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*Research Article*

# Del Pezzo Singularities and SUSY Breaking

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An analytic construction of compact Calabi-Yau manifolds with del Pezzo singularities is found. We present complete intersection CY manifolds for all del Pezzo singularities and study the complex deformations of these singularities. An example of the quintic CY manifold with del Pezzo 6 singularity and some number of conifold singularities is studied in detail. The possibilities for the "geometric" and ISS mechanisms of dynamical SUSY breaking are discussed. As an example, we construct the ISS vacuum for the del Pezzo 6 singularity.

## 1. Motivation

Recently, there has been a substantial progress in Model building involving the D-branes at the singularities of noncompact Calabi-Yau manifolds. On the one hand, the singularities provide enough flexibility to find phenomenologically acceptable extensions of the Standard Model [1, 2] and solve some problems such as finding metastable susy breaking vacua [3, 4]. On the other hand, the presence of the singularity eliminates certain massless moduli, such as the adjoint fields on the branes wrapping rigid cycles [1, 5].

The main purpose of this paper is to study the del Pezzo and conifold singularities on compact CY manifolds that may be useful for the compactifications of dynamical SUSY breaking mechanisms. The stringy reallizations of metastable SUSY breaking vacua have been known for some time [6, 7]. We will focus on the two recent approaches to the dynamical SUSY breaking: on the "geometrical" approach of [8, 9] and on the ISS construction [10]. One of the main goals will be to study the topological conditions for the compactification of the above constructions.

An important topological property of "geometrical" mechanism is the presence of several homologous rigid two-cycles. This is not difficult to achieve in the case of conifold singularities. For example, in the geometric transitions on compact CY manifolds [11, 12],

several conifolds may be resolved by a single Kahler modulus, that is, the two-cycles at the tip of these conifolds are homologous to each other. However, this is not always true for the del Pezzo singularities, that is, the two-cycles in the resolution of del Pezzo singularity may have no homologous rigid two-cycles on the compact CY. In the paper, we explicitly construct a compact CY manifold with del Pezzo 6 singularity and a number of conifolds such that some two-cycles on the del Pezzo are homologous to the two-cycles of the conifolds. This construction opens up the road for the generalization of geometrical SUSY breaking in the case of del Pezzo singularities, where one may hope to use the richness of deformations of these singularity for phenomenological applications.

A more direct way towards phenomenology is provided by the ISS mechanism. The realization of ISS construction for del Pezzo 5 and 8 singularities was considered in [4]. As an example, we find an ISS vacuum for the del Pezzo 6 singularity. The del Pezzo 6 surface can be embedded in $\mathbb{P}^3$ by a degree 3 polynomial. This is one of the most simple analytical representations of del Pezzo surfaces, which enables us to find an analytical embedding of the corresponding del Pezzo 6 singularity in a compact Calabi Yau manifold, the quintic CY embedded in $\mathbb{P}^4$ by a degree 5 polynomial.

A nice feature of the del Pezzo singularities is that they are isolated. Thus, the fractional branes, that one typically introduces in these models, are naturally stabilized against moving away from the singularity. But, for example, in the models involving quotients of conifolds [3, 13], the singularities are not isolated and one needs to pay special attention to stabilize the fractional branes against moving along the singular curves.

Apart from the application to SUSY breaking, the construction of compact CY manifolds with del Pezzo singularities may be useful for the study of deformations of these singularities. In particular, we will be interested in the D-brane interpretation of deformations.

In general, a singularity can be smoothed out in two different ways, it can be either deformed or resolved (blown up). The former corresponds to the deformations of the complex structure, described by the elements of $H^{2,1}$; the latter corresponds to Kähler deformations given by the elements of $H^{1,1}$ [14–16]. In terms of the cycles, the resolution corresponds to blowing up some two-cycles (four-cycles), while the complex deformations correspond to the deformations of the three-cycles. For example, the conifold can be either deformed by placing an $S^3$ at the tip of the conifold or resolved by placing an $S^2$ [17]. The process where some three-cycles shrink to form a singularity and after that the singularity is blown up is called the geometric transition [11, 12]. For the conifold, the geometric transition has a nice interpretation in terms of the branes. The deformation of the conifold is induced by wrapping the D5-branes around the vanishing $S^2$ at the tip [18]. The resolution of the conifold corresponds to giving a vev to a baryonic operator, that can be interpreted in terms of the D3-branes wrapping the vanishing $S^3$ at the tip of the conifold [19].

The example of the conifold encourages to conjecture that any geometric transition can be interpreted in terms of the branes. The nonanomalous (fractional) branes produce the fluxes that deform the three-cycles. The massless/tensionless branes correspond to baryonic operators whose vevs are interpreted as the blow-up modes.

However, there are a few puzzles with the above interpretation. In some cases, there are less deformations than nonanomalous fractional branes; in the other cases there are deformations but no fractional branes, The quiver gauge theory on the del Pezzo 1 singularity has a nonanomalous fractional brane; moreover, it has a cascading behavior [20] similar to the conifold cascade. But it is known that there are no complex deformations of the cone over

$dP_1$ [21–23]. The relevant observation [24] is that there are no geometric transitions for the cone over $dP_1$. From the point of view of gauge theory, there is a runaway behavior at the bottom of the cascade and no finite vacuum [25].

On the other side of the puzzle, there are more complex deformations of higher del Pezzo singularities, than there are possible fractional branes. It is known that the cone over del Pezzo $n$ surface has $c^\vee(E_n) - 1$ complex deformations [24], where $c^\vee(E_n)$ is the dual Coxeter number of the corresponding Lie group. For instance, the cone over $dP_8$ has 29 deformations. But there are only 8 nonanomalous combinations of fractional branes [1].

We believe that these puzzles can be managed more effectively if there were more examples of compact CY manifolds with local del Pezzo singularities. The advantage of working with compact manifolds is that they have finite a number of deformations and well-defined cohomology (there are no noncompact cycles).

The organization of the paper is as follows. In Section 2, we study the singularities on compact CY manifolds using the quintic CY manifold as an example. We restrict our attention to isolated singularities that admit crepant resolution, that is, their resolution does not affect the CY condition. There are two types of primitive isolated singularities on CY 3-folds: small contractions or conifold singularities, and del Pezzo singularities [26, 27]. We will study the example of del Pezzo 6 singularity and some number of conifolds on the quintic. The presence of conifold singularities is important if we want to put fractional branes at the del Pezzo singularity. Without conifolds, the nonanomalous two-cycles on del Pezzo (i.e., the ones that do not intersect the canonical class) are trivial within the CY manifold. It is impossible to put the fractional branes on such "cycles", because the corresponding RR fluxes have "nowhere to go." In the presence of conifolds, some of the two-cycles on del Pezzo may become homologous to the two-cycles of the conifolds (this will be the case in our example). Then we can put some number of D5-branes on the two-cycles of del Pezzo and some number of anti-D5-branes on the two-cycles of the conifolds. Such configuration of branes and antibranes is a first step in the geometrical SUSY breaking [8, 28]. Also the possibility to introduce the fractional branes will be crucial for the D-brane realizations of ISS construction.

In Section 3, we discuss the compactification of the geometrical SUSY breaking and the ISS model and find an ISS SUSY breaking vacuum in a quiver gauge theory for the $dP_6$ singularity.

In Section 4, we formulate the general construction of compact CY manifolds with del Pezzo singularities and discuss the complex deformations of these singularities. We observe that the number of deformations depends on the global properties of the two-cycles on del Pezzo that do not intersect the canonical class and have self-intersection $(-2)$. Suppose all such cycles are trivial within the CY, then the singularity has the maximal number of deformations. This will be the case for our embeddings of del Pezzo 5, 6, 7, and 8 singularities and for the cone over $\mathbb{P}^1 \times \mathbb{P}^1$. In the case of $dP_0 = \mathbb{P}^2$ and $dP_1$ singularities, we do not expect to find any deformations. In the case of del Pezzo 2, 3, and 4, our embedding leaves some of the $(-2)$ two-cycles nontrivial within the CY; accordingly, we find less complex deformations. This result can be expected, since it is known that the del Pezzo singularities for $n \leq 4$ in general cannot be represented as complete intersections [27, 29]. In our case, the del Pezzo singularities are complete intersections but they are not generic. Specific equations for embedding of del Pezzo singularities and their deformations are provided in the appendix.

Section 5 contains discussion and conclusions.

## 2. Del Pezzo 6 and Conifold Singularities on the Quintic CY

The CY manifolds can have two types of primitive isolated singularities: conifold singularities and del Pezzo singularities [26, 27]. Correspondingly, we will have two types of geometric transitions.

(1) Type I, or conifold transitions: several $\mathbb{P}^1$'s shrink to form conifold singularities and then these singularities are deformed.

(2) Type II, or del Pezzo transition: a del Pezzo shrinks to a point and the corresponding singularity is deformed.

In order to illustrate the geometric transitions, we will study a particular example of transitions on the quintic CY. The example is summarized in the diagram in Figure 1. The type I transitions are horizontal, whereas the type II transitions are vertical. It is known [24] that the maximal number of deformations of a cone over $dP_6$ is $c^\vee(E_6) - 1 = 11$, where $c^\vee(E_6) = 12$ is the dual Coxeter number of $E_6$. Going along the left vertical arrow we recover all complex deformations of the cone over $dP_6$. In this case, all the two-cycles that do not intersect the canonical class on $dP_6$ are trivial within the CY.

For the CY with both del Pezzo and conifold singularities, the deformation of the del Pezzo singularity has only 7 parameters (right vertical arrow). The del Pezzo surface is not generic in this case. It has a two-cycle that is nontrivial within the full CY and does not intersect the canonical class inside del Pezzo. As a general rule, the existence of nontrivial two-cycles reduces the number of possible complex deformations.

