and so forth.

The diagonalization of the mass matrix is done via three steps. In the first step, it is in the base of $(A_\mu, Z_\mu, Z'_\mu, W_{4\mu})$, where the two remaining gauge vectors are given by

$$\begin{aligned} Z_\mu &= c_W W_{3\mu} - s_W\left(-\frac{t_W}{\sqrt{3}}W_{8\mu} + \sqrt{1-\frac{t_W^2}{3}}B_\mu\right), \\ Z'_\mu &= \sqrt{1-\frac{t_W^2}{3}}W_{8\mu} + \frac{t_W}{\sqrt{3}}B_\mu. \end{aligned} \tag{2.36}$$

In this basis, the mass matrix M^2 becomes

$$M'^2 = \frac{g^2}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{u^2+v^2}{c_W^2} & \dfrac{c_{2W}u^2 - v^2}{c_W^2\sqrt{3-4s_W^2}} & \dfrac{2u\omega}{c_W} \\ 0 & \dfrac{c_{2W}u^2 - v^2}{c_W^2\sqrt{3-4s_W^2}} & \dfrac{v^2 + 4c_W^4\omega^2 + c_{2W}^2u^2}{c_W^2(3-4s_W^2)} & -\dfrac{2u\omega}{c_W\sqrt{3-4s_W^2}} \\ 0 & \dfrac{2u\omega}{c_W} & -\dfrac{2u\omega}{c_W\sqrt{3-4s_W^2}} & u^2+\omega^2 \end{pmatrix}. \tag{2.37}$$

Also, in the limit $u \to 0$, $W_{4\mu}$ does not mix with Z_μ, Z'_μ. The eigenstate $W'_{4\mu}$ is now defined by

$$W'_{4\mu} = \frac{t_{2\theta}}{c_W\sqrt{1+4t_{2\theta}^2}}Z_\mu + \frac{\sqrt{4c_W^2-1}\,t_{2\theta}}{c_W\sqrt{1+4t_{2\theta}^2}}Z'_\mu + \frac{1}{\sqrt{1+4t_{2\theta}^2}}W_{4\mu}. \tag{2.38}$$

We turn to the second step. To see explicitly that the following basis is orthogonal and normalized, let us put

$$s_{\theta'} \equiv \frac{t_{2\theta}}{c_W\sqrt{1+4t_{2\theta}^2}}, \tag{2.39}$$

which leads to

$$W'_{4\mu} = s_{\theta'}Z_\mu + c_{\theta'}\left[t_{\theta'}\sqrt{4c_W^2-1}Z'_\mu + \sqrt{1-t_{\theta'}^2(4c_W^2-1)}W_{4\mu}\right]. \tag{2.40}$$

Note that the mixing angle in this step θ' is the same order as the mixing angle in the charged gauge boson sector. Taking into account [3] $s_W^2 \simeq 0.231$, from (2.39) we get $s_{\theta'} \simeq 2.28 s_\theta$. It is now easy to choose two remaining gauge vectors orthogonal to $W'_{4\mu}$:

$$\begin{aligned} \mathcal{Z}_\mu &= c_{\theta'}Z_\mu - s_{\theta'}\left[t_{\theta'}\sqrt{4c_W^2-1}Z'_\mu + \sqrt{1-t_{\theta'}^2(4c_W^2-1)}W_{4\mu}\right], \\ \mathcal{Z}'_\mu &= \sqrt{1-t_{\theta'}^2(4c_W^2-1)}Z'_\mu - t_{\theta'}\sqrt{4c_W^2-1}W_{4\mu}. \end{aligned} \tag{2.41}$$

Therefore, in the base of $(A_\mu, \mathfrak{Z}_\mu, \mathfrak{Z}'_\mu, W'_{4\mu})$, the mass matrix M'^2 has a quasi-diagonal form:

$$M''^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & m^2_{\mathfrak{Z}} & m^2_{\mathfrak{Z}\mathfrak{Z}'} & 0 \\ 0 & m^2_{\mathfrak{Z}\mathfrak{Z}'} & m^2_{\mathfrak{Z}'} & 0 \\ 0 & 0 & 0 & \frac{g^2}{4}(u^2+\omega^2) \end{pmatrix} \tag{2.42}$$

with

$$\begin{aligned} m^2_{\mathfrak{Z}} &= \frac{(1+3t^2_{2\theta})u^2 + (1+4t^2_{2\theta})v^2 - t^2_{2\theta}\omega^2}{4g^{-2}[c^2_W + (3-4s^2_W)t^2_{2\theta}]}, \\ m^2_{\mathfrak{Z}\mathfrak{Z}'} &= \frac{\sqrt{1+4t^2_{2\theta}}\{[c_{2W} + (3-4s^2_W)t^2_{2\theta}]u^2 - v^2 - (3-4s^2_W)t^2_{2\theta}\omega^2\}}{4g^{-2}\sqrt{3-4s^2_W}[c^2_W + (3-4s^2_W)t^2_{2\theta}]}, \\ m^2_{\mathfrak{Z}'} &= \frac{[c^2_{2W} + (3-4s^2_{2W})t^2_{2\theta}]u^2 + v^2 + [4c^4_W + (1+4c^2_W)(3-4s^2_W)t^2_{2\theta}]\omega^2}{4g^{-2}(3-4s^2_W)[c^2_W + (3-4s^2_W)t^2_{2\theta}]}. \end{aligned} \tag{2.43}$$

In the last step, it is trivial to diagonalize the mass matrix in (2.42). The two remaining mass eigenstates are given by

$$Z^1_\mu = c_\varphi \mathfrak{Z}_\mu - s_\varphi \mathfrak{Z}'_\mu, \qquad Z^2_\mu = s_\varphi \mathfrak{Z}_\mu + c_\varphi \mathfrak{Z}'_\mu, \tag{2.44}$$

where the mixing angle φ between $\mathfrak{Z}$ and $\mathfrak{Z}'$ is defined by

$$\begin{aligned} t_{2\varphi} = {} & \{[(3-4s^2_W)(1+4t^2_{2\theta})]^{1/2}\{[c_{2W} + (3-4s^2_W)t^2_{2\theta}]u^2 - v^2 - (3-4s^2_W)t^2_{2\theta}\omega^2\}\} \\ & \times \{[2s^4_W - 1 + (8s^4_W - 2s^2_W - 3)t^2_{2\theta}]u^2 - [c_{2W} + 2(3-4s^2_W)t^2_{2\theta}]v^2 \\ & + [2c^4_W + (8s^4_W + 9c_{2W})t^2_{2\theta}]\omega^2\}^{-1}. \end{aligned} \tag{2.45}$$

The physical mass eigenvalues are defined by

$$\begin{aligned} M^2_{Z^1} = {} & [2g^{-2}(3-4s^2_W)]^{-1}\Big\{c^2_W(u^2+\omega^2) + v^2 \\ & - \sqrt{[c^2_W(u^2+\omega^2)+v^2]^2 + (3-4s^2_W)(3u^2\omega^2 - u^2v^2 - v^2\omega^2)}\Big\}, \\ M^2_{Z^2} = {} & [2g^{-2}(3-4s^2_W)]^{-1}\Big\{c^2_W(u^2+\omega^2) + v^2 \\ & + \sqrt{[c^2_W(u^2+\omega^2)+v^2]^2 + (3-4s^2_W)(3u^2\omega^2 - u^2v^2 - v^2\omega^2)}\Big\}. \end{aligned} \tag{2.46}$$

Because of the condition (2.8), the angle φ has to be very small:

$$t_{2\varphi} \simeq -\frac{\sqrt{3-4s_W^2}\left[v^2+(11-14s_W^2)u^2\right]}{2c_W^4\omega^2}. \tag{2.47}$$

In this approximation, the above physical states have masses:

$$M_{Z^1}^2 \simeq \frac{g^2}{4c_W^2}(v^2-3u^2), \tag{2.48}$$

$$M_{Z^2}^2 \simeq \frac{g^2c_W^2\omega^2}{3-4s_W^2}. \tag{2.49}$$

Consequently, Z^1 can be identified as the Z boson in the standard model, and Z^2 being the new neutral (Hermitian) gauge boson. It is important to note that in the limit $u \to 0$ the mixing angle φ between $\mathcal{Z}$ and $\mathcal{Z}'$ is always nonvanishing. This differs from the mixing angle θ between the W boson of the standard model and the singly-charged bilepton Y. Phenomenology of the mentioned mixing is quite similar to the $W_L - W_R$ mixing in the left-right symmetric model based on the $\mathrm{SU}(2)_R \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_{B-L}$ group (the interested reader can find in [90]).

2.3. Currents

The interaction among fermions with gauge bosons arises in part from

$$i\overline{\psi}\gamma_\mu D^\mu\psi = \text{kinematic terms} + H^{\mathrm{CC}} + H^{\mathrm{NC}}. \tag{2.50}$$

2.3.1. Charged currents

Despite neutrality, the gauge bosons X^0, X^{0*} belong to this section by their nature. Because of the mixing among the standard model W boson and the charged bilepton Y as well as among $(X^0 + X^{0*})$ with (W_3, W_8, B), the new interaction terms exist as follows:

$$H^{\mathrm{CC}} = \frac{g}{\sqrt{2}}\left(J_W^{\mu-}W_\mu^+ + J_Y^{\mu-}Y_\mu^+ + J_X^{\mu 0*}X_\mu^0 + \mathrm{H.c.}\right), \tag{2.51}$$

where

$$J_W^{\mu-} = c_\theta\left(\overline{\nu}_{aL}\gamma^\mu l_{aL} + \overline{u}_{aL}\gamma^\mu d_{aL}\right) - s_\theta\left(\overline{\nu}_{aL}^c\gamma^\mu l_{aL} + \overline{U}_L\gamma^\mu d_{1L} + \overline{u}_{\alpha L}\gamma^\mu D_{\alpha L}\right), \tag{2.52}$$

$$J_Y^{\mu-} = c_\theta\left(\overline{\nu}_{aL}^c\gamma^\mu l_{aL} + \overline{U}_L\gamma^\mu d_{1L} + \overline{u}_{\alpha L}\gamma^\mu D_{\alpha L}\right) + s_\theta\left(\overline{\nu}_{aL}\gamma^\mu l_{aL} + \overline{u}_{aL}\gamma^\mu d_{aL}\right), \tag{2.53}$$

$$\begin{aligned} J_X^{\mu 0*} \simeq{}& \left(1-t_{2\theta}^2\right)\left(\overline{\nu}_{aL}\gamma^\mu\nu_{aL}^c + \overline{u}_{1L}\gamma^\mu U_L - \overline{D}_{\alpha L}\gamma^\mu d_{\alpha L}\right) \\ &- t_{2\theta}^2\left(\overline{\nu}_{aL}^c\gamma^\mu\nu_{aL} + \overline{U}_L\gamma^\mu u_{1L} - \overline{d}_{\alpha L}\gamma^\mu D_{\alpha L}\right) + \frac{t_{2\theta}}{\sqrt{1+4t_{2\theta}^2}} \\ &\times\left(\overline{\nu}_a\gamma^\mu\nu_a + \overline{u}_{1L}\gamma^\mu u_{1L} - \overline{U}_L\gamma^\mu U_L - \overline{d}_{\alpha L}\gamma^\mu d_{\alpha L} + \overline{D}_{\alpha L}\gamma^\mu D_{\alpha L}\right). \end{aligned} \tag{2.54}$$

Comparing with the charged currents in the usual 3-3-1 model with right-handed neutrinos [34, 35], we get the following discrepancies:

(1) the second term in (2.52),

(2) the second term in (2.53),

(3) the second and the third terms in (2.54).

All above-mentioned interactions are lepton-number violating and weak (proportional to $\sin\theta$ or its square $\sin^2\theta$). However, these couplings lead to lepton-number violations only in the neutrino sector.

2.3.2. Neutral currents

As before, in this model, a real part of the non-Hermitian neutral X'^0 mixes with the real neutral ones such as Z and Z'. This gives the *unusual* term as follows:

$$H^{\mathrm{NC}} = eA^{\mu}J_{\mu}^{\mathrm{EM}} + \mathcal{L}^{\mathrm{NC}} + \mathcal{L}_{\mathrm{unnormal}}^{\mathrm{NC}}. \tag{2.55}$$

Despite the mixing among $W_3,\ W_8,\ B,\ W_4$, the electromagnetic interactions *remain* the same as in the standard model and the usual 3-3-1 model with right-handed neutrinos, that is,

$$J_{\mu}^{\mathrm{EM}} = \sum_f q_f \overline{f}\gamma_{\mu} f, \tag{2.56}$$

where f runs among all the fermions of the model.

Interactions of the neutral currents with fermions have a common form:

$$\mathcal{L}^{\mathrm{NC}} = \frac{g}{2c_W}\overline{f}\gamma^{\mu}\left[g_{kV}(f) - g_{kA}(f)\gamma^5\right] f Z_{\mu}^{k}, \quad k = 1,2, \tag{2.57}$$

where

$$\begin{aligned}
g_{1V}(f) &= \frac{c_{\varphi}\{T_3(f_L) - 3t_{2\theta}^2 X(f_L) + [(3-8s_W^2)t_{2\theta}^2 - 2s_W^2]Q(f)\}}{\sqrt{(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}} \\
&\quad - \frac{s_{\varphi}[(4c_W^2-1)T_3(f_L) + 3c_W^2 X(f_L) - (3-5s_W^2)Q(f)]}{\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}, \\
g_{1A}(f) &= \frac{c_{\varphi}[T_3(f_L) - 3t_{2\theta}^2(X-Q)(f_L)]}{\sqrt{(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}} \\
&\quad - \frac{s_{\varphi}[(4c_W^2-1)T_3(f_L) + 3c_W^2(X-Q)(f_L)]}{\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}, \\
g_{2V}(f) &= g_{1V}(f)(c_{\varphi} \longrightarrow s_{\varphi},\ s_{\varphi} \longrightarrow -c_{\varphi}), \\
g_{2A}(f) &= g_{1A}(f)(c_{\varphi} \longrightarrow s_{\varphi},\ s_{\varphi} \longrightarrow -c_{\varphi}).
\end{aligned} \tag{2.58}$$

Table 3: The $Z_\mu^1 \to \overline{f} f$ couplings.

f	$g_{1V}(f)$	$g_{1A}(f)$
ν_a	$\dfrac{c_\varphi - s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}}{2\sqrt{(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}+s_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
l_a	$\dfrac{(3-4c_W^2)[c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}+s_\varphi]}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}+s_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
u_1	$\dfrac{c_\varphi\sqrt{4c_W^2-1}[3(1+2t_{2\theta}^2)-8s_W^2(1+4t_{2\theta}^2)]-s_\varphi(3+2s_W^2)\sqrt{1+4t_{2\theta}^2}}{6\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{c_\varphi\sqrt{4c_W^2-1}(1+2t_{2\theta}^2)-s_\varphi c_{2W}\sqrt{1+4t_{2\theta}^2}}{2\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$
d_1	$\dfrac{(1-4c_W^2)[c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}+s_\varphi]}{6\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}+s_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
u_α	$\dfrac{(3-8s_W^2)[c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}+s_\varphi]}{6\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}+s_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
d_α	$\dfrac{c_\varphi\sqrt{4c_W^2-1}[(1-4c_W^2)(1+4t_{2\theta}^2)+6t_{2\theta}^2]+s_\varphi(1+2c_W^2)\sqrt{1+4t_{2\theta}^2}}{6\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{c_\varphi\sqrt{4c_W^2-1}(1+2t_{2\theta}^2)-s_\varphi c_{2W}\sqrt{1+4t_{2\theta}^2}}{2\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$
U	$\dfrac{c_\varphi\sqrt{4c_W^2-1}[3t_{2\theta}^2-4s_W^2(1+4t_{2\theta}^2)]+s_\varphi(3-7s_W^2)\sqrt{1+4t_{2\theta}^2}}{3\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{c_\varphi\sqrt{4c_W^2-1}t_{2\theta}^2+s_\varphi c_W^2\sqrt{1+4t_{2\theta}^2}}{\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$
D_α	$\dfrac{c_\varphi\sqrt{4c_W^2-1}[2s_W^2(1+4t_{2\theta}^2)-3t_{2\theta}^2]-s_\varphi(3-5s_W^2)\sqrt{1+4t_{2\theta}^2}}{3\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{c_\varphi\sqrt{4c_W^2-1}t_{2\theta}^2+s_\varphi c_W^2\sqrt{1+4t_{2\theta}^2}}{\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$

Here, $T_3(f_L)$, $X(f_L)$, and $Q(f)$ are, respectively, the third component of the weak isospin, the $U(1)_X$ charge, and the electric charge of the fermion f_L. Note that the isospin for the $SU(2)_L$ fermion singlet (in the bottom of triplets) vanishes: $T_3(f_L) = 0$. The values of $g_{1V}(f)$, $g_{1A}(f)$ and $g_{2V}(f)$ are listed in Tables 3 and 4.

Because of the above-mentioned mixing, the lepton-number violating interactions mediated by neutral gauge bosons Z^1 and Z^2 exist in the *neutrino and the exotic quark sectors*:

$$\mathcal{L}_{\text{unnormal}}^{\text{NC}} = -\frac{g t_{2\theta} g_{kV}(\nu)}{2}\left(\overline{\nu}_{aL}\gamma^\mu \nu_{aL}^c + \overline{u}_{1L}\gamma^\mu U_L - \overline{D}_{\alpha L}\gamma^\mu d_{\alpha L}\right) Z_\mu^k + \text{H.c.} \tag{2.59}$$

Again, these interactions are very weak and proportional to $\sin\theta$. From (2.52)–(2.54) and (2.59), we conclude that all lepton-number violating interactions are expressed in the terms dependent only in the mixing angle between the charged gauge bosons.

2.4. Phenomenology

First of all, we should find some constraints on the parameters of the model. There are many ways to get constraints on the mixing angle θ and the charged bilepton mass M_Y. Below we present a simple one. In our model, the W boson has the following *normal main* decay modes:

$$\begin{aligned} W^- &\longrightarrow l\,\tilde{\nu}_l \quad (l = e, \mu, \tau), \\ &\searrow u^c d,\ u^c s,\ u^c b, \quad (u \longrightarrow c), \end{aligned} \tag{2.60}$$

which are the same as in the standard model and in the 3-3-1 model with right-handed neutrinos. Beside the above MODES, there are additional ones which are lepton-number violating ($\Delta L = 2$)—the model's specific feature:

$$W^- \longrightarrow l\,\nu_l \quad (l = e, \mu, \tau). \tag{2.61}$$

It is easy to compute the tree-level decay widths as follows [91, 92]:

$$\begin{aligned} \Gamma^{\text{Born}}(W \longrightarrow l\tilde{\nu}_l) &= \frac{g^2 c_\theta^2}{8}\frac{M_W}{6\pi}(1-x)\left(1-\frac{x}{2}-\frac{x^2}{2}\right) \simeq \frac{c_\theta^2 \alpha M_W}{12 s_W^2}, \\ \Gamma^{\text{Born}}(W \longrightarrow l\,\nu_l) &= \frac{g^2 s_\theta^2}{8}\frac{M_W}{6\pi}(1-x)\left(1-\frac{x}{2}-\frac{x^2}{2}\right) \simeq \frac{s_\theta^2 \alpha M_W}{12 s_W^2}, \quad x \equiv \frac{m_l^2}{M_W^2}, \\ \sum_{\text{color}} \Gamma^{\text{Born}}(W \longrightarrow u_i^c d_j) &= \frac{3 g^2 c_\theta^2}{8}\frac{M_W}{6\pi}\left|V_{ij}\right|^2\left[1-2(x+\overline{x})+(x-\overline{x})^2\right]^{1/2} \\ &\quad \times \left[1-\frac{x+\overline{x}}{2}-\frac{(x-\overline{x})^2}{2}\right] \simeq \frac{c_\theta^2 \alpha M_W}{4 s_W^2}\left|V_{ij}\right|^2, \quad x \equiv \frac{m_{d_j}^2}{M_W^2},\ \overline{x} \equiv \frac{m_{u_i^c}^2}{M_W^2}. \end{aligned} \tag{2.62}$$

Quantum chromodynamics radiative corrections modify (2.62) by a multiplicative factor [3, 91, 92]:

$$\delta_{\text{QCQ}} = \frac{1+\alpha_s(M_Z)}{\pi} + \frac{1.409\alpha_s^2}{\pi^2} - \frac{12.77\alpha_s^3}{\pi^3} \simeq 1.04, \tag{2.63}$$

which is estimated from $\alpha_s(M_Z) \simeq 0.12138$. All the state masses can be ignored, the predicted total width for W decay into fermions is

$$\Gamma_W^{\text{tot}} = 1.04\frac{\alpha M_W}{2 s_W^2}(1-s_\theta^2) + \frac{\alpha M_W}{4 s_W^2}. \tag{2.64}$$

Taking $\alpha(M_Z) \simeq 1/128$, $M_W = 80.425\,\text{GeV}$, $s_W^2 = 0.2312$, and $\Gamma_W^{\text{tot}} = 2.124 \pm 0.041\,\text{GeV}$ [3], in Figure 1, we have plotted Γ_W^{tot} as function of s_θ. From the figure we get an upper limit:

$$\sin\theta \le 0.08. \tag{2.65}$$

It is important to note that this limit value on the LNV parameter u/ω is much larger than those in [50, 93, 94].

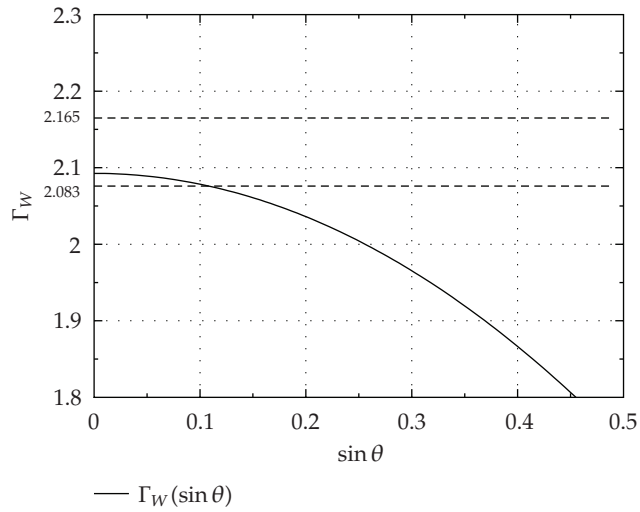


Figure 1: W width as function of $\sin\theta$, and the horizontal lines are an upper and a lower limit.

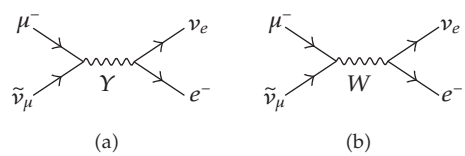


Figure 2: Feynman diagram for the wrong muon decay $\mu^- \to e^-\nu_e\tilde{\nu}_\mu$.

Since one of the VEVs is closely to the those in the standard model: $v \simeq v_{\text{weak}} = 246\,\text{GeV}$, therefore only two free VEVs exist in the considering model, namely, u and ω. The bilepton mass limit can be obtained from the "wrong" muon decay:

$$\mu^- \longrightarrow e^-\nu_e\tilde{\nu}_\mu, \tag{2.66}$$

mediated at the tree level, by both the standard model W and the singly-charged bilepton Y (see Figure 2). Remind that in the 3-3-1 model with right-handed neutrinos, at the lowest order, this decay is mediated only by the singly-charged bilepton Y. In our case, the second diagram in Figure 2 gives main contribution. Taking into account of the famous experimental data [3]

$$R_{\text{muon}} \equiv \frac{\Gamma(\mu^- \longrightarrow e^-\nu_e\tilde{\nu}_\mu)}{\Gamma(\mu^- \longrightarrow e^-\tilde{\nu}_e\nu_\mu)} < 1.2\%\ 90\%\ \text{CL}, \tag{2.67}$$

we get the constraint: $R_{\text{muon}} \simeq M_W^4/M_Y^4$. Therefore, it follows that $M_Y \geq 230\,\text{GeV}$.

However, the stronger bilepton mass bound of 440 GeV has been derived from consideration of experimental limit on lepton-number violating charged lepton decays [85].

