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Review Article

Calabi-Yau Threefolds in Weighted Flag Varieties

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We review the construction of families of projective varieties, in particular Calabi-Yau threefolds, as quasilinear sections in weighted flag varieties. We also describe a construction of tautological orbibundles on these varieties, which may be of interest in heterotic model building.

1. Introduction

The classical flag varieties $\Sigma = G/P$ are projective varieties which are homogeneous spaces under complex reductive Lie groups G; the stabilizer P of a point in Σ is a parabolic subgroup P of G. The simplest example is projective space $\mathbb{P}^n$ itself, which is a homogeneous space under the complex Lie group $\mathrm{GL}(n+1)$. Weighted flag varieties $w\Sigma$, which are the analogues of weighted projective space in this more general context, were defined by Corti and Reid [1] following unpublished work of Grojnowski. They admit a Plücker-style embedding

$$w\Sigma \subset \mathbb{P}[w_0, \dots, w_n] \tag{1.1}$$

into a weighted projective space. In this paper, we review the construction of Calabi-Yau threefolds X that arise as complete intersections within $w\Sigma$ of some hypersurfaces of weighted projective space [1–3]:

$$X \subset w\Sigma \subset \mathbb{P}[w_0, \dots, w_n]. \tag{1.2}$$

To be more precise, our examples are going to be quasilinear sections in $w\Sigma$, where the degree of each equation agrees with one of the w_i. The varieties X will have standard threefold singularities similar to complete intersections in weighted projective spaces; they have crepant desingularizations $Y \to X$ by standard theory.

We start by computing the Hilbert series of a weighted flag variety $w\Sigma$ of a given type. By numerical considerations, we get candidate degrees for possible Calabi-Yau complete intersection families. To prove the existence of a particular family, in particular to check that general members of the family only have mild quotient singularities, we need equations for the Plücker style embedding. It turns out that the equations of $w\Sigma$ in the weighted projective space, which are the same as the equations of the straight flag variety Σ in its natural embedding, can be computed relatively easily using computer algebra [2].

The smooth Calabi-Yau models Y that arise from this method may be new, though it is probably difficult to tell. One problem we do not treat in general is the determination of topological invariants such as Betti and Hodge numbers of Y. Some Hodge number calculations for varieties constructed using a related method are performed in [4], via explicit birational maps to complete intersections in weighted projective spaces; the Hodge numbers of such varieties can be computed by standard methods. Such maps are hard to construct in general. A better route would be to first compute the Hodge structure of $w\Sigma$ then deduce the invariants of their quasilinear sections X and finally their resolutions Y. See, for example, [5] for analogous work for hypersurfaces in toric varieties. We leave the development of such an approach for future work.

We conclude our paper with the outline of a possible application of our construction: by its definition, the weighted flag variety $w\Sigma$ and thus its quasilinear section X carry natural orbibundles; these are the analogues of $\mathcal{O}(1)$ on (weighted) projective space. It is possible that these can be used to construct interesting bundles on the resolution Y which may be relevant in heterotic compactifications. Again, we have no conclusive results.

2. Weighted Flag Varieties

2.1. The Main Definition

We start by recalling the notion of weighted flag variety due to Corti and Reid [1]. Fix a reductive Lie group G and a highest weight $\lambda \in \Lambda_W$ in the weight lattice of G, the lattice of characters of the maximal torus T of G. Then we have a corresponding parabolic subgroup P_λ, well defined up to conjugation. The quotient $\Sigma = G/P_\lambda$ is a homogeneous variety called a *(generalized) flag variety*, a projective subvariety of $\mathbb{P}V_\lambda$, where V_λ is the irreducible representation of G with highest weight λ.

Let Λ_W^* denote the lattice of one-parameter subgroups of T, dual to the weight lattice Λ_W. Choose $\mu \in \Lambda_W^*$ and an integer $u \in \mathbb{Z}$ such that

$$\langle w\lambda, \mu \rangle + u > 0 \tag{2.1}$$

for all elements w of the Weyl group of the Lie group G, where $\langle , \rangle$ denotes the perfect pairing between Λ_W and Λ_W^*.