The horizontal arrows represent the conifold transitions. In our example, we have 36 conifold singularities on the quintic CY. These singularities have 35 complex deformations. In the presence of $dP_6$ singularity, there will be only 32 conifolds that have, respectively, 31 complex deformations. (It may seem puzzling that we need exactly 36 or 32 conifolds. One can easily find the examples of quintic CY with fewer conifold singularities. But it is impossible to blow up these singularities unless we have a specific number of them at specific locations. In the example considered in [11, 12], the quintic CY has 16 conifolds placed at a $\mathbb{P}^2$ inside the CY.)

In general, the del Pezzo singularity and the conifold singularities are away from each other but they still affect the number of complex deformations, that is, the presence of conifolds reduces the number of deformations of del Pezzo singularity and vice versa. The diagram in Figure 1 is commutative, and the total number of complex deformations of the CY with the del Pezzo singularity and 32 conifold singularities is 42. But the interpretation of these deformations changes whether we first deform the del Pezzo singularity or we first deform the conifold singularities.

Before we go to the calculations, let us clarify what we mean by the deformations of the del Pezzo singularity. We will distinguish three kinds of deformations. The deformations of the shape of the cone, the deformations of the blown up del Pezzo with fixed canonical class and deformations that smooth out the singularity.

The first kind of deformations corresponds to the general deformations of del Pezzo surface at the base of the cone. Recall that the $dP_n$ surface for $n > 4$ has $2n - 8$ deformations that parameterize the superpotential of the corresponding quiver gauge theory [5].

The second kind of deformations is obtained by blowing up the singularity and fixing the canonical class on the del Pezzo. In this case, the deformations of del Pezzo $n$ surface can be described as the deformations of $E_n$ singularity on the del Pezzo [30]. The deformations of this singularity have $n$ parameters, corresponding to the $n$ two-cycles that do not intersect

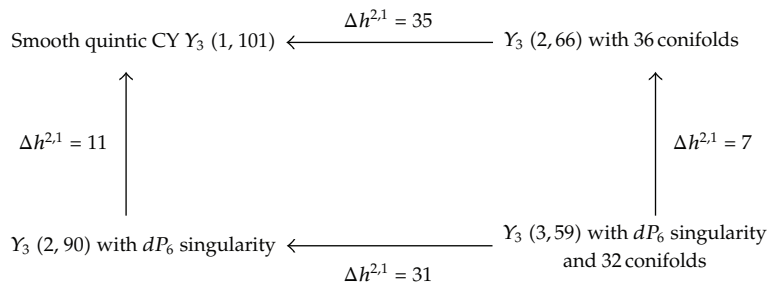


**Figure 1:** Possible geometric transitions of quintic CY. The numbers in parentheses denote the dimensions $(h^{1,1}, h^{2,1})$.

the canonical class. Note that the intersection matrix of these two-cycles is (minus) the Cartan matrix of $E_n$. The $E_n$ singularity on the del Pezzo is an example of du Val surface singularity [31] (also known as an ADE singularity or a Kleinian singularity). A three-dimensional singularity that has a du Val singularity in a hyperplane section is called compound du Val (cDV) [26, 31]. The conifold is an example of cDV singularity since it has the $A_1$ singularity in a hyperplane section. The generalized conifolds [32, 33] also have an ADE singularity in a hyperplane section, that is, from the 3-dimensional point of view they correspond to some cDV singularities. In terms of the large $N$ gauge/string duality, the deformation of the $E_n$ generalized conifold singularity corresponds to putting some combination of fractional branes on the zero size two-cycles at the singularity. Hence, the deformtion of cDV singularity that restricts to $E_n$ singularity on the del Pezzo can be considered as a generalized type I transition.

We will be mainly interested in the the third type of deformations that correspond to smoothing of del Pezzo singularities. These deformations make the canonical class of del Pezzo surface trivial within the CY. If we put some number of nonanomalous fractional D-branes at the singularity, then the corresponding geometric transition smooths the singularity [24]. But not all the deformations can be described in this way.

In order to get some intuition about possible interpretations of these deformations, we will consider the del Pezzo 6 singularity. It is known that the $dP_6$ singularity has 11 complex deformations [21, 34] but there are only 6 nonanomalous fractional branes in the corresponding quiver gauge theory and there are only 6 two-cycles that do not intersect the canonical class [24]. It will prove helpful to start with a quintic CY that has 36 conifold singularities. The del Pezzo 6 singularity can be obtained by merging four conifolds at one point. There are 7 deformations of del Pezzo 6 singularity that separate these four conifolds (right vertical arrow). The remaining 4 deformations of $dP_6$ cone correspond to 4 deformations of the four "hidden" conifolds at the singularity. Note that the total number of deformations is 11 (left vertical arrow).

## *2.1. Quintic CY*

The description of the quintic CY is well known [16]. Here, we repeat it in order to recall the methods [16] of finding the topology and deformations that we use later in more difficult situations.

The quintic CY manifold $Y_3$ is given by a degree five equation in $\mathbb{P}^4$

$$Q_5(z_i) = 0, \tag{2.1}$$

where $(z_0, z_1, z_2, z_3, z_4) \in \mathbb{P}^4$. The total Chern class of this manifold is

$$c(Y_3) = \frac{(1+H)^5}{1+5H} = 1 + 10H^2 - 40H^3 \tag{2.2}$$

and the first Chern class $c_1(Y_3) = 0$.

Let us calculate the number of complex deformations. The complex structures are parameterized by the coefficients in (2.1) up to the change of coordinates in $\mathbb{P}^4$. The number of coefficients in a homogeneous polynomial of degree $n$ in $k$ variables is

$$\binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}. \tag{2.3}$$

In the case of the quintic in $\mathbb{P}^4$, the number of coefficients is

$$\binom{9}{5} = \frac{9!}{5!4!} = 126. \tag{2.4}$$

The number of reparametrizations of $\mathbb{P}^4$ is equal to $\dim Gl(5) = 25$. Thus, the dimension of the space of complex deformations is 101.

The number of complex deformations of CY threefolds is equal to the dimension of $H^{2,1}$ cohomology group

$$h^{2,1} = h^{1,1} - \frac{\chi}{2}, \tag{2.5}$$

where $h^{1,1}$ can be found via the Lefschetz hyperplane theorem [16, 35]

$$h^{1,1}(Y_3) = h^{1,1}\left(\mathbb{P}^4\right) = 1 \tag{2.6}$$

and the Euler characteristic is given by the integral of the highest Chern class over $Y_3$

$$\chi = \int_{Y_3} c_3 = \int_{\mathbb{P}^4} -40H^3 \wedge 5H = -200, \tag{2.7}$$

here, we have used that $5H$ is the Poincare dual class to $Y_3$ inside $\mathbb{P}^4$. Consequently, $h^{2,1} = 101$ which is consistent with the number of complex deformations found before.

### 2.2. Quintic CY with $dP_6$ Singularity

Suppose that the quintic equation is not generic but has a degree three zero at the point $(w_0, w_1, w_2, w_3, w_4) = (0,0,0,0,1)$,

$$P_3(w_0,\ldots,w_3)w_4^2 + P_4(w_0,\ldots,w_3)w_4 + P_5(w_0,\ldots,w_3) = 0, \tag{2.8}$$

where $P_n$'s denote degree $n$ polynomials. The shape of the singularity is determined by $P_3(w_0,\ldots,w_3)$ (we will see that this polynomial defines the del Pezzo at the tip of the cone). The deformations that smooth out the singularity correspond to adding less singular terms to (2.8), that is, the terms that have bigger powers of $w_4$.

The resolution of the singularity in (2.8) can be obtained by blowing up the point $(0,0,0,0,1) \in \mathbb{P}^4$. Away from the blowup, we can use the following coordinates on $\mathbb{P}^4$:

$$(w_0,\ldots,w_3,w_4) = (tz_0,\ldots,tz_3,s), \tag{2.9}$$

where $(s,t) \in \mathbb{P}^1$ and $(z_0,\ldots,z_3) \in \mathbb{P}^3$. The blowup of the point at $t = 0$ corresponds to inserting the $\mathbb{P}^3$ instead of this point. Hence, the points on the blown up $\mathbb{P}^4$ can be parameterized globally by $(z_0,\ldots,z_3) \in \mathbb{P}^3$ and $(s,t) \in \mathbb{P}^1$. The projective invariance $(s,t) \sim (\lambda s, \lambda t)$ corresponds to the projective invariance in the original $\mathbb{P}^4$. In order to compensate for the projective invariance of $\mathbb{P}^3$, we need to assume that locally the coordinates on $\mathbb{P}^1$ belong to the following line bundles over $\mathbb{P}^3$, $s \in \mathcal{O}$ and $t \in \mathcal{O}(-H)$. Thus, the blowup of $\mathbb{P}^4$ at a point is a $\mathbb{P}^1$ bundle over $\mathbb{P}^3$ obtained by projectivization of the direct sum of $\mathcal{O}_{\mathbb{P}^3}$ and $\mathcal{O}_{\mathbb{P}^3}(-H)$ bundles, $\widetilde{\mathbb{P}}^4 = P(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-H))$ (for more details on projective bundles see, e.g., [36, 37]). In working with projective bundles, we will use the technics similar to [37].

Using parametrization (2.9), we can write the equation on the blown up $\mathbb{P}^4$ as

$$P_3(z_0,\ldots,z_3)s^2 + P_4(z_0,\ldots,z_3)st + P_5(z_0,\ldots,z_3)t^2 = 0. \tag{2.10}$$

This equation is homogeneous of degree two in the coordinates on $\mathbb{P}^1$ and degree three in the $z_i$'s. Note that $t \in \mathcal{O}(-H)$, that is, it has degree $(-1)$ in the $z_i$'s and $s \in \mathcal{O}$ has degree zero.

Let us prove that the manifold defined by (2.10) has vanishing first Chern class, that is, it is a CY manifold. Let $H$ be the hyperplane class in $\mathbb{P}^3$ and $G$ the hyperplane class on the $\mathbb{P}^1$ fibers. Let $M = P(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-H))$ denote the $\mathbb{P}^1$ bundle over $\mathbb{P}^3$. The total Chern class of $M$ is

$$c(M) = (1+H)^4(1+G)(1+G-H), \tag{2.11}$$

where $(1+H)^4$ is the total Chern class of $\mathbb{P}^3$, $(1+G)$ corresponds to $s \in \mathcal{O}_{\mathbb{P}^3}$, and $(1+G-H)$ corresponds to $t \in \mathcal{O}_{\mathbb{P}^3}(-H)$. Note that $G(G-H) = 0$ on this $\mathbb{P}^1$ bundle and, as usual, $H^4 = 0$ on the $\mathbb{P}^3$.