In the case of $u \to 0$, analyzing the Z decay width [37, 95, 96], the $Z - Z'$ mixing angle is constrained by $-0.0015 \leq \varphi \leq 0.001$. From atomic parity violation in cesium, bounds for mass of the new exotic Z' and the $Z - Z'$ mixing angles, again in the limit $u \to 0$, are given [37, 95, 96]:

$$-0.00156 \leq \varphi \leq 0.00105, \qquad M_{Z_2} \geq 2.1\,\text{TeV}. \tag{2.68}$$

Table 4: The $Z_\mu^2 \to \overline{f} f$ couplings.

f	$g_{2V}(f)$	$g_{2A}(f)$
ν_a	$\dfrac{s_\varphi + c_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}}{2\sqrt{(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)} - c_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
l_a	$\dfrac{(3-4c_W^2)[s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)} - c_\varphi]}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)} - c_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
u_1	$\dfrac{s_\varphi\sqrt{4c_W^2-1}[3(1+2t_{2\theta}^2)-8s_W^2(1+4t_{2\theta}^2)]+c_\varphi(3+2s_W^2)\sqrt{1+4t_{2\theta}^2}}{6\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{s_\varphi\sqrt{4c_W^2-1}(1+2t_{2\theta}^2)+c_\varphi c_{2W}\sqrt{1+4t_{2\theta}^2}}{2\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$
d_1	$\dfrac{(1-4c_W^2)[s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)} - c_\varphi]}{6\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)} - c_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
u_α	$\dfrac{(3-8s_W^2)[s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)} - c_\varphi]}{6\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)} - c_\varphi}{2\sqrt{(4c_W^2-1)[1+(3-t_W^2)t_{2\theta}^2]}}$
d_α	$\dfrac{s_\varphi\sqrt{4c_W^2-1}[(1-4c_W^2)(1+4t_{2\theta}^2)+6t_{2\theta}^2]-c_\varphi(1+2c_W^2)\sqrt{1+4t_{2\theta}^2}}{6\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{s_\varphi\sqrt{4c_W^2-1}(1+2t_{2\theta}^2)+c_\varphi c_{2W}\sqrt{1+4t_{2\theta}^2}}{2\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$
U	$\dfrac{s_\varphi\sqrt{4c_W^2-1}[3t_{2\theta}^2-4s_W^2(1+4t_{2\theta}^2)]-c_\varphi(3-7s_W^2)\sqrt{1+4t_{2\theta}^2}}{3\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$\dfrac{s_\varphi\sqrt{4c_W^2-1}t_{2\theta}^2-c_\varphi c_W^2\sqrt{1+4t_{2\theta}^2}}{\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$
D_α	$\dfrac{s_\varphi\sqrt{4c_W^2-1}[2s_W^2(1+4t_{2\theta}^2)-3t_{2\theta}^2]+c_\varphi(3-5s_W^2)\sqrt{1+4t_{2\theta}^2}}{3\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$	$-\dfrac{s_\varphi\sqrt{4c_W^2-1}t_{2\theta}^2-c_\varphi c_W^2\sqrt{1+4t_{2\theta}^2}}{\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)[1+(3-t_W^2)t_{2\theta}^2]}}$

These values coincide with the bounds in the usual 3-3-1 model with right-handed neutrinos [97]. The interested reader can find in [40] for the general case $u \neq 0$ of the constraints.

For our purpose, we consider the ρ parameter—one of the most important quantities of the standard model, having a leading contribution in terms of the T parameter, is very useful to get the new-physics effects. It is well-known relation between ρ and T parameter:

$$\rho = 1 + \alpha T. \tag{2.69}$$

In the usual 3-3-1 model with right-handed neutrinos, T gets contribution from the oblique correction and the $Z - Z'$ mixing [88]:

$$T_{\text{RHN}} = T_{ZZ'} + T_{\text{oblique}}, \tag{2.70}$$

where $T_{ZZ'} \simeq (\tan^2\varphi/\alpha)(M_{Z_2}^2/M_{Z_1}^2 - 1)$ is negligible for $M_{Z'}$ less than 1 TeV; T_{oblique} depends on masses of the top quark and the standard model Higgs boson. Again, at the tree level

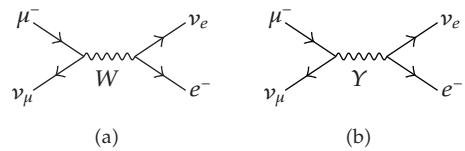


Figure 3: Feynman diagram for $\mu^- \to e^-\nu_e\nu_\mu$.

and the limit (2.8), from (2.22) and (2.48) we get an expression for the ρ parameter in the considering model:

$$\rho = \frac{M_W^2}{c_W^2 M_{Z^1}^2} = \frac{v^2}{v^2 - 3u^2} \simeq 1 + \frac{3u^2}{v^2}. \tag{2.71}$$

Note that (2.71) has only one free parameter u, since v is very close to the VEV in the standard model. Neglecting the contribution from the usual 3-3-1 model with right-handed neutrinos and taking into account the experimental data [3] $\rho = 0.9987 \pm 0.0016$, we get the constraint on u parameter by $u/v \leq 0.01$ which leads to $u \leq 2.46\,\text{GeV}$. This means that u is much smaller than v, as expected.

It seems that the ρ parameter, at the tree level, in this model, is favorable to be bigger than one and this is similar to the case of the models contained heavy Z' [98].

The interesting new physics compared with other 3-3-1 models is the neutrino physics. Due to lepton-number violating couplings, we have the following interesting consequences.

(1) Processes with $\Delta L = \pm 2$

From the charged currents, we have the following lepton-number violating $\Delta L = \pm 2$ decays such as

$$\begin{aligned} &\mu^- \longrightarrow e^-\nu_e\nu_\mu, \\ &\mu^- \longrightarrow e^-\tilde{\nu}_e\tilde{\nu}_\mu, \quad (\mu \text{ can be replaced by } \tau) \end{aligned} \tag{2.72}$$

in which both the standard model W boson and charged bilepton Y_μ^- are in intermediate states (see Figure 3). Here, the main contribution arises from the first diagram. Note that the wrong muon decay violates only *family* lepton-number, that is, $\Delta L = 0$, but not lepton number at all as in (2.72). The decay rates are given by

$$R_{\text{rare}} \equiv \frac{\Gamma(\mu^- \longrightarrow e^-\nu_e\nu_\mu)}{\Gamma(\mu^- \longrightarrow e^-\tilde{\nu}_e\nu_\mu)} = \frac{\Gamma(\mu^- \longrightarrow e^-\tilde{\nu}_e\tilde{\nu}_\mu)}{\Gamma(\mu^- \longrightarrow e^-\tilde{\nu}_e\nu_\mu)} \simeq s_\theta^2. \tag{2.73}$$

Taking $s_\theta = 0.08$, we get $R_{\text{rare}} \simeq 6 \times 10^{-3}$. This rate is the same as the wrong muon decay one. Interesting to note that, the family lepton-number violating processes

$$\nu_i\nu_i \longrightarrow \nu_j\nu_j, \quad (i \neq j) \tag{2.74}$$

are mediated not only by the non-Hermitian bilepton X but also by the Hermitian neutral Z^1, Z^2 (see Figure 4).

The first diagram in Figure 4 exists also in the 3-3-1 model with right-handed neutrinos, but the second one does not appear there.

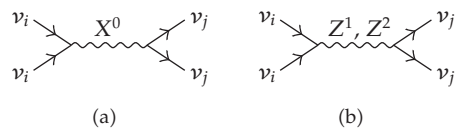


Figure 4: Feynman diagram for $\nu_i\nu_i \to \nu_j\nu_j$ $(i \neq j = e, \mu, \tau)$.

(2) Lepton-number violating kaon decays

Next, let us consider the lepton-number violating decay [3]:

$$K^+ \longrightarrow \pi^0 + e^+\tilde{\nu}_e < 3 \times 10^{-3} \quad \text{at } 90\% \text{ CL}. \tag{2.75}$$

This decay can be explained in the considering model as the subprocess given below:

$$\tilde{s} \longrightarrow \tilde{u} + e^+\tilde{\nu}_e. \tag{2.76}$$

This process is mediated by the standard model W boson and the charged bilepton Y. Amplitude of the considered process is proportional to $\sin\theta$:

$$M(\tilde{s} \longrightarrow \tilde{u} + e^+\tilde{\nu}_e) \simeq \frac{\sin 2\theta}{2M_W^2}\left(1 - \frac{M_W^2}{M_Y^2}\right). \tag{2.77}$$

Next, let us consider the "normal decay" [3]:

$$K^+ \longrightarrow \pi^0 + e^+\nu_e \quad (4.87 \pm 0.06)\% \tag{2.78}$$

with amplitude

$$M(\tilde{s} \longrightarrow \tilde{u} + e^+\nu_e) \simeq \frac{1}{M_W^2}. \tag{2.79}$$

From (2.77) and (2.79), we get

$$R_{\text{kaon}} \equiv \frac{\Gamma(\tilde{s} \longrightarrow \tilde{u} + e^+\tilde{\nu}_e)}{\Gamma(\tilde{s} \longrightarrow \tilde{u} + e^+\nu_e)} \simeq \sin^2\theta. \tag{2.80}$$

In the framework of this model, we derive the following decay modes with rates:

$$R_{\text{kaon}} = \frac{\Gamma(K^+ \longrightarrow \pi^0 + e^+\tilde{\nu}_e)}{\Gamma(K^+ \longrightarrow \pi^0 + e^+\nu_e)} \simeq \frac{\Gamma(K^+ \longrightarrow \pi^0 + \mu^+\tilde{\nu}_\mu)}{\Gamma(K^+ \longrightarrow \pi^0 + \mu^+\nu_\mu)} \simeq \sin^2\theta \leq 6 \times 10^{-3}. \tag{2.81}$$

Note that the similar lepton-number violating processes exist in the $\mathrm{SU}(2)_R \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_{B-L}$ model (for details, see [90]).

2.5. Summary

In this section, we have presented the 3-3-1 model with the minimal scalar sector (only two Higgs triplets). This version belongs to the 3-3-1 model without exotic charges (charges of the exotic quarks are 2/3 and −1/3). The spontaneous symmetry breakdown is achieved with only two Higgs triplets. One of the VEVs u is a source of lepton-number violations and a reason for the mixing between the charged gauge bosons—the standard model W and the singly-charged bilepton gauge bosons as well as between neutral non-Hermitian X^0 and neutral gauge bosons: the Z and the new exotic Z'. At the tree level, masses of the charged gauge bosons satisfy the law of Pythagoras $M_Y^2 = M_X^2 + M_W^2$ and in the limit $\omega \gg u, v$, the ρ parameter gets additional contribution dependent only on u/v. Thus, this leads to $u \ll v$, and there are three quite different scales for the VEVs of the model: one is very small $u \simeq O(1)$ GeV—a lepton-number violating parameter; the second v is close to the standard model one: $v \simeq v_{\text{weak}} = 246$ GeV; and the last is in the range of new physics scale about $O(1)$ TeV.

In difference with the usual 3-3-1 model with right-handed neutrinos, in this model the first family of quarks should be distinctive of the two others.

The exact diagonalization of the neutral gauge boson sector is derived. Because of the parameter u, the lepton-number violation happens only in neutrino but not in charged lepton sector. It is interesting to note that despite the above-mentioned mixing, the electromagnetic current remains unchanged. In this model, the lepton-number changing ($\Delta L = \pm 2$) processes exist but only in the neutrino sector.

It is worth mentioning on the advantage of the considered model: the new mixing angle between the charged gauge bosons θ is connected with one of the VEVs u—the parameter of lepton-number violations. There is no new parameter, but it contains very simple Higgs sector, hence the significant number of free parameters is reduced.

The model contains of new kinds of interactions in the neutrino sector. Hence, neutrino physics in this model is very rich. We will turn to further studies on neutrino masses and mixing in Section 4.

3. Higgs-gauge boson interactions

We first obtain the scalar fields and mass spectra. The couplings of the scalar fields with the ordinary gauge bosons are presented then. Cross section for the production of the charged Higgs boson at LHC is calculated.

3.1. Higgs potential

The Higgs potential in the model under consideration is given by (2.11). Let us first shift the Higgs fields into physical ones:

$$\chi = \begin{pmatrix} \chi_1^{P0} + \dfrac{u}{\sqrt{2}} \\ \chi_2^- \\ \chi_3^{P0} + \dfrac{\omega}{\sqrt{2}} \end{pmatrix}, \qquad \phi = \begin{pmatrix} \phi_1^+ \\ \phi_2^{P0} + \dfrac{v}{\sqrt{2}} \\ \phi_3^+ \end{pmatrix}. \tag{3.1}$$

The subscript P denotes *physical* fields as in the usual treatment. However, in the following, this subscript will be dropped. By substitution of (3.1) into (2.11), the potential becomes

$$\begin{aligned}
V(\chi,\phi) = {} & \mu_1^2\left[\left(\chi_1^{0*}+\frac{u}{\sqrt{2}}\right)\left(\chi_1^0+\frac{u}{\sqrt{2}}\right)+\chi_2^+\chi_2^-+\left(\chi_3^{0*}+\frac{\omega}{\sqrt{2}}\right)\left(\chi_3^0+\frac{\omega}{\sqrt{2}}\right)\right] \\
& +\mu_2^2\left[\phi_1^-\phi_1^++\left(\phi_2^{0*}+\frac{v}{\sqrt{2}}\right)\left(\phi_2^0+\frac{v}{\sqrt{2}}\right)+\phi_3^-\phi_3^+\right] \\
& +\lambda_1\left[\left(\chi_1^{0*}+\frac{u}{\sqrt{2}}\right)\left(\chi_1^0+\frac{u}{\sqrt{2}}\right)+\chi_2^+\chi_2^-+\left(\chi_3^{0*}+\frac{\omega}{\sqrt{2}}\right)\left(\chi_3^0+\frac{\omega}{\sqrt{2}}\right)\right]^2 \\
& +\lambda_2\left[\phi_1^-\phi_1^++\left(\phi_2^{0*}+\frac{v}{\sqrt{2}}\right)\left(\phi_2^0+\frac{v}{\sqrt{2}}\right)+\phi_3^-\phi_3^+\right]^2 \\
& +\lambda_3\left[\left(\chi_1^{0*}+\frac{u}{\sqrt{2}}\right)\left(\chi_1^0+\frac{u}{\sqrt{2}}\right)+\chi_2^+\chi_2^-+\left(\chi_3^{0*}+\frac{\omega}{\sqrt{2}}\right)\left(\chi_3^0+\frac{\omega}{\sqrt{2}}\right)\right] \\
& \times\left[\phi_1^-\phi_1^++\left(\phi_2^{0*}+\frac{v}{\sqrt{2}}\right)\left(\phi_2^0+\frac{v}{\sqrt{2}}\right)+\phi_3^-\phi_3^+\right] \\
& +\lambda_4\left[\left(\chi_1^{0*}+\frac{u}{\sqrt{2}}\right)\phi_1^++\chi_2^+\left(\phi_2^0+\frac{v}{\sqrt{2}}\right)+\left(\chi_3^{0*}+\frac{\omega}{\sqrt{2}}\right)\phi_3^+\right] \\
& \times\left[\phi_1^-\left(\chi_1^0+\frac{u}{\sqrt{2}}\right)+\left(\phi_2^{0*}+\frac{v}{\sqrt{2}}\right)\chi_2^-+\phi_3^-\left(\chi_3^0+\frac{\omega}{\sqrt{2}}\right)\right].
\end{aligned} \tag{3.2}$$

From the above expression, we get constraint equations at the tree level:

$$\begin{aligned}
\mu_1^2+\lambda_1\left(u^2+\omega^2\right)+\lambda_3\frac{v^2}{2} &= 0, \\
\mu_2^2+\lambda_2 v^2+\lambda_3\frac{\left(u^2+\omega^2\right)}{2} &= 0.
\end{aligned} \tag{3.3}$$

The nonzero values of χ and ϕ at the potential minimum as mentioned can be easily derived from these equations to yield the given (2.12) and (2.13).

Since u is a parameter of lepton-number violation, therefore the terms linear in u violate the latter. Applying the constraint (3.3), we get the minimum value, mass terms, lepton-number conserving, and violating interactions as follows:

$$V(\chi,\phi) = V_{\min}+V_{\text{mass}}^{\text{N}}+V_{\text{mass}}^{\text{C}}+V_{\text{LNC}}+V_{\text{LNV}}, \tag{3.4}$$

where

$$\begin{aligned}
V_{\min} &= -\frac{\lambda_2}{4}v^4-\frac{1}{4}\left(u^2+\omega^2\right)\left[\lambda_1\left(u^2+\omega^2\right)+\lambda_3 v^2\right], \\
V_{\text{mass}}^{\text{N}} &= \lambda_1\left(uS_1+\omega S_3\right)^2+\lambda_2 v^2 S_2^2+\lambda_3 v\left(uS_1+\omega S_3\right)S_2,
\end{aligned} \tag{3.5}$$

$$V_{\text{mass}}^{\text{C}} = \frac{\lambda_4}{2}\left(u\phi_1^++v\chi_2^++\omega\phi_3^+\right)\left(u\phi_1^-+v\chi_2^-+\omega\phi_3^-\right), \tag{3.6}$$

$$\begin{aligned} V_{\mathrm{LNC}} = {} & \lambda_1(\chi^\dagger\chi)^2 + \lambda_2(\phi^\dagger\phi)^2 + \lambda_3(\chi^\dagger\chi)(\phi^\dagger\phi) + \lambda_4(\chi^\dagger\phi)(\phi^\dagger\chi) + 2\lambda_1\omega S_3(\chi^\dagger\chi) \\ & + 2\lambda_2 v S_2(\phi^\dagger\phi) + \lambda_3 v S_2(\chi^\dagger\chi) + \lambda_3\omega S_3(\phi^\dagger\phi) + \frac{\lambda_4}{\sqrt{2}}(v\chi_2^- + \omega\phi_3^-)(\chi^\dagger\phi) \\ & + \frac{\lambda_4}{\sqrt{2}}(v\chi_2^+ + \omega\phi_3^+)(\phi^\dagger\chi), \end{aligned} \tag{3.7}$$

$$V_{\mathrm{LNV}} = 2\lambda_1 u S_1(\chi^\dagger\chi) + \lambda_3 u S_1(\phi^\dagger\phi) + \frac{\lambda_4}{\sqrt{2}}u[\phi_1^-(\chi^\dagger\phi) + \phi_1^+(\phi^\dagger\chi)]. \tag{3.8}$$

In the above equations, we have dropped the subscript P and used $\chi = (\chi_1^0, \chi_2^-, \chi_3^0)^T$, $\phi = (\phi_1^+, \phi_2^0, \phi_3^+)^T$. Moreover, we have expanded the neutral Higgs fields as

$$\chi_1^0 = \frac{S_1 + iA_1}{\sqrt{2}}, \qquad \chi_3^0 = \frac{S_3 + iA_3}{\sqrt{2}}, \qquad \phi_2^0 = \frac{S_2 + iA_2}{\sqrt{2}}. \tag{3.9}$$

In the literature, the real parts (S_i, $i = 1,2,3$) are also called CP-even scalar and the imaginary part (A_i, $i = 1,2,3$)—CP-odd scalar. In this paper, for short, we call them scalar and pseudoscalar field, respectively. As expected, the lepton-number violating part V_{LNC} is linear in u and trilinear in scalar fields. These couplings will be also a source for lepton-number violations such as the mass spectra of quarks including exotic ones as well as neutrino Majorana masses, but given at higher-order corrections.

In the pseudoscalar sector, all the fields are Goldstone bosons: $G_1 = A_1$, $G_2 = A_2$, and $G_3 = A_3$ (cf. (3.5)). The scalar fields S_1, S_2, and S_3 gain masses via (3.5), thus we get one Goldstone boson G_4 and two neutral physical fields—the standard model H^0 and the new H_1^0 with masses:

$$m_{H^0}^2 = \lambda_2 v^2 + \lambda_1(u^2+\omega^2) - \sqrt{[\lambda_2 v^2 - \lambda_1(u^2+\omega^2)]^2 + \lambda_3^2 v^2(u^2+\omega^2)} \simeq \frac{4\lambda_1\lambda_2 - \lambda_3^2}{2\lambda_1}v^2, \tag{3.10}$$

$$M_{H_1^0}^2 = \lambda_2 v^2 + \lambda_1(u^2+\omega^2) + \sqrt{[\lambda_2 v^2 - \lambda_1(u^2+\omega^2)]^2 + \lambda_3^2 v^2(u^2+\omega^2)} \simeq 2\lambda_1\omega^2. \tag{3.11}$$

In term of original fields, the Goldstone and Higgs fields are given by

$$\begin{aligned} G_4 &= \frac{1}{\sqrt{1+t_\theta^2}}(S_1 - t_\theta S_3), \\ H^0 &= c_\zeta S_2 - \frac{s_\zeta}{\sqrt{1+t_\theta^2}}(t_\theta S_1 + S_3), \\ H_1^0 &= s_\zeta S_2 + \frac{c_\zeta}{\sqrt{1+t_\theta^2}}(t_\theta S_1 + S_3), \end{aligned} \tag{3.12}$$

where

$$t_{2\zeta} \equiv \frac{\lambda_3 M_W M_X}{\lambda_1 M_X^2 - \lambda_2 M_W^2}. \tag{3.13}$$

From (3.11), it follows that mass of the new Higgs boson $M_{H_1^0}$ is related to mass of the bilepton gauge X^0 (or $Y^\pm$ via the law of Pythagoras) through

$$M_{H_1^0}^2 = \frac{8\lambda_1}{g^2} M_X^2 \left[1 + \mathcal{O}\left(\frac{M_W^2}{M_X^2}\right)\right] = \frac{2\lambda_1 s_W^2}{\pi\alpha} M_X^2 \left[1 + \mathcal{O}\left(\frac{M_W^2}{M_X^2}\right)\right] \approx 18.8\lambda_1 M_X^2. \tag{3.14}$$

Here, we have used $\alpha = 1/128$ and $s_W^2 = 0.231$.

In the charged Higgs sector, the mass terms for (ϕ_1, χ_2, ϕ_3) are given by (3.6), thus there are two Goldstone bosons and one physical scalar field:

$$H_2^+ \equiv \frac{1}{\sqrt{u^2 + v^2 + \omega^2}} \left(u\phi_1^+ + v\chi_2^+ + \omega\phi_3^+\right) \tag{3.15}$$

with mass

$$M_{H_2^+}^2 = \frac{\lambda_4}{2}\left(u^2 + v^2 + \omega^2\right) = 2\lambda_4 \frac{M_Y^2}{g^2} = \frac{s_W^2 \lambda_4}{2\pi\alpha} M_Y^2 \simeq 4.7\lambda_4 M_Y^2. \tag{3.16}$$

The two remaining Goldstone bosons are

$$\begin{aligned} G_5^+ &= \frac{1}{\sqrt{1 + t_\theta^2}} \left(\phi_1^+ - t_\theta \phi_3^+\right), \\ G_6^+ &= \frac{1}{\sqrt{\left(1 + t_\theta^2\right)\left(u^2 + v^2 + \omega^2\right)}} \left[v\left(t_\theta \phi_1^+ + \phi_3^+\right) - \omega\left(1 + t_\theta^2\right)\chi_2^+\right]. \end{aligned} \tag{3.17}$$

Thus, all the pseudoscalars are eigenstates and massless (Goldstone). Other fields are related to the scalars in the weak basis by the linear transformations:

$$\begin{gathered} \begin{pmatrix} H^0 \\ H_1^0 \\ G_4 \end{pmatrix} = \begin{pmatrix} -s_\zeta s_\theta & c_\zeta & -s_\zeta c_\theta \\ c_\zeta s_\theta & s_\zeta & c_\zeta c_\theta \\ c_\theta & 0 & -s_\theta \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}, \\ \begin{pmatrix} H_2^+ \\ G_5^+ \\ G_6^+ \end{pmatrix} = \frac{1}{\sqrt{\omega^2 + c_\theta^2 v^2}} \begin{pmatrix} \omega s_\theta & v c_\theta & \omega c_\theta \\ c_\theta \sqrt{\omega^2 + c_\theta^2 v^2} & 0 & -s_\theta \sqrt{\omega^2 + c_\theta^2 v^2} \\ \dfrac{v s_{2\theta}}{2} & -\omega & v c_\theta^2 \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \chi_2^+ \\ \phi_3^+ \end{pmatrix}. \end{gathered} \tag{3.18}$$

With the two Higgs triplets of the model, there are twelve real scalar components. Eight of the gauge symmetries of $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ are spontaneously broken, which eliminates just eight Goldstone bosons associated with these fields. It leaves over just four massive scalar particles as obtained (one charged and two natural). There is no Majoron field in this model which contrasts to the 3-3-1 model with right-handed neutrinos [99, 100]. Let us remind the reader that among the Goldstone bosons there are four fields carrying the lepton number, but they can be gauged away by an unitary transformation [87].

From (3.10) and (3.11), we come to the previous result in [36]:

$$\lambda_1 > 0,\ \lambda_2 > 0, \quad 4\lambda_1\lambda_2 > \lambda_3^2. \tag{3.19}$$

Equation (3.16) shows that the mass of the charged Higgs boson $H_2^{\pm}$ is proportional to those of the charged bilepton Y through a coefficient of Higgs self-interaction $\lambda_4 > 0$. Analogously, this happens for the standard-model-like Higgs boson H^0 ($M_{H^0} \sim M_W$) and the new H_1^0 ($M_{H_1^0} \sim M_X$). Combining (3.19) with the constraint (3.3), we get a consequence: λ_3 is negative ($\lambda_3 < 0$). Let us remind the reader that the couplings $\lambda_{4,1,2}$ are fixed by the Higgs boson masses and λ_3, where λ_3 defines the splitting $\Delta m_H^2 \simeq -[\lambda_3^2/(2\lambda_1)]v^2$ from the standard model prediction.

To finish this section, let us comment on our physical Higgs bosons. In the effective approximation $w \gg v, u$, from (3.18), it follows that

$$\begin{aligned} &H^0 \sim S_2, \qquad H_1^0 \sim S_3, \qquad G_4 \sim S_1, \\ &H_2^+ \sim \phi_3^+, \qquad G_5^+ \sim \phi_1^+, \qquad G_6^+ \sim \chi_2^+. \end{aligned} \tag{3.20}$$

This means that, in the effective approximation, the charged boson H_2^- is a scalar bilepton (with lepton number $L = 2$), while the neutral scalar bosons H^0 and H_1^0 do not carry lepton number (with $L = 0$).