Consider the affine cone $\widetilde{\Sigma} \subset V_\lambda$ of the embedding $\Sigma \hookrightarrow \mathbb{P}V_\lambda$. There is a $\mathbb{C}^*$-action on $V_\lambda \setminus \{0\}$ given by

$$(\varepsilon \in \mathbb{C}^*) \longmapsto \left(v \longmapsto \varepsilon^u \left(\mu(\varepsilon) \circ v\right)\right) \tag{2.2}$$

which induces an action on $\widetilde{\Sigma}$. Inequality (2.1) ensures that all the $\mathbb{C}^*$-weights on V_λ are positive, leading to a well-defined quotient

$$w\mathbb{P}V_\lambda = V_\lambda \setminus \{0\}/\mathbb{C}^*. \tag{2.3}$$

This weighted projective space has weights

$$\{\langle \alpha, \mu\rangle + u \mid \alpha \in \nabla(V_\lambda)\}, \tag{2.4}$$

where $\nabla(V_\lambda)$ denotes the set of weights (understood with multiplicities) appearing in the weight space decomposition of the representation V_λ. Inside this weighted projective space, we consider the projective quotient

$$w\Sigma = \widetilde{\Sigma} \setminus \{0\}/\mathbb{C}^* \subset w\mathbb{P}V_\lambda. \tag{2.5}$$

We call $w\Sigma$ a *weighted flag variety*. By definition, $w\Sigma$ quasismooth, that is, its affine cone $\widetilde{\Sigma}$ is nonsingular outside its vertex $\underline{0}$. Hence it only has finite quotient singularities.

The weighted flag variety $w\Sigma$ is called *well formed* [6], if no $(n-1)$ of weights w_i have a common factor, and moreover $w\Sigma$ does not contain any codimension $c+1$ singular stratum of $w\mathbb{P}V_\lambda$, where c is the codimension of $w\Sigma$.

2.2. The Hilbert Series of a Weighted Flag Variety

Consider the embedding $w\Sigma \subset w\mathbb{P}V_\lambda$. The restriction of the line (orbi)bundle of degree one Weil divisors $\mathcal{O}_{w\mathbb{P}V_\lambda}(1)$ gives a polarization $\mathcal{O}_{w\Sigma}(1)$ on $w\Sigma$, a $\mathbb{Q}$-ample line orbibundle some tensor power of which is a very ample line bundle. Powers of $\mathcal{O}_{w\Sigma}(1)$ have well-defined spaces of sections $H^0(w\Sigma, \mathcal{O}_{w\Sigma}(m))$. The *Hilbert series* of the pair $(w\Sigma, \mathcal{O}_{w\Sigma}(1))$ is the power series given by

$$P_{w\Sigma}(t) = \sum_{m\geq 0} \dim H^0(w\Sigma, \mathcal{O}_{w\Sigma}(m))t^m. \tag{2.6}$$

Theorem 2.1 (see [2, Theorem 3.1]). *The Hilbert series $P_{w\Sigma}(t)$ has the closed form*

$$P_{w\Sigma}(t) = \frac{\sum_{w\in W}(-1)^w\left(t^{\langle w\rho,\mu\rangle}/\left(1-t^{\langle w\lambda,\mu\rangle+u}\right)\right)}{\sum_{w\in W}(-1)^w t^{\langle w\rho,\mu\rangle}}. \tag{2.7}$$

Here ρ is the Weyl vector, half the sum of the positive roots of G, and $(-1)^w = 1$ or -1 depending on whether w consists of an even or odd number of simple reflections in the Weyl group W.

The right hand side of (2.7) can be converted into a form

$$P_{w\Sigma}(t) = \frac{N(t)}{\prod_{\alpha\in\nabla(V_\lambda)}\left(1-t^{\langle\alpha,\mu\rangle+u}\right)}, \tag{2.8}$$

where, as before, $\nabla(V_\lambda)$ denotes the set of weights of the representation V_λ. The numerator is a polynomial $N(t)$, the *Hilbert numerator*. Since (2.7) involves summing over the Weyl group, it is best to use a computer algebra system for explicit computations.

A well-formed weighted flag variety is *projectively Gorenstein*, which means that

(i) $H^i(w\Sigma, \mathcal{O}_{w\Sigma}(m)) = 0$ for all m and $0 < i < \dim(w\Sigma)$;

(ii) the Hilbert numerator $N(t)$ is a palindromic symmetric polynomial of degree q, called the *adjunction number* of $w\Sigma$;

(iii) the canonical divisor of $w\Sigma$ is given by

$$K_{w\Sigma} \sim \mathcal{O}_{w\Sigma}\left(q - \sum w_i\right), \tag{2.9}$$

where, as above, the w_i are the weights of the projective space $w\mathbb{P}V_\lambda$; the integer $k = q - \sum w_i$ is called the *canonical weight*.

2.3. Equations of Flag Varieties

The flag variety $\Sigma = G/P \hookrightarrow \mathbb{P}V_\lambda$ is defined by an ideal $I = \langle Q \rangle$ of quadratic equations generating a linear subspace $Q \subset Z = S^2 V_\lambda^*$ of the second symmetric power of the contragradient representation V_λ^*. The G-representation Z has a decomposition

$$Z = V_{2\nu} \oplus V_1 \oplus \cdots \oplus V_n \tag{2.10}$$

into irreducible direct summands, with ν being the highest weight of the representation V_λ^*. As discussed in [7, 2.1], the subspace Q in fact consists of all the summands except $V_{2\nu}$. The equations of $w\Sigma$ can be readily computed from this information using computer algebra [2].