Let $Y_3$ denote the surface embedded in $M$ by (2.10). Since the equation has degree 3 in $z_i$ and degree two in $(s,t)$, the class Poincare dual to $Y_3 \subset M$ is $3H + 2G$ and the total Chern class is

$$c(Y_3) = \frac{(1+H)^4(1+G)(1+G-H)}{1+3H+2G}. \tag{2.12}$$

Expanding $c(Y_3)$, it is easy to check that $c_1(Y_3) = 0$.

The intersection of $Y_3$ with the blown up $\mathbb{P}^3$ at $t = 0$ is given by the degree three equation $P_3(z_0, \ldots, z_3) = 0$ in $\mathbb{P}^3$. The surface $B$ defined by this equation is the del Pezzo 6 surface [16, 35]. The total Chern class and the Euler character of $B$

$$c(B) = \frac{(1+H)^4}{1+3H} = 1 + H + 3H^2, \tag{2.13}$$

$$\chi(B) = \int_B c_2(B) = \int_{\mathbb{P}^3} 3H^2 \wedge 3H = 9. \tag{2.14}$$

In the calculation of $\chi(B)$, we have used that $3H$ is the Poincare dual class to $B$ inside $\mathbb{P}^3$.

It is known that the normal bundle to contractable del Pezzo in a CY manifold is the canonical bundle on del Pezzo [38]. Let us check this statement in our example. The canonical class is minus the first Chern class that can be found from (2.13) (Slightly abusing the notations, we denote by $H$ both the class of $\mathbb{P}^3$ and the restriction of this class to $B \in \mathbb{P}^3$.)

$$K(B) = -H. \tag{2.15}$$

The coordinate $t$ describes the normal direction to $B$ inside $Y_3$. Since $t \in \mathcal{O}_{\mathbb{P}^3}(-H)$, restricting to $B$ we find that $t$ belongs to the canonical bundle over $B$. Hence locally, near $t = 0$, the CY threefold $Y_3$ has the structure of the CY cone over the del Pezzo 6 surface.

The smoothing of the singularity corresponds to adding less singular terms in (2.8). These terms have 15 parameters, but also we get back 4 reparametrizations (now, we can add $w_4$ to the other coordinates). Hence, smoothing of the singularity corresponds to 11 complex structure deformations that is the maximal expected number of deformations of $dP_6$ singularity.

In view of applications in Section 4, let us describe the geometric transition between the CY with the resolved $dP_6$ singularity and a smooth quintic CY in more details. As we have shown above, the CY with the blown up $dP_6$ singularity can be described by the following equation in the $\mathbb{P}^1$ bundle over $\mathbb{P}^3$:

$$P_3(z_0, \ldots, z_3)s^2 + P_4(z_0, \ldots, z_3)st + P_5(z_0, \ldots, z_3)t^2 = 0. \tag{2.16}$$

This equation can be rewritten as

$$P_3(tz_0, \ldots, tz_3)s^2 + P_4(tz_0, \ldots, tz_3)s + P_5(tz_0, \ldots, tz_3) = 0. \tag{2.17}$$

Next, we note that, being a projective bundle, $M$ is equivalent [35, 36] to $P(\mathcal{O}_{\mathbb{P}^3}(H) \oplus \mathcal{O}_{\mathbb{P}^3})$, where locally $s$ and $t$ are sections of $\mathcal{O}_{\mathbb{P}^3}(H)$ and $\mathcal{O}_{\mathbb{P}^3}$, respectively. We further observe that $tz_i$, $i = 0 \cdots 3$ are also sections of $\mathcal{O}_{\mathbb{P}^3}(H)$ and the equivalence $(t, s) \sim (\lambda t, \lambda s)$ induces the equivalence $(tz_0, \ldots, tz_i, s) \sim (\lambda tz_0, \ldots, \lambda tz_i, \lambda s)$. Consequently, if we blow down the section $t = 0$ of $M$, then $(tz_0, \ldots, tz_i, s) \in \mathbb{P}^4$. Now, we define $(w_0, \ldots, w_3, w_4) = (tz_0, \ldots, tz_3, s)$ and rewrite (2.17) as

$$P_3(w_0, \ldots, w_3)w_4^2 + P_4(w_0, \ldots, w_3)w_4 + P_5(w_0, \ldots, w_3) = 0. \tag{2.18}$$

Not surprisingly, we get back (2.8).

Above we have found that there are 11 complex deformations of the $dP_6$ singularity embedded in the quintic CY manifold. In the view of further applications, let us rederive the number of complex deformations by calculating the dimension of $H^{2,1}$.

Expanding (2.12), we get the third Chern class

$$c_3(Y_3) = -2G^3 - 13HG^2 - 17H^2G - 8H^3. \tag{2.19}$$

The Poincare dual class to $Y_3 \in M$ is $3H + 2G$ and

$$\chi(Y_3) = \int_{Y_3} c_3(Y_3) = \int_M c_3(Y_3) \wedge (3H + 2G). \tag{2.20}$$

In calculating this integral, one needs to take into account that $G(G - H) = 0$ on $M$. Finally, we get

$$\begin{gathered} \chi(Y_3) = -176, \\ h^{2,1} = h^{1,1} - \frac{\chi}{2} = 90. \end{gathered} \tag{2.21}$$

The number of complex deformations of the del Pezzo singularity is $101 - 90 = 11$, which is consistent with the number found above.

### 2.3. Quintic CY with 36 Conifold Singularities

In this subsection, we use the methods of geometric transitions [11, 12, 16] to find the quintic CY with conifold singularities, that is, we describe the upper horizontal arrow in Figure 1. Consider the system of two equations in $\mathbb{P}^4 \times \mathbb{P}^1$

$$\begin{gathered} P_3 u + R_3 v = 0, \\ P_2 u + R_2 v = 0, \end{gathered} \tag{2.22}$$

where $(u, v) \in \mathbb{P}^1$ and $P_n$, $R_n$ denote polynomials of degree $n$ in $\mathbb{P}^4$.

Suppose that at least one of the polynomials $P_3$, $R_3$, $P_2$, and $R_2$ is nonzero, then we can solve for $u$, $v$ and substitute in the second equation, where we get

$$P_3 R_2 - R_3 P_2 = 0, \tag{2.23}$$

a nongeneric quintic in $\mathbb{P}^4$. The points where $P_3 = R_3 = P_2 = R_2 = 0$ (but otherwise generic) have conifold singularities. There are $3 \cdot 3 \cdot 2 \cdot 2 = 36$ such points. The system (2.22) describes the blowup of the singularities, since every singular point is replaced by the $\mathbb{P}^1$ and the resulting manifold is non singular.

Let $H$ be the hyperplane class of $\mathbb{P}^4$ and $G$ by the hyperplane class of $\mathbb{P}^1$, then the total Chern class of $Y_3$ is

$$c = \frac{(1+H)^5(1+G)^2}{(1+3H+G)(1+2H+G)}, \tag{2.24}$$

since $c_1 = 0$, $Y_3$ is a CY.

By Lefschetz hyperplane theorem $h^{1,1}(Y_3) = h^{1,1}(\mathbb{P}^4 \times \mathbb{P}^1) = 2$, there are only two independent Kahler deformations in $Y_3$. One of them is the overall size of $Y_3$ and the other is the size of the blown up $\mathbb{P}^1$'s. Thus, the 36 $\mathbb{P}^1$'s are not independent but homologous to each other and represent only one class in $H_2(Y_3)$. If we shrink the size of blown up $\mathbb{P}^1$'s to zero, then we can deform the singularities of (2.23) to get a generic quintic CY. In this case, the 35 three chains that where connecting the 36 $\mathbb{P}^1$'s become independent three cycles. Thus, we expect the general quintic CY to have 35 more complex deformations than the quintic with 36 conifold singularities.

Calculating the Euler character similarly to the previous subsections, we find

$$h^{2,1} = 66. \tag{2.25}$$

Recall that the smooth quintic has 101 complex deformations. Thus, the quintic with 36 conifold singularities has $101 - 66 = 35$ less complex deformations than the generic one.

### 2.4. Quintic CY with $dP_6$ Singularity and 32 Conifold Singularities

The equation for the quintic CY manifold with the blown up $dP_6$ singularity was found in (2.10). Here, we reproduce it for convenience

$$P_3(z_i)s^2 + P_4(z_i)st + P_5(z_i)t^2 = 0. \tag{2.26}$$

This equation describes an embedding of the CY manifold in the $\mathbb{P}^1$ bundle $M = P(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-H))$. As before, $(z_0, \ldots, z_3) \in \mathbb{P}^3$ and $(s,t)$ are the coordinates on the $\mathbb{P}^1$ fibers over $\mathbb{P}^3$.

In order to have more Kahler deformations, we need to embed (2.26) in a space with more independent two-cycles. For example, we can consider a system of two equations in the product ($\mathbb{P}^1$ bundle over $\mathbb{P}^3$) $\times$ $\mathbb{P}^1$

$$\begin{aligned} (P_1 s + P_2 t)u + (Q_1 s + Q_2 t)v &= 0, \\ (R_2 s + R_3 t)u + (S_2 s + S_3 t)v &= 0, \end{aligned} \tag{2.27}$$

where $(u,v)$ are the coordinates on the additional $\mathbb{P}^1$. Let $G$, $H$, and $K$ be the hyperplane classes on the $\mathbb{P}^1$ fibers, on the $\mathbb{P}^3$, and on the additional $\mathbb{P}^1$, respectively. Then, the total Chern class of $Y_3$ is

$$c = \frac{(1+H)^4(1+G)(1+G-H)(1+K)^2}{(1+H+G+K)(1+2H+G+K)}, \tag{2.28}$$

and it is easy to see that the first Chern class is zero.