3.2. Higgs-standard model gauge couplings

There are a total of 9 gauge bosons in the $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ group and 8 of them are massive. As shown in the previous section, we have got just 8 massless Goldstone bosons—the justified number for the model. One of the neutral scalars is identified with the standard model Higgs boson; therefore, its couplings to ordinary gauge bosons such as the photon, the Z, and the $W^{\pm}$ bosons have to have, in the effective limit, usual known forms. To search Higgs bosons at future high-energy colliders, one needs their couplings with ordinary particles, specially with the gauge bosons in the standard model.

The interactions among the gauge bosons and the Higgs bosons arise in part from

$$\sum_{Y=\chi,\,\phi} (D_\mu Y)^\dagger (D^\mu Y). \tag{3.21}$$

In the following, the summation over Y is default and only the terms giving interested couplings are explicitly displayed. The covariant derivative is given by (2.14):

$$D_\mu = \partial_\mu - i\mathcal{P}_\mu \equiv \partial_\mu - i\mathcal{P}_\mu^{\mathrm{NC}} - i\mathcal{P}_\mu^{\mathrm{CC}}, \tag{3.22}$$

where the matrices $\mathcal{P}_\mu^{\mathrm{NC}}$ and $\mathcal{P}_\mu^{\mathrm{CC}}$ are written as

$$\mathcal{P}_\mu^{\mathrm{NC}} = \frac{g}{2}\begin{pmatrix} W_{3\mu} + \dfrac{W_{8\mu}}{\sqrt{3}} + t\sqrt{\dfrac{2}{3}}XB_\mu & 0 & y_\mu \\ 0 & -W_{3\mu} + \dfrac{W_{8\mu}}{\sqrt{3}} + t\sqrt{\dfrac{2}{3}}XB_\mu & 0 \\ y_\mu & 0 & -\dfrac{2W_{8\mu}}{\sqrt{3}} + t\sqrt{\dfrac{2}{3}}XB_\mu \end{pmatrix}, \tag{3.23}$$

$$\mathcal{P}_\mu^{\mathrm{CC}} = \frac{g}{\sqrt{2}}\begin{pmatrix} 0 & c_\theta W_\mu^+ + s_\theta Y_\mu^+ & X_\mu^0 \\ c_\theta W_\mu^- + s_\theta Y_\mu^- & 0 & c_\theta Y_\mu^- - s_\theta W_\mu^- \\ X_\mu^{0*} & c_\theta Y_\mu^+ - s_\theta W_\mu^+ & 0 \end{pmatrix}. \tag{3.24}$$

Table 5: Trilinear coupling constants of W^+W^- with neutral Higgs bosons.

Vertex	Coupling
W^+W^-H	$\frac{g^2}{2}vc_\zeta$
$W^+W^-H_1^0$	$\frac{g^2}{2}vs_\zeta$

Let us recall that $t = g_X/g = 3\sqrt{2}s_W/\sqrt{3-4s_W^2}$, $\tan\theta = u/\omega$, and $W_\mu^\pm$, $Y_\mu^\pm$, and X_μ^0 are the physical fields. The existence of y^μ is a consequence of mixing among the real part $(X_\mu^{0*} + X_\mu^0)$ with $W_{3\mu}$, $W_{8\mu}$, and B_μ; and its expression is determined from the mixing matrix U given in Appendix A.1:

$$y_\mu \equiv U_{42}Z_\mu + U_{43}Z'_\mu + (U_{44}-1)\frac{(X_\mu^{0*} + X_\mu^0)}{\sqrt{2}}, \tag{3.25}$$

where

$$\begin{aligned} U_{42} &= -t_{\theta'}\left(c_\varphi\sqrt{1-4s_{\theta'}^2c_W^2} - s_\varphi\sqrt{4c_W^2-1}\right), \\ U_{43} &= -t_{\theta'}\left(s_\varphi\sqrt{1-4s_{\theta'}^2c_W^2} + c_\varphi\sqrt{4c_W^2-1}\right), \\ U_{44} &= \sqrt{1-4s_{\theta'}^2c_W^2}. \end{aligned} \tag{3.26}$$

First, we consider the relevant couplings of the standard model W boson with the Higgs and Goldstone bosons. The trilinear couplings of the pair W^+W^- with the neutral scalars are given by

$$(p_\mu^{CC}\langle\chi\rangle)^\dagger(p^{CC\mu}\chi) + (p_\mu^{CC}\langle\phi\rangle)^\dagger(p^{CC\mu}\phi) + \text{H.c.} = \frac{g^2v}{2}W_\mu^+W^{-\mu}S_2. \tag{3.27}$$

Because of S_2 is a combination of only H and H_1^0, therefore, there are two couplings which are given in Table 5.

Couplings of the single W with two Higgs bosons exist in

$$\begin{aligned} i(\Upsilon^\dagger p_\mu^{CC}\partial^\mu\Upsilon - \partial^\mu\Upsilon^\dagger p_\mu^{CC}\Upsilon) &= \frac{ig}{\sqrt{2}}W_\mu^-\left[\Upsilon_2^*(c_\theta\partial^\mu\Upsilon_1 - s_\theta\partial^\mu\Upsilon_3) - \partial^\mu\Upsilon_2^*(c_\theta\Upsilon_1 - s_\theta\Upsilon_3)\right] + \text{H.c.} \\ &= \frac{ig}{\sqrt{2}}W_\mu^-\left[\chi_2^+(c_\theta\partial^\mu\chi_1^0 - s_\theta\partial^\mu\chi_3^0) - \partial^\mu\chi_2^+(c_\theta\chi_1^0 - s_\theta\chi_3^0)\right. \\ &\qquad \left. + \phi_2^{0*}(c_\theta\partial^\mu\phi_1^+ - s_\theta\partial^\mu\phi_3^+) - \partial^\mu\phi_2^{0*}(c_\theta\phi_1^+ - s_\theta\phi_3^+)\right] + \text{H.c.} \end{aligned} \tag{3.28}$$

The resulting couplings of the single W boson with two scalar fields are listed in Table 6, where we have used a notation $A\overleftrightarrow{\partial}_\mu B = A(\partial_\mu B) - (\partial_\mu A)B$. Vanishing couplings are

$$\begin{aligned} \mathcal{V}(W^-H_2^+H^0) &= \mathcal{V}(W^-H_2^+H_1^0) = \mathcal{V}(W^-H^0G_6^+) \\ &= \mathcal{V}(W^-H_1^0G_6^+) = \mathcal{V}(W^-H_2^+G_2) = \mathcal{V}(W^-G_6^+G_2) = 0. \end{aligned} \tag{3.29}$$

Quartic couplings of W^+W^- with two scalar fields arise in part from

$$(\mathcal{D}_\mu^{\mathrm{CC}}\Upsilon)^+(\mathcal{D}^{\mathrm{CC}\mu}\Upsilon) = \frac{g^2}{2}W_\mu^+W^{-\mu}\left[\chi_2^+\chi_2^- + c_\theta^2\chi_1^{0*}\chi_1^0 + s_\theta^2\chi_3^{0*}\chi_3^0 - c_\theta s_\theta\left(\chi_1^{0*}\chi_3^0 + \chi_1^0\chi_3^{0*}\right)\right.$$
$$\left. + \phi_2^{0*}\phi_2^0 + c_\theta^2\phi_1^-\phi_1^+ + s_\theta^2\phi_3^-\phi_3^+ - c_\theta s_\theta\left(\phi_1^+\phi_3^- + \phi_1^-\phi_3^+\right)\right]. \tag{3.30}$$

With the help of (A.3) and (A.4), we get the interested couplings of W^+W^- with two scalars which are listed in Table 7. Our calculation give following vanishing couplings:

$$\mathcal{V}(W^+W^-H_2^+G_5^-) = \mathcal{V}(W^+W^-G_5^+G_6^-) = \mathcal{V}(W^+W^-H^0G_4^0) = \mathcal{V}(W^+W^-H_1^0G_4^0) = 0. \tag{3.31}$$

Now, we turn to the couplings of neutral gauge bosons with Higgs bosons. In this case, the interested couplings exist in

$$\begin{aligned}
&i\left(\Upsilon^\dagger\mathcal{D}_\mu^{\mathrm{NC}}\partial^\mu\Upsilon - \partial^\mu\Upsilon^\dagger\mathcal{D}_\mu^{\mathrm{NC}}\Upsilon\right)\\
&\quad = -\frac{ig}{2}\left\{W_3^\mu\left(\partial_\mu\chi_1^{0*}\chi_1^0 - \partial_\mu\chi_2^+\chi_2^- + \partial_\mu\phi_1^-\phi_1^+ - \partial_\mu\phi_2^{0*}\phi_2^0\right)\right.\\
&\qquad + \frac{W_8^\mu}{\sqrt{3}}\left(\partial_\mu\chi_1^{0*}\chi_1^0 + \partial_\mu\chi_2^+\chi_2^- + \partial_\mu\phi_1^-\phi_1^+ + \partial_\mu\phi_2^{0*}\phi_2^0 - 2\partial_\mu\chi_3^{0*}\chi_3^0 - 2\partial_\mu\phi_3^-\phi_3^+\right)\\
&\qquad + t\sqrt{\frac{2}{3}}B^\mu\left[-\frac{1}{3}\left(\partial_\mu\chi_1^{0*}\chi_1^0 + \partial_\mu\chi_2^+\chi_2^- + \partial_\mu\chi_3^{0*}\chi_3^0\right) + \frac{2}{3}\left(\partial_\mu\phi_1^-\phi_1^+ + \partial_\mu\phi_2^{0*}\phi_2^0 + \partial_\mu\phi_3^-\phi_3^+\right)\right]\\
&\qquad \left. + y^\mu\left(\partial_\mu\chi_1^{0*}\chi_3^0 + \partial_\mu\chi_3^{0*}\chi_1^0 + \partial_\mu\phi_1^-\phi_3^+ + \partial_\mu\phi_3^-\phi_1^+\right)\right\} + \mathrm{H.c.}
\end{aligned} \tag{3.32}$$

It can be checked that, as expected, the photon A_μ does not interact with neutral Higgs bosons. Other vanishing couplings are

$$\begin{aligned}
\mathcal{V}(AH_2^+G_5^-) &= \mathcal{V}(AH_2^+G_6^-) = \mathcal{V}(AG_6^+G_5^-) = 0,\\
\mathcal{V}(AAH^0) &= \mathcal{V}(AAH_1^0) = \mathcal{V}(AAG_4) = 0,\\
\mathcal{V}(AZH^0) &= \mathcal{V}(AZH_1^0) = \mathcal{V}(AZG_4) = 0,\\
\mathcal{V}(AZ'H^0) &= \mathcal{V}(AZ'H_1^0) = \mathcal{V}(AZ'G_4) = 0.
\end{aligned} \tag{3.33}$$

The nonzero electromagnetic couplings are listed in Table 8. It should be noticed that the electromagnetic interaction is diagonal, that is, the nonzero couplings in this model always have a form:

$$ieq_H A^\mu H^* \overleftrightarrow{\partial}_\mu H. \tag{3.34}$$

For the Z bosons, the following observation is useful:

$$\begin{aligned}
W_3^\mu &= U_{12}Z^\mu + \cdots, \qquad & W_8^\mu &= U_{22}Z^\mu + \cdots,\\
B^\mu &= U_{32}Z^\mu + \cdots, \qquad & y^\mu &= U_{42}Z^\mu + \cdots.
\end{aligned} \tag{3.35}$$

Here,

$$\begin{aligned}
U_{12} &= c_\varphi c_{\theta'} c_W,\\
U_{22} &= \frac{c_\varphi\left(s_W^2 - 3c_W^2 s_{\theta'}^2\right) - s_\varphi\sqrt{\left(1 - 4s_{\theta'}^2 c_W^2\right)\left(4c_W^2 - 1\right)}}{\sqrt{3}c_W c_{\theta'}},\\
U_{32} &= -\frac{t_W\left(c_\varphi\sqrt{4c_W^2 - 1} + s_\varphi\sqrt{1 - 4s_{\theta'}^2 c_W^2}\right)}{\sqrt{3}c_{\theta'}}
\end{aligned} \tag{3.36}$$

Table 6: Trilinear coupling constants of W^- with two Higgs bosons.

Vertex	Coupling	Vertex	Coupling
$W^{\mu-}H_2^+\overleftrightarrow{\partial}_\mu G_4$	$\dfrac{igvc_\theta}{2\sqrt{\omega^2+c_\theta^2v^2}}$	$W^{\mu-}G_6^+\overleftrightarrow{\partial}_\mu G_1$	$\dfrac{gc_\theta\omega}{2\sqrt{\omega^2+c_\theta^2v^2}}$
$W^{\mu-}G_5^+\overleftrightarrow{\partial}_\mu H$	$-\dfrac{igc_\zeta}{2}$	$W^{\mu-}G_5^+\overleftrightarrow{\partial}_\mu G_2$	$-\dfrac{g}{2}$
$W^{\mu-}G_6^+\overleftrightarrow{\partial}_\mu G_4$	$\dfrac{ig\omega}{2\sqrt{\omega^2+c_\theta^2v^2}}$	$W^{\mu-}G_5^+\overleftrightarrow{\partial}_\mu H_1^0$	$-\dfrac{ig}{2}s_\zeta$
$W^{\mu-}H_2^+\overleftrightarrow{\partial}_\mu G_1$	$-\dfrac{gvc_\theta^2}{2\sqrt{\omega^2+c_\theta^2v^2}}$	$W^{\mu-}G_6^+\overleftrightarrow{\partial}_\mu G_3^0$	$-\dfrac{gs_\theta\omega}{2\sqrt{\omega^2+c_\theta^2v^2}}$
$W^{\mu-}H_2^+\overleftrightarrow{\partial}_\mu G_3$	$\dfrac{gvs_{2\theta}}{4\sqrt{\omega^2+c_\theta^2v^2}}$		

are elements in the mixing matrix of the neutral gauge bosons given in Appendix A.1. From (3.32) and (3.35), it follows that the trilinear couplings of the single Z with charged Higgs bosons exist in part from the Lagrangian terms:

$$\begin{aligned}-\frac{ig}{2}Z^\mu\Bigg[&\left(U_{12}-\frac{U_{22}}{\sqrt{3}}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\partial_\mu\chi_2^-\chi_2^+ + \left(U_{12}+\frac{U_{22}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\partial_\mu\phi_1^-\phi_1^+\\ &+\left(-\frac{2}{\sqrt{3}}U_{22}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\partial_\mu\phi_3^-\phi_3^+ + U_{42}\left(\partial_\mu\phi_1^-\phi_3^+ + \partial_\mu\phi_3^-\phi_1^+\right)\Bigg] + \text{H.c.}\end{aligned} \tag{3.37}$$

From (3.37), we get trilinear couplings of the Z with the charged Higgs bosons which are listed in Table 9. The limit sign ($\rightarrow$) in the Tables is the effective one.

In the effective limit, the ZG_5G_5 vertex gets an exact expression as in the standard model. Hence, G_5 can be identified with the charged Goldstone boson in the standard model (G_{W^+}).

Now, we search couplings of the single Z_μ boson with neutral scalar fields. With the help of the following equations,

$$\begin{gathered}\chi_1^0\overleftrightarrow{\partial}_\mu\chi_1^{0*} = iG_1\overleftrightarrow{\partial}_\mu S_1, \qquad \chi_3^0\overleftrightarrow{\partial}_\mu\chi_3^{0*} = iG_3\overleftrightarrow{\partial}_\mu S_3, \qquad \phi_2^0\overleftrightarrow{\partial}_\mu\phi_2^{0*} = iG_2\overleftrightarrow{\partial}_\mu S_2,\\ \partial_\mu\chi_1^{0*}\chi_3^0 + \partial_\mu\chi_3^{0*}\chi_1^0 = \frac{1}{2}\left[\partial_\mu S_1S_3 + \partial_\mu S_3S_1 + \partial_\mu G_1G_3 + \partial_\mu G_3G_1 + iG_3\overleftrightarrow{\partial}_\mu S_1 + iG_1\overleftrightarrow{\partial}_\mu S_3\right],\end{gathered} \tag{3.38}$$

the necessary parts of Lagrangian are

$$\begin{aligned}\frac{g}{2}\Bigg[&\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)G_1\overleftrightarrow{\partial}_\mu S_1 + U_{42}G_1\overleftrightarrow{\partial}_\mu S_3 + \left(-\frac{2}{\sqrt{3}}U_{22}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\\ &\times G_3\overleftrightarrow{\partial}_\mu S_3 + U_{42}G_3\overleftrightarrow{\partial}_\mu S_1 + \left(-U_{12}+\frac{U_{22}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)G_2\overleftrightarrow{\partial}_\mu S_2\Bigg].\end{aligned} \tag{3.39}$$

Table 7: Nonzero quartic coupling constants of W^+W^- with Higgs bosons.

Vertex	Coupling	Vertex	Coupling
$W^+W^-H_2^+H_2^-$	$\dfrac{g^2c_\theta^2v^2}{2(\omega^2+v^2c_\theta^2)}$	$W^+W^-G_1^0G_1^0$	$\dfrac{g^2c_\theta^2}{2}$
$W^+W^-G_5^+G_5^-$	$\dfrac{g^2}{2}$	$W^+W^-G_3^0G_3^0$	$\dfrac{g^2s_\theta^2}{2}$
$W^+W^-G_6^+G_6^-$	$\dfrac{g^2\omega^2}{2(\omega^2+c_\theta^2v^2)}$	$W^+W^-G_4^0G_4^0$	$\dfrac{g^2}{2}$
$W^+W^-H_2^+G_6^-$	$-\dfrac{g^2c_\theta v\omega}{2(\omega^2+c_\theta^2v^2)}$	$W^+W^-HH_1^0$	$\dfrac{g^2s_{2\zeta}}{4}$
W^+W^-HH	$\dfrac{g^2c_\zeta^2}{2}$	$W^+W^-G_1^0G_3^0$	$-\dfrac{g^2s_{2\theta}}{4}$
$W^+W^-H_1^0H_1^0$	$\dfrac{g^2s_\zeta^2}{2}$	$W^+W^-G_2^0G_2^0$	$\dfrac{g^2}{2}$

Table 8: Trilinear electromagnetic coupling constants of A^μ with two Higgs bosons.

Vertex	$A^\mu H_2^- \overleftrightarrow{\partial}_\mu H_2^+$	$A^\mu G_5^- \overleftrightarrow{\partial}_\mu G_5^+$	$A^\mu G_6^- \overleftrightarrow{\partial}_\mu G_6^+$
Coupling	ie	ie	ie

The resulting couplings are listed in Table 10. we conclude that G_2 should be identified to G_Z in the standard model. For the Z' boson, the following remark is again helpful:

$$\begin{aligned} W_3^\mu &= U_{13}Z'^\mu + \cdots, \qquad & W_8^\mu &= U_{23}Z'^\mu + \cdots, \\ B^\mu &= U_{33}Z'^\mu + \cdots, \qquad & y^\mu &= U_{43}Z'^\mu + \cdots, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} U_{13} &= s_\varphi c_{\theta'} c_W, \\ U_{23} &= \frac{s_\varphi\left(s_W^2 - 3c_W^2 s_{\theta'}^2\right) + c_\varphi\sqrt{\left(1-4s_{\theta'}^2c_W^2\right)\left(4c_W^2-1\right)}}{\sqrt{3}c_W c_{\theta'}}, \\ U_{33} &= -\frac{t_W\left(s_\varphi\sqrt{4c_W^2-1} - c_\varphi\sqrt{1-4s_{\theta'}^2c_W^2}\right)}{\sqrt{3}c_{\theta'}}. \end{aligned} \tag{3.41}$$

Thus, with the replacement $Z \to Z'$ one just replaces column 2 by 3, for example, trilinear coupling constants of the Z'_μ with two neutral Higgs bosons are given in Table 11.

Next, we search couplings of two neutral gauge bosons with scalar fields which arise in part from

$$\begin{aligned}
\Upsilon^{+}\mathcal{D}_{\mu}^{\mathrm{NC}}\mathcal{D}^{\mathrm{NC}\mu}\Upsilon &= \frac{g^2}{4}\{[\Upsilon_1^*(A_{11}^{\mu}A_{11\mu}+y_\mu y^\mu)+\Upsilon_3^*(A_{11\mu}y^\mu+A_{33\mu}y^\mu)]\Upsilon_1+A_{22}^{\mu}A_{22\mu}\times\Upsilon_2^*\Upsilon_2 \\
&\quad+[\Upsilon_1^*(A_{11\mu}y^\mu+A_{33\mu}y^\mu)+\Upsilon_3^*(A_{33}^{\mu}A_{33\mu}+y_\mu y^\mu)]\Upsilon_3\} \\
&= \frac{g^2}{4}\{[\chi_1^{0*}(A_{11}^{\mu\chi}A_{11\mu}^{\chi}+y_\mu y^\mu)+\chi_3^{0*}(A_{11\mu}^{\chi}y^\mu+A_{33\mu}^{\chi}y^\mu)]\chi_1^0 \\
&\quad+[\chi_1^{0*}(A_{11\mu}^{\chi}y^\mu+A_{33\mu}^{\chi}y^\mu)+\chi_3^{0*}(A_{33}^{\mu\chi}A_{33\mu}^{\chi}+y_\mu y^\mu)]\chi_3^0 \\
&\quad+[\phi_1^-(A_{11}^{\mu\phi}A_{11\mu}^{\phi}+y_\mu y^\mu)+\phi_3^-(A_{11\mu}^{\phi}y^\mu+A_{33\mu}^{\phi}y^\mu)]\phi_1^+ \\
&\quad+[\phi_1^-(A_{11\mu}^{\phi}y^\mu+A_{33\mu}^{\phi}y^\mu)+\phi_3^-(A_{33}^{\mu\phi}A_{33\mu}^{\phi}+y_\mu y^\mu)]\phi_3^+ \\
&\quad+(A_{22}^{\mu\chi}A_{22\mu}^{\chi})\chi_2^+\chi_2^-+(A_{22}^{\mu\phi}A_{22\mu}^{\phi})\phi_2^{0*}\phi_2^0\}.
\end{aligned} \tag{3.42}$$

Here, A_{ii}^{μ} $(i=1,2,3)$ is a diagonal element in the matrix $(2/g)\mathcal{D}_{\mu}^{\mathrm{NC}}$ which is dependent on the $\mathrm{U}(1)_X$ charge:

$$\begin{aligned}
A_{11}^{\mu\chi} &= W_3^\mu+\frac{W_8^\mu}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}B^\mu, & A_{11}^{\mu\phi} &= W_3^\mu+\frac{W_8^\mu}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}B^\mu, \\
A_{22}^{\mu\chi} &= -W_3^\mu+\frac{W_8^\mu}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}B^\mu, & A_{22}^{\mu\phi} &= -W_3^\mu+\frac{W_8^\mu}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}B^\mu, \\
A_{33}^{\mu\chi} &= -2\frac{W_8^\mu}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}B^\mu, & A_{33}^{\mu\phi} &= -2\frac{W_8^\mu}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}B^\mu.
\end{aligned} \tag{3.43}$$

Quartic couplings of two Z with neutral scalar fields are given by

$$\begin{aligned}
&\frac{g^2}{4}\{[\chi_1^{0*}(A_{11}^{\mu\chi}A_{11\mu}^{\chi}+y_\mu y^\mu)+\chi_3^{0*}(A_{11\mu}^{\chi}y^\mu+A_{33\mu}^{\chi}y^\mu)]\chi_1^0 \\
&\quad+[\chi_1^{0*}(A_{11\mu}^{\chi}y^\mu+A_{33\mu}^{\chi}y^\mu)+\chi_3^{0*}(A_{33}^{\mu\chi}A_{33\mu}^{\chi}+y_\mu y^\mu)]\chi_3^0+(A_{22}^{\mu\phi}A_{22\mu}^{\phi})\phi_2^{0*}\phi_2^0\} \\
&=\frac{g^2}{4}\{(A_{11}^{\mu\chi}A_{11\mu}^{\chi}+y_\mu y^\mu)\chi_1^{0*}\chi_1^0+(A_{33}^{\mu\chi}A_{33\mu}^{\chi}+y_\mu y^\mu)\chi_3^{0*}\chi_3^0 \\
&\quad+(A_{11\mu}^{\chi}y^\mu+A_{33\mu}^{\chi}y^\mu)(\chi_1^{0*}\chi_3^0+\chi_3^{0*}\chi_1^0)+(A_{22}^{\mu\phi}A_{22\mu}^{\phi})\phi_2^{0*}\phi_2^0\}.
\end{aligned} \tag{3.44}$$

In this case, the couplings are listed in Table 12.

Trilinear couplings of the pair ZZ with one scalar field are obtained via the following terms:

$$\begin{aligned}
&\frac{g^2}{4}[vS_2A_{22\mu}^{\phi}A_{22}^{\mu\phi}+uS_1A_{11\mu}^{\chi}A_{11}^{\mu\chi}+\omega S_3A_{33\mu}^{\chi}A_{33}^{\mu\chi} \\
&\qquad+(uS_1+\omega S_3)y_\mu y^\mu-(\omega S_1+uS_3)y^\mu A_{22\mu}^{\phi}].
\end{aligned} \tag{3.45}$$

The obtained couplings are given in Table 13.

Because of (3.40), for the ZZ' couplings with scalar fields, the above manipulation is good enough. For example, Table 12 is replaced by Table 14.