2.4. Constructing Calabi-Yau Threefolds

We recall the different steps in the construction of Calabi-Yau threefolds as quasilinear sections of weighted flag varieties.

(1) Choose Embedding

We choose a reductive Lie group G and a G-representation V_λ of dimension n with highest weight λ. We get a straight flag variety $\Sigma = G/P_\lambda \hookrightarrow \mathbb{P}V_\lambda$ of computable dimension d and codimension $c = n - 1 - d$. We choose $\mu \in \Lambda_W^*$ and $u \in \mathbb{Z}$ to get an embedding $w\Sigma \hookrightarrow w\mathbb{P}V_\lambda = \mathbb{P}^{n-1}[\langle \alpha_i, \mu \rangle + u]$, with $\alpha_i \in \nabla(V_\lambda)$ being the weights of the representation V_λ. The equations, the Hilbert series, and the canonical class of $w\Sigma \subset w\mathbb{P}$ can be found as described above.

(2) Take Threefold Calabi-Yau Section of $w\Sigma$

We take a quasilinear complete intersection

$$X = w\Sigma \cap (w_{i_1}) \cap \cdots \cap (w_{i_l}) \tag{2.11}$$

of l generic hypersurfaces of degrees equal to some of the weights w_i. We choose values so that $\dim(X) = d - l = 3$ and $k + \sum_{j=1}^{l} w_{i_j} = 0$, thus $K_X \sim \mathcal{O}_X$. After relabelling the weights, this gives an embedding $X \hookrightarrow \mathbb{P}^s[w_0, \ldots, w_s]$, with $s = n - l - 1$, of codimension c, polarized by the ample $\mathbb{Q}$-Cartier divisor D with $\mathcal{O}_X(D) = \mathcal{O}_{w\Sigma}(1)|_X$. More generally, as in [1], we can take complete intersections inside projective cones over $w\Sigma$, adding weight one variables to the coordinate ring which are not involved in any relation.

(3) Check Singularities

We are interested in quasismooth Calabi-Yau threefolds, subvarieties of $w\Sigma$ all of whose singularities are induced by the weights of $\mathbb{P}^s[w_i]$. Singular strata S of $\mathbb{P}^s[w_i]$ correspond to sets of weights $w_{i_0}, \ldots, w_{i_p}$ with

$$\gcd\left(w_{i_0}, \ldots, w_{i_p}\right) = r \tag{2.12}$$

nontrivial. If the intersection $X \cap S$ is nonempty, it has to be a singular point $P \in X$ or a curve $C \subset X$ of quotient singularities, and we need to find local coordinates in a neighbourhood of the point of P, respectively of points of C, to check the local transversal structure. Since we are interested in Calabi-Yau varieties which admit crepant resolutions, singular points P have to be quotient singularities of the form $(1/r)(a, b, c)$ with $a + b + c$ divisible by r, whereas the transversal singularity along a singular curve C has to be of the form $(1/r)(a, r - a)$ of type A_{r-1}.

(4) Find Projective Invariants and Check Consistency

The orbifold Riemann-Roch formula of [4, Section 3] determines the Hilbert series of a polarized Calabi-Yau threefold (X, D) with quotient singularities in terms of the projective invariants D^3 and $D \cdot c_2(X)$, as well as for each curve, the degree $\deg D|_C$ of the polarization, and an extra invariant γ_C related to the normal bundle of C in X. Using the Riemann-Roch formula, we can determine the invariants of a given family from the first few values of $h^0(nD)$ and verify that the same Hilbert series can be recovered.

2.5. Explicit Examples

In the next two sections, we find families of Calabi-Yau threefolds admitting crepant resolutions using this programme. We illustrate the method using two embeddings, corresponding to the Lie groups of type G_2 and A_5, leading to Calabi-Yau families of codimension 8, 6, respectively. Further examples for the Lie groups of type C_3 and A_3, in codimensions 7 and 9, are discussed in [3].

3. The Codimension Eight Weighted Flag Variety

3.1. Generalities

Consider the simple Lie group of type G_2. Denote by $\alpha_1, \alpha_2 \in \Lambda_W$ a pair of simple roots of the root system ∇ of G_2, taking α_1 to be the short simple root and α_2 the long one.

The fundamental weights are $\omega_1 = 2\alpha_1 + \alpha_2$ and $\omega_2 = 3\alpha_1 + 2\alpha_2$. The sum of the fundamental weights, which is equal to half the sum of the positive roots, is $\rho = 5\alpha_1 + 3\alpha_2$. We partition the set of roots into long and short roots as $\nabla = \nabla_l \cup \nabla_s \subset \Lambda_W$. Let $\{\beta_1, \beta_2\}$ be the basis of the lattice Λ_W^* dual to $\{\alpha_1, \alpha_2\}$.