For generic points on the $\mathbb{P}^1$ bundle over $\mathbb{P}^3$, at least one of the functions in front of $u$ or $v$ is nonzero. Thus, we can find a point $(u, v)$ and substitute it in the second equation, which becomes a nongeneric equation similar to (2.26)

$$(P_1S_2 - Q_1R_2)s^2 + (P_1S_3 + P_2S_2 - Q_1R_3 - Q_2R_2)st + (P_2S_3 - Q_2R_3)t^2 = 0. \tag{2.29}$$

The CY manifold defined in (2.27) has the following characteristics:

$$\begin{aligned} \chi &= \int_{Y_3} c_3 = -112, \\ h^{1,1} &= 3, \\ h^{2,1} &= h^{1,1} - \frac{\chi}{2} = 59. \end{aligned} \tag{2.30}$$

Recall that the number of complex deformations on the quintic with the del Pezzo 6 singularity is 90. Since we lose 31 complex deformations, we expect that the corresponding three-cycles become the three chains that connect 32 $\mathbb{P}^1$'s at the blowups of the singularities in (2.29). These singularities occur when all four equations in (2.27) vanish

$$\begin{aligned} R_2s + R_3t &= 0, \\ S_2s + S_3t &= 0, \\ P_1s + P_2t &= 0, \\ Q_1s + Q_2t &= 0. \end{aligned} \tag{2.31}$$

The number of solutions equals the number of intersections of the corresponding classes $\int_M (2H + G)^2 (H + G) = 32$, where $M$ is the $\mathbb{P}^1$ bundle over $\mathbb{P}^3$ and $G(G - H) = 0$.

The right vertical arrow corresponds to smoothing of del Pezzo singularity in the presence of conifold singularities. Before the transition, the CY has $h^{2,1} = 59$ deformations and after the transition it has $h^{2,1} = 66$ deformations. Hence, the number of complex deformations of $dP_6$ singularity is $66 - 59 = 7$ which is less than $c^\vee(E_6) - 1 = 11$. This is related to the fact that the del Pezzo at the tip of the cone is not generic. The equation of the del Pezzo can be found by restricting (2.27) to $t = 0$, $s = 1$ section

$$\begin{aligned} P_1u + Q_1v &= 0, \\ R_2u + S_2v &= 0. \end{aligned} \tag{2.32}$$

This del Pezzo contains a two-cycle $\alpha$ that is nontrivial within the full CY and does not intersect the canonical class inside $dP_6$.

In the rest of this subsection, we will argue that $\alpha$ is homologous to four $\mathbb{P}^1$'s at the tip of the conifolds. The heuristic argument is the following. The formation of $dP_6$ singularity on the CY manifold with 36 conifolds reduces the number of conifolds to 32. Let us show that the deformation of the del Pezzo singularity that preserves the conifold singularities corresponds

to separating 4 conifolds hidden in the del Pezzo singularity. The CY that has a $dP_6$ singularity and 32 resolved conifolds can be found from (2.27) by the following coordinate redefinition $(w_0, \ldots, w_3, w_4) = (tz_0, \ldots, tz_3, s)$ (compare to the discussion after (2.17)):

$$\begin{aligned} (P_1 w_4 + P_2)u + (Q_1 w_4 + Q_2)v = 0, \\ (R_2 w_4 + R_3)u + (S_2 w_4 + S_3)v = 0. \end{aligned} \tag{2.33}$$

If we blow down the $\mathbb{P}^1$, then we get the quintic CY with 32 conifold singularities and a $dP_6$ singularity. For a finite size $\mathbb{P}^1$, the conifold singularities and one of the two-cycles in the $dP_6$ are blown up. The deformations of $dP_6$ singularity correspond to adding terms with higher power of $w_4$. After the deformation, the degree two zeros of $R_2$ and $S_2$ will split into four degree one zeros that correspond to the four conifolds "hidden" in the $dP_6$ singularity. The blown up two-cycle of $dP_6$ is homologous to the two-cycles on the four conifolds. (Formally, we can prove this by calculating the corresponding Poincaré dual classes. The Poincaré dual of $\mathbb{P}^1$ on the blown up conifold is $H^3G$—this is the $\mathbb{P}^1$ parameterized by $(u, v)$. The Poincaré dual of the canonical class on $dP_6$ is $(G-H)(H+K)(2H+K)(-H)$, where $(G-H)$ restricts to $t = 0$ section of the $\mathbb{P}^1$ bundle, $(H+K)(2H+K)$ restricts to $dP_6$ in (2.32), while the restriction of $(-H)$ is the canonical class on $dP_6$ (see (2.15)). The class that does not intersect $(-H)$ inside $dP_6$ is dual to $(G-H)(H+K)(2H+K)(2H-3G) = 4H^3G$, q.e.d.)

## 3. SUSY Breaking

In this paper, we compare two mechanisms for dynamical SUSY breaking: the "geometrical" approach of Aganagic et al. [8] and a more "physical" approach of ISS [10].

In both approaches, there is a confinement in the microscopic gauge theory leading to the SUSY breaking in the effective theory. But the particular mechanisms and the effective theories are quite different. In the "geometrical" approach the effective theory is a non-SUSY analog of Veneziano-Yankielowicz superpotential [39] for the gaugino bilinear field $S$. This potential has an interpretation as the GVW superpotential [40] for the complex structure moduli of the CY manifold. The original Veneziano-Yankielowicz potential [39] is derived for the pure YM theory without any flavors. It has a number of isolated vacua and no massless fields. This is a nice feature for the (meta) stability of the vacuum but, since all the fields are massive, the applications of this potential in the low-energy effective theories are limited (see, e.g., the discussion in [41]).

In the ISS construction, the number of flavors is bigger than the number of colors $N_c < N_f < 3/2N_c$ (and probably $N_f = N_c$). After the confinement, the low-energy effective theory contains classically massless fields that get some masses only at 1 loop. Hence, this theory is a more genuine effective theory but the geometric interpretation is harder to achieve [3]. Moreover, the geometric constructions similar to [3] generally have D5-branes wrapping vanishing cycles. In any compactification of these models, one has to put the O-planes or anti D5-branes somewhere else in the geometry, that is, the analysis of [8, 9] becomes inevitable.

In summary, it seems that the ISS construction is more useful for immediate applications to SUSY breaking in the low-energy effective theories, whereas more global geometric analysis of [8, 9] becomes inevitable in the compactifications.

In the previous section, we constructed the compact CY with del Pezzo 6 singularity and some number of conifold singularities. We have shown that it is possible to make some

two-cycles on del Pezzo homologous to the two-cycles on the conifolds. This is the first step in the geometric analysis of [8]. In the next subsection, we show how the ISS story can be represented in the del Pezzo 6 quiver gauge theories.

### *3.1. ISS Vacuum for the $dP_6$ Singularity*

Consider the quiver gauge theory for the cone over $dP_6$ represented in Figure 2. This quiver can be found by the standard methods [1] from the three-block exceptional collection of sheaves [42]. But, in order to prove the existence of this quiver, it is easier to do the Seiberg dualities on the nodes 4, 5, 6, and 1 and reduce it to the known $dP_6$ quiver [2].

In the compact CY manifold, one can put the D5-branes only on cycles that are nontrivial globally. A deformation of the $dP_6$ singularity in (2.33) leaves four conifold singularities. We will assume that after joining the 4 conifolds to form a $dP_6$ singularity the two-cycles remain nontrivial. We also expect that these two-cycles are represented by the four two-cycles on del Pezzo that have self-intersection $(-2)$ and do not intersect with each other. Note that the total number of nonanomalous fractional branes and the number of $(-2)$ two-cycles is 6, but there are only 4 two-cycles that do not intersect with each other and with the canonical class. (It is interesting to note the similarity between the branes wrapping the non intersecting cycles on $dP_6$ and the deformation D-branes in [3, 23].)

Let $A_i$ denote the two-cycle corresponding to the D5-brane charge [1] of the bound state of branes at the $i$th node in Figure 2. Then, the four non intersecting $(-2)$ two-cycles can be chosen as $A_2$-$A_3$, $A_4$-$A_5$, $A_6$-$A_7$, and $A_8$-$A_9$. Now, we would like to add $K$ fractional branes to $A_4$-$A_5$ and $N$ fractional branes to $A_6$-$A_7$ and to $A_8$-$A_9$. The corresponding quiver is depicted in Figure 3.

The gauge groups at the nodes 6 and 8 have $N_f = N_c$. Consider the Seiberg duality in the strong coupling limit of these gauge groups. The moduli space consists of the mesonic and the baryonic branches [43, 44]. Suppose we are on the baryonic branch. For the generic Yukawa couplings, the two mesons $\Phi = BC$ couple linearly to the fields $A$ and become massive together with two of the $A$ fields.

An important question is whether the baryons for the gauge groups in nodes 6 and 8 remain massless. The baryons are charged under the baryonic $U(1)_B$ symmetries. In the noncompact setting, these $U(1)_B$ symmetries are global [45]. If the baryons get vevs, then the symmetries are broken spontaneously and there are massless goldston bosons. But for the compact CY manifold the $U(1)_B$ symmetries are gauged and the goldstone bosons become massive [13, 45] through the Higgs mechanism. Integrating out the massive fields, we get the quiver in Figure 4.

Next, we assume that the strong coupling scale for the gauge group $SU(N+K)$ at node 4 is bigger than the scale for the $SU(2N)$. This assumption does not include a lot of tuning especially if $K \lesssim N$. The number of flavors for the gauge group $SU(N+K)$ is $N_f = 2N > N_c = N + K$. Consequently, we can assume that the mesons do not get VEVs after the confinement of $SU(N+K)$ and remain massless. The corresponding quiver is shown in Figure 5. The subscripts of the bifundamental fields denote the gauge groups at the ends of the corresponding link. The subscript $k = 2, 3$ labels the two $U(N)$ gauge groups on the left. For example, $A_{k1}$ denotes both the field $A_{21}$ going from the node 2 to the node 1 and $A_{31}$ going from 3 to 1.