Now, we turn to the interested coupling $ZW^{\pm}H_2^{\mp}$ arisen in part from

$$\begin{aligned}
&Y^{+}p_{\mu}^{\mathrm{NC}}p^{\mathrm{CC}\mu}Y + \mathrm{H.c.} \\
&\quad = \frac{g^2}{2\sqrt{2}}\left\{W_{\mu}^{-}A_{22}^{\mu}Y_2^{*}\left(c_{\theta}Y_1 - s_{\theta}Y_3\right) + W_{\mu}^{+}\left[\left(c_{\theta}A_{11}^{\mu} - s_{\theta}y^{\mu}\right)Y_1^{*} + \left(c_{\theta}y^{\mu} - s_{\theta}A_{33}^{\mu}\right)Y_3^{*}\right]Y_2\right\} + \mathrm{H.c.}
\end{aligned} \tag{3.46}$$

For our Higgs triplets, one gets

$$\begin{aligned}
&\frac{g^2}{2\sqrt{2}}\left\{W_{\mu}^{-}\left[A_{22}^{\chi\mu}\chi_2^{+}\left(c_{\theta}\chi_1^{0} - s_{\theta}\chi_3^{0}\right) + A_{22}^{\phi\mu}\phi_2^{0*}\left(c_{\theta}\phi_1^{+} - s_{\theta}\phi_3^{+}\right)\right]\right. \\
&\qquad + W_{\mu}^{+}\chi_2^{-}\left[\left(c_{\theta}A_{11}^{\chi\mu} - s_{\theta}y^{\mu}\right)\chi_1^{0*} + \left(c_{\theta}y^{\mu} - s_{\theta}A_{33}^{\chi\mu}\right)\chi_3^{0*}\right] \\
&\qquad \left. + W_{\mu}^{+}\phi_2^{0}\left[\left(c_{\theta}A_{11}^{\phi\mu} - s_{\theta}y^{\mu}\right)\phi_1^{-} + \left(c_{\theta}y^{\mu} - s_{\theta}A_{33}^{\phi\mu}\right)\phi_3^{-}\right]\right\} + \mathrm{H.c.}
\end{aligned} \tag{3.47}$$

From (3.47), the trilinear couplings of the W boson with one scalar and one neutral gauge bosons exist in a part:

$$\begin{aligned}
&\frac{g^2}{4}W_{\mu}^{+}\left\{v\phi_1^{-}\left[c_{\theta}\left(\frac{2}{\sqrt{3}}W_8^{\mu} + \frac{4t}{3}\sqrt{\frac{2}{3}}B^{\mu}\right) - s_{\theta}y^{\mu}\right]\right. \\
&\qquad + v\phi_3^{-}\left[c_{\theta}y^{\mu} - s_{\theta}\left(-W_3^{\mu} - \frac{W_8^{\mu}}{\sqrt{3}} + \frac{4t}{3}\sqrt{\frac{2}{3}}B^{\mu}\right)\right] \\
&\qquad \left. + \omega\chi_2^{-}\left[s_{\theta}\left(W_3^{\mu} + \sqrt{3}W_8^{\mu}\right) + \frac{c_{2\theta}}{c_{\theta}}y^{\mu}\right]\right\} + \mathrm{H.c.}
\end{aligned} \tag{3.48}$$

From the above equation, we get necessary nonzero couplings, which are listed in Table 15. Vanishing couplings are

$$\mathcal{V}\left(AW^{+}H_2^{-}\right) = \mathcal{V}\left(AW^{+}G_6^{-}\right) = 0. \tag{3.49}$$

Equation (3.49) is consistent with an evaluation in [53], where authors neglected the diagrams with the $\gamma W^{\pm}H^{\mp}$ vertex.

From (3.24), it follows that, to get couplings of the bilepton gauge boson Y^{+} with ZH_2^{-}, one just makes in (3.48) the replacement $c_{\theta} \to -s_{\theta},\ s_{\theta} \to c_{\theta}$.

Finally, we can identify the scalar fields in the considered model with that in the standard model as follows:

$$H \longleftrightarrow h, \qquad G_5^{+} \longleftrightarrow G_{W^{+}}, \qquad G_2 \longleftrightarrow G_Z. \tag{3.50}$$

In the effective limit $\omega \gg v, u$ our Higgs can be represented as

$$\chi = \begin{pmatrix} \frac{1}{\sqrt{2}}u + G_{X^0} \\ G_{Y^-} \\ \frac{1}{\sqrt{2}}\left(\omega + H_1^0 + iG_{Z'}\right) \end{pmatrix}, \qquad \phi = \begin{pmatrix} G_{W^+} \\ \frac{1}{\sqrt{2}}\left(v + h + iG_Z\right) \\ H_2^{+} \end{pmatrix}, \tag{3.51}$$

where $G_3 \sim G_{Z'}$, $G_6^- \sim G_{Y^-}$ and

$$G_4 + i\,G_1 \sim \sqrt{2}\,G_{X^0} \tag{3.52}$$

are the Goldstone boson of the massive gauge bosons Z', Y^-, and X^0, respectively. Note that identification in (3.52) is possible due to the fact that both scalar and pseudoscalar parts of χ_1^0 are massless. In addition, the pseudoscalar part is decoupled from others, while its scalar part mixes in the same as in the gauge boson sector.

We emphasize again that, in the effective approximation, all Higgs gauge boson couplings in the standard model are *recovered* (see Table 16). In contradiction with the previous analysis in [36], the condition $u \sim v$ or introduction of the third triplet are not necessary.

3.3. Production of $H_2^\pm$ via WZ fusion at LHC

The possibility to detect the neutral Higgs boson in the minimal version at e^+e^- colliders was considered in [101] and production of the standard model-like neutral Higgs boson at LHC was considered in [52]. This section is devoted to production of the charged $H_2^\pm$ at the CERN LHC.

Let us firstly discuss on the mass of this Higgs boson. Equation (3.16) gives us a connection between its mass and those of the singly-charged bilepton Y through the coefficient of Higgs self-coupling λ_4. Note that in the considered model the neutrino Majorana masses exist only in the loop levels. To keep these masses in the experimental range, the mass of $M_{H_2^\pm}$ can be taken in the electroweak scale with $\lambda_4 \sim 0.01$ (see the next section). From (3.16), taking the lower limit for M_Y to be 1 TeV, the mass of $H_2^\pm$ is in range of 200 GeV.

Taking into account that, in the effective approximation, H_2^- is the bilepton, we get the dominant decay channels as follows:

$$\begin{aligned} H_2^- &\longrightarrow l\nu_l,\ \tilde{U}d_a,\ D_\alpha\tilde{u}_a, \\ &\searrow ZW^-,\ Z'W^-,\ XW^-,\ ZY^-. \end{aligned} \tag{3.53}$$

Assuming that masses of the exotic quarks (U, D_α) are larger than $M_{H_2^\pm}$, we come to the fact that the hadron modes are absent in decay of the charged Higgs boson. Due to that the Yukawa couplings of $H_2^\pm l^\mp \nu$ are very small, the main decay modes of the are in the second line of (3.53). Note that the charged Higgs bosons in doublet models such as two-Higgs doublet model or minimal supersymmetric standard model have both hadronic and leptonic modes [54]. This is a specific feature of the model under consideration.

Because of the exotic X, Y, Z' gauge bosons are heavy, the coupling of a singly-charged Higgs boson $(H_2^\pm)$ with the weak gauge bosons, $H_2^\pm W^\mp Z$, may dominate. Here, it is of particular importance for the electroweak symmetry breaking. Its magnitude is directly related to the structure of the extended Higgs sector under global symmetries [102–106]. This coupling can appear at the tree level in models with scalar triplets, while it is induced at the loop level in multiscalar doublet models. The coupling, in our model, differs from zero at the tree level due to the fact that the $H_2^\pm$ belongs to a triplet.

Thus, for the charged Higgs boson $H_2^\pm$, it is important to study the couplings given by the interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = f_{ZWH} H_2^\pm W_\mu^\mp Z^\mu, \tag{3.54}$$

Table 9: Trilinear coupling constants of Z^μ with two charged Higgs bosons.

Vertex	Coupling
$Z^\mu H_2^- \overleftrightarrow{\partial}_\mu H_2^+$	$\frac{ig}{2(\omega^2+v^2c_\theta^2)}\Big\{(v^2c_\theta^2+\omega^2s_\theta^2)U_{12}+[\omega^2(1-3c_\theta^2)-v^2c_\theta^2]\frac{U_{22}}{\sqrt{3}}$ $+(v^2c_\theta^2+2\omega^2)\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}+\omega^2s_{2\theta}U_{42}\Big\}\longrightarrow -igs_Wt_W$
$Z^\mu G_5^- \overleftrightarrow{\partial}_\mu G_5^+$	$\frac{ig}{2}\left[c_\theta^2U_{12}+(1-3s_\theta^2)\frac{U_{22}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}-s_{2\theta}U_{42}\right]\longrightarrow\frac{ig}{2c_W}(1-2s_W^2)$
$Z^\mu G_6^- \overleftrightarrow{\partial}_\mu G_6^+$	$\frac{ig}{2(\omega^2+c_\theta^2v^2)}\Big\{(\omega^2+v^2s_\theta^2c_\theta^2)U_{12}+[v^2c_\theta^2(1-3c_\theta^2)-\omega^2]\frac{U_{22}}{\sqrt{3}}$ $+\frac{t}{3}\sqrt{\frac{2}{3}}(\omega^2+2v^2c_\theta^2)U_{32}+2v^2s_\theta c_\theta^3U_{42}\Big\}\longrightarrow\frac{ig}{2c_W}(1-2s_W^2)$
$Z^\mu H_2^- \overleftrightarrow{\partial}_\mu G_5^+$	$\frac{ig\omega}{4\sqrt{\omega^2+c_\theta^2v^2}}(s_{2\theta}U_{12}+\sqrt{3}s_{2\theta}U_{22}+2c_{2\theta}U_{42})\longrightarrow 0$
$Z^\mu H_2^- \overleftrightarrow{\partial}_\mu G_6^+$	$\frac{ig\omega vc_\theta}{2(\omega^2+c_\theta^2v^2)}\left[-c_\theta^2U_{12}+(2-3c_\theta^2)\frac{U_{22}}{\sqrt{3}}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}+s_{2\theta}U_{42}\right]\longrightarrow 0$
$Z^\mu G_5^- \overleftrightarrow{\partial}_\mu G_6^+$	$\frac{igvc_\theta}{4\sqrt{\omega^2+c_\theta^2v^2}}(s_{2\theta}U_{12}+\sqrt{3}s_{2\theta}U_{22}+2c_{2\theta}U_{42})\longrightarrow 0$

where f_{ZWH}, at tree level, is given in Table 15. The same as in [53], the dominant rate is due to the diagram connected with the W and Z bosons. Putting necessary matrix elements in Table 15 , we get

$$f_{ZWH}=-\frac{g^2v\omega s_{2\theta}}{4\sqrt{\omega^2+c_\theta^2v^2}}\frac{c_\varphi-s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}}{\sqrt{(1+4t_{2\theta}^2)[c_W^2+(4c_W^2-1)t_{2\theta}^2]}}. \tag{3.55}$$

Thus, the form factor, at the tree-level, is obtained by

$$F\equiv\frac{f_{ZWH}}{gM_W}=-\frac{\omega s_{2\theta}\left[c_\varphi-s_\varphi\sqrt{(4c_W^2-1)(1+4t_{2\theta}^2)}\right]}{2\sqrt{(\omega^2+c_\theta^2v^2)(1+4t_{2\theta}^2)[c_W^2+(4c_W^2-1)t_{2\theta}^2]}}. \tag{3.56}$$

The decay width of $H_2^\pm\to W_i^\pm Z_i$, where $i=L,T$ representing, respectively, the longitudinal and transverse polarizations is given by [53]:

$$\Gamma(H_2^\pm\longrightarrow W_i^\pm Z_i)=M_{H_2^\pm}\frac{\lambda^{1/2}(1,w,z)}{16\pi}\left|M_{ii}\right|^2, \tag{3.57}$$

where $\lambda(1,w,z)=(1-w-z)^2-4wz$, $w=M_W^2/M_{H_2^\pm}^2$ and $z=M_Z^2/M_{H_2^\pm}^2$. The longitudinal and transverse contributions are given in terms of F by

$$\left|M_{LL}\right|^2=\frac{g^2}{4z}(1-w-z)^2|F|^2,\qquad \left|M_{TT}\right|^2=2g^2w|F|^2. \tag{3.58}$$

For the case of $M_{H_2^\pm} \gg M_Z$, we have $|M_{TT}|^2/|M_{LL}|^2 \sim 8M_W^2 M_Z^2/M_{H_2^\pm}^4$ which implies that the decay into a longitudinally polarized weak boson pair dominates that into a transversely polarized one. The form factor F and mixing angle t_φ are presented in Table 17, where we have used $s_W^2 = 0.2312$, $v = 246\,\text{GeV}$, $\omega = 3$ TeV (or $M_Y = 1\,\text{TeV}$) as the typical values to get five cases corresponding with the s_θ values under the constraint (2.65).

Next, let us study the impact of the $H_2^\pm W^\mp Z$ vertex on the production cross section of $pp \to W^{\pm *} Z^* X \to H_2^\pm X$ which is a pure electroweak process with high p_T jets going into the forward and backward directions from the decay of the produced scalar boson without color flow in the central region. The hadronic cross section for $pp \to H_2^\pm X$ via $W^\pm Z$ fusion is expressed in the effective vector boson approximation [107–109] by

$$\sigma_{\text{eff}}\left(s, M_{H_2^\pm}^2\right) \simeq \frac{16\pi^2}{\lambda(1,w,z) M_{H_2^\pm}^3} \sum_{\lambda=T,L} \Gamma\left(H_2^\pm \longrightarrow W_\lambda^\pm Z_\lambda\right) \tau \frac{d\mathcal{L}}{d\tau}\bigg|_{pp/W_\lambda^\pm Z_\lambda}, \tag{3.59}$$

where $\tau = M_{H_2^\pm}^2/s$, and

$$\frac{d\mathcal{L}}{d\tau}\bigg|_{pp/W_\lambda^\pm Z_\lambda} = \sum_{ij} \int_\tau^1 \frac{d\tau'}{\tau'} \int_{\tau'}^1 \frac{dx}{x} f_i(x) f_j\left(\frac{\tau'}{x}\right) \frac{d\mathcal{L}}{d\xi}\bigg|_{q_i q_j/W_\lambda^\pm Z_\lambda} \tag{3.60}$$

with $\tau' = \hat{s}/s$ and $\xi = \tau/\tau'$. Here, $f_i(x)$ is the parton structure function for the ith quark, and

$$\begin{aligned} \frac{d\mathcal{L}}{d\xi}\bigg|_{q_i q_j/W_T^\pm Z_T} &= \frac{c}{64\pi^4} \frac{1}{\xi} \ln\left(\frac{\hat{s}}{M_W^2}\right) \ln\left(\frac{\hat{s}}{M_Z^2}\right) \left[(2+\xi)^2 \ln\left(\frac{1}{\xi}\right) - 2(1-\xi)(3+\xi)\right], \\ \frac{d\mathcal{L}}{d\xi}\bigg|_{q_i q_j/W_L^\pm Z_L} &= \frac{c}{16\pi^4} \frac{1}{\xi} \left[(1+\xi) \ln\left(\frac{1}{\xi}\right) + 2(\xi - 1)\right], \end{aligned} \tag{3.61}$$

where $c = (g^4 c_\theta^2/16c_W^2)[g_{1V}^2(q_j) + g_{1A}^2(q_j)]$ with $g_{1V}(q_j)$, $g_{1A}(q_j)$ for quark q_j is given in [38, Table I]. Using CTEQ6L [110], in Figure 5, we have plotted $\sigma_{\text{eff}}(s, M_{H_2^\pm}^2)$ at $\sqrt{s} = 14\,\text{TeV}$, as a function of the Higgs boson mass corresponding five cases in Table 17.

Assuming discovery limit of 25 events corresponding to the horizontal line, and taking the integrated luminosity of $300\,\text{fb}^{-1}$ [111], from the figure, we come to conclusion that, for $s_\theta = 0.08$ (the line on top), the charged Higgs boson $H_2^\pm$ with mass larger than 1700 GeV, cannot be seen at the LHC. These limiting masses are denoted by $M_{H_2^\pm}^{\max}$ and listed in Table 17. If the mass of the above-mentioned Higgs boson is in range of 200 GeV and $s_\theta = 0.08$, the cross section can exeed 260 fb, that is, 78000 of $H_2^\pm$ can be produced at the integrated LHC luminosity of $300\,\text{fb}^{-1}$. This production rate is about ten times larger than those in [53]. The cross sections decrease rapidly as mass of the Higgs boson increases from 200 GeV to 400 GeV.

3.4. Summary

In this section, we have considered the scalar sector in the economical 3-3-1 model. The model contains eight Goldstone bosons—the justified number of the massless ones eaten by the massive gauge bosons. Couplings of the standard model-like gauge bosons such as of the photon, the Z, and the new Z' gauge bosons with physical Higgs ones are also given. From these couplings, the standard model-like Higgs boson as well as Goldstone ones are identified. In the effective approximation, full content of scalar sector can be recognized. The

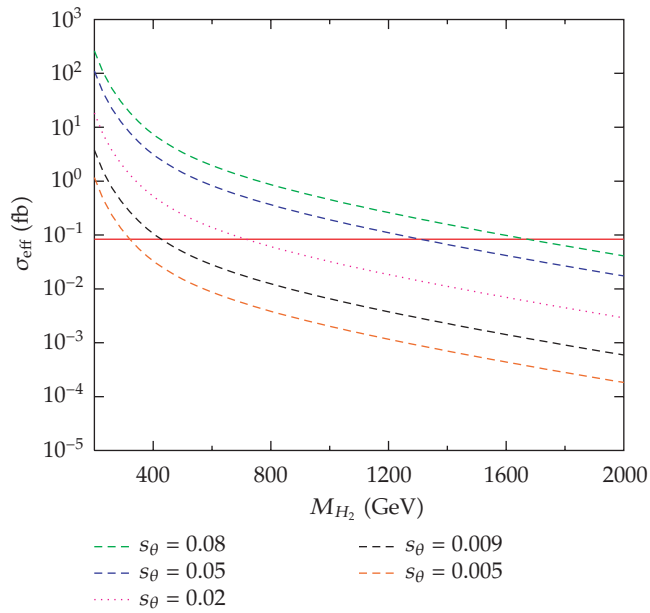


Figure 5: Hadronic cross section of $W^{\pm}Z$ fusion process as a function of the charged Higgs boson mass for five cases of $\sin\theta$. Horizontal line is discovery limit (25 events).

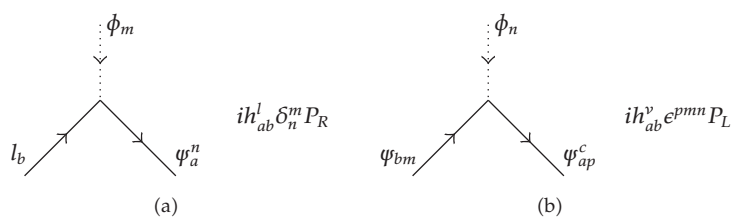


Figure 6: Lepton Yukawa couplings.

CP-odd part of Goldstone associated with the neutral non-Hermitian bilepton gauge bosons G_{X^0} is decoupled, while its CP-even counterpart has the mixing by the same way in the gauge boson sector. Despite the mixing among the photon with the non-Hermitian neutral bilepton X^0 as well as with the Z and the Z' gauge bosons, the electromagnetic couplings remain unchanged.

It is worth mentioning that masses of all physical Higgs bosons are related to that of gauge bosons through the coefficients of Higgs self-interactions. All gauge scalar boson couplings in the standard model are recovered. The coupling of the photon with the Higgs bosons are diagonal.

It should be mentioned that in [36], to get nonzero coupling ZZh at the tree level, the authors suggested the following solution: (i) $u \sim v$ or (ii) by introducing the third Higgs scalar with VEV ($\sim v$). This problem does not happen in our consideration.

After all we focused attention to the singly-charged Higgs boson $H_2^{\pm}$ with mass proportional to the bilepton mass M_Y through the coefficient λ_4. Mass of the $H_2^{\pm}$ is estimated in a range of 200 GeV. This boson, in difference with those arisen in the Higgs doublet models, does not have the hadronic and leptonic decay modes. The trilinear coupling $ZW^{\pm}H_2^{\mp}$ which

Table 10: Trilinear coupling constants of Z_μ with two neutral Higgs bosons.

Vertex	Coupling
$Z^\mu G_1 \overleftrightarrow{\partial}_\mu H$	$-\frac{g s_\zeta}{2}\left[\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)s_\theta+U_{42}c_\theta\right]\longrightarrow 0$
$Z^\mu G_2 \overleftrightarrow{\partial}_\mu H$	$\frac{g}{2}\left(-U_{12}+\frac{U_{22}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)c_\zeta\longrightarrow -\frac{g}{2c_W}$
$Z^\mu G_3 \overleftrightarrow{\partial}_\mu H$	$\frac{g s_\zeta}{2}\left[\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)c_\theta-U_{42}s_\theta\right]\longrightarrow 0$
$Z^\mu G_1 \overleftrightarrow{\partial}_\mu H_1^0$	$\frac{g c_\zeta}{2}\left[\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)s_\theta+U_{42}c_\theta\right]\longrightarrow 0$
$Z^\mu G_2 \overleftrightarrow{\partial}_\mu H_1^0$	$\frac{g}{2}\left(-U_{12}+\frac{U_{22}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)s_\zeta\longrightarrow 0$
$Z^\mu G_3 \overleftrightarrow{\partial}_\mu H_1^0$	$-\frac{g c_\zeta}{2}\left[\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)c_\theta-U_{42}s_\theta\right]\longrightarrow 0$
$Z^\mu G_1 \overleftrightarrow{\partial}_\mu G_4$	$\frac{g}{2}\left[\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)c_\theta-U_{42}s_\theta\right]\longrightarrow \frac{g}{2c_W}$
$Z^\mu G_2 \overleftrightarrow{\partial}_\mu G_4$	0
$Z^\mu G_3 \overleftrightarrow{\partial}_\mu G_4$	$\frac{g}{2}\left[\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)s_\theta+U_{42}c_\theta\right]\longrightarrow 0$

differs, at the tree level, while the similar coupling of the photon $\gamma W^\pm H_2^\mp$ as expected, vanishes. In the model under consideration, the charged Higgs boson $H_2^\pm$ with mass larger than 1700 GeV cannot be seen at the LHC. If the mass of the above-mentioned Higgs boson is in range of 200 GeV, however, the cross section can exceed 260 fb, that is, 78000 of $H_2^\pm$ can be produced at the LHC for the luminosity of 300 fb^{-1}. By measuring this process, we can obtain useful information to determine the structure of the Higgs sector.

4. Fermion masses

We first give some comments on the charged lepton masses and set conventions. The neutrino and quark masses are correspondingly considered.

4.1. Charged-lepton masses

The charged leptons ($l = e, \mu, \tau$) gain masses via the following couplings:

$$\mathcal{L}_Y^l = h_{ab}^l \overline{\psi}_{aL}\phi l_{bR} + \text{H.c.} \tag{4.1}$$

Table 11: Trilinear coupling constants of Z'_μ with two neutral Higgs bosons.

Vertex	Coupling
$Z'^\mu G_1 \overleftrightarrow{\partial}_\mu H$	$-\frac{g s_\zeta}{2}\left[\left(U_{13}+\frac{U_{23}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\right)s_\theta+U_{43}c_\theta\right]\longrightarrow 0$
$Z'^\mu G_2 \overleftrightarrow{\partial}_\mu H$	$\frac{g}{2}\left(-U_{13}+\frac{U_{23}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{33}\right)c_\zeta\longrightarrow\frac{g}{2c_W\sqrt{4c_W^2-1}}$
$Z'^\mu G_3 \overleftrightarrow{\partial}_\mu H$	$\frac{g s_\zeta}{2}\left[\left(\frac{2}{\sqrt{3}}U_{23}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\right)c_\theta-U_{43}s_\theta\right]\longrightarrow 0$
$Z'^\mu G_1 \overleftrightarrow{\partial}_\mu H_1^0$	$\frac{g c_\zeta}{2}\left[\left(U_{13}+\frac{U_{23}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\right)s_\theta+U_{43}c_\theta\right]\longrightarrow 0$
$Z'^\mu G_2 \overleftrightarrow{\partial}_\mu H_1^0$	$\frac{g}{2}\left(-U_{13}+\frac{U_{23}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{33}\right)s_\zeta\longrightarrow 0$
$Z'^\mu G_3 \overleftrightarrow{\partial}_\mu H_1^0$	$-\frac{g c_\zeta}{2}\left[\left(\frac{2}{\sqrt{3}}U_{23}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\right)c_\theta-U_{43}s_\theta\right]\longrightarrow-\frac{g c_W}{\sqrt{4c_W^2-1}}$
$Z'^\mu G_1 \overleftrightarrow{\partial}_\mu G_4$	$\frac{g}{2}\left[\left(U_{13}+\frac{U_{23}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\right)c_\theta-U_{43}s_\theta\right]\longrightarrow\frac{g c_{2W}}{2c_W\sqrt{4c_W^2-1}}$
$Z'^\mu G_2 \overleftrightarrow{\partial}_\mu G_4$	0
$Z'^\mu G_3 \overleftrightarrow{\partial}_\mu G_4$	$\frac{g}{2}\left[\left(\frac{2}{\sqrt{3}}U_{23}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\right)s_\theta+U_{43}c_\theta\right]\longrightarrow 0$

The mass matrix is, therefore, followed by

$$M_l=-\frac{v}{\sqrt{2}}\begin{pmatrix} h_{11}^l & h_{12}^l & h_{13}^l \\ h_{21}^l & h_{22}^l & h_{23}^l \\ h_{31}^l & h_{32}^l & h_{33}^l \end{pmatrix}, \tag{4.2}$$

which of course is the same as in the standard model and thus gives consistent masses for the charged leptons [37].