We consider the G_2-representation with highest weight $\lambda = \omega_2 = 3\alpha_1 + 2\alpha_2$. The dimension of V_λ is 14 [8, Chapter 22]. The homogeneous variety $\Sigma \subset \mathbb{P}V_\lambda$ is five-dimensional, so we have an embedding $\Sigma^5 \hookrightarrow \mathbb{P}^{13}$ of codimension 8. To work out the weighted version in this case, take $\mu = a\beta_1 + b\beta_2 \in \Lambda_W^*$ and $u \in \mathbb{Z}$.

Proposition 3.1. *The Hilbert series of the codimension eight weighted G_2 flag variety is given by*

$$P_{w\Sigma}(t) = \frac{1 - \left(4 + 2\sum_{\alpha\in\nabla_s} t^{\langle\alpha,\mu\rangle} + \sum_{\alpha\in\nabla_s} t^{2\langle\alpha,\mu\rangle} + \sum_{\alpha\in\nabla_l} t^{\langle\alpha,\mu\rangle}\right)t^{2u} + \cdots + t^{11u}}{(1-t^u)^2 \prod_{\alpha\in\nabla}\left(1 - t^{\langle\alpha,\mu\rangle + u}\right)}. \quad (3.1)$$

Moreover, if $w\Sigma$ is well-formed, then the canonical bundle is $K_{w\Sigma} \sim \mathcal{O}_{w\Sigma}(-3u)$.

The Hilbert series of the straight flag variety $\Sigma \hookrightarrow \mathbb{P}^{13}$ can be computed to be

$$P_\Sigma(t) = \frac{1 - 28t^2 + 105t^3 - \cdots + 105t^8 - 28t^9 - t^{11}}{(1-t)^{14}}. \quad (3.2)$$

The image is defined by 28 quadratic equations, listed in the appendix of [2].

3.2. Examples

Example 3.2. Consider the following initial data.

(i) Input: $\mu = (-1, 1), u = 3$.

(ii) Plücker embedding: $w\Sigma \subset \mathbb{P}^{13}[1, 2^4, 3^4, 4^4, 5]$.

(iii) Hilbert numerator: $1 - 3t^4 - 6t^5 - 8t^6 + 6t^7 + 21t^8 + \cdots + 6t^{26} - 8t^{27} - 6t^{28} - 3t^{29} + t^{33}$.

(iv) Canonical divisor: $K_{w\Sigma} \sim \mathcal{O}_{w\Sigma}(33 - \sum_i w_i) = \mathcal{O}(-9)$, as $w\Sigma$ is well formed.

(v) Variables on weighted projective space together with their weights x_i:

Variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
Weights	2	4	3	2	1	2	4	2	3	4	5	4	3	3

(3.3)

The reason for the curious ordering of the variables is that these variables are exactly those appearing in the defining equations of this weighted flag variety given in [2, Appendix].

Consider the threefold quasilinear section

$$X = w\Sigma \cap \{f_4(x_i) = 0\} \cap \{g_5(x_i) = 0\} \subset \mathbb{P}^{11}\left[1, 2^4, 3^4, 4^3\right], \quad (3.4)$$

where the intersection is taken with general forms f_4, g_5 of degrees four and five, respectively. The canonical divisor class of X is

$$K_X \sim \mathcal{O}_X(-9 + (5 + 4)) = \mathcal{O}_X. \tag{3.5}$$

To determine the singularities of the general threefold X, we need to consider sets of variables whose weights have a greatest common divisor greater than one.

(i) $1/4$ singularities: this singular stratum is defined by setting those variables to zero whose degrees are not divisible by 4. We also have the equations of [2, Appendix]; only (A5), (A23), and (A24) from that list survive to give

$$S = \left\{ \begin{array}{l} \frac{1}{9}x_7x_{10} + x_2x_{12} = 0 \\ -\frac{1}{3}x_{10}^2 + x_7x_{12} = 0 \\ \frac{1}{3}x_7^2 + x_2x_{10} = 0 \end{array} \right\} \subset \mathbb{P}^3_{x_2,x_7,x_{10},x_{12}}. \tag{3.6}$$

In this case, it is easy to see by hand (or certainly using Macaulay) that $S \subset \mathbb{P}^3$ is in fact a twisted cubic curve isomorphic to $\mathbb{P}^1$. We then need to intersect this with the general X; the quintic equation will not give anything new, since x_2, x_7, x_{10}, x_{12} are degree 4 variables, but the quartic equation will give a linear relation between them. Thus $S \cap X$ consists of three points, the three points of $1/4$ singularities. A little further work gives that they are all of type $(1/4)(3,3,2)$.