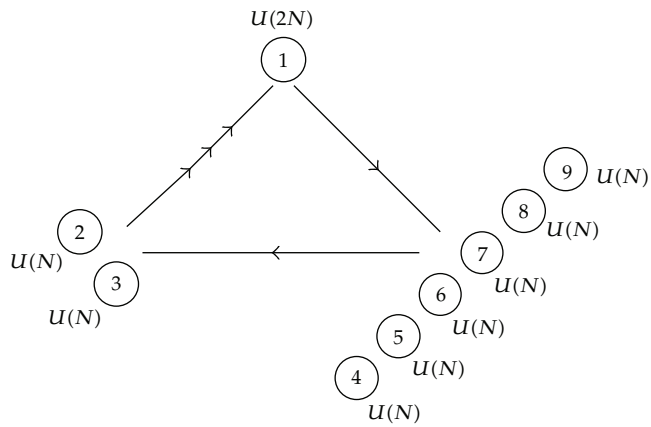


**Figure 2:** Quiver gauge theory for the cone over $dP_6$.

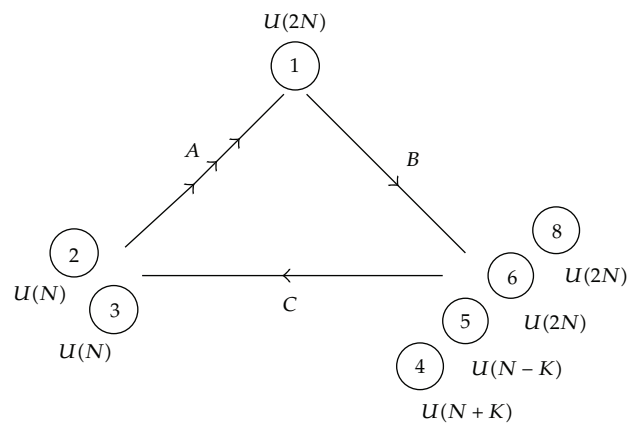


**Figure 3:** Quiver gauge theory for the cone over $dP_6$ after adding the fractional branes.

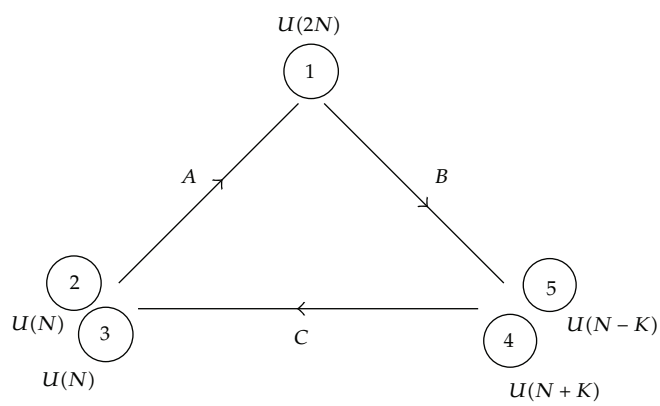


**Figure 4:** Quiver gauge theory for the cone over $dP_6$ after confinement of nodes 6 and 8.

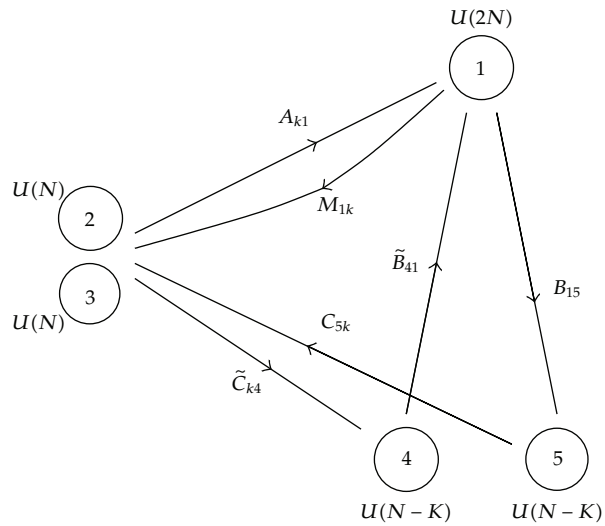


**Figure 5:** Quiver gauge theory for the cone over $dP_6$ after Seiberg duality on node 4.

The superpotential of the quiver gauge theory in Figure 5 has the form

$$\begin{aligned} W &= \mathrm{Tr}(mA_{21}M_{12} + mA_{31}M_{13}) \\ &\quad + \mathrm{Tr}\Big(\lambda M_{12}\tilde{C}_{24}\tilde{B}_{41} + \lambda A_{21}B_{15}C_{52} + \lambda M_{13}\tilde{C}_{34}\tilde{B}_{41} + \lambda A_{31}B_{15}C_{53}\Big). \end{aligned} \tag{3.1}$$

In order to make the notations shorter, we do not write the subscripts of the couplings. (The couplings are different but have the same order of magnitude.)

If $\Lambda_1$ for the $SU(2N)$ gauge group at node 1 is close to $\Lambda_4$ for $SU(N+K)$ at node 4 in Figure 4, then it is natural to assume that for small values of corresponding Yukawa couplings the mass parameters $m$ satisfy $m \ll \Lambda_1$. Now, we note that the $SU(2N)$ gauge group has $N_c = 2N$ and $N_f = 3N - K$, that is, $N_c + 1 \leq N_f < 3/2N_c$. This group is a good candidate for the the microscopic gauge group in the ISS construction. After the Seiberg duality, the magnetic gauge group has $\widetilde{N}_c = N - K$. The superpotential of the dual theory is

$$\begin{aligned} \widetilde{W} &= \mathrm{Tr}(mM_{22} + mM_{33}) \\ &\quad + \mathrm{Tr}\Big(\lambda M_{22}\widetilde{M}_{21}\tilde{A}_{12} + \lambda M_{33}\widetilde{M}_{31}\tilde{A}_{13}\Big) \\ &\quad + \mathrm{Tr}\Big(mM_{42}\tilde{C}_{24} + mM_{25}C_{52} + mM_{43}\tilde{C}_{34} + mM_{35}C_{53}\Big) \\ &\quad + \mathrm{Tr}\Big(\lambda M_{42}\widetilde{M}_{21}\tilde{\tilde{B}}_{14} + \lambda M_{25}\tilde{B}_{51}\tilde{A}_{12} + \lambda M_{43}\widetilde{M}_{31}\tilde{\tilde{B}}_{14} + \lambda M_{35}\tilde{B}_{51}\tilde{A}_{13}\Big). \end{aligned} \tag{3.2}$$

The indices of the meson fields correspond to the two gauge groups under which they transform. In our case, this leads to unambiguous identifications, for example, $M_{22} = A_{21}M_{12}$,

$M_{33} = A_{31}M_{13}$, $M_{42} = \widetilde{B}_{41}M_{12}$, and so forth. The mesons $M_{22}$ and $M_{33}$ are in adjoint representation of $SU(N)_2$ and $SU(N)_3$, and their F-term equations read

$$\begin{aligned} m \cdot \mathbf{1} + \lambda \widetilde{M}_{21}\widetilde{A}_{12} &= 0, \\ m \cdot \mathbf{1} + \lambda \widetilde{M}_{31}\widetilde{A}_{13} &= 0, \end{aligned} \tag{3.3}$$

where $\mathbf{1}$ is the $N \times N$ identity matrix. The Seiberg dual gauge group at node 1 is $SU(N-K)$; hence the rank of the matrices $\widetilde{M}_{21}$ and so forth, is at most $N-K$ and the SUSY is broken by the rank condition of [10]. Classically, there are massless excitations around the vacua in (3.3). In order to prove that the vacuum is metastable, one has to check that these fields acquire a positive mass at 1 loop. Similarly to [10], we expect this to be true, but a more detailed study is necessary.

As a summary, in this section we have found an example of dymanical SUSY breaking in the quiver gauge theory on del Pezzo singularity. An interesting property of this example is that there are massless chiral fields after the SUSY breaking. This behavoir seems to be quite generic, and we expect that similar constructions are possible for other del Pezzo singularities.

## 4. Compact CY Manifolds with Del Pezzo Singularities

Noncompact CY singularities are useful in constructing local geometries that enable SUSY breaking configurations of D-branes. However, for a consistent embedding of these constructions in string theory, one needs to find compact CY manifolds that posses the corresponding singularities.

The noncompact CY manifolds with del Pezzo singularities are known [27, 29]. The $dP_n$ singularities for $5 \leq n \leq 8$ and for the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ can be represented as complete intersections. (Note that in the mathematics literature, the del Pezzo surfaces are classified by their degree $k = 9-n$, where $n$ is the number of blown up points in $\mathbb{P}^2$.) The CY cones over $\mathbb{P}^2$ and $dP_n$ for $1 \leq n \leq 4$ are not complete intersections. The compact CY manifolds for complete intersection singularities where presented in [34].

Our construction is different from [34]. It enables one to construct the complete intersection compact CY manifolds for all del Pezzo singularities. This construction does not contradict the statement that for $n \leq 4$ the del Pezzo singularities are not complete intersections. The price we have to pay is that these singularities will not be generic, that is, they will not have the maximal number of complex deformations. Whereas for the del Pezzo singularities with $n \geq 5$ and for $\mathbb{P}^1 \times \mathbb{P}^1$ we will represent all complex deformations in our construction.

### *4.1. General Construction*

At first, we present the construction in the case of $dP_6$ singularity and, then, give a more general formulation.

The input data is the embedding of $dP_6$ surface in $\mathbb{P}^3$ via a degree three equation. The problem is to find a CY threefold such that it has a local $dP_6$ singularity. The solution has several steps.

(1) Find the canonical class on $B = dP_6$ in terms of a restriction of a class on $\mathbb{P}^3$. Let us denote this class as $K \in H^{1,1}(\mathbb{P}^3)$. $K$ can be found from expanding the total Chern class of $B$

$$c(B) = \frac{(1+H)^4}{1+3H} = 1 + H + \cdots . \tag{4.1}$$

Thus, $K = -c_1(B) = -H$.