For the sake of simplicity, in the following, we can suppose that the Yukawa coupling of charged leptons h^l is flavor diagonal, thus l_a $(a=1,2,3)$ are mass eigenstates respective to the mass eigenvalues $m_a=-(v/\sqrt{2})h_{aa}^l$.

For convenience in further reading, we present the Yukawa interactions of (2.6) and (2.7) in terms by Feynman diagrams in Figures 6, 7, and 8, where the Hermitian adjoint ones are not displayed. The Higgs boson self-couplings are depicted in Figure 9.

4.2. Neutrino masses

First, we present mass mechanisms for the neutrinos. Next, detailed calculations and analysis of the neutrino mass spectrum are given. The experimental constraints on the coupling h^ν are also considered.

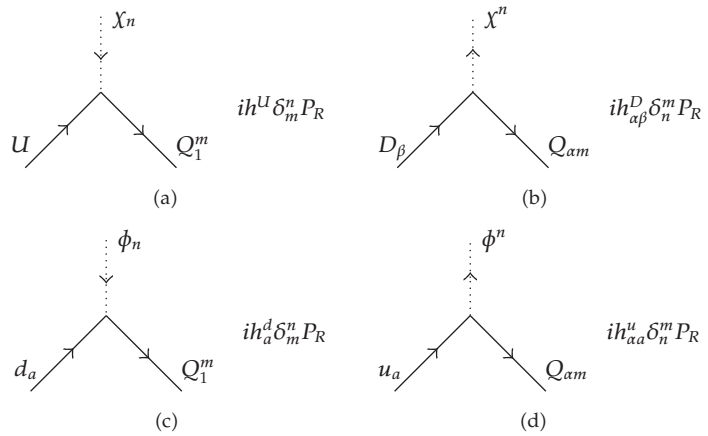


Figure 7: Relevant lepton-number conserving quark Yukawa couplings.

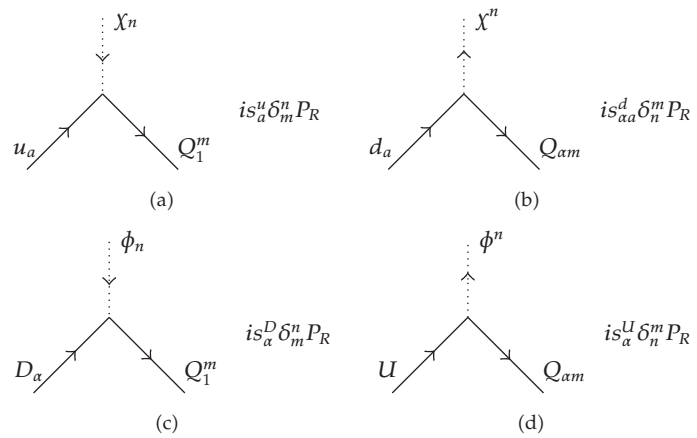


Figure 8: Lepton-number violating quark Yukawa couplings.

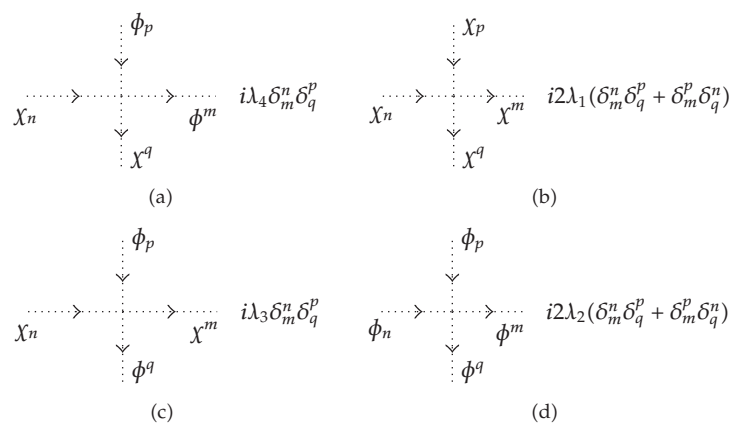


Figure 9: Higgs boson self-couplings.

Table 12: Quartic coupling constants of ZZ with two scalar bosons.

Vertex	Coupling
ZZG_1G_1	$\frac{g^2}{2}\left[\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+U_{42}^2\right]\longrightarrow\frac{g^2}{2c_W^2}$
ZZG_2G_2	$\frac{g^2}{2}\left(-U_{12}+\frac{U_{22}}{\sqrt{3}}+\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2\longrightarrow\frac{g^2}{2c_W^2}$
ZZG_3G_3	$\frac{g^2}{2}\left[\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+U_{42}^2\right]\longrightarrow 0$
ZZG_1G_3	$\frac{g^2}{2}\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)U_{42}\longrightarrow 0$
$ZZHH$	$\frac{g^2}{2}\left\{s_\zeta^2\left[s_\theta^2\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+c_\theta^2\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+U_{42}^2\right.\right.$ $\left.\left.+s_{2\theta}U_{42}\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\right]+c_\zeta^2\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2\right\}\longrightarrow\frac{g^2}{2c_W^2}$
$ZZH_1^0H_1^0$	$\frac{g^2}{2}\left\{c_\zeta^2\left[s_\theta^2\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+c_\theta^2\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+U_{42}^2\right.\right.$ $\left.\left.+s_{2\theta}U_{42}\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\right]+s_\zeta^2\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2\right\}\longrightarrow 0$
ZZG_4G_4	$\frac{g^2}{2}\left[c_\theta^2\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+s_\theta^2\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2\right.$ $\left.-s_{2\theta}\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)U_{42}+U_{42}^2\right]\longrightarrow\frac{g^2}{2c_W^2}$
$ZZHH_1$	$-\frac{g^2s_{2\zeta}}{4}\left[s_\theta^2\left(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+c_\theta^2\left(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+U_{42}^2\right.$ $\left.-\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)^2+s_{2\theta}\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)U_{42}\right]\longrightarrow 0$
$ZZHG_4$	$-\frac{g^2s_\zeta}{4}\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\left[2c_{2\theta}U_{42}+s_{2\theta}\left(U_{12}+\sqrt{3}U_{22}\right)\right]\longrightarrow 0$
ZZH_1G_4	$\frac{g^2c_\zeta}{4}\left(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\right)\left[2c_{2\theta}U_{42}+s_{2\theta}\left(U_{12}+\sqrt{3}U_{22}\right)\right]\longrightarrow 0$

Table 13: Trilinear coupling constants of ZZ with one scalar bosons.

Vertex	Coupling
ZZH	$\frac{g^2}{2}\Big[vc_\zeta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)^2-us_\zeta s_\theta\Big(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)^2-\omega\frac{s_\zeta}{c_\theta}U_{42}^2$ $-\omega s_\zeta c_\theta\Big(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)^2-2\omega s_\zeta s_\theta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)U_{42}\Big]\longrightarrow\frac{g^2v}{2c_W^2}$
ZZH_1^0	$\frac{g^2}{2}\Big[vs_\zeta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)^2+uc_\zeta s_\theta\Big(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)^2+\omega\frac{c_\zeta}{c_\theta}U_{42}^2$ $+\omega c_\zeta c_\theta\Big(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)^2+2\omega c_\zeta s_\theta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)U_{42}\Big]\longrightarrow 0$
ZZG_4	$\frac{g^2\omega}{2}\Big[s_\theta(U_{12}+\sqrt{3}U_{22})+\frac{c_{2\theta}}{c_\theta}U_{42}\Big]\Big[U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big]\longrightarrow 0$

Table 14: Trilinear coupling constants of ZZ' with one scalar bosons.

Vertex	Coupling
$ZZ'H$	$\frac{g^2}{2}\Big[vc_\zeta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)\Big(U_{13}-\frac{U_{23}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)-us_\zeta s_\theta$ $\times\Big(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)\Big(U_{13}+\frac{U_{23}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)-\omega s_\zeta c_\theta\Big(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)$ $\times\Big(\frac{2}{\sqrt{3}}U_{23}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)-\omega\frac{s_\zeta}{c_\theta}U_{42}U_{43}-\omega s_\zeta s_\theta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)U_{43}$ $-\omega s_\zeta s_\theta\Big(U_{13}-\frac{U_{23}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)U_{42}\Big]\longrightarrow\frac{g^2vc_{2W}}{2c_W\sqrt{4c_W^2-1}}$
$ZZ'H_1^0$	$\frac{g^2}{2}\Big[vs_\zeta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)\Big(U_{13}-\frac{U_{23}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)+uc_\zeta s_\theta$ $\times\Big(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)\Big(U_{13}+\frac{U_{23}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)+\omega c_\zeta c_\theta\Big(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)$ $\times\Big(\frac{2}{\sqrt{3}}U_{23}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)+\omega\frac{c_\zeta}{c_\theta}U_{42}U_{43}+\omega c_\zeta s_\theta\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)U_{43}$ $+\omega c_\zeta s_\theta\Big(U_{13}-\frac{U_{23}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)U_{42}\Big]\longrightarrow 0$
$ZZ'G_4$	$\frac{g^2\omega s_\theta}{2}\Big[\Big(U_{12}+\frac{U_{22}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)\Big(U_{13}+\frac{U_{23}}{\sqrt{3}}-\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)$ $-\Big(\frac{2}{\sqrt{3}}U_{22}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)\Big(\frac{2}{\sqrt{3}}U_{23}+\frac{t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)+\cot_{2\theta}U_{42}$ $\times\Big(U_{13}-\frac{U_{23}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{33}\Big)+\cot_{2\theta}U_{43}\Big(U_{12}-\frac{U_{22}}{\sqrt{3}}-\frac{2t}{3}\sqrt{\frac{2}{3}}U_{32}\Big)\Big]\longrightarrow 0$

Table 15: Trilinear coupling constants of neutral gauge bosons with W^+ and the charged scalar boson.

Vertex	Coupling
$AW^+G_5^-$	$\frac{g^2}{2}vs_W$
$ZW^+H_2^-$	$\frac{g^2 v\omega}{2\sqrt{\omega^2+c_\theta^2 v^2}}\left[s_\theta c_\theta\left(U_{12}+\sqrt{3}U_{22}\right)+c_{2\theta}U_{42}\right]$
$Z'W^+H_2^-$	$\frac{g^2 v\omega}{2\sqrt{\omega^2+c_\theta^2 v^2}}\left[s_\theta c_\theta\left(U_{13}+\sqrt{3}U_{23}\right)+c_{2\theta}U_{43}\right]\longrightarrow 0$
$ZW^+G_5^-$	$\frac{g^2 v}{4}\left[-s_\theta^2 U_{12}+\left(2-3s_\theta^2\right)\frac{U_{22}}{\sqrt{3}}+\frac{4t}{3}\sqrt{\frac{2}{3}}U_{32}-s_{2\theta}U_{42}\right]\longrightarrow -\frac{g^2}{2}vs_W t_W$
$ZW^+G_6^-$	$\frac{g^2\left(v^2c_\theta^2-\omega^2\right)}{8c_\theta\sqrt{\omega^2+c_\theta^2 v^2}}\left[s_{2\theta}\left(U_{12}+\sqrt{3}U_{22}\right)+2c_{2\theta}U_{42}\right]\longrightarrow 0$

Table 16: The standard model coupling constants in the effective limit.

Vertex	Coupling	Vertex	Coupling
$WWhh$	$\frac{g^2}{2}$	$G_W G_W A$	ie
WWh	$\frac{g^2}{2}v$	$WWG_Z G_Z$	$\frac{g^2}{2}$
$WG_W h$	$-\frac{ig}{2}$	$WWG_W G_W$	$\frac{g^2}{2}$
$WG_W G_Z$	$\frac{g}{2}$	ZZh	$\frac{g^2}{2c_W^2}v$
$ZZhh$	$\frac{g^2}{2c_W^2}$	$ZZG_Z G_Z$	$\frac{g^2}{2c_W^2}$
AWG_W	$\frac{g^2}{2}vs_W$	ZWG_W	$-\frac{g^2}{2}vs_W t_W$
$ZG_Z h$	$-\frac{g}{2c_W}$	$ZG_W G_W$	$\frac{ig}{2c_W}\left(1-2s_W^2\right)$

4.2.1. Neutrino mass mechanisms

In the considering model, the possible different mass mechanisms for the neutrinos can be summarized through the three dominant $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$-invariant effective

Table 17: Values of F, t_φ, and $M_{H_2^\pm}^{\max}$ for given s_θ.

s_θ	0.08	0.05	0.02	0.009	0.005
t_φ	−0.0329698	−0.0156778	−0.00598729	−0.00449063	−0.00422721
F	−0.087481	−0.0561693	−0.022803	−0.0102847	−0.00571598
$M_{H_2^\pm}^{\max}$ [GeV]	1700	1300	700	420	320

operators as follows [112, 113]:

$$O_{ab}^{\mathrm{LNC}} = \overline{\varphi}_{aL}^{c}\varphi_{bL}\phi, \tag{4.3}$$

$$O_{ab}^{\mathrm{LNV}} = (\chi^*\overline{\varphi}_{aL}^{c})(\chi^*\varphi_{bL}), \tag{4.4}$$

$$O_{ab}^{\mathrm{SLB}} = (\chi^*\overline{\varphi}_{aL}^{c})(\varphi_{bL}\phi\chi), \tag{4.5}$$

where the Hermitian adjoint operators are not displayed. It is worth noting that they are also all the performable operators with the mass dimensionality $d \leq 6$ responsible for the neutrino masses. The difference among the mass mechanisms can be verified through the operators. Both (4.3) and (4.5) conserve $\mathcal{L}$, while (4.4) violates this charge with two units. Since $d(O^{\mathrm{LNC}}) = 4$ and $L\langle\phi\rangle = 0$, (4.3) provides only Dirac masses for the neutrinos which can be obtained at the tree level through the Yukawa couplings in (2.6). Since $d(O^{\mathrm{SLB}}) = 6$ and $(L\langle\chi\rangle)_p \neq 0$ for $p = 1$ vanish for other cases, (4.5) provides both Dirac and Majorana masses for the neutrinos through radiative corrections mediated by the model particles. The masses induced by (4.3) are given by the standard $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ symmetry breaking via the VEV v. However, those by (4.5) are obtained from both the stages of $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ breaking achieved by the VEVs u, ω, and v.

Note that the LNV interactions in (2.7) are due to quarks. Hence, they do not give contribution to LNV of the leptons such as of the neutrinos. Except the LNV couplings of (2.7), all the remaining interactions of the model (lepton Yukawa couplings (2.6), Higgs self-couplings (2.11), etc.) conserve $\mathcal{L}$. This means that the operator (4.4) of LNV cannot be mediated by particles of the model; in other words, it must be introduced by hands. As a fact, the economical 3-3-1 model including the alternative versions [17–22] are only extensions beyond the standard model in the scales of orders of TeV [40, 114, 115]. Hence, it is expected that the operator in (4.4) has to be mediated by heavy particles of an underlined new physics at a scale $\mathcal{M}$ much greater than ω which have been followed in various of grand unified theories (GUTs) [37, 112, 113, 116–125]. Thus, in this model the neutrinos can get mass from three very different sources widely ranging over the mass scales: $u \sim \mathcal{O}(1)$ GeV, $v \approx 246$ GeV, $\omega \sim \mathcal{O}(1)$ TeV, and $\mathcal{M} \sim \mathcal{O}(10^{16})$ GeV.

We remind that, in the former version [20–22], the authors in [126] have considered operators of the type (4.4), however, under a discrete symmetry [22, 37]. As shown in Section 4, the current model is realistic, and such a discrete symmetry is not needed because as a fact that the model will fail if it is enforced. In addition, if such discrete symmetries are not discarded, the important mass contributions for the neutrinos mediated by model particles are then suppressed. For example, in this case the remaining operators (4.3) and (4.5) will be removed. With the only operator (4.4), the three active neutrinos will get effective zero masses under a type II seesaw [55–62] (see below). However, this operator occupies a particular importance in this version.

Alternatively, in such model, the authors in [49] have examined two-loop corrections to (4.4) by the aid of explicit LNV Higgs self-couplings, and using a fine tuning for the tree-level Dirac masses of (4.3) down to current values. However, as mentioned, this is not the case in the considering model because our Higgs potential (2.11) conserves $\mathcal{L}$. We know that one of the problems of the 3-3-1 model with RH neutrinos is associated with the Dirac mass term of neutrinos. In the following, we will show that if such a fine tuning is done to get small values for these terms, then the mass generation of neutrinos mediated by model particles is not able, or the results will be trivial. This is in contradiction with [49]. In the next, the large bare Dirac masses for the neutrinos, which are as of charged fermions of a natural result from standard symmetry breaking, will be studied.

4.2.2. Neutrino mass matrix

The operators O^{LNC}, O^{SLB}, and O^{LNV} (including their Hermitian adjoint) will provide the masses for the neutrinos: the first responsible for tree-level masses, the second for one-loop corrections, and the third for contributions of heavy particles.

Tree-Level Dirac Masses

From the Yukawa couplings in (2.6), the tree-level mass Lagrangian for the neutrinos is obtained by [127, 128]:

$$\begin{aligned}
\mathcal{L}_{\mathrm{mass}}^{\mathrm{LNC}} &= h_{ab}^{\nu}\overline{\nu}_{aR}\nu_{bL}\langle\phi_2^0\rangle - h_{ab}^{\nu}\overline{\nu}_{aL}^{c}\nu_{bR}^{c}\langle\phi_2^0\rangle + \mathrm{H.c.} \\
&= 2\langle\phi_2^0\rangle h_{ab}^{\nu}\overline{\nu}_{aR}\nu_{bL} + \mathrm{H.c.} \\
&= -\left(M_D\right)_{ab}\overline{\nu}_{aR}\nu_{bL} + \mathrm{H.c.} \\
&= -\frac{1}{2}\left(\overline{\nu}_{aL}^{c},\ \overline{\nu}_{aR}\right)\begin{pmatrix} 0 & \left(M_D^T\right)_{ab} \\ \left(M_D\right)_{ab} & 0 \end{pmatrix}\begin{pmatrix} \nu_{bL} \\ \nu_{bR}^{c} \end{pmatrix} + \mathrm{H.c.} \\
&= -\frac{1}{2}\overline{X}_L^{c}M_{\nu}X_L + \mathrm{H.c.},
\end{aligned} \tag{4.6}$$

where $h_{ab}^{\nu} = -h_{ba}^{\nu}$ is due to Fermi statistics. The M_D is the mass matrix for the Dirac neutrinos:

$$\left(M_D\right)_{ab} \equiv -\sqrt{2}vh_{ab}^{\nu} = \left(-M_D^T\right)_{ab} = \begin{pmatrix} 0 & -A & -B \\ A & 0 & -C \\ B & C & 0 \end{pmatrix}, \tag{4.7}$$

where

$$A \equiv \sqrt{2}h_{e\mu}^{\nu}v, \qquad B \equiv \sqrt{2}h_{e\tau}^{\nu}v, \qquad C \equiv \sqrt{2}h_{\mu\tau}^{\nu}v. \tag{4.8}$$

This mass matrix has been rewritten in a general basis $X_L^T \equiv (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}, \nu_{eR}^{c}, \nu_{\mu R}^{c}, \nu_{\tau R}^{c})$:

$$M_{\nu} \equiv \begin{pmatrix} 0 & M_D^T \\ M_D & 0 \end{pmatrix}. \tag{4.9}$$

The tree-level neutrino spectrum, therefore, consists of only Dirac fermions. Since h^{ν}_{ab} is antisymmetric in a and b, the mass matrix M_D gives one neutrino massless and two others degenerate in mass 0, $-m_D$, m_D, where $m_D \equiv (A^2+B^2+C^2)^{1/2}$. This mass spectrum is not realistic under the data; however, it will be severely changed by the quantum corrections; the most general mass matrix can then be written as follows:

$$M_\nu = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}, \tag{4.10}$$

where $M_{L,R}$ (vanished at the tree level) and M_D get possible corrections.

If such a tree-level contribution dominates the resulting mass matrix (after corrections), the model will provide an explanation about a large splitting either $\Delta m^2_{\text{atm}} \gg \Delta m^2_{\text{sol}}$ or $\Delta m^2_{\text{LSND}} \gg \Delta m^2_{\text{atm, sol}}$ [3] (see also [49]). Hence, we need a fine-tuning at the tree level [49] either $m_D \sim (\Delta m^2_{\text{atm}})^{1/2} (\sim 5\times 10^{-2}\,\text{eV})$ or $m_D \sim (\Delta m^2_{\text{LSND}})^{1/2} (\sim \text{eV})$ [3]. Without loss of generality, assuming that $h^{\nu}_{e\mu} \sim h^{\nu}_{e\tau} \sim h^{\nu}_{\mu\tau}$, we get then $h^{\nu} \sim 10^{-13}$ (or 10^{-12}). The coupling h^{ν} in this case is so small and, therefore, this fine tuning is not natural [129, 130]. Indeed, as shown below, since h^{ν} enters the dominant corrections from (4.5) for $M_{L,R}$, these terms $M_{L,R}$ get very small values which are not large enough to split the degenerate neutrino masses into a realistic spectrum. (The largest degenerate splitting in squared mass is still much smaller than $\Delta m^2_{\text{sol}} \sim 8\times 10^{-5}\text{eV}^2$ [3].) In addition, in this case, the Dirac masses get corrections trivially.

The above problem can be solved *just* by the LNV operator (4.4), and then the operator (4.5) obtaining the contributions from particles in the model is suppressed (for details, see [126]). However, we do not consider the above solution in this work. This implies that the tree-level Dirac mass term for the neutrinos by its naturalness should be treated as those as of the usual charged fermions resulted of the standard symmetry breaking, say, $h^{\nu} \sim h^{e}$ ($\sim 10^{-6}$) [129, 130]. It turns out that this term is regarded as a large bare quantity and unphysical. Under the interactions, they will of course change to physical masses. In the following, we will obtain such *finite renormalizations* (for more details, see [131]) in the masses of neutrinos.

One-Loop Level Dirac and Majorana masses

The operator (4.5) and its Hermitian adjoint arise from the radiative corrections mediated by the model particles and give contributions to Majorana and Dirac mass terms M_L, M_R, and M_D for the neutrinos. The Yukawa couplings of the leptons in (2.6) and the relevant Higgs self-couplings in (2.11) are explicitly rewritten as follows:

$$\begin{aligned}
\mathcal{L}^{\text{lept}}_{\text{Y}} &= 2h^{\nu}_{ab}\overline{\nu}^{c}_{aL} l_{bL}\phi^{+}_{3} - 2h^{\nu}_{ab}\overline{\nu}_{aR} l_{bL}\phi^{+}_{1} + h^{l}_{ab}\overline{\nu}_{aL} l_{bR}\phi^{+}_{1} + h^{l}_{ab}\overline{\nu}^{c}_{aR} l_{bR}\phi^{+}_{3} + h^{l}_{ab}\bar{l}_{aL} l_{bR}\phi^{0}_{2} + \text{H.c.},\\
\mathcal{L}^{\text{relv}}_{\text{H}} &= \lambda_3\phi^{-}_{1}\phi^{+}_{1}\left(\chi^{0*}_{1}\chi^{0}_{1} + \chi^{0*}_{3}\chi^{0}_{3}\right) + \lambda_3\phi^{-}_{3}\phi^{+}_{3}\left(\chi^{0*}_{1}\chi^{0}_{1} + \chi^{0*}_{3}\chi^{0}_{3}\right) + \lambda_4\phi^{-}_{1}\phi^{+}_{1}\chi^{0*}_{1}\chi^{0}_{1} + \lambda_4\phi^{-}_{3}\phi^{+}_{3}\chi^{0*}_{3}\chi^{0}_{3}\\
&\quad + \lambda_4\phi^{-}_{3}\phi^{+}_{1}\chi^{0*}_{1}\chi^{0}_{3} + \lambda_4\phi^{-}_{1}\phi^{+}_{3}\chi^{0*}_{3}\chi^{0}_{1}.
\end{aligned} \tag{4.11}$$

The one-loop corrections to the mass matrices M_L of ν_L, M_R of ν_R, and M_D of ν are, therefore, given in Figures 10, 11, and 12, respectively.

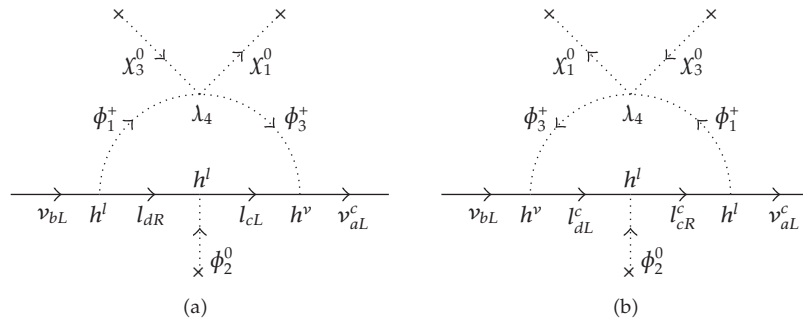


Figure 10: The one-loop corrections for the mass matrix M_L.