(ii) $1/3$ singularities: the general X does not intersect this singular stratum; the equations from [2, Appendix] in the degree three variables give the empty locus; this is easiest to check by Macaulay.

(iii) $1/2$ singularities: the intersection of X with this singular stratum is a rational curve $C \subset X$ containing the $1/4$ singular points; again, Macaulay computes this without difficulty. At each other point of the curve we can check that the transverse singularity is $(1/2)(1,1)$.

Thus (X, D) is a Calabi-Yau threefold with three singular points of type $(1/4)(3,3,2)$ and a rational curve C of singularities of type $(1/2)(1,1)$ containing them. Comparing with the orbifold Riemann-Roch formula of [4, Section 3], feeding in the first few known values of $h^0(X, nD)$ from the Hilbert series gives that the projective invariants of this family are

$$D^3 = \frac{9}{8}, \qquad D \cdot c_2(X) = 21, \qquad \deg D|_C = \frac{9}{4}, \qquad \gamma_C = 1. \tag{3.7}$$

Example 3.3. In this example, we consider the same initial data as in Example 3.2. To construct a new family of Calabi-Yau threefolds, we take a projective cone over $w\Sigma$. Therefore we get the embedding

$$\mathcal{C}w\Sigma \subset \mathbb{P}^{14}\left[1^2, 2^4, 3^4, 4^4, 5\right]. \tag{3.8}$$

The canonical divisor class of $\mathcal{C}w\Sigma$ is $K_{\mathcal{C}w\Sigma} \sim \mathcal{O}_{\mathcal{C}w\Sigma}(-10)$. Consider the threefold quasilinear section

$$X = \mathcal{C}w\Sigma \cap (5) \cap (3) \cap (2) \subset \mathbb{P}^{11}\left[1^2, 2^3, 3^3, 4^4, 5\right] \tag{3.9}$$

with $K_X \sim \mathcal{O}_X$; brackets (w_i) denote a general hypersurface of degree w_i.

(i) 1/4 singularities: since there is no quartic equation this time, the whole twisted cubic curve $C \subset \mathbb{P}^3[x_2, x_7, x_{10}, x_{12}]$, found above, is contained in the general X and is a rational curve of singularities of type $(1/4)(1,3)$.

(ii) 1/3 singularities: the general X does not intersect this singular stratum.

(iii) 1/2 singularities: the intersection of X with this singular strata defines a further rational curve E of singularities. On each point of the curve we check that local transverse parameters have odd weight. Therefore E is a curve of type $(1/2)(1,1)$.

Thus (X, D) is a Calabi-Yau threefold with two disjoint rational curves of singularities C and E of type $(1/4)(1,3)$ and $(1/2)(1,1)$, respectively. The rest of the invariants of this family are

$$D^3 = \frac{27}{16}, \qquad D \cdot c_2(X) = 21, \qquad \deg D|_C = \frac{3}{4}, \qquad \gamma_C = 2, \qquad \deg D|_E = \frac{3}{4}, \qquad \gamma_E = 1. \tag{3.10}$$

Example 3.4. The next example is obtained by a slight generalization of the method described so far. The computation of the canonical class $K_{w\Sigma}$, as the basic line bundle $\mathcal{O}_{w\Sigma}(1)$ raised to the power equal to the difference of the adjunction number and the sum of the weights on $w\mathbb{P}^n$, only works if $w\Sigma$ is well formed. In this example, we will make our ambient weighted homogeneous variety not well formed. We then turn it into a well formed variety by taking projective cones over it. We finally take a quasilinear section to construct a Calabi-Yau threefold (X, D).

(i) Input: $\mu = (0,0)$, $u = 2$.

(ii) Plücker embedding: $w\Sigma \subset \mathbb{P}^{13}[2^{14}]$, not well formed.

(iii) Hilbert numerator: $1 - 28t^4 + 105t^6 - 162t^8 + 84t^{10} + 84t^{12} - 162t^{14} + 105t^{16} - 28t^{18} + t^{22}$.

We take a double projective cone over $w\Sigma$, by introducing two new variables x_{15} and x_{16} of weight one, which are not involved in any of the defining equations of $w\Sigma$. We get a seven-dimensional well-formed and quasismooth variety

$$\mathcal{C}\mathcal{C}w\Sigma \subset \mathbb{P}^{15}\left[1^2, 2^{14}\right] \tag{3.11}$$

with canonical class $K_{\mathcal{C}\mathcal{C}w\Sigma} \sim \mathcal{O}_{\mathcal{C}\mathcal{C}w\Sigma}(-8)$.