(2) Construct the $\mathbb{P}^1$ fiber bundle over $\mathbb{P}^3$ as the projectivisation $M = P(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(K))$.

(3) The Calabi-Yau $Y_3$ is given by an equation of degree 3 in $\mathbb{P}^3$ and degree 2 in the coordinates on the fiber. The total Chern class of $Y_3$ is

$$c(Y_3) = \frac{(1+H)^4(1+G-H)(1+G)}{1+3H+2G}. \tag{4.2}$$

This has a vanishing first Chern class. By construction, this Calabi-Yau has a del Pezzo singularity at $t = 0$.

This construction has a generalization for the other del Pezzo surfaces. Let $B$ denote a del Pezzo surface embedded in $X$ as a complete intersection of a system of equations [16]. Assume, for concreteness, that the system contains two equations and denote by $L_1$ and $L_2$ the classes corresponding to the divisors for these two equations in $X$. The case of other number of equations can be obtained as a straightforward generalization.

(1) First, we find the canonical class of surface $B \subset X$, defined in terms of two equations with the corresponding classes $L_1, L_2 \in H^{1,1}(X)$,

$$c(B) = \frac{c(X)}{(1+L_1)(1+L_2)} = 1 + c_1(X) - L_1 - L_2 + \cdots . \tag{4.3}$$

Thus, the canonical class of $X$ is obtained by the restriction of $K = L_1 + L_2 - c_1(X)$.

(2) Second, we construct the $\mathbb{P}^1$ fiber bundle over $X$ as the projectivisation $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(K))$.

(3) In the case of two equations, the Calabi-Yau manifold $Y_3 \subset M$ is not unique. Let $G$ be the hyperplane class in the fibers, then we can write three different systems of equations that define a CY manifold: the classes for the equations in the first system are $L_1+2G$ and $L_2$, the second one has $L_1+G$ and $L_2+G$, and the third one has $L_1$ and $L_2 + 2G$ (here $L_1, L_2 \in H^{1,1}(M)$ are defined via the pull back of the corresponding classes in $H^{1,1}(X)$ with respect to the projection of $\mathbb{P}^1$ the fibers $\pi : M \to X$).

As an example, let us describe the first system. The first equation in this system is given by $L_1$ in $X$ and has degree 2 in the coordinates on the fibers. The second equation is $L_2$ in $X$. The total Chern class is

$$c(Y_3) = \frac{c(X)(1+G+K)(1+G)}{(1+L_1+2G)(1+L_2)}. \tag{4.4}$$

Since $K = L_1 + L_2 - c_1(X)$, it is straightforward to check that the first Chern class is trivial.

Let us show how this program works in an example of a CY cone over $B = \mathbb{P}^1 \times \mathbb{P}^1$. The $\mathbb{P}^1 \times \mathbb{P}^1$ surface can be embedded in $\mathbb{P}^3$ by a generic degree two polynomial equation [16, 35]

$$P_2(z_i) = 0, \tag{4.5}$$

where $(z_0, \ldots, z_3) \in \mathbb{P}^3$. (By coordinate redefinition in $\mathbb{P}^3$ one can represent the equation as $z_0 z_3 = z_1 z_2$. The solutions of this equation can be parameterized by the points $(x_1, y_1) \times (x_2, y_2) \in \mathbb{P}^1 \times \mathbb{P}^1$ as $(z_0, z_1, z_2, z_3) = (x_1 x_2, x_1 y_2, y_1 x_2, y_1 y_2)$. This is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.)

The first step of the program is to find the canonical class of $B$ in terms of a class in $\mathbb{P}^3$. Let $H$ be the hyperplane class of $\mathbb{P}^3$. Then, the total Chern class of $B$ is

$$c(B) = \frac{(1+H)^4}{1+2H} = 1 + 2H + 2H^2. \tag{4.6}$$

The canonical class is

$$K(B) = -c_1(B) = -2H. \tag{4.7}$$

Next, we construct the $\mathbb{P}^1$ bundle $M = P(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(K))$ with the coordinates $(s, t)$ along the fibers, where locally $s \in \mathcal{O}_{\mathbb{P}^3}$ and $t \in \mathcal{O}_{\mathbb{P}^3}(-2H)$. The equation that describes the embedding of the CY manifold $Y_3$ in $M$ is

$$P_2(z_i)s^2 + P_4(z_i)st + P_6(z_i)t^2 = 0. \tag{4.8}$$

This equation is homogeneous in $z_i$ of degree two, since $t$ has degree $-2$.

The section of $M$ at $t = 0$ is contractable, and the intersection with the $Y_3$ is $P_2(z_i) = 0$, that is, $Y_3$ is the CY cone over $\mathbb{P}^1 \times \mathbb{P}^1$ near $t = 0$.

The total Chern class of $Y_3$ is

$$c(Y_3) = \frac{(1+H)^4(1+G)(1+G-2H)}{1+2H+2G}. \tag{4.9}$$

It is easy to check that $c_1(Y_3) = 0$.

### *4.2. A Discussion of Deformations*

In this subsection, we will discuss the deformations of the del Pezzo singularities in the compact CY spaces. The explicit description of the singularities and their deformations can be found in the appendix.

The procedure is similar to the deformation of the $dP_6$ singularity described in Section 2. As before, let $Y_3 \subset M$ be an embedding of the CY threefold $Y_3$ in $M$, a $\mathbb{P}^1$ bundle over products of (weighted) projective spaces. If we blowdown the section of the $\mathbb{P}^1$ bundle

that contains the del Pezzo, then $M$ becomes a toric variety that we denote by $V$. After the blowdown, equation for the CY in $M$ becomes a singular equation for a CY embedded in $V$. The last step is to deform the equation in $V$ to get a generic CY. (In the example of $dP_6$ singularity on the quintic, the projective bundle is $M = P(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-H))$, the manifold $V$, obtained by blowing down the exceptional $\mathbb{P}^3$ in $M$, is $\mathbb{P}^4$, and the singular equation is the singular quintic in $\mathbb{P}^4$.)

Let $n$ denote the number of two-cycles on del Pezzo with self-intersection $(-2)$. The intersection matrix of these cycles is minus the Cartan matrix of the corresponding Lie algebra $E_n$.

The maximal number of complex deformations of del Pezzo singularity is $c^\vee(E_n) - 1$, where $c^\vee(E_n)$ is the dual Coxeter number of $E_n$. These deformations can be performed only if the del Pezzo has a zero size. As a result of these deformations, the canonical class on the del Pezzo becomes trivial within the CY and the del Pezzo singularity is partially or completely smoothed out. In the generic situation, we expect that all $(-2)$ two-cycles on del Pezzo are trivial within the CY, then the number of complex deformations is maximal (this will be the case for $\mathbb{P}^1 \times \mathbb{P}^1$, $dP_5$, $dP_6$, $dP_7$, $dP_8$). If some of the $(-2)$ two-cycles become nontrivial within the CY, then the number of complex deformations of the corresponding cone is smaller. We will observe this for our embedding of $dP_2$, $dP_3$, and $dP_4$. This reduction of the number of complex deformations depends on the particular embedding of del Pezzo cone. In [8], the generic deformations of the cones over $dP_2$ and $dP_3$ were constructed (Tables 1 and 2).

## 5. Conclusions and Outlook

In this paper, we have constructed a class of compact Calabi-Yau manifolds that have del Pezzo singularities. The construction is analytic, that is, the CY manifolds are described by a system of equations in the $\mathbb{P}^1$ bundles over the projective spaces.

We argue that this construction can be used for the geometrical SUSY breaking [8] as well as for the compactification of ISS [10]. As an example, we find a compact CY manifold with del Pezzo 6 singularity and some conifolds such that some 2-cycles on del Pezzo are homologous to the 2-cycles on the conifolds, that is, this manifold can be used for the geometrical SUSY breaking. Also we find an ISS vacuum in the quiver gauge theory for $dP_6$ singularity.

In order to have a consistent string theory representation of the SUSY breaking vacua, one needs to find compact CY manifolds that have the necessary local singularities. In the last section, we present embedding of del Pezzo singularities in complete intersection CY manifolds and study the complex deformations of the singularities. The del Pezzo $n$ surface corresponds to the Lie group $E_n$. The expected number of complex deformations for the cone over del Pezzo is $c^\vee(E_n) - 1$, where $c^\vee$ is the dual Coxeter number for the Lie group $E_n$. In the studied examples, the cones over $\mathbb{P}^1 \times \mathbb{P}^1$ and over $dP_5$, $dP_6$, $dP_7$, and $dP_8$ have generic deformations. But the cones over $dP_2$, $dP_3$, and $dP_4$ have less deformations, that is, these cones do not describe the most generic embedding of the corresponding del Pezzo singularities. (It is known that the generic embeddings of del Pezzo $n$ singularities for $n \leq 4$ (or rank $k = 9 - n \geq 5$) cannot be represented as complete intersections [27, 29], in our construction the del Pezzo singularities are nongeneric complete intersections.)