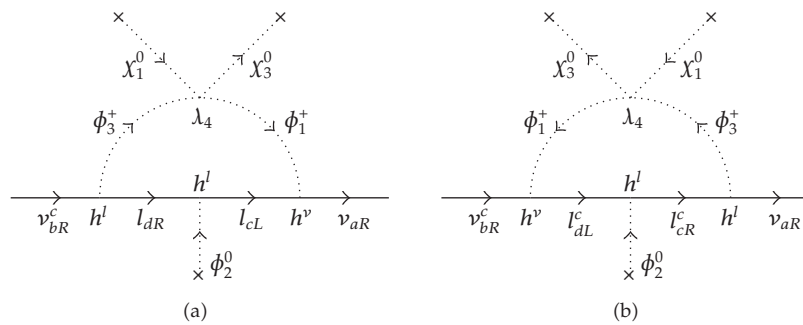


Figure 11: The one-loop corrections for the mass matrix M_R.

Radiative Corrections to M_L and M_R

With the Feynman rules at hand [127, 128], M_L is obtained by

$$\begin{aligned}
-i(M_L)_{ab}P_L = &\int \frac{d^4p}{(2\pi)^4}(i2h_{ac}^\nu P_L)\frac{i(\not{p}+m_c)}{p^2-m_c^2}\left(ih_{cd}^l\frac{v}{\sqrt{2}}P_R\right)\frac{i(\not{p}+m_d)}{p^2-m_d^2} \\
&\times (ih_{bd}^{l*}P_L)\frac{-1}{(p^2-m_{\phi_1}^2)(p^2-m_{\phi_3}^2)}\left(i\lambda_4\frac{u\omega}{2}\right) \\
&+\int \frac{d^4p}{(2\pi)^4}(ih_{ac}^{l*}P_L)\frac{i(-\not{p}+m_c)}{p^2-m_c^2}\left(ih_{dc}^l\frac{v}{\sqrt{2}}P_R\right)\frac{i(-\not{p}+m_d)}{p^2-m_d^2} \\
&\times (i2h_{bd}^\nu P_L)\frac{-1}{(p^2-m_{\phi_1}^2)(p^2-m_{\phi_3}^2)}\left(i\lambda_4\frac{u\omega}{2}\right).
\end{aligned} \tag{4.12}$$

Because the Yukawa couplings of the charged leptons are flavor diagonal, (4.12) becomes

$$(M_L)_{ab} = \frac{i\sqrt{2}\lambda_4 u\omega}{v}h_{ab}^\nu\left[m_b^2 I(m_b^2, m_{\phi_3}^2, m_{\phi_1}^2) - m_a^2 I(m_a^2, m_{\phi_3}^2, m_{\phi_1}^2)\right], \quad (a, b \text{ not summed}), \tag{4.13}$$

where the integral $I(a,b,c)$ is given in Appendix B.

In the effective approximation (2.8), identifications are given by $\phi_3^\pm \sim H_2^\pm$ and $\phi_1^\pm \sim G_W^\pm$ [39], where $H_2^\pm$ and $G_W^\pm$ as above mentioned are the charged bilepton Higgs boson and the Goldstone boson associated with $W^\pm$ boson, respectively. For the masses, we have also $m_{\phi_3}^2 \simeq m_{H_2}^2 (\simeq (\lambda_4/2)\omega^2)$ and $m_{\phi_1}^2 \simeq 0$. Using (B.5), the integrals are given by

$$I\left(m_a^2, m_{\phi_3}^2, m_{\phi_1}^2\right) \simeq -\frac{i}{16\pi^2}\frac{1}{m_a^2 - m_{H_2}^2}\left[1 - \frac{m_{H_2}^2}{m_a^2 - m_{H_2}^2}\ln\frac{m_a^2}{m_{H_2}^2}\right], \quad a = e, \mu, \tau. \tag{4.14}$$

Consequently, the mass matrix (4.13) becomes

$$\begin{aligned}(M_L)_{ab} &\simeq \frac{\sqrt{2}\lambda_4 u\omega h_{ab}^\nu}{16\pi^2 v}\left[\frac{m_{H_2}^2\left(m_a^2 - m_b^2\right)}{\left(m_b^2 - m_{H_2}^2\right)\left(m_a^2 - m_{H_2}^2\right)} + \frac{m_a^2 m_{H_2}^2}{\left(m_a^2 - m_{H_2}^2\right)^2}\ln\frac{m_a^2}{m_{H_2}^2} - \frac{m_b^2 m_{H_2}^2}{\left(m_b^2 - m_{H_2}^2\right)^2}\ln\frac{m_b^2}{m_{H_2}^2}\right] \\ &\simeq \frac{\sqrt{2}\lambda_4 u\omega h_{ab}^\nu}{16\pi^2 v m_{H_2}^2}\left[m_a^2\left(1 + \ln\frac{m_a^2}{m_{H_2}^2}\right) - m_b^2\left(1 + \ln\frac{m_b^2}{m_{H_2}^2}\right)\right],\end{aligned} \tag{4.15}$$

where the last approximation (4.15) is kept in the orders up to $\mathcal{O}[(m_{a,b}^2/m_{H_2}^2)^2]$. Since $m_{H_2}^2 \simeq (\lambda_4/2)\omega^2$, it is worth noting that the resulting M_L is not explicitly dependent on λ_4, however, proportional to $t_\theta = u/\omega$ (the mixing angle between the W boson and the singly-charged bilepton gauge boson Y [38]), $\sqrt{2}vh_{ab}^\nu$ (the tree-level Dirac mass term of neutrinos), and m_{H_2} in the logarithm scale. Here, the VEV $v \approx v_{\text{weak}}$ and the charged-lepton masses m_a $(a = e, \mu, \tau)$ have the well-known values. Let us note that M_L is symmetric and has vanishing diagonal elements.

For the corrections to M_R, it is easily to check that the relationship $(M_R)_{ab} = -(M_L)_{ab}$ is exact at the one-loop level. (This result can be derived from Figure 11 in a general case without imposing any additional condition on h^l, h^ν, and further.) Combining this result with (4.15), the mass matrices are explicitly rewritten as follows:

$$(M_L)_{ab} = -(M_R)_{ab} \simeq \begin{pmatrix} 0 & f & r \\ f & 0 & t \\ r & t & 0 \end{pmatrix}, \tag{4.16}$$

where the elements are obtained by

$$\begin{aligned} f &\equiv \left(\sqrt{2}vh_{e\mu}^\nu\right)\left\{\left(\frac{t_\theta}{8\pi^2 v^2}\right)\left[m_e^2\left(1 + \ln\frac{m_e^2}{m_{H_2}^2}\right) - m_\mu^2\left(1 + \ln\frac{m_\mu^2}{m_{H_2}^2}\right)\right]\right\}, \\ r &\equiv \left(\sqrt{2}vh_{e\tau}^\nu\right)\left\{\left(\frac{t_\theta}{8\pi^2 v^2}\right)\left[m_e^2\left(1 + \ln\frac{m_e^2}{m_{H_2}^2}\right) - m_\tau^2\left(1 + \ln\frac{m_\tau^2}{m_{H_2}^2}\right)\right]\right\}, \\ t &\equiv \left(\sqrt{2}vh_{\mu\tau}^\nu\right)\left\{\left(\frac{t_\theta}{8\pi^2 v^2}\right)\left[m_\mu^2\left(1 + \ln\frac{m_\mu^2}{m_{H_2}^2}\right) - m_\tau^2\left(1 + \ln\frac{m_\tau^2}{m_{H_2}^2}\right)\right]\right\}. \end{aligned} \tag{4.17}$$

It can be checked that f, r, t are much smaller than those of M_D. To see this, we can take $m_e \simeq 0.51099\,\text{MeV}$, $m_\mu \simeq 105.65835\,\text{MeV}$, $m_\tau \simeq 1777\,\text{MeV}$, $v \simeq 246\,\text{GeV}$, $u \simeq 2.46\,\text{GeV}$, $\omega \simeq 3000\,\text{GeV}$, and $m_{H_2} \simeq 700\,\text{GeV}$ $(\lambda_4 \sim 0.11)$ [38–40], which give us then

$$f \simeq \left(\sqrt{2}vh^{\nu}_{e\mu}\right)\left(3.18\times 10^{-11}\right), \qquad r \simeq \left(\sqrt{2}vh^{\nu}_{e\tau}\right)\left(5.93\times 10^{-9}\right), \qquad t \simeq \left(\sqrt{2}vh^{\nu}_{\mu\tau}\right)\left(5.90\times 10^{-9}\right), \tag{4.18}$$

where the second factors rescale negligibly with $\omega \sim 1$–$10\,\text{TeV}$ and $m_{H_2} \sim 200$–$2000\,\text{GeV}$. This thus implies that

$$\frac{\left|M_{L,R}\right|}{\left|M_D\right|} \sim 10^{-9}, \tag{4.19}$$

which can be checked with the help of $|M| \equiv \left(M^{\dagger}M\right)^{1/2}$. In other words, the constraint is given as follows

$$\left|M_{L,R}\right| \ll \left|M_D\right|. \tag{4.20}$$

With the above results at hand, we can then get the masses by studying diagonalization of the mass matrix (4.10), in which, the submatrices $M_{L,R}$ and M_D satisfying the constraint (4.20) are given by (4.16) and (4.7), respectively. In calculation, let us note that since M_D has one vanishing eigenvalue, M_ν does not possess the pseudo-Dirac property in all three generations [132], despite being very close to those because the remaining eigenvalues do. As a fact, we will see that M_ν contains a combined framework of the seesaw [55–62] and the pseudo-Dirac [132–142]. To get mass, we can suppose that h^ν is real and, therefore, the matrix iM_D is Hermitian: $(iM_D)^\dagger = iM_D$ (4.7). The Hermitianity for $M_{L,R}$ is also followed by (4.16). Because the dominant matrix is M_D (4.20), we first diagonalize it by biunitary transformation [131]:

$$\overline{\nu}_{aR} = \overline{\nu}_{iR}(-iU)^{\dagger}_{ia}, \quad \nu_{bL} = U_{bj}\nu_{jL}, \quad (i,j = 1,2,3), \tag{4.21}$$

$$M_{\text{diag}} \equiv \text{diag}\left(0,-m_D,m_D\right) = (-iU)^{\dagger}M_D U, \quad m_D = \sqrt{A^2+B^2+C^2}, \tag{4.22}$$

where the matrix U is easily obtained by

$$U = \frac{1}{m_D\sqrt{2\left(A^2+C^2\right)}}\begin{pmatrix} C\sqrt{2\left(A^2+C^2\right)} & iBC - Am_D & BC - iAm_D \\ -B\sqrt{2\left(A^2+C^2\right)} & i\left(A^2+C^2\right) & \left(A^2+C^2\right) \\ A\sqrt{2\left(A^2+C^2\right)} & iAB + Cm_D & AB + iCm_D \end{pmatrix}. \tag{4.23}$$

Resulted by the anti-Hermitianity of M_D, it is worth noting that M_ν in the case of vanishing $M_{L,R}$ (4.9) is indeed diagonalized by the following unitary transformation:

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} U & U \\ -iU & iU \end{pmatrix}. \tag{4.24}$$

A new basis $(\nu_1,\nu_2,\ldots,\nu_6)^T_L \equiv V^\dagger X^T_L$, which is different from $(\nu_{jL},\nu^c_{iR})^T$ of (4.21), is therefore performed. The neutrino mass matrix (4.10) in this basis becomes

$$V^\dagger M_\nu V = \begin{pmatrix} M_{\text{diag}} & \epsilon \\ \epsilon & -M_{\text{diag}} \end{pmatrix}, \tag{4.25}$$

$$\epsilon \equiv U^\dagger M_L U, \; \epsilon^\dagger = \epsilon, \tag{4.26}$$

where the elements of ϵ are obtained by

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0, \tag{4.27}$$

$$\begin{aligned}\epsilon_{12} = i\epsilon_{13}^{*} = \{&[ABm_D + iC(A^2 - B^2 + C^2)]f + [(C^2 - A^2)m_D + 2iABC]r \\ &+ [iA(A^2 - B^2 + C^2) - BCm_D]t\}\left[m_D^2\sqrt{2(A^2 + C^2)}\right]^{-1},\end{aligned} \tag{4.28}$$

$$\begin{aligned}\epsilon_{23} = \{&(A^2 + C^2)[(Cm_D - iAB)t - (Am_D + iBC)f] \\ &- [B(A^2 - C^2)m_D + iAC(A^2 + 2B^2 + C^2)]r\}\left[m_D^2(A^2 + C^2)\right]^{-1}.\end{aligned} \tag{4.29}$$

Let us remind the reader that (4.27) is exactly given at the one-loop level M_L (4.13) without imposing any approximation on this mass matrix. Interchanging the positions of component fields in the basis $(\nu_1, \nu_2, \ldots, \nu_6)_L^T$ by a permutation transformation $P^\dagger \equiv P_{23}P_{34}$, that is, $\nu_p \to (P^\dagger)_{pq}\nu_q$ $(p, q = 1, 2, \ldots, 6)$ with

$$P^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.30}$$

the mass matrix (4.25) can be rewritten as follows:

$$P^\dagger(V^\dagger M_\nu V)P = \left(\begin{array}{cc|cccc} 0 & 0 & 0 & 0 & \epsilon_{12} & \epsilon_{13} \\ 0 & 0 & \epsilon_{12} & \epsilon_{13} & 0 & 0 \\ \hline 0 & \epsilon_{21} & -m_D & 0 & 0 & \epsilon_{23} \\ 0 & \epsilon_{31} & 0 & m_D & \epsilon_{32} & 0 \\ \epsilon_{21} & 0 & 0 & \epsilon_{23} & m_D & 0 \\ \epsilon_{31} & 0 & \epsilon_{32} & 0 & 0 & -m_D \end{array}\right). \tag{4.31}$$

It is worth noting that in (4.31), all the off-diagonal components $|\epsilon|$ are much smaller than the eigenvalues $|\pm m_D|$ due to Condition (4.20). The degenerate eigenvalues 0, $-m_D$, and $+m_D$ (each twice) are now splitting into three pairs with six different values, two light and four heavy. The two neutrinos of first pair resulted by the 0 splitting have very small masses as a result of exactly what a seesaw does [55–62], that is, the off-diagonal block contributions to these masses are suppressed by the large pseudo-Dirac masses of the lower-right block. The suppression in this case is different from the usual ones [55–62] because it needs only the pseudo-Dirac particles [132–142] with the masses m_D of the electroweak scale instead of extremely heavy RH Majorana fields, and that the Dirac masses in those mechanisms are now played by loop-induced f, r, t (4.17) as a result of the SLB u/ω. Therefore, the mass matrix (4.31) is effectively decomposed into M_{S} for the first pair of light neutrinos (ν_{S}) and M_{P} for the last two pairs of heavy pseudo-Dirac neutrinos (ν_{P}):

$$\begin{gathered}(\nu_1, \nu_4, \nu_2, \nu_3, \nu_5, \nu_6)_L^T \longrightarrow (\nu_{\mathrm{S}}, \nu_{\mathrm{P}})_L^T = V_{\mathrm{eff}}^\dagger(\nu_1, \nu_4, \nu_2, \nu_3, \nu_5, \nu_6)_L^T, \\ V_{\mathrm{eff}}^\dagger(P^\dagger V^\dagger M_\nu V P)V_{\mathrm{eff}} = \mathrm{diag}(M_{\mathrm{S}}, M_{\mathrm{P}}),\end{gathered} \tag{4.32}$$

Table 18: The values for h^ν and two largest splittings in squared mass.

Fine tuning	h^ν	m_D^2 (eV^2)	$4m_D\lvert\epsilon_{23}\rvert$ (eV^2)
$m_D^2 \sim \Delta m^2_{\text{atm}}$	8.30×10^{-14}	2.50×10^{-3}	3.95×10^{-11}
$m_D^2 \sim \Delta m^2_{\text{LSND}}$	1.66×10^{-12}	1.00	1.58×10^{-8}

where V_{eff}, M_{S}, and M_{P} get the approximations:

$$V_{\text{eff}} \simeq \begin{pmatrix} 1 & \mathcal{E} \\ -\mathcal{E}^+ & 1 \end{pmatrix}, \quad \mathcal{E} \equiv \begin{pmatrix} 0 & 0 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{13} & 0 & 0 \end{pmatrix} \begin{pmatrix} -m_D & 0 & 0 & \epsilon_{23} \\ 0 & m_D & \epsilon_{32} & 0 \\ 0 & \epsilon_{23} & m_D & 0 \\ \epsilon_{32} & 0 & 0 & -m_D \end{pmatrix}^{-1},$$
$$M_{\text{S}} \simeq -\mathcal{E} \begin{pmatrix} 0 & \epsilon_{21} \\ 0 & \epsilon_{31} \\ \epsilon_{21} & 0 \\ \epsilon_{31} & 0 \end{pmatrix}, \qquad M_{\text{P}} \simeq \begin{pmatrix} -m_D & 0 & 0 & \epsilon_{23} \\ 0 & m_D & \epsilon_{32} & 0 \\ 0 & \epsilon_{23} & m_D & 0 \\ \epsilon_{32} & 0 & 0 & -m_D \end{pmatrix}. \tag{4.33}$$

The mass matrices M_{S} and M_{P}, respectively, give exact eigenvalues as follows:

$$m_{\text{S}\pm} = \pm\frac{2\text{Im}(\epsilon_{13}\epsilon_{13}\epsilon_{32})}{m_D^2 - \epsilon_{23}^2} \simeq \pm 2\text{Im}\left(\frac{\epsilon_{13}\epsilon_{13}\epsilon_{32}}{m_D^2}\right), \tag{4.34}$$

$$m_{\text{P}\pm} = -m_D \pm |\epsilon_{23}|, \qquad m_{\text{P}'\pm} = m_D \pm |\epsilon_{23}|. \tag{4.35}$$

In this case, the mixing matrices are collected into $(\nu_{\text{S}\pm}, \nu_{\text{P}\pm}, \nu_{\text{P}'\pm})_L^T = V_\pm^\dagger(\nu_{\text{S}}, \nu_{\text{P}})_L^T$, where the $V_\pm$ is obtained by

$$V_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \kappa & -\kappa \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad \kappa \equiv \frac{\epsilon_{23}}{|\epsilon_{23}|} = \exp(i \arg \epsilon_{23}). \tag{4.36}$$

It is to be noted that the degeneration in the Dirac one $|\pm m_D|$ is now splitting severally.

From (4.35), we see that the four large pseudo-Dirac masses for the neutrinos are almost degenerate. In addition, the resulting spectrum (4.34), (4.35) yields two largest squared mass splittings, respectively, proportional to m_D^2 and $4m_D|\epsilon_{23}|$. From (4.29) and (4.18), we can evaluate $|\epsilon_{23}| \simeq 3.95 \times 10^{-9}\, m_D \ll m_D$ (where $A \sim B \sim C \sim m_D/\sqrt{3}$ is understood). Because the splitting $4m_D|\epsilon_{23}|$ is still much smaller than Δm^2_{sol}, this therefore implies that the fine tuning, as mentioned, is not realistic. (In detail, in Table 18, we give the numerical values of these fine tunings, where the parameters are given as before (4.18).)

Similarly, for the two small masses, we can also evaluate $|m_{\text{S}\pm}| \simeq 4.29 \times 10^{-28}\, m_D$. This shows that the masses $m_{\text{S}\pm}$ are very much smaller than the splitting $|\epsilon_{23}|$. This also implies that the two light neutrinos in this case are hidden for any m_D value of pseudo-Dirac neutrinos. Let us see the sources of the problem why these masses are so small: (i) vanishing of all the elements of left-upper block of (4.31); (ii) in (4.34) the resulting masses are proportional

to $|\epsilon|^3/m_D^2$, but not to $|\epsilon|^2/m_D$ as expected from (4.31). It turns out that this is due to the antisymmetric of h^ν_{ab} enforcing on the tree-level Dirac-mass matrix and the degenerate of $M_R = -M_L$ of the one-loop level left-handed (LH) and RH Majorana-mass matrices. It can be easily checked that such degeneration in Majorana masses remains up to higher-order radiative corrections as a result of treating the LH and RH neutrinos in the same gauge triplets with the model Higgs content. For example, by the aid of (4.5) the degeneration retains up to any higher-order loop.

Radiative corrections to M_D

As mentioned, the mass matrix M_D requires the one-loop corrections as given in Figure 12, and the contributions are easily obtained as follows:

$$\begin{aligned}
-i\left(M_D^{\text{rad}}\right)_{ab}P_L = &\int \frac{d^4p}{(2\pi)^4}\left(-i2h^\nu_{ac}P_L\right)\frac{i(\not{p}+m_c)}{p^2-m_c^2}\left(ih^l_{cd}\frac{v}{\sqrt{2}}P_R\right)\frac{i(\not{p}+m_d)}{p^2-m_d^2}\\
&\times\left(ih^{l*}_{bd}P_L\right)\frac{-1}{\left(p^2-m_{\phi_1}^2\right)^2}\left(i\lambda_3\frac{u^2+\omega^2}{2}+i\lambda_4\frac{u^2}{2}\right)\\
&+\int\frac{d^4p}{(2\pi)^4}\left(ih^{l*}_{ac}P_L\right)\frac{i(-\not{p}+m_c)}{p^2-m_c^2}\left(ih^l_{dc}\frac{v}{\sqrt{2}}P_R\right)\frac{i(-\not{p}+m_d)}{p^2-m_d^2}\\
&\times\left(i2h^\nu_{bd}P_L\right)\frac{-1}{\left(p^2-m_{\phi_3}^2\right)^2}\left(i\lambda_3\frac{u^2+\omega^2}{2}+i\lambda_4\frac{\omega^2}{2}\right).
\end{aligned}\tag{4.37}$$

We rewrite

$$\begin{aligned}
\left(M_D^{\text{rad}}\right)_{ab} = -\frac{i\sqrt{2}h^\nu_{ab}}{v}\Big\{&\left[\lambda_3\left(u^2+\omega^2\right)+\lambda_4u^2\right]m_b^2I\left(m_b^2,m_{\phi_1}^2\right)\\
&+\left[\lambda_3\left(u^2+\omega^2\right)+\lambda_4\omega^2\right]m_a^2I\left(m_a^2,m_{\phi_3}^2\right)\Big\},\quad (a,b \text{ not summed}),
\end{aligned}\tag{4.38}$$

where $I(a,b)$ is given in (B.13). With the help of (B.14), the approximation for (4.38) is obtained by

$$\begin{aligned}
\left(M_D^{\text{rad}}\right)_{ab} &\simeq -\frac{h^\nu_{ab}}{8\sqrt{2}\pi^2v}\left\{\left[\lambda_3\left(u^2+\omega^2\right)+\lambda_4u^2\right]+\left[\lambda_3\left(u^2+\omega^2\right)+\lambda_4\omega^2\right]\frac{m_a^2}{m_{H_2}^2}\right\}\\
&= -\sqrt{2}h^\nu_{ab}\left(\frac{\lambda_3\omega^2}{16\pi^2v}\right)\left[1+\left(1+\frac{\lambda_4}{\lambda_3}\right)\left(\frac{u^2}{\omega^2}+\frac{m_a^2}{m_{H_2}^2}\right)+\mathcal{O}\left(\frac{u^4}{\omega^4},\frac{m_{a,b}^4}{m_{H_2}^4}\right)\right].
\end{aligned}\tag{4.39}$$

Because of the constraint (2.8), the higher-order corrections $\mathcal{O}(\cdots)$ can be neglected; thus M_D^{rad} is rewritten as follows:

$$\left(M_D^{\text{rad}}\right)_{ab} = -\sqrt{2}h^\nu_{ab}\left(\frac{\lambda_3\omega^2}{16\pi^2v}\right)(1+\delta_a),\quad \delta_a \equiv \left(1+\frac{\lambda_4}{\lambda_3}\right)\left(\frac{u^2}{\omega^2}+\frac{m_a^2}{m_{H_2}^2}\right),\tag{4.40}$$

where δ_a is of course an infinitesimal coefficient, that is, $|\delta| \ll 1$. Again, this implies also that if the fine tuning is done the resulting Dirac-mass matrix get trivially. It is due to the fact that the contribution of the term associated with δ_a in (4.40) is then very small and neglected;

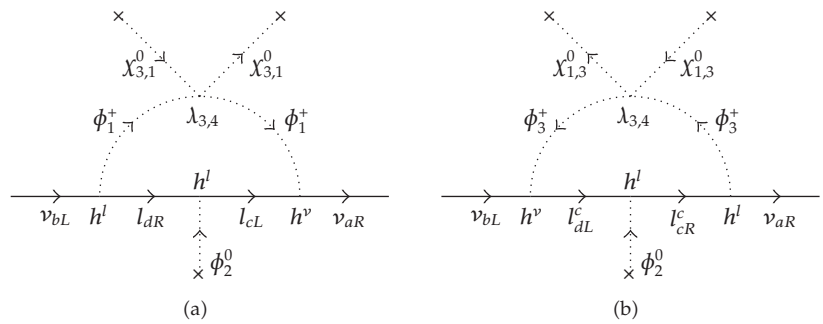


Figure 12: The one-loop corrections for the mass matrix M_D.

the remaining term gives an antisymmetric resulting Dirac-mass matrix, that is, therefore, unrealistic under the data.