Consider the threefold quasilinear section

$$X = \mathcal{C}\mathcal{C}w\Sigma \cap (2)^4 \subset \mathbb{P}^{11}\left[1^2, 2^{10}\right]. \tag{3.12}$$

The canonical class K_X becomes trivial. Since $w\Sigma$ is a five-dimensional variety and we are taking a complete intersection with four generic hypersurfaces of degree two inside $\mathbb{P}^{15}[1^2,2^{14}]$, the singular locus defined by weight two variables defines a curve in $\mathbb{P}^{11}[1^2,2^{10}]$. Thus (X,D) is a Calabi-Yau threefold with a curve of singularities of type $(1/2)(1,1)$. The rest of the invariants of (X,D) are given as follows:

$$D^3 = \frac{9}{2}, \qquad D \cdot c_2(X) = 42, \qquad \deg D|_C = 9, \qquad \gamma_C = 1. \tag{3.13}$$

Example 3.5. Our final initial data in this section consists of the following.

(i) Input: $\mu = (-1,1)$, $u = 5$.

(ii) Plücker embedding: $w\Sigma \subset \mathbb{P}^{13}[3,4^4,5^4,6^4,7]$.

(iii) Hilbert numerator: $1 - 3t^8 - 6t^9 - 10t^{10} - 6t^{11} - t^{12} + 12t^{13} + \cdots + t^{55}$.

(iv) Canonical class: $K_{w\Sigma} \sim \mathcal{O}_{w\Sigma}(-15)$, as $w\Sigma$ is well formed.

We take a projective cone over $w\Sigma$ to get the embedding

$$\mathcal{C}w\Sigma \subset \mathbb{P}^{14}\left[1,3,4^4,5^4,6^4,7\right] \tag{3.14}$$

with $K_{\mathcal{C}w\Sigma} \sim \mathcal{O}_{\mathcal{C}w\Sigma}(-16)$. We take a complete intersection inside $\mathcal{C}w\Sigma$, with three general forms of degree seven, five, and four in $w\mathbb{P}^{14}$. Therefore we get a threefold

$$X = \mathcal{C}w\Sigma \cap (7) \cap (5) \cap (4) \hookrightarrow \mathbb{P}^{11}\left[1,3,4^3,5^3,6^4\right], \tag{3.15}$$

with trivial canonical divisor class. To work out the singularities, we work through the singular strata to find that (X,D) is a polarised Calabi-Yau threefold containing three dissident singular points of type $(1/4)(1,1,2)$, a rational curve of singularities C of type $(1/6)(1,5)$ containing them, and a further isolated singular point of type $(1/3)(1,1,1)$. The rest of the invariants are

$$D^3 = \frac{5}{24}, \qquad D \cdot c_2(X) = 17, \qquad \deg D|_C = \frac{5}{4}, \qquad \gamma_C = 9. \tag{3.16}$$

4. The Codimension 6 Weighted Grassmannian Variety

4.1. The Weighted Flag Variety

We take G to be the reductive Lie group of type $\mathrm{GL}(6,\mathbb{C})$. The five simple roots are $\alpha_i = e_i - e_{i+1} \in \Lambda_W$, the weight lattice with basis $e_1,\ldots,e_6$. The Weyl vector can be taken to be

$$\rho = 5e_1 + 4e_2 + 3e_3 + 2e_4 + e_5. \tag{4.1}$$

Consider the irreducible G-representation V_λ, with $\lambda = e_1 + e_2$. Then V_λ is 15-dimensional, and all of the weights appear with multiplicity one. The highest weight orbit space

$\Sigma = G/P_\lambda \subset \mathbb{P}V_\lambda = \mathbb{P}^{14}$ is eight-dimensional. This flag variety can be identified with the Grassmannian of 2-planes in a 6-dimensional vector space, a codimension 6 variety

$$\Sigma^8 = \mathrm{Gr}(2,6) \hookrightarrow \mathbb{P}V_\lambda = \mathbb{P}^{14}. \tag{4.2}$$

Let $\{f_i, 1 \le i \le 6\}$ be the dual basis of the dual lattice Λ_W^*. We choose

$$\mu = \sum_{i=1}^{6} a_i f_i \in \Lambda_W^*, \tag{4.3}$$

$u \in \mathbb{Z}$, to get the weighted version of $\mathrm{Gr}(2,6)$,

$$w\Sigma(\mu,u) = w\,\mathrm{Gr}(2,6)_{(\mu,u)} \hookrightarrow w\mathbb{P}^{14}. \tag{4.4}$$

The set of weights on our projective space is $\{\langle \lambda_i, \mu\rangle + u\}$, where λ_i are weights appearing in the G-representation V_λ. As a convention we will write an element of dual lattice as row vector, that is, $\mu = (a_1, a_2, \ldots, a_6)$.