We propose that for the generic embedding the two-cycles on del Pezzo with self-intersection $(-2)$ are trivial within the full Calabi-Yau geometry. The nontrivial two cycles with self-intersection $(-2)$ impose restrictions on the complex deformations. This proposal

**Table 1:** Some characteristics of del Pezzo surfaces.

| Del Pezzo | No. two-cycles | No. (−2) two-cycles | Dynkin's diagram | $c^\vee - 1$ |
|---|---|---|---|---|
| $\mathbb{P}^2$ | 1 | 0 | 0 | 0 |
| $\mathbb{P}^1 \times \mathbb{P}^1$ | 2 | 1 | $A_1$ | 1 |
| $dP_1$ | 2 | 0 | 0 | 0 |
| $dP_2$ | 3 | 1 | $A_1$ | 1 |
| $dP_3$ | 4 | 3 | $A_2 \times A_1$ | 3 |
| $dP_4$ | 5 | 4 | $A_4$ | 4 |
| $dP_5$ | 6 | 5 | $D_5$ | 7 |
| $dP_6$ | 7 | 6 | $E_6$ | 11 |
| $dP_7$ | 8 | 7 | $E_7$ | 17 |
| $dP_8$ | 9 | 8 | $E_8$ | 29 |

**Table 2:** Complex deformations of del Pezzo singularities studied in the paper.

| Del Pezzo | No. (−2) two-cycles | No. trivial (−2) two-cycles | $c^\vee - 1$ | No. complex deforms |
|---|---|---|---|---|
| $\mathbb{P}^2$ | 0 | 0 | 0 | 0 |
| $\mathbb{P}^1 \times \mathbb{P}^1$ | 1 | 1 | 1 | 1 |
| $dP_1$ | 0 | 0 | 0 | 0 |
| $dP_2$ | 1 | 0 | 1 | 0 |
| $dP_3$ | 3 | 1 | 3 | 1 |
| $dP_4$ | 4 | 3 | 4 | 3 |
| $dP_5$ | 5 | 5 | 7 | 7 |
| $dP_6$ | 6 | 6 | 11 | 11 |
| $dP_7$ | 7 | 7 | 17 | 17 |
| $dP_8$ | 8 | 8 | 29 | 29 |

agrees with the above examples of the embeddings of del Pezzo singularities. Also we get a similar conclusion when the CY has some number of conifolds in addition to the del Pezzo singularity. Although the conifolds are away from the del Pezzo and the del Pezzo itself is not singular, it acquires a nontrivial two-cycle and the number of deformations is reduced.

Sometimes the F-theory/orientifolds point of view has advantages compared to the type IIB theory. Our construction of CY threefolds can be generalized to find the 3-dimensional base spaces of elliptic fibrations in F-theory with the necessary del Pezzo singularities. Also we expect this construction to be useful as a first step in finding the warped deformations of the del Pezzo singularities and in the studies of the Landscape of string compactifications.

## Appendix

## A List of Compact CY with Del Pezzo Singularities

In the appendix, we construct the embeddings of all del Pezzo singularities in compact CY manifolds and describe the complex deformations of these embeddings. This description follows the general construction in Section 4.

In the following, $B$ denotes the two-dimensional del Pezzo surface and $X$ denotes the space where we embed $B$. The space $X$ will be either a product of projective spaces or a

weighted projective space. For example, if $B \subset X = \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^k$, then the coordinates on the three projective spaces will be denoted as $(z_0, \ldots, z_n)$, $(u_0, \ldots, u_m)$, and $(v_0, \ldots, v_k)$, respectively. The hyperplane classes of the three projective spaces will be denoted by $H$, $K$, $R$, respectively.

A polynomial of degree $q$ in $z_i$, degree $r$ in $u_j$, and degree $s$ in $v_l$ will be denoted by $P_{q,r,s}(z_i; u_j; v_l)$.

If there are only two or one projective space, then we will use the first two or the first one projective spaces in the above definitions.

For the weighted projective spaces, we will use the notations of [30]. For example, consider the space $W\mathbb{P}^3_{11pq}$, where $p, q \in \mathbb{N}$. The dimension of this space is 3, the subscripts $(1, 1, p, q)$ denote the weights of the coordinates with respect to the projective identifications $(z_0, z_1, z_2, z_3) \sim (\lambda z_0, \lambda z_1, \lambda^p z_2, \lambda^q z_3)$.

The $\mathbb{P}^1$ bundles over $X$ will be denoted as $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(K))$, where $K$ is the class on $X$ that restricts to the canonical class on $B$. The coordinates on the fibers will be $(s, t)$ so that locally $s \in \mathcal{O}_X$ and $t \in \mathcal{O}_X(K)$. The hyperplane class of the fibers will be denoted by $G$, it satisfies the property $G(G + K) = 0$ for $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(K))$. In the construction of the $\mathbb{P}^1$ bundles, we will use the fact that $K(B) = -c_1(B)$ and will not calculate $K(B)$ separately.

The deformations of some del Pezzo singularities will be described via embedding in particular toric varieties. We will call them generalized weighted projective spaces. Consider, for example, the following notation:

$$\begin{array}{l} GW\mathbb{P}^5 \\ \quad 11100002 \\ \quad 00011001 \\ \quad 00000111 \end{array} \tag{A.1}$$

The number 5 is the dimension of the space. This space is obtained from $\mathbb{C}^{8*}$ by taking the classes of equivalence with respect to three identifications. The numbers in the three rows correspond to the charges under these identifications

$$\begin{gathered} (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \sim \left(\lambda_1 z_1, \lambda_1 z_2, \lambda_1 z_3, z_4, z_5, z_6, z_7, \lambda_1^2 z_8\right), \\ (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \sim (z_1, z_2, z_3, \lambda_2 z_4, \lambda_2 z_5, z_6, z_7, \lambda_2 z_8), \\ (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \sim (z_1, z_2, z_3, z_4, z_5, \lambda_3 z_6, \lambda_3 z_7, \lambda_3 z_8). \end{gathered} \tag{A.2}$$

(1) $B = \mathbb{P}^2 \subset X = \mathbb{P}^3$.

The equation for $B$

$$P_1(z_i) = 0. \tag{A.3}$$

The total Chern class of $B$

$$c(B) = (1 + H)^3 = 1 + 3H + 3H^2. \tag{A.4}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-3H))$. The equation for the Calabi-Yau threefold $Y_3$

$$P_1(z_i)s^2 + P_4(z_i)st + P_7(z_i)t^2 = 0. \tag{A.5}$$

The embedding space $V = W\mathbb{P}^4_{11113}$ has the coordinates $(z_0, \ldots, z_3;\ w)$ and the singular CY is

$$P_1(z_0, \ldots, z_3)w^2 + P_4(z_0, \ldots, z_3)w + P_7(z_0, \ldots, z_3) = 0. \tag{A.6}$$

This is already the most general equation, that is, there are no additional complex deformations.

(2) $B = \mathbb{P}^1 \times \mathbb{P}^1 \subset X = \mathbb{P}^3$.

The equation for $B$

$$P_2(z_i) = 0. \tag{A.7}$$

The total Chern class of $B$

$$c(B) = \frac{(1+H)^4}{1+2H} = 1 + 2H + 2H^2. \tag{A.8}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-2H))$. The equation for the Calabi-Yau threefold $Y_3$

$$P_2(z_i)s^2 + P_4(z_i)st + P_6(z_i)t^2 = 0. \tag{A.9}$$

The embedding space $V = W\mathbb{P}^4_{11112}$ has the coordinates $(z_0, \ldots, z_3;\ w)$ and the singular CY is

$$P_2(z_i)w^2 + P_4(z_i)w + P_6(z_i) = 0. \tag{A.10}$$

This equation has one deformation $kw^3$, and the spaces $M$ and $V$ have the same number of coordinate redefinitions. Thus, the space of complex deformations is one dimensional.

(3) $B = dP_1 \subset X = \mathbb{P}^2 \times \mathbb{P}^1$.

The equation defining $B$ has degree one in $z_i$ and degree one in $u_j$

$$P_1(z_i)u_0 + Q_1(z_i)u_1 = 0. \tag{A.11}$$

The total Chern class of $B$

$$c(B) = \frac{(1+H)^3(1+K)^2}{1+H+K} = 1 + 2H + K + H^2 + 3HK. \tag{A.12}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-2H - K))$. The equation for the Calabi-Yau threefold $Y_3$ is

$$P_{1,1}(z_i; u_j)s^2 + P_{3,2}(z_i; u_j)st + P_{5,3}(z_i; u_j)t^2 = 0. \tag{A.13}$$

The embedding space $V = GW\mathbb{P}^4_{\substack{111002\\000111}}$ has the coordinates $(z_0, z_1, z_2;\ u_0, u_1;\ w)$ and the singular CY is

$$P_{1,1}(z_i; u_j)w^2 + P_{3,2}(z_i; u_j)w + P_{5,3}(z_i; u_j) = 0. \tag{A.14}$$

There are no complex deformations of this equation.

(4) $B = dP_2 \subset X = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.

The del Pezzo surface is defined by a system of two equations. The first equation has degree one in $z_i$ and degree one in $u_k$. The second equation has degree one in $z_i$ and degree one in $v_k$

$$\begin{aligned} P_1(z_i)u_0 + Q_1(z_i)u_1 &= 0, \\ R_1(z_i)v_0 + S_1(z_i)v_1 &= 0. \end{aligned} \tag{A.15}$$

The total Chern class of $B$

$$c(B) = \frac{(1+H)^3(1+K)^2(1+R)^2}{(1+H+K)(1+H+R)} = 1 + 2H + K + R + 2H(K+R) + KR. \tag{A.16}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-2H - K - R))$. The system of equations for the Calabi-Yau threefold $Y_3$ can be written as

$$\begin{gathered} P_{1,1,0}(z_i; u_k; v_k)s^2 + P_{3,2,1}(z_i; u_k; v_k)st + P_{5,3,2}(z_i; u_k; v_k)t^2 = 0, \\ Q_{1,0,1}(z_i; u_k; v_k) = 0. \end{gathered} \tag{A.17}$$

The space $V = GW\mathbb{P}^5_{\substack{11100002\\00011001\\00000111}}$ has the coordinates $(z_0, z_1, z_2;\ u_0, u_1;\ v_0, v_1;\ w)$, and the singular CY is

$$\begin{gathered} P_{1,1,0}(z_i; u_k; v_k)w^2 + P_{3,2,1}(z_i; u_k; v_k)w + P_{5,3,2}(z_i; u_k; v_k) = 0, \\ Q_{1,0,1}(z_i; u_k; v_k) = 0. \end{gathered} \tag{A.18}$$

There are no complex deformations of this equation. This is in contradiction with the general expectation of one complex deformation, that is, the embedding is not the most general. This is connected to the fact that all the two-cycles on the del Pezzo are nontrivial within the CY.