With this result, it is worth noting that the scale

$$\left|\frac{\lambda_3\omega^2}{16\pi^2 v}\right| \tag{4.41}$$

of the radiative Dirac masses (4.40) is in the orders of the scale v of the tree-level Dirac masses (4.7). Indeed, if one puts $|(\lambda_3\omega^2)/(16\pi^2 v)| = v$ and takes $|\lambda_3| \sim 0.1$–1, then $\omega \sim 3$–10 TeV as expected in the constraints [40, 114, 115]. The resulting Dirac-mass matrix which is combined of (4.7) and (4.40), therefore, gets two typical examples of the bounds: (i) $(\lambda_3\omega^2)/(16\pi^2 v) + v \sim \mathcal{O}(v)$; (ii) $(\lambda_3\omega^2)/(16\pi^2 v) + v \sim \mathcal{O}(0)$. The first case (i) yields that the status on the masses of neutrinos as given above is remained unchanged and therefore is also trivial as mentioned. In the last case (ii), the combination of (4.7) and (4.40) gives

$$(M_D)_{ab} = \sqrt{2}h^\nu_{ab}(v\delta_a). \tag{4.42}$$

It is interesting that in this case the scale v for the Dirac masses (4.7) gets naturally a large reduction, and we argue that this is not a fine tuning. Because the large radiative mass term in (4.40) is canceled by the tree-level Dirac masses, we mean this as a finite renormalization in the masses of neutrinos. It is also noteworthy that, unlike the case of the tree-level mass term (4.7), the mass matrix (4.42) is now *nonantisymmetric* in a and b. Among the three eigenvalues of this matrix, we can check that one vanishes (since $\det M_D = 0$) and two others massive are now *nondegenerate* (splitting). Let us recall that in the first case (i) the degeneration of the two nonzero eigenvalues are, however, retained because the combination of (4.7) and (4.40) is proportional to $h^\nu_{ab}v$.

In contrast to (4.19), in this case there is no large hierarchy between $M_{L,R}$ and M_D. To see this explicitly, let us take the values of the parameters as given before (4.18), thus $\lambda_3 \simeq -1.06$ and the coefficients δ_a are evaluated by

$$\delta_e \simeq 6.03\times 10^{-7}, \qquad \delta_\mu \simeq 6.23\times 10^{-7}, \qquad \delta_\tau \simeq 6.28\times 10^{-6}. \tag{4.43}$$

Hence, we get

$$\frac{|M_{L,R}|}{|M_D|} \sim 10^{-2}\text{–}10^{-3}. \tag{4.44}$$

With the values given in (4.43), the quantities h^ν and m_D can be evaluated through the mass term (4.42); the neutrino data imply that h^ν and m_D are *in the orders of* h^e *and* m_e—the Yukawa coupling and mass of electron, respectively.

Because of the Condition (4.44) and the vanishing of one eigenvalue of M_D, we can repeat the procedure as given above to diagonalize the full matrix M_ν with M_D given by (4.42) and $M_{L,R}$ by (4.16). First, we can easily find a mixing matrix V as in (4.24); Second, in the new basis we obtain the seesaw form as in (4.31); Finally, the resulting mixing matrix and masses for the neutrinos are derived. It is worth checking that the two largest squared mass splittings as given before can be approximately applied on this case of (4.44) such as $(m_D|\delta|)^2$ and $4(m_D|\delta|)|\epsilon|$, and seeing that they fit *naturally* the data.

Mass contributions from heavy particles

There remain now two questions not yet answered: (i) the degeneration of $M_R = -M_L$; (ii) the hierarchy of $M_{L,R}$ and M_D (4.44) can be continuously reduced? As mentioned, we will prove that the new physics gives us the solution.

The mass Lagrangian for the neutrinos given by the operator (4.4) can be explicitly written as follows:

$$\begin{aligned}\mathcal{L}_{\text{mass}}^{\text{LNV}} &= s_{ab}^{\nu}\mathcal{M}^{-1}\left(\langle\chi^{\dagger}\rangle\overline{\varphi}_{aL}^{c}\right)\left(\langle\chi^{\dagger}\rangle\varphi_{bL}\right) + \text{H.c.}\\ &= s_{ab}^{\nu}\mathcal{M}^{-1}\left(\frac{u}{\sqrt{2}}\overline{\nu}_{aL}^{c} + \frac{\omega}{\sqrt{2}}\overline{\nu}_{aR}\right)\left(\frac{u}{\sqrt{2}}\nu_{bL} + \frac{\omega}{\sqrt{2}}\nu_{bR}^{c}\right) + \text{H.c.}\\ &= -\frac{1}{2}\overline{\mathrm{X}}_{L}^{c}M_{\nu}^{\text{new}}\mathrm{X}_{L} + \text{H.c.},\end{aligned} \tag{4.45}$$

where the mass matrix for the neutrinos is obtained by

$$M_{\nu}^{\text{new}} \equiv -\begin{pmatrix} \dfrac{u^2}{\mathcal{M}}s^{\nu} & \dfrac{u\omega}{\mathcal{M}}s^{\nu} \\ \dfrac{u\omega}{\mathcal{M}}s^{\nu} & \dfrac{\omega^2}{\mathcal{M}}s^{\nu} \end{pmatrix}, \tag{4.46}$$

in which the coupling s_{ab}^{ν} is symmetric in a and b. For convenience in reading, let us define the submatrices of (4.46) to be M_L^{new}, M_D^{new}, and M_R^{new} similar to that of (4.10). Because of the condition $u^2 \ll u\omega \ll \omega^2$, the corresponding submatrices M_L^{new}, M_D^{new}, and M_R^{new} of (4.46) get the right hierarchies and the two questions as mentioned are *solved simultaneously* .

Intriguing comparisons between s^ν and h^ν are given in order:

(1) h^ν conserves the lepton number; s^ν violates this charge;

(2) h^ν is antisymmetric and enforcing on the Dirac-mass matrix; s^ν is symmetric and breaks this property;

(3) h^ν preserves the degeneration of $M_R = -M_L$; s^ν breaks the $M_R = -M_L$;

(4) a pair of (s^ν, h^ν) in the lepton sector will complete the rule played by the quark couplings (s^q, h^q) (see below);

(5) h^ν defines the interactions in the standard model scale v; s^ν gives those in the GUT scale $\mathcal{M}$.

Let us now take the values $\mathcal{M} \simeq 10^{16}\,\text{GeV}$, $\omega \simeq 3000\,\text{GeV}$, $u \simeq 2.46\,\text{GeV}$, and $s^{\nu} \sim \mathcal{O}(1)$ (perhaps smaller), the submatrices $M_L^{\text{new}} \simeq -6.05 \times 10^{-7} s^{\nu}\,\text{eV}$ and $M_D^{\text{new}} \simeq -7.38 \times 10^{-4} s^{\nu}\,\text{eV}$ can give contributions (to the diagonal components of M_L and M_D, resp.) but very small. It is noteworthy that the last one $M_R^{\text{new}} \simeq -0.9 s^{\nu}\,\text{eV}$ can *dominate* M_R.

To summarize, in this model the neutrino mass matrix is combined by $M_{\nu} + M_{\nu}^{\text{new}}$, where the first term is defined by (4.10) and the last term by (4.46); the submatrices of M_{ν} are given in (4.16) and (4.42), respectively. Depending on the strength of the new physics coupling s^{ν}, the submatrices of the last term, M_L^{new} and M_D^{new}, are included or removed.

4.2.3. Some remarks from experimental constraints

Conventional neutrino oscillations are insensitive to the absolute scale of neutrino masses. Although the latter will be tested directly in high-sensitivity tritium beta decay studies and neutrinoless double beta decay ($0\nu\beta\beta$) as well as by its effects on the cosmic microwave background and the large-scale structure of the Universe [143, 144]. With the present of sterile neutrinos in this model, the experimental constraints on their masses may be also important and give us bounds on several parameters such as the coupling h^{ν} and δ_a.

If the liquid scintillator neutrino detector experiment is confirmed, the sterile-neutrino masses will get some values in range of eV. In this case, the coupling h^{ν} is also in orders of h^{e}. The X-ray measurements yield an upper limit of sterile neutrino mass [145] $m_s < 6.3\,\text{keV}$. For all the other cosmological constraints, the sterile neutrino masses are in the range [146, 147] $2\,\text{keV} < m_s < 8\,\text{keV}$. In such cases, the coupling h^{ν} will get bounds in orders of $h^{\mu,\tau}$.

It is well known that the radiative mass generation can also induce the large lepton flavor violating processes such as $\mu \to e\gamma$ as the similar one-loop effect. The possible one-loop diagrams for this process are depicted in Figure 13. Suppose that $m_Y^2, m_{H_2}^2 \gg m_W^2 = g^2 v^2/4$ [38], we get the approximation [148]:

$$\text{Br}(\mu \longrightarrow e\gamma) \equiv \frac{\Gamma(\mu \longrightarrow e\gamma)}{\Gamma(\mu \longrightarrow e\tilde{\nu}_e \nu_{\mu})} \simeq \frac{3 s_W^4}{8\pi^3 \alpha} \left(h_{\mu\tau}^{\nu *} h_{e\tau}^{\nu}\right)^2. \tag{4.47}$$

Since $\text{Br}(\mu \to e\gamma) < 1.2 \times 10^{-11}$, $\alpha = 1/128$ and $s_W^2 = 0.2312$ [3], the coupling h^{ν} is bounded by $h^{\nu} < 3.47 \times 10^{-3}$, where $h^{\nu} \equiv h_{e\tau}^{\nu} = h_{\mu\tau}^{\nu}$ set is understood. Our above result, $h^{\nu} \sim h^{e}$, satisfies this constraint. It can be shown that the value for h^{ν} also satisfies constraints from such processes as $\mu \to 3e$ and μe conversion (for more details, see [149]).

4.3. Quark masses

First, we present the general quark mass spectrum. Some details on the one-loop quark masses are given then.

4.3.1. Quark mass spectra

Note that in [37], the authors have considered the fermion mass spectrum under the Z_2 discrete symmetry which discards the LNV interactions. Here, the couplings of (2.7) in such case are forbidden. Then, it can be checked that some quarks remain massless up to two-loop level. To solve the mass problem of the quarks, the authors in [37] have shown that one-third scalar triplet has to be added to the resulting model. In the following, we show that it is not

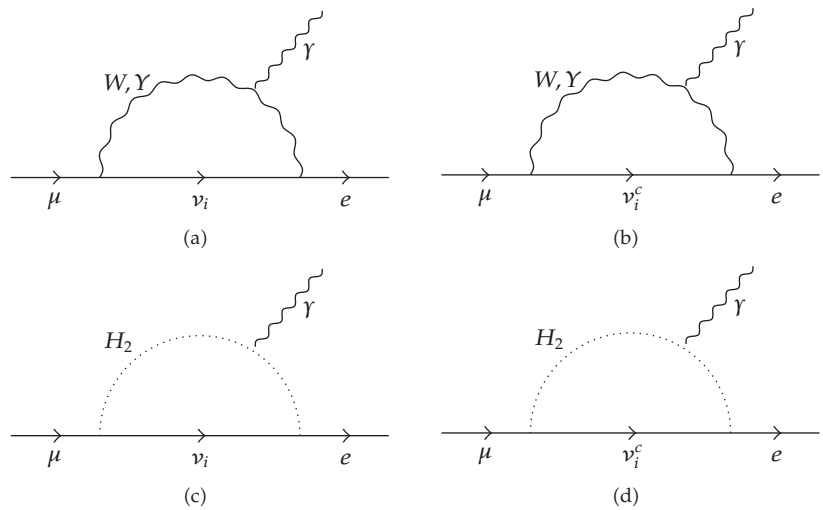


Figure 13: One-loop contributions to the lepton flavor violating decay $\mu \to e\gamma$.

necessary. The Z_2 is not introduced and thus the third one is not required. The LNV Yukawa couplings are vital for the economical 3-3-1 model.

The Yukawa couplings in (2.6) and (2.7) give the mass Lagrangian for the upquarks (quark sector with electric charge $q_{\text{up}} = 2/3$):

$$\mathcal{L}_{\text{up}}^{\text{mass}} = \frac{h^U}{\sqrt{2}}\left(\overline{u}_{1L}u + \overline{U}_L\omega\right)U_R + \frac{s_a^u}{\sqrt{2}}\left(\overline{u}_{1L}u + \overline{U}_L\omega\right)u_{aR} - \frac{v}{\sqrt{2}}\overline{u}_{\alpha L}\left(h_{\alpha a}^u u_{aR} + s_\alpha^U U_R\right) + \text{H.c.} \tag{4.48}$$

Consequently, we obtain the mass matrix for the upquarks (u_1, u_2, u_3, U) as follows:

$$M_{\text{up}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -s_1^u u & -s_2^u u & -s_3^u u & -h^U u \\ h_{21}^u v & h_{22}^u v & h_{23}^u v & s_2^U v \\ h_{31}^u v & h_{32}^u v & h_{33}^u v & s_3^U v \\ -s_1^u \omega & -s_2^u \omega & -s_3^u \omega & -h^U \omega \end{pmatrix}. \tag{4.49}$$

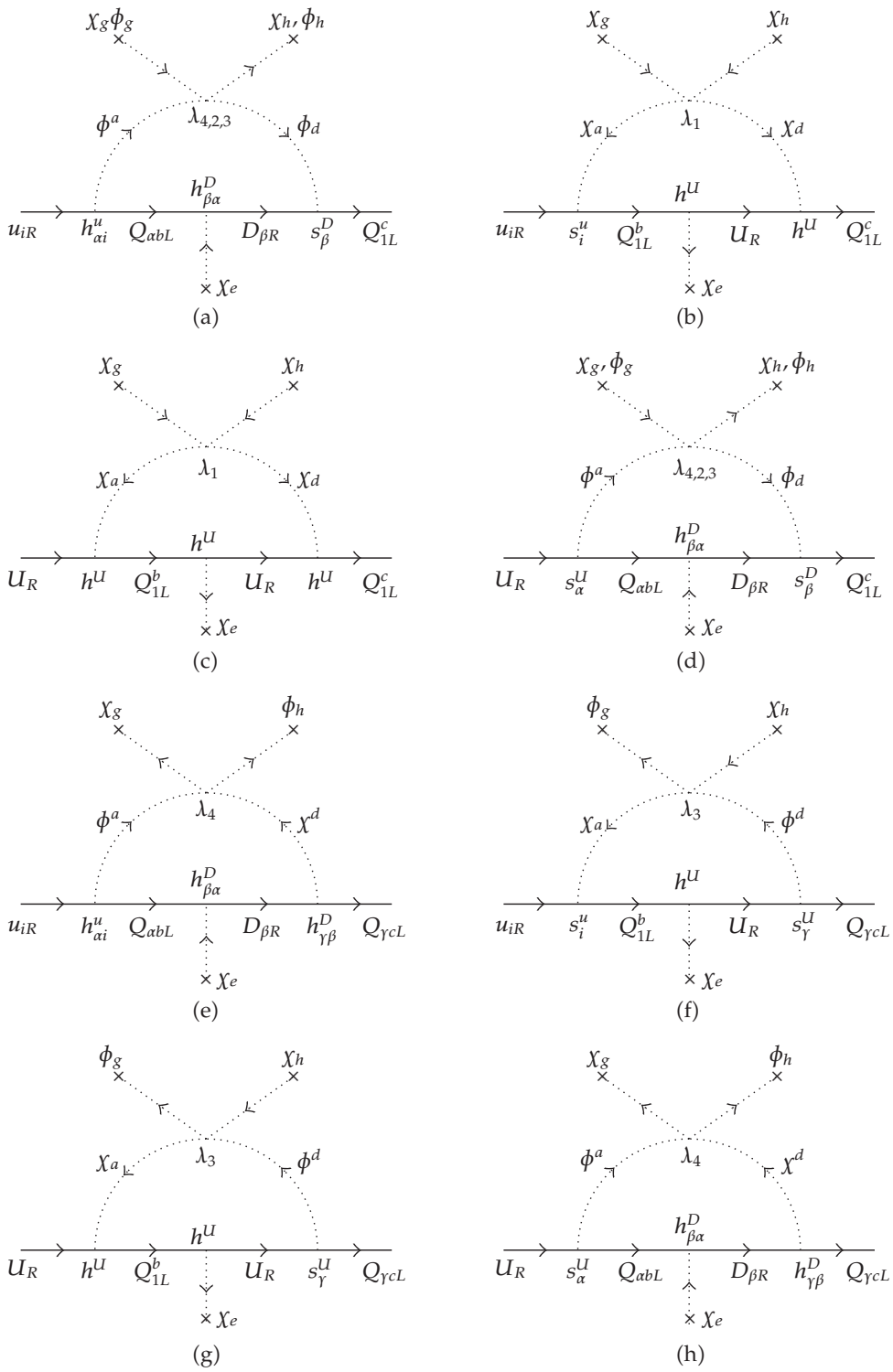
Because the first and the last rows of the matrix (4.49) are proportional, the tree-level upquark spectrum contains a massless one!

Similarly, for the downquarks ($q_{\text{down}} = -1/3$), we get the following mass Lagrangian:

$$\mathcal{L}_{\text{down}}^{\text{mass}} = \frac{h_{\alpha\beta}^D}{\sqrt{2}}\left(\overline{d}_{\alpha L}u + \overline{D}_{\alpha L}\omega\right)D_{\beta R} + \frac{s_{\alpha a}^d}{\sqrt{2}}\left(\overline{d}_{\alpha L}u + \overline{D}_{\alpha L}\omega\right)d_{aR} + \frac{v}{\sqrt{2}}\overline{d}_{1L}\left(h_a^d d_{aR} + s_\alpha^D D_{\alpha R}\right) + \text{H.c.} \tag{4.50}$$

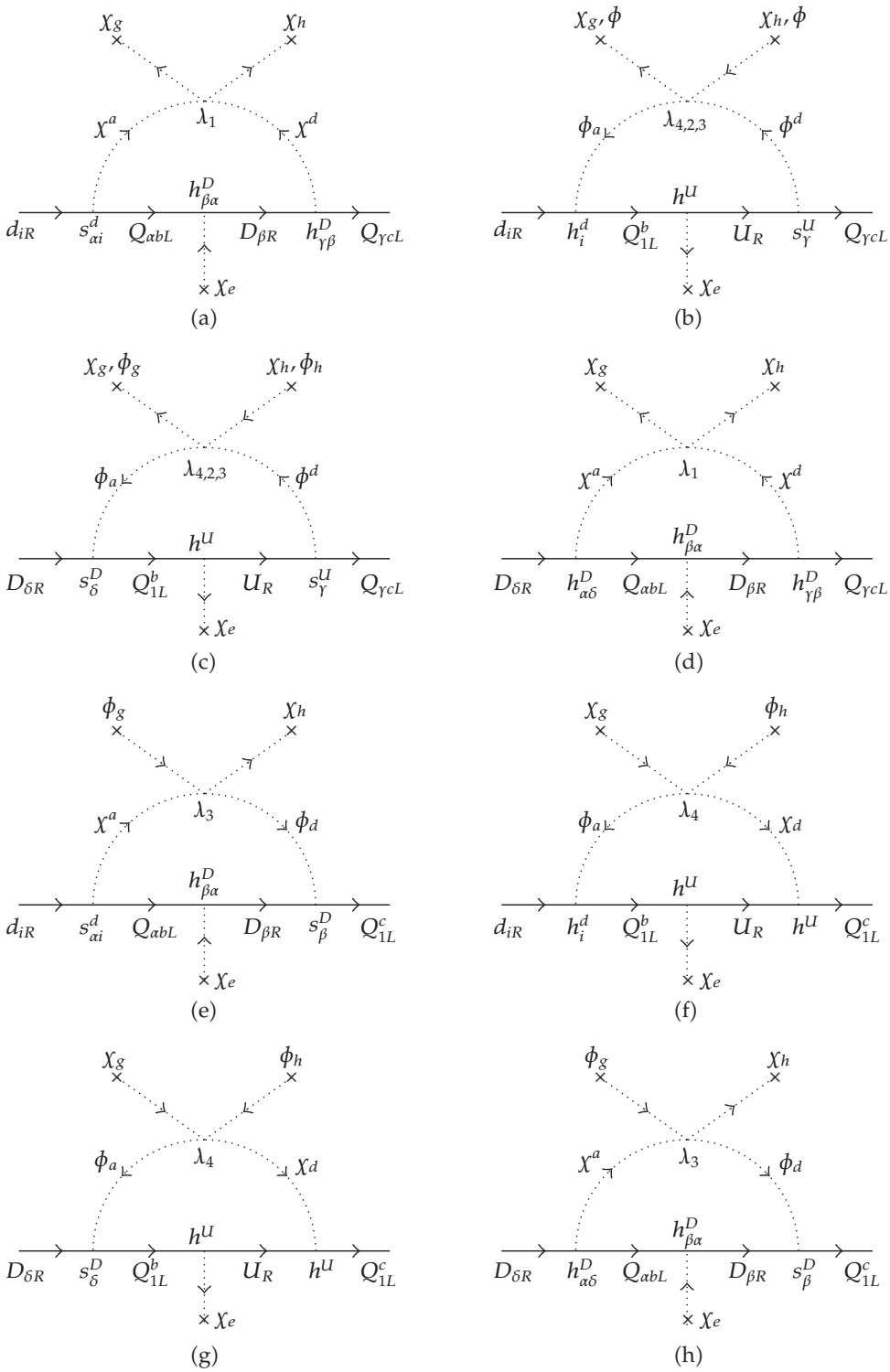
Hence, we get mass matrix for the downquarks $(d_1, d_2, d_3, D_2, D_3)$:

$$M_{\text{down}} = -\frac{1}{\sqrt{2}}\begin{pmatrix} h_1^d v & h_2^d v & h_3^d v & s_2^D v & s_3^D v \\ s_{21}^d u & s_{22}^d u & s_{23}^d u & h_{22}^D u & h_{23}^D u \\ s_{31}^d u & s_{32}^d u & s_{33}^d u & h_{32}^D u & h_{33}^D u \\ s_{21}^d \omega & s_{22}^d \omega & s_{23}^d \omega & h_{22}^D \omega & h_{23}^D \omega \\ s_{31}^d \omega & s_{32}^d \omega & s_{33}^d \omega & h_{32}^D \omega & h_{33}^D \omega \end{pmatrix}. \tag{4.51}$$



+ 16 graphs with smaller contributions

Figure 14: One-loop contributions to the upquark mass matrix (4.49).



+ 16 graphs with smaller contributions

Figure 15: One-loop contributions to the downquark mass matrix (4.51).

We see that the second and fourth rows of matrix in (4.51) are proportional, while the third and the last are the same. Hence, in this case there are two massless eigenstates.

The masslessness of the tree-level quarks in both the sectors calls radiative corrections (the so-called mass problem of quarks). These corrections start at the one-loop level. The diagrams in Figure 14 contribute the upquark spectrum, while Figure 15 gives the downquarks. Let us note the reader that the quarks also get some one-loop contributions in the case of the Z_2 symmetry enforcing [37]. The careful study of this radiative mechanism shows that the one-loop quark spectrum is consistent.

4.3.2. Typical examples of the one-loop corrections

To provide the quarks masses, in the following we can suppose that the Yukawa couplings are flavor diagonal. Then, the u_2 and u_3 states are mass eigenstates corresponding to the mass eigenvalues:

$$m_2 = h^u_{22}\frac{v}{\sqrt{2}}, \qquad m_3 = h^u_{33}\frac{v}{\sqrt{2}}. \tag{4.52}$$

The u_1 state mixes with the exotic U in terms of one submatrix of the mass matrix (4.48):

$$M_{uU} = -\frac{1}{\sqrt{2}}\begin{pmatrix} s^u_1 u & h^U u \\ s^u_1 \omega & h^U \omega \end{pmatrix}. \tag{4.53}$$

This matrix contains one massless quark $\sim u_1$, $m_1 = 0$, and the remaining exotic quark $\sim U$ with the mass of the scale ω.

Similarly, for the downquarks, the d_1 state is a mass eigenstate corresponding to the eigenvalue:

$$m'_1 = -h^d_1\frac{v}{\sqrt{2}}. \tag{4.54}$$

The pairs (d_2, D_2) and (d_3, D_3) are decoupled, while the quarks of each pair mix via the mass submatrices, respectively,

$$M_{d_2D_2} = -\frac{1}{\sqrt{2}}\begin{pmatrix} s^d_{22} u & h^D_{22} u \\ s^d_{22}\omega & h^D_{22}\omega \end{pmatrix}, \tag{4.55}$$

$$M_{d_3D_3} = -\frac{1}{\sqrt{2}}\begin{pmatrix} s^d_{33} u & h^D_{33} u \\ s^d_{33}\omega & h^D_{33}\omega \end{pmatrix}. \tag{4.56}$$

These matrices contain the massless quarks $\sim d_2$ and d_3 corresponding to $m'_2 = 0$ and $m'_3 = 0$, and two exotic quarks $\sim D_2$ and D_3 with the masses of the scale ω.

With the help of the constraint (2.8), we identify m_1, m_2, and m_3, respectively, to those of the $u_1 = u$, $u_2 = c$, and $u_3 = t$ quarks. The downquarks d_1, d_2, and d_3 are, therefore, corresponding to d, s, and b quarks. Unlike the usual 3-3-1 model with right-handed neutrinos, where the third family of quarks should be discriminating [28], in the model under consideration the *first* family has to be different from the two others.

The mass matrices (4.53), (4.55), and (4.56) remain the tree-level properties for the quark spectra—one massless in the upquark sector and two in the downquark sector. From these matrices, it is easily to verify that the conditions in (2.8) and (2.10) are satisfied. First, we consider radiative corrections to the upquark masses.

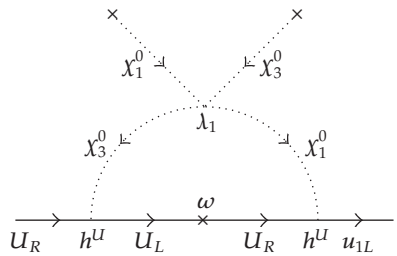


Figure 16: One-loop contribution under Z_2 to the upquark mass matrix (4.57).