We expand formula (2.7) for the given values of λ, μ to get the following formula for the Hilbert series of $w\,\mathrm{Gr}(2,6)$:

$$P_{w\,\mathrm{Gr}(2,6)}(t) = \frac{1 - Q_1(t)t^{2u} + Q_2(t)t^{3u} - Q_3(t)t^{4u} - Q_4(t)t^{5u} + Q_5(t)t^{6u} - Q_6(t)t^{7u} + t^{3s+9u}}{\prod_{1\le i<j\le 6}\left(1 - t^{a_i+a_j+u}\right)}. \tag{4.5}$$

Here

$$\begin{aligned}
Q_1(t) &= \sum_{1\le i<j\le 6} t^{s-(a_i+a_j)},\\
Q_2(t) &= \sum_{1\le (i,j)\le 6} t^{s+(a_i-a_j)} - t^s,\\
Q_3(t) &= \sum_{1\le i\le j\le 6} t^{s+(a_i+a_j)},\\
Q_4(t) &= \sum_{1\le i\le j\le 6} t^{2s-(a_i+a_j)},\\
Q_5(t) &= \sum_{1\le (i,j)\le 6} t^{2s+(a_i-a_j)} - t^{2s},\\
Q_6(t) &= \sum_{1\le i\le j\le 6} t^{2s+(a_i+a_j)}.
\end{aligned} \tag{4.6}$$

In particular, if $w\,\mathrm{Gr}(2,6) \hookrightarrow \mathbb{P}^{14}[\langle w_i, \mu\rangle + u]$ is well formed, then its canonical bundle is $K_{w\,\mathrm{Gr}(2,6)} \sim \mathcal{O}_{w\,\mathrm{Gr}(2,6)}(-2s - 6u)$, with $s = \sum_{i=1}^{6} a_i$.

The defining equations for $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$ are well known to be the 4×4 Pfaffians obtained by deleting two rows and the corresponding columns of the 6×6 skew symmetric matrix

$$A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ & 0 & x_6 & x_7 & x_8 & x_9 \\ & & 0 & x_{10} & x_{11} & x_{12} \\ & & & 0 & x_{13} & x_{14} \\ & & & & 0 & x_{15} \\ & & & & & 0 \end{bmatrix}. \tag{4.7}$$

4.2. Examples

Example 4.1. Consider the following data.

(i) Input: $\mu = (2,1,0,0,-1,-2), u = 4$.

(ii) Plücker embedding: $w\,\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}[1,2^2,3^3,4^3,5^3,6^2,7]$.

(iii) Hilbert numerator: $1 - t^5 - 2t^6 - 3t^7 - 2t^8 - t^9 + \cdots + t^{36}$.

(iv) Canonical class: $K_{w\,\mathrm{Gr}(2,6)} \sim \mathcal{O}_{w\,\mathrm{Gr}(2,6)}(-24)$.

Consider the threefold quasilinear section

$$X = w\,\mathrm{Gr}(2,6) \cap (7) \cap (6) \cap (5) \cap (4) \cap (2) \subset \mathbb{P}^9\left[1,2,3^3,4^2,5^2,6\right]. \tag{4.8}$$

Then K_X is trivial, and X is a Calabi-Yau 3-fold with a singular point of type $(1/6)(5,4,3)$, lying on the intersection of two curves, C of type $(1/3)(1,2)$ and E of type $(1/2)(1,1)$. There is an additional isolated singular point of type $(1/5)(4,3,3)$. The rest of the invariants of this variety are

$$D^3 = \frac{11}{30}, \qquad D \cdot c_2(X) = \frac{68}{5}, \qquad \deg D|_C = \frac{1}{3}, \qquad \gamma_C = \frac{-15}{2}, \qquad \deg D|_E = \frac{1}{2}, \qquad \gamma_E = 1. \tag{4.9}$$

Example 4.2. We take the following.

(i) Input: $\mu = (2,1,1,1,1,0), u = 0$.

(ii) Plücker embedding: $w\,\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}[1^4,2^7,3^4]$.

(iii) Hilbert numerator: $1 - 4t^3 - 6t^4 + 4t^5 + \cdots + t^{18}$.

(iv) Canonical class: $K_{w\,\mathrm{Gr}(2,6)} \sim \mathcal{O}_{w\,\mathrm{Gr}(2,6)}(-12)$, as $w\Sigma$ is well formed.