(5) $B = dP_3 \subset X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

The del Pezzo surface is defined by an equation of degree one in $z_i$, degree one in $u_j$ and degree one in $v_k$

$$P_{1,1,1}(z_i; u_j; v_k) = 0. \tag{A.19}$$

The total Chern class of $B$

$$c(B) = \frac{(1+H)^2(1+K)^2(1+R)^2}{(1+H+K+R)} = 1 + (H+K+R) + 2(HK+HR+KR), \tag{A.20}$$

where $H$, $K$, and $R$ are the hyperplane classes on the three $\mathbb{P}^1$'s. The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-H-K-R))$. The equation for the Calabi-Yau threefold $Y_3$ is

$$P_{1,1,1}(z_i; u_j; v_k)s^2 + P_{2,2,2}(z_i; u_j; v_k)st + P_{3,3,3}(z_i; u_j; v_k)t^2 = 0. \tag{A.21}$$

The embedding space $V = GW\mathbb{P}^4_{\substack{1100001\\0011001\\0000111}}$ has the coordinates $(z_0, z_1;\ u_0, u_1;\ v_0, v_1;\ w)$, and the singular CY is

$$P_{1,1,1}(z_i; u_j; v_k)w^2 + P_{2,2,2}(z_i; u_j; v_k)w + P_{3,3,3}(z_i; u_j; v_k) = 0. \tag{A.22}$$

This equation has one deformation $kw^3$, and the spaces $M$ and $V$ have the same number of reparameterizations. Consequently, there is one complex deformation of the cone. This is related to the fact that 3 out of 4 two-cycles on $dP_3$ are independent within the CY and there is only one $(-2)$ two-cycle on $dP_3$ that is trivial within the CY.

(6) $B = dP_4 \subset X = \mathbb{P}^2 \times \mathbb{P}^1$.

Equation defining $B$ has degree two in $z_i$ and degree one in $u_j$

$$P_2(z_i)u_0 + Q_2(z_i)u_1 = 0. \tag{A.23}$$

The total Chern class of $B$

$$c(B) = \frac{(1+H)^3(1+K)^2}{1+2H+K} = 1 + H + K + H^2 + 3HK, \tag{A.24}$$

where $H$ and $K$ are the hyperplane classes on $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively. The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-H-K))$. The equation for the Calabi-Yau threefold $Y_3$ is

$$P_{2,1}(z_i; u_j)s^2 + P_{3,2}(z_i; u_j)st + P_{4,3}(z_i; u_j)t^2 = 0. \tag{A.25}$$

The embedding space $V = GW\mathbb{P}^4_{\substack{111001\\000111}}$ has the coordinates $(z_0, z_1, z_3;\ u_0, u_1;\ w)$, and the singular CY is

$$P_{2,1}(z_i; u_j)w^2 + P_{3,2}(z_i; u_j)w + P_{4,3}(z_i; u_j) = 0. \tag{A.26}$$

The deformations of the singularity have the form of degree one polynomial in $z_0, z_1, z_2$ times $w^3$. Consequently, there are three deformation parameters and the spaces $V$ and $M$ have the same reparameterizations. In this case, we have three complex deformations and three $(-2)$ two-cycles on $dP_4$ that are trivial within CY.

(7) $B = dP_5 \subset X = \mathbb{P}^4$.

The del Pezzo surface is defined by a system of two equations. Both equation have degree 2 in $z_i$

$$\begin{aligned} P_2(z_i) &= 0, \\ R_2(z_i) &= 0. \end{aligned} \tag{A.27}$$

The total Chern class of $B$

$$c(B) = \frac{(1+H)^5}{(1+2H)^2} = 1 + H + 2H^2. \tag{A.28}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-H))$. The system of equations for the first possible Calabi-Yau threefold $Y_3$ is

$$\begin{aligned} P_2(z_i)s^2 + P_3(z_i)st + P_4(z_i)t^2 &= 0, \\ R_2(z_i) &= 0. \end{aligned} \tag{A.29}$$

It has the following characteristics:

$$\begin{aligned} \chi(Y_3) &= -160, \\ h^{1,1}(Y_3) &= 2, \\ h^{2,1} &= 82. \end{aligned} \tag{A.30}$$

Now we find the deformations of this cone over $dP_5$. The $\mathbb{P}^1$ bundle $M$ is, in fact, the $\mathbb{P}^5$ blown up at one point. By blowing down the $t = 0$ section of $M$, we get $\mathbb{P}^5$. The CY three-fold with the $dP_5$ singularity is embedded in $\mathbb{P}^5$ by the system of two equations

$$\begin{gathered} P_2(z_i)w^2 + P_3(z_i)w + P_4(z_i) = 0, \\ R_2(z_i) = 0. \end{gathered} \tag{A.31}$$

The deformations of the singularity correspond to taking a general degree four polynomial in the first equation. This general CY has

$$\begin{gathered} \chi = -176, \\ h^{1,1}(Y_3) = 1, \\ h^{2,1} = 89. \end{gathered} \tag{A.32}$$

Since the system (A.31) has only the $dP_5$ singularity and the general CY manifold has $89-82 = 7$ more complex deformations, we interpret these extra 7 deformations as the deformations of the cone over $dP_5$. This number is consistent with the general expectation, since $c^\vee(D5)-1 = 7$, where $c^\vee(D5) = 8$ is the dual Coxeter number for $D5$.

The second CY with the $dP_5$ singularity is described by

$$\begin{gathered} P_2(z_i)s + P_3(z_i)t = 0, \\ R_2(z_i)s + R_3(z_i)t = 0. \end{gathered} \tag{A.33}$$

Using the same methods as for the first CY, one can show that this singularity also has 7 complex deformations.

(8) $B = dP_6 \subset X = \mathbb{P}^3$.

The case of $dP_6$ was described in details in Section 2; here we just repeat the general results.

The equation defining $dP_6 \subset \mathbb{P}^3$

$$P_3(z_i) = 0. \tag{A.34}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-H))$.

The equation for the Calabi-Yau threefold $Y_3$

$$P_3(z_i)s^2 + P_4(z_i)st + P_5(z_i)t^2 = 0. \tag{A.35}$$

The total Chern class of $Y_3$

$$c(Y_3) = \frac{(1+H)^4(1+G)(1+G-H)}{1+3H+2G}. \tag{A.36}$$

The Euler number and the cohomologies for the CY with the $dP_6$ singularity are

$$\begin{aligned} \chi &= -176, \\ h^{1,1} &= 2, \\ h^{2,1} &= 90. \end{aligned} \tag{A.37}$$

The deformation of this singularity is a quintic in $\mathbb{P}^4$, that has

$$h^{2,1} = 101 \tag{A.38}$$

complex deformations. The difference between the number of complex deformations is $101 - 90 = 11$, which is consistent with $c^{\vee}(E6) - 1 = 11$.

(9) $B = dP_7 \subset X = W\mathbb{P}^3_{1112}$.

The equation defining $B$ is homogeneous of degree four in $z_i$'s

$$P_4(z_i) = 0. \tag{A.39}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-H))$. The equation for the Calabi-Yau threefold $Y_3$

$$P_4(z_i)s^2 + P_5(z_i)st + P_6(z_i)t^2 = 0. \tag{A.40}$$

The total Chern class of $Y_3$

$$c(Y_3) = \frac{(1+H)^3(1+2H)(1+G)(1+G-H)}{1+4H+2G}. \tag{A.41}$$

The Euler number and the cohomologies for the CY with the $dP_6$ singularity are

$$\begin{aligned} \chi &= -168, \\ h^{1,1} &= 2, \\ h^{2,1} &= 86. \end{aligned} \tag{A.42}$$

Blowing down the $t = 0$ section of $M$, we get $V = W\mathbb{P}^4_{11112}$. The general CY is given by the degree six equation in $V$. The total Chern class of this CY is

$$c = \frac{(1+H)^4(1+2H)}{(1+6H)}. \tag{A.43}$$

And the number of complex deformations

$$h^{2,1} = 103. \tag{A.44}$$

The difference $103 - 86 = 17$ is equal to $c^{\vee}(E7) - 1 = 17$, where $c^{\vee}(E7) = 18$ is the dual Coxeter number of $E7$. Consequently, we can represent all complex deformations of $dP_7$ singularity in this embedding.

(10) $B = dP_8 \subset X = W\mathbb{P}^3_{1123}$.

The equation defining $B$ has degree six

$$P_6(z_i) = 0. \tag{A.45}$$

The $\mathbb{P}^1$ bundle is $M = P(\mathcal{O}_X \oplus \mathcal{O}_X(-H))$. The equation for the Calabi-Yau threefold $Y_3$

$$P_6(z_i)s^2 + P_7(z_i)st + P_8(z_i)t^2 = 0. \tag{A.46}$$

The total Chern class of $Y_3$

$$c(Y_3) = \frac{(1+H)^2(1+2H)(1+3H)(1+G)(1+G-H)}{1+6H+2G}. \tag{A.47}$$

The problem with this CY is that for any polynomials $P_6$, $P_7$, and $P_8$, it has a singularity at $s = z_0 = z_1 = z_2 = z_3 = 0$ and $z_4 = 1$. As a consequence, the naive calculation of the Euler number gives a fractional number

$$\chi = -150\frac{2}{3}. \tag{A.48}$$

The good feature of this singularity is that it is away from the del Pezzo; thus one can argue that, this singularity should not affect the deformation of the $dP_8$ cone. In order to justify that we will calculate the number of complex deformations of the CY manifold with $dP_8$ singulariy by calculating the number of coefficients in the equation minus the number of reparamterizations of $M$. The result is

$$h^{2,1} = 77. \tag{A.49}$$

Blowing down the $t = 0$ section of $M$, we get $V = W\mathbb{P}^4_{11123}$. The general CY is given by the degree eight equation in $V$. The number of coefficients minus the number of reparamterizations of $V = W\mathbb{P}^4_{11123}$ is

$$h^{2,1} = 106. \tag{A.50}$$

The difference $106 - 77 = 29$ is equal to $c^{\vee}(E8) - 1 = 29$, where $c^{\vee}(E8) = 30$ is the dual Coxeter number of $E8$. Thus, all complex deformations of $dP_8$ singularity can be realized in this embedding.

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