Upquarks

In the previous studies [19, 37, 84–86], the LNV interactions have often been excluded, commonly by the adoption of an appropriate discrete symmetry. Let us remind that there is no reason within the 3-3-1 model to ignore such interactions. The experimental limits on processes which do not conserve total lepton numbers such as neutrinoless double beta decay [150, 151] require them to be small.

If the Yukawa Lagrangian is restricted to $\mathcal{L}_{\mathrm{LNC}}$ [37], then the mass matrix (4.53) becomes

$$M_{uU} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & h^U u \\ 0 & h^U \omega \end{pmatrix}. \tag{4.57}$$

In this case, only the element $(M_{uU})_{12}$ gets an one-loop correction defined by Figure 16. Other elements remain unchanged under this one-loop effect.

The Feynman rules gives us

$$\begin{aligned} -i(M_{uU})_{12}P_R = \int \frac{d^4p}{(2\pi)^4}(ih^U P_R)\frac{i(\not{p}+M_U)}{p^2-M_U^2}(-iM_U P_L)\frac{i(\not{p}+M_U)}{p^2-M_U^2}(ih^U P_R) \\ \times \frac{-1}{(p^2-M_{\chi_1}^2)(p^2-M_{\chi_3}^2)}(i4\lambda_1)\frac{u\omega}{2}. \end{aligned} \tag{4.58}$$

Thus, we get

$$\begin{aligned} (M_{uU})_{12} &= -2iu\omega\lambda_1 M_U (h^U)^2 \int \frac{d^4p}{(2\pi)^4}\frac{p^2}{(p^2-M_U^2)^2(p^2-M_{\chi_3}^2)(p^2-M_{\chi_1}^2)} \\ &\equiv -2iu\omega\lambda_1 M_U (h^U)^2 I(M_U^2, M_{\chi_3}^2, M_{\chi_1}^2). \end{aligned} \tag{4.59}$$

The integral $I(a,b,c)$ with $a,b \gg c$ is given in the B. Following [39], we conclude that in an effective approximation, M_U^2, $M_{\chi_3}^2 \gg M_{\chi_1}^2$. Hence, we have

$$(M_{uU})_{12} \simeq -\frac{\lambda_1 t_\theta M_U^3}{4\pi^2}\left[\frac{M_U^2 - M_{\chi_3}^2 + M_{\chi_3}^2 \ln(M_{\chi_3}^2/M_U^2)}{(M_U^2 - M_{\chi_3}^2)^2}\right] \sim u \equiv -\frac{1}{\sqrt{2}}R(M_U). \tag{4.60}$$

The resulting mass matrix is given by

$$M_{uU} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & h^U u + R \\ 0 & h^U \omega \end{pmatrix}. \tag{4.61}$$

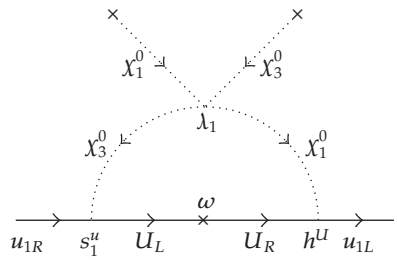


Figure 17: One-loop contribution to the upquark mass matrix (4.53).

Table 19: Mass for the u quark as function of (s_1^u, h^U).

h^U	2	1.5	1	0.5	0.1
s_1^u	0.0002	0.0003	0.0004	0.001	0.01
m_u [MeV]	2.207	2.565	2.246	2.375	2.025

We see that one quark remains massless as the case of the tree-level spectrum. This result keeps up to two-loop level and can be applied to the downquark sector as well as in the cases of nondiagonal Yukawa couplings. Therefore, under the Z_2, it is not able to provide consistent masses for the quarks.

If the full Yukawa Lagrangian is used, the LNV couplings must be enough small in comparison with the usual couplings [see (2.10)]. Combining (2.8) and (2.10), we have

$$h^U\omega \gg h^U u, \qquad s_1^u\omega \gg s_1^u u. \tag{4.62}$$

In this case, the element $(M_{uU})_{11}$ of (4.53) gets the radiative correction depicted in Figure 17. The resulting mass matrix is obtained by

$$M_{uU} = -\frac{1}{\sqrt{2}}\begin{pmatrix} s_1^u\left(u + \dfrac{R}{h^U}\right) & h^U u \\ s_1^u\omega & h^U\omega \end{pmatrix}. \tag{4.63}$$

In contradiction with the first case, the mass of u quark is now nonzero and given by

$$m_u \simeq \frac{s_1^u}{\sqrt{2}h^U}R. \tag{4.64}$$

Let us note that the matrix (4.63) gives an eigenvalue in the scale of $(1/\sqrt{2})h^U\omega$ which can be identified with that of the exotic quark U. In effective approximation [39], the mass for the Higgs χ_3 is defined by $M_{\chi_3}^2 \simeq 2\lambda_1\omega^2$. Hereafter, for the parameters, we use the following values $\lambda_1 = 2.0$, $t_\theta = 0.08$ as mentioned, and $\omega = 10\,\text{TeV}$. The mass value for the u quark is as function of s_1^u and h^U. Some values of the pair (s_1^u, h^U) which give consistent masses for the u quark are listed in Table 19.

Note that the mass values in Table 19 for the u quark are in good consistence with the data given in [3]: $m_u \in 1.5 \div 4\,\text{MeV}$.

Downquarks

For the downquarks, the constraint,

$$h_{\alpha\alpha}^D\omega \gg h_{\alpha\alpha}^D u, \qquad s_{\alpha\alpha}^d\omega \gg s_{\alpha\alpha}^d u, \tag{4.65}$$

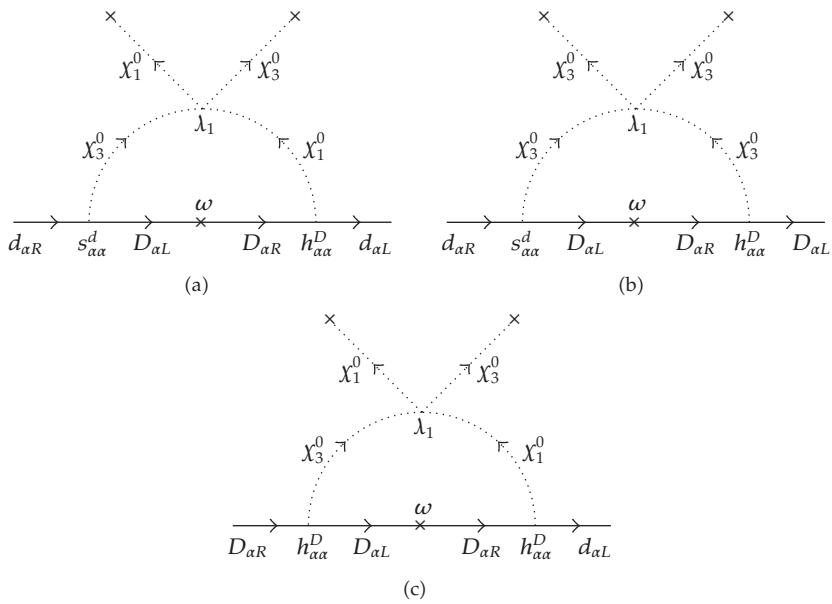


Figure 18: One-loop contributions to the downquark mass matrix (4.55) or (4.56).

should be applied. In this case, three elements $(M_{d_\alpha D_\alpha})_{11}$, $(M_{d_\alpha D_\alpha})_{12}$, and $(M_{d_\alpha D_\alpha})_{21}$ will get radiative corrections. The relevant diagrams are depicted in Figure 18. It is worth noting that Diagram 18(c) exists even in the case of the Z_2 symmetry. The contributions are given by

$$\begin{aligned}
(M_{d_\alpha D_\alpha})_{11} &= -\frac{s^d_{\alpha\alpha}}{\sqrt{2}h^D_{\alpha\alpha}} R(M_{D_\alpha}),\\
(M_{d_\alpha D_\alpha})_{21} &= -4i\lambda_1 \frac{s^d_{\alpha\alpha}}{h^D_{\alpha\alpha}} M^3_{D_\alpha} I(M^2_{D_\alpha}, M^2_{\chi_3}, M^2_{\chi_3})\\
&= -\frac{\lambda_1 s^d_{\alpha\alpha} M^3_{D_\alpha}}{4\pi^2 h^D_{\alpha\alpha}} \left[\frac{M^2_{D_\alpha} + M^2_{\chi_3}}{(M^2_{D_\alpha} - M^2_{\chi_3})^2} - \frac{2M^2_{D_\alpha} M^2_{\chi_3}}{(M^2_{D_\alpha} - M^2_{\chi_3})^3} \ln \frac{M^2_{D_\alpha}}{M^2_{\chi_3}}\right]\\
&\equiv -\frac{1}{\sqrt{2}} R'(M_{D_\alpha}),\\
(M_{d_\alpha D_\alpha})_{12} &= -\frac{1}{\sqrt{2}} R(M_{D_\alpha}).
\end{aligned} \tag{4.66}$$

We see that two last terms are much larger than the first one. This is responsible for the masses of the quarks d_2 and d_3. At the one-loop level, the mass matrix for the downquarks is given by

$$M_{d_\alpha D_\alpha} = -\frac{1}{\sqrt{2}} \begin{pmatrix} s^d_{\alpha\alpha}\left(u + \dfrac{R}{h^D_{\alpha\alpha}}\right) & h^D_{\alpha\alpha} u + R \\ s^d_{\alpha\alpha}\omega + R' & h^D_{\alpha\alpha}\omega \end{pmatrix}. \tag{4.67}$$

We remind the reader that a matrix (see also [131])

$$\begin{pmatrix} a & c \\ b & D \end{pmatrix} \tag{4.68}$$

with $D \gg b$, $c \gg a$ has two eigenvalues:

$$x_1 \simeq \left[a^2 - \frac{2bca}{D} + \frac{b^2c^2 - (b^2+c^2)a^2}{D^2}\right]^{1/2}, \qquad x_2 \simeq D. \tag{4.69}$$

Therefore, the mass matrix in (4.67) gives an eigenvalue in the scale of $D \equiv (1/\sqrt{2})h^D_{\alpha\alpha}\omega$ which is of the exotic quark D'_α. Here, we have another eigenvalue for the mass of d'_α:

$$m_{d'_\alpha} = \frac{h^D_{\alpha\alpha}u + R}{\sqrt{2}h^D_{\alpha\alpha}\omega}\left\{R'^2 - \frac{(s^d_{\alpha\alpha})^2}{(h^D_{\alpha\alpha})^2}\left[(s^d_{\alpha\alpha}\omega + R')^2 + (h^D_{\alpha\alpha}u + R)^2\right]\right\}^{1/2}. \tag{4.70}$$

Let us remember that $M^2_{\chi_3} \simeq 2\lambda_1\omega^2$ and the parameters $\lambda_1 = 2.0$, $t_\theta = 0.08$ and $\omega = 10\,\text{TeV}$ as given above are used in this case; m_{d_α} is a function of $s^d_{\alpha\alpha}$ and $h^D_{\alpha\alpha}$. We take the value $h^D_{\alpha\alpha} = 2.0$ for both the sectors, $\alpha = 2$ and $\alpha = 3$. If $s^d_{22} = 0.015$, we get then the mass of the so-called s quark:

$$m_s = 99.3\,\text{MeV}. \tag{4.71}$$

For the downquark of the third family, we put $s^d_{33} = 0.7$. Then, the mass of the b quark is obtained by

$$m_b = 4.4\,\text{GeV}. \tag{4.72}$$

We emphasize again that (4.71) and (4.72) are in good consistence with the data given in [3]: $m_s \sim 95 \pm 25\,\text{MeV}$ and $m_b \sim 4.70 \pm 0.07\,\text{GeV}$.

4.4. Summary

The basic motivation of this section is to present the answer to one of the most crucial questions: whether within the framework of the model based on $\text{SU}(3)_C \otimes \text{SU}(3)_L \otimes \text{U}(1)_X$ gauge group contained minimal Higgs sector with right-handed neutrinos, all fermions including quarks and neutrinos can gain the consistent masses.

In this model, the masses of neutrinos are given by three different sources widely ranging over the mass scales including the GUT's and the small VEV u of spontaneous lepton breaking. At the tree level, there are three Dirac neutrinos: one massless and two degenerate with the masses in the order of the electron mass. At the one-loop level, a possible framework for the finite renormalization of the neutrino masses is obtained. The Dirac masses obtain a large reduction; the Majorana mass types get degenerate in $M_R = -M_L$; all these masses are in the bound of the data. It is emphasized that the above degeneration is a consequence of the fact that the left-handed and right-handed neutrinos in this model are in the same gauge triplets. The new physics including the 3-3-1 model is strongly signified. The degenerations and hierarchies among the mass types are completely removed by heavy particles.

The resulting mass matrix for the neutrinos consists of two parts $M_\nu + M^{\text{new}}_\nu$: the first is mediated by the model particles, and the last is due to the new physics. Upon the contributions of M^{new}_ν, the different realistic mass textures can be produced. For example, neglecting the last term, the pseudo-Dirac patterns can be obtained. In another scenario, that the bare coupling h^ν of Dirac masses gets higher values, for example, in orders of $h^{\mu,\tau}$, the

VEV ω can be picked up to an enough large value ($\sim \mathcal{O}(10^4$–$10^5)$ TeV) so that the type II seesaw spectrum is obtained. Such features deserve further study. We have also shown that the lepton flavor violating processes such as $\mu \to e\gamma$, $\mu \to 3e$, and μe conversion get the consistent values in the bounds of the current experiments.

In the first section, we have shown that, in the considered model, there are three quite different scales of vacuum expectation values: $\omega \sim \mathcal{O}(1)$ TeV, $v \approx 246$ GeV, and $u \sim \mathcal{O}(1)$ GeV. In this section, we have added a new characteristic property, namely, there are two types of Yukawa couplings with different strengths: the LNC coupling h's and the LNV ones s's satisfying the condition $s \ll h$. With the help of these key properties, the mass spectrum of quarks is consistent without introducing the third scalar triplet. With the given set of parameters, the numerical evaluation shows that in this model, masses of the exotic quarks also have different scales, namely, the U exotic quark ($q_U = 2/3$) gains mass $m_U \approx 700$ GeV, while the D_α exotic quarks ($q_{D_\alpha} = -1/3$) have masses in the TeV scale: $m_{D_\alpha} \in 10 \div 80$ TeV.

Let us summarize our results.

(1) *At the tree level.*

(a) All charged leptons gain masses similar to that in the standard model.
(b) One neutrino is massless and the other two are degenerate in masses.
(c) Three quarks u_1, d_2, d_3 are massless.
(d) Allexotic quarks gain masses proportional to ω—the VEV of the first step of symmetry breaking.

(2) *At the one-loop level.*

(a) All above-mentioned fermions gain masses.
(b) The light quarks gain masses proportional to u—the LNV parameter.
(c) The exotic quark masses are separated: $m_U \approx 700$ GeV, $m_{D_\alpha} \in 10 \div 80$ TeV.
(d) There exist two types of Yukawa couplings: the LNC and LNV with quite different strengths.

With the *positive* answer, the economical version becomes one of the very attractive models beyond the standard model.

5. Conclusion

Finally, this is the time to mention some developments of the model as reported on this work [36–42]. The idea to give VEVs at the top and bottom elements of χ triplet was given in [36]. Some consequences such as the atomic parity violation, $Z-Z'$ mixing angle and Z' mass were studied [37]. However, in the above-mentioned works, the $W-Y$ and W_4-Z-Z' mixings were excluded. To solve the difficulties such as the standard model coupling ZZh or quark masses, the third scalar triplet was introduced. Thus, the scalar sector was no longer minimal and the economical in this sense was unrealistic!

In the beginning of the last year, there was a new step in development of the model. In [38], the correct identification of non-Hermitian bilepton gauge boson X^0 was established. The $W-Y$ mixing as well as W_4, Z, Z' one were also entered into couplings among fermions and gauge bosons. The lepton-number violating interactions exist in both charged and neutral gauge boson sectors. However, the lepton-number violation happens only in the neutrino and exotic quarks sectors, but not in the charged lepton sector. The scalar sector

was studied in [39], and all gauge-Higgs couplings were presented, and all similar ones in the standard model were recovered. The Higgs sector contains eight Goldstone bosons—the needed number for massive gauge ones of the model. Interesting to note that the CP-odd part of Goldstone associated with the neutral non-Hermitian gauge boson G_{X^0} is decoupled, while its CP-even counterpart has the mixing by the same way in the gauge boson sector.

In [40], the deviation δQ_{W} of the weak charge from its standard model prediction due to the mixing of the W boson with the charged bilepton Y as well as of the Z boson with the neutral Z' and the real part of the non-Hermitian neutral bilepton X^0 is established.

The model is consistent with the effective theory and new experiments because it can provide all fermions including the quarks and neutrinos with the consistent masses [41, 42]. The exotic quarks and new bosons get masses in order of TeV. There are two different scales of exotic quark masses: $m_U \approx 700\,\mathrm{GeV}$, $m_{D_\alpha} \in 10 \div 80\,\mathrm{TeV}$.

It is worth mentioning on advantage of the model: the new mixing angle between the charged gauge bosons θ is connected with one of the VEVs u—the parameter of lepton-number violations. There is no new parameter, but it contains very simple Higgs sector, hence the significant number of free parameters is reduced. The Higgs self-couplings $\lambda_{1,2,4}$ are constrained by the scalar masses, but the remainder λ_3 is fixed by the neutrino masses [42]. This means also that the generation of the neutrino masses leads to a shift in mass of the Higgs boson from the standard model prediction.

The model is rich in physics because it includes the right-handed neutrinos, exotic quarks, and new bosons and also gives an possible explanation of the generation question, electric charge quantization, and current neutrino mass problem. The supersymmetric version has been considered [43–46]. The new physics is at TeV scale, therefore, the results can be verified in the next generation of collides such as LHC and ILC.

Appendices

A. Mixing matrices

For convenience in calculating, in this appendix we give the mixing matrices of the gauge and Higgs sectors.

A.1. Neutral gauge bosons

$$\begin{pmatrix} W_3 \\ W_8 \\ B \\ W_4 \end{pmatrix} = \begin{pmatrix} s_W & c_\varphi c_{\theta'} c_W & s_\varphi c_{\theta'} c_W & s_{\theta'} c_W \\ -\dfrac{s_W}{\sqrt{3}} & \dfrac{c_\varphi\left(s_W^2 - 3c_W^2 s_{\theta'}^2\right) - s_\varphi \lambda \kappa}{\sqrt{3} c_W c_{\theta'}} & \dfrac{s_\varphi\left(s_W^2 - 3c_W^2 s_{\theta'}^2\right) + c_\varphi \lambda \kappa}{\sqrt{3} c_W c_{\theta'}} & \sqrt{3} s_{\theta'} c_W \\ \dfrac{\kappa}{\sqrt{3}} & -\dfrac{t_W\left(c_\varphi \kappa + s_\varphi \lambda\right)}{\sqrt{3} c_{\theta'}} & -\dfrac{t_W\left(s_\varphi \kappa - c_\varphi \lambda\right)}{\sqrt{3} c_{\theta'}} & 0 \\ 0 & -t_{\theta'}\left(c_\varphi \lambda - s_\varphi \kappa\right) & -t_{\theta'}\left(s_\varphi \lambda + c_\varphi \kappa\right) & \lambda \end{pmatrix} \begin{pmatrix} A \\ Z^1 \\ Z^2 \\ W_4' \end{pmatrix}, \tag{A.1}$$

where we have denoted

$$s_{\theta'} \equiv \frac{t_{2\theta}}{\left(c_W \sqrt{1 + 4t_{2\theta}^2}\right)}, \qquad \kappa \equiv \sqrt{4c_W^2 - 1}, \qquad \lambda \equiv \sqrt{1 - 4s_{\theta'}^2 c_W^2}. \tag{A.2}$$

A.2. Neutral scalar bosons

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} -s_\zeta s_\theta & c_\zeta s_\theta & c_\theta \\ c_\zeta & s_\zeta & 0 \\ -s_\zeta c_\theta & c_\zeta c_\theta & -s_\theta \end{pmatrix} \begin{pmatrix} H \\ H_1^0 \\ G_4 \end{pmatrix}. \tag{A.3}$$

A.3. Singly-charged scalar bosons

$$\begin{pmatrix} \phi_1^+ \\ \chi_2^+ \\ \phi_3^+ \end{pmatrix} = \frac{1}{\sqrt{\omega^2 + c_\theta^2 v^2}} \begin{pmatrix} \omega s_\theta & c_\theta \sqrt{\omega^2 + c_\theta^2 v^2} & \dfrac{v s_{2\theta}}{2} \\ v c_\theta & 0 & -\omega \\ \omega c_\theta & -s_\theta \sqrt{\omega^2 + c_\theta^2 v^2} & v c_\theta^2 \end{pmatrix} \begin{pmatrix} H_2^+ \\ G_5^+ \\ G_6^+ \end{pmatrix}. \tag{A.4}$$

B. Feynman integrations

In this appendix, we present evaluation of the integral:

$$I(a,b,c) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{(p^2-a)^2(p^2-b)(p^2-c)}, \tag{B.1}$$

where $a,b,c > 0$ and $I(a,b,c) = I(a,c,b)$.

B.1. Case of $b \neq c$ and $b, c \neq a$

We first introduce a well-known integral as follows:

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2-a)(p^2-b)(p^2-c)} = \frac{-i}{16\pi^2} \left\{ \frac{a \ln a}{(a-b)(a-c)} + \frac{b \ln b}{(b-a)(b-c)} + \frac{c \ln c}{(c-b)(c-a)} \right\}. \tag{B.2}$$

Differentiating two sides of this equation with respect to a, we have

$$\begin{aligned} &\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2-a)^2(p^2-b)(p^2-c)} \\ &\quad = \frac{-i}{16\pi^2} \left\{ \frac{\ln a + 1}{(a-b)(a-c)} - \frac{a(2a-b-c)\ln a}{(a-b)^2(a-c)^2} + \frac{b \ln b}{(b-a)^2(b-c)} + \frac{c \ln c}{(c-a)^2(c-b)} \right\}. \end{aligned} \tag{B.3}$$

Combining (B.2) and (B.3), the integral (B.1) becomes

$$\begin{aligned} I(a,b,c) &= \int \frac{d^4p}{(2\pi)^4} \left[\frac{1}{(p^2-a)(p^2-b)(p^2-c)} + \frac{a}{(p^2-a)^2(p^2-b)(p^2-c)} \right] \\ &= \frac{-i}{16\pi^2} \left\{ \frac{a(2\ln a + 1)}{(a-b)(a-c)} - \frac{a^2(2a-b-c)\ln a}{(a-b)^2(a-c)^2} + \frac{b^2 \ln b}{(b-a)^2(b-c)} + \frac{c^2 \ln c}{(c-a)^2(c-b)} \right\}. \end{aligned} \tag{B.4}$$

If $a, b \gg c$ or $c \simeq 0$, we have an approximation as follows:

$$I(a,b,c) \simeq -\frac{i}{16\pi^2} \frac{1}{a-b} \left[1 - \frac{b}{a-b} \ln \frac{a}{b} \right]. \tag{B.5}$$

B.2. Case of $b = c$ ***and*** $b \neq a$

We put

$$I(a,b) \equiv I(a,b,b) = \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{(p^2-a)^2(p^2-b)^2}, \tag{B.6}$$

where $I(a,b) = I(b,a)$.

Using the Feynman's parametrization,

$$\frac{1}{A^2B^2} = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} \int_0^1 dx \frac{x(1-x)}{[xA+(1-x)B]^4}, \tag{B.7}$$

we have

$$\frac{1}{(p^2-a)^2(p^2-b)^2} = 6\int_0^1 dx \frac{x(1-x)}{(p^2-M^2)^4}, \tag{B.8}$$

where $M^2 \equiv xa + (1-x)b$. Equation (B.6), therefore, becomes

$$I(a,b) = 6\int_0^1 dx x(1-x) \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{(p^2-M^2)^4}. \tag{B.9}$$

With the help of

$$\int \frac{d^4p}{(2\pi)^4} \frac{p^2}{(p^2-M^2)^4} = \frac{-i}{3(4\pi)^2} \frac{1}{M^2}, \tag{B.10}$$

(B.9) is given by

$$I(a,b) = \frac{-2i}{(4\pi)^2} \int_0^1 dx \frac{x(1-x)}{xa+(1-x)b}. \tag{B.11}$$

To obtain the integral, we can put $t = xa + (1-x)b$; (B.11) is then rewritten

$$I(a,b) = \frac{2i}{(4\pi)^2(a-b)^3} \int_b^a dt \left[t - (a+b) + \frac{ab}{t}\right]. \tag{B.12}$$

Therefore, we get

$$I(a,b) = -\frac{i}{16\pi^2}\left[\frac{a+b}{(a-b)^2} - \frac{2ab}{(a-b)^3}\ln\frac{a}{b}\right]. \tag{B.13}$$

If $b \gg a$ or $a \simeq 0$, we have the following approximation:

$$I(a,b) \simeq -\frac{i}{16\pi^2 b}. \tag{B.14}$$

Let us note that the above approximations $aI(a,b,c)$, (or $bI(a,b,c)$), and $bI(a,b)$ are kept in the orders up to $\mathcal{O}(c/a, c/b)$ and $\mathcal{O}(a/b)$, respectively.

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