Consider the quasilinear section

$$X = w\,\mathrm{Gr}(2,6) \cap (3)^2 \cap (2)^3 \subset \mathbb{P}^9\left[1^4,2^4,3^2\right], \tag{4.10}$$

then

$$K_X = \mathcal{O}_X(-12 + (2 \times 3 + 3 \times 2)) = \mathcal{O}_X. \tag{4.11}$$

The variety (X, D) is a well-formed and quasismooth Calabi-Yau 3-fold. Its singularities consist of two rational curves C and E of singularities of type $(1/3)(1,2)$ and $(1/2)(1,1)$, respectively. The rest of the invariants are

$$D^3 = \frac{97}{18}, \qquad D \cdot c_2(X) = 42, \qquad \deg D|_C = \frac{1}{3}, \qquad \gamma_C = 2, \qquad \deg D|_E = 1, \qquad \gamma_E = 1. \tag{4.12}$$

5. Tautological (Orbi)bundles

5.1. The Classical Story

Let $\Sigma = G/P$ be a flag variety. A representation V of the parabolic subgroup P gives rise to a vector bundle $\mathcal{E}$ on Σ as follows:

$$\begin{array}{c} \mathcal{E} = G \times_P V \\ \downarrow \\ \Sigma = G/P. \end{array} \tag{5.1}$$

In other words, the total space of $\mathcal{E}$ consists of pairs $(g, e) \in G \times V$ modulo the equivalence

$$(gp, e) \sim (g, pe), \quad \text{for } p \in P. \tag{5.2}$$

The fiber of $\mathcal{E}$ over each point Σ is isomorphic to the vector space underlying V.

Example 5.1. The simplest example is $\Sigma = \mathbb{P}^{n-1}$, a homogeneous variety G/P with $G = \mathrm{GL}(n)$ and P the parabolic subgroup consisting of matrices of the form

$$A = \begin{pmatrix} \alpha & * & \cdots & * \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}. \tag{5.3}$$

We obtain a one-dimensional representation of P by mapping A to α. The associated line bundle is just the tautological line bundle on $\mathbb{P}^{n-1}$, the dual of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Example 5.2. More generally, consider $\Sigma = \mathrm{Gr}(k,n)$, the Grassmannian of k-planes in $\mathbb{C}^n$. Then $G = \mathrm{GL}(n)$ and the corresponding parabolic is the subgroup of matrices of the form

$$A = \begin{pmatrix} B_1 & * \\ 0 & B_2 \end{pmatrix}, \tag{5.4}$$

with B_1, B_2 of size $k \times k$ and $(n-k) \times (n-k)$, respectively. The representations of P defined by $A \mapsto B_1$, $A \mapsto B_2$, respectively, give the standard tautological sub- and quotient bundles $\mathcal{S}$ and $\mathcal{Q}$ on the Grassmannian $\mathrm{Gr}(k,n)$, fitting into the exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^{\oplus n}_{\mathrm{Gr}(k,n)} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{5.5}$$

Example 5.3. Finally consider the G_2-variety $\Sigma = G/P$ studied in Section 3. The smallest representations of the corresponding P have dimensions 2 and 5. The corresponding tautological bundles are easiest to describe using an embedding $\Sigma \hookrightarrow \mathrm{Gr}(2,7)$, mapping the G_2 flag variety into the Grassmannian of 2-planes in a 7-dimensional vector space, the space $\mathrm{Im}\ \mathbb{O}$ of imaginary octonions. Then the tautological bundles on the G_2-variety Σ are the restrictions of the tautological sub- and quotient bundle from $\mathrm{Gr}(2,7)$.

5.2. Orbibundles on Calabi-Yau Sections

Recall that weighted flag varieties are constructed by first considering the $\mathbb{C}^*$-covering $\widetilde{\Sigma} \setminus \{0\} \rightarrow \Sigma$ and then dividing $\widetilde{\Sigma} \setminus \{0\}$ by a different $\mathbb{C}^*$-action given by the weights. A tautological vector bundle $\mathcal{E}$ on Σ pulls back to a vector bundle $\widetilde{\mathcal{E}}$ on $\widetilde{\Sigma} \setminus \{0\}$. This can then be pushed forward to a weighted flag variety $w\Sigma$ along the quotient map $\widetilde{\Sigma} \setminus \{0\} \rightarrow w\Sigma$. Because of the finite stabilizers that exist under this second action, the resulting object $w\mathcal{E}$ is not a vector bundle, but an orbibundle [9, Section 4.2], which trivializes on local orbifold covers with compatible transition maps. If X is a Calabi-Yau threefold inside $w\Sigma$, then we can define an orbi-bundle on X by restricting $w\mathcal{E}$ to X.

In the constructions of Sections 3 and 4, the Calabi-Yau sections therefore carry possibly interesting orbibundles of ranks 2 and 5, respectively 4. We have not investigated the question whether these orbibundles can be pulled back to vector bundles on a resolution $Y \rightarrow X$, but this seems to be of some interest. If so, stability properties of the resulting vector bundles may deserve some investigation, in view of their possible use in heterotic model building [10, 11].

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