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Research Article

Spin and Pseudospin Symmetries with Trigonometric Pöschl-Teller Potential including Tensor Coupling

M. Hamzavi and A. A. Rajabi

Physics Department, Shahrood University of Technology, P.O. Box 3619995161, Shahrood, Iran

Correspondence should be addressed to M. Hamzavi; majid.hamzavi@gmail.com

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We study approximate analytical solutions of the Dirac equation with the trigonometric Pöschl-Teller (tPT) potential and a Coulomb-like tensor potential for arbitrary spin-orbit quantum number κ under the presence of exact spin and pseudospin (p-spin) symmetries. The bound state energy eigenvalues and the corresponding two-component wave functions of the Dirac particle are obtained using the parametric generalization of the Nikiforov-Uvarov (NU) method. We show that tensor interaction removes degeneracies between spin and pseudospin doublets. The case of nonrelativistic limit is studied too.

1. Introduction

The Dirac equation, which describes the motion of a spin-1/2 particle, has been used in solving many problems of nuclear and high-energy physics. The spin and the p-spin symmetries of the Dirac Hamiltonian had been discovered many years ago; however, these symmetries have recently been recognized empirically in nuclear and hadronic spectroscopes [1]. Within the framework of Dirac equation, p-spin symmetry used to feature the deformed nuclei and the super deformation to establish an effective shell-model [2–4], whereas spin symmetry is relevant for mesons [5]. The spin symmetry occurs when the scalar potential $S(r)$ is nearly equal to the vector potential $V(r)$ or equivalently $S(r) \approx V(r)$, and p-spin symmetry occurs when $S(r) \approx -V(r)$ [6, 7]. The p-spin symmetry refers to a quasidegeneracy of single nucleon doublets with nonrelativistic quantum number $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l, and j are single nucleon radial, orbital, and total angular quantum numbers, respectively [8, 9]. The total angular momentum is given by $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ pseudoangular momentum and $\tilde{s}$ is p-spin angular momentum [10, 11]. Liang et al. [12] investigated the symmetries of the Dirac Hamiltonian and their breaking in realistic nuclei in the framework of perturbation theory. Guo [13] used the similarity renormalization group to transform the spherical Dirac operator into a diagonal form, and then the upper (lower) diagonal element became an operator describing Dirac (anti-)particle, which holds the form of the Schrödinger-like operator with the singularity disappearing in every component. Chen and Guo [14] investigated the evolution toward the nonrelativistic limit from the solutions of the Dirac equation by a continuous transformation of the Compton wavelength λ. Lu et al. [15] recently showed that the p-spin symmetry in single particle resonant states in nuclei is conserved when the attractive scalar and repulsive vector potentials have the same magnitude but opposite sign.

Tensor potentials were introduced into the Dirac equation with the substitution $\vec{p} \rightarrow \vec{p} - im\omega\, \beta \cdot \hat{r}\, U(r)$, and a spin-orbit coupling is added to the Dirac Hamiltonian [16, 17]. Lisboa et al. [18] have studied a generalized relativistic harmonic oscillator for spin-1/2 particles by considering a Dirac Hamiltonian that contains quadratic vector and scalar potentials together with a linear tensor potential, under the conditions of pseudospin and spin symmetry. Alberto et al. [19] studied the contribution of the isoscalar tensor coupling to the realization of pseudospin symmetry in nuclei. Akcay showed that the Dirac equation with scalar and vector quadratic potentials and a Coulomb-like tensor potential can be solved exactly [20]; also, he exactly solved Dirac equation with tensor potential containing a linear and Coulomb-like terms too [21]. Aydoğdu and Sever obtained

exact solution of Dirac equation for the pseudoharmonic potential in the presence of linear tensor potential under the pseudospin symmetry and showed that tensor interactions remove all degeneracies between members of pseudospin doublets [22]. Zhou et al. solved Dirac equation approximately for Hulthén potential including Coulomb-like tensor potential with arbitrary spin-orbit coupling number κ under spin and pseudospin symmetry limit [10]. Aydoğdu and Sever solved approximately for the Woods-Saxon potential and a tensor potential with the arbitrary spin-orbit coupling quantum number κ under pseudospin and spin symmetry [23]. Very recently, Hamzavi et al. gave exact solutions of the Dirac equation for Mie-type potential and position-dependent mass Coulomb potential with a Coulomb-like tensor potential [24, 25] and pseudoharmonic potential with linear plus Coulomb-like tensor potential [26].

The tPT potential has been proposed for the first time by Pöschl and Teller [27] in 1933 to describe the diatomic molecular vibration. Chen [28] and Zhang and Wang [29] have studied the relativistic bound state solutions for the tPT potential and hyperbolical PT (Second PT) potential, respectively. Liu et al. [30] studied the tPT potential within the framework of the Dirac theory. Recently, Candemir [31] investigated the analytical $\widetilde{s}$-wave solutions of Dirac equation for tPT potential under the p-spin symmetry condition. Very recently, Hamzavi and Rajabi [32] studied the exact s-wave solution ($l = 0$) of the Schrödinger equation for the vibrational tPT potential. The tPT takes the form:

$$V_{\mathrm{tPT}}(r) = \frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}, \quad V_1 > 0,\ V_2 > 0, \tag{1}$$

where the parameters V_1 and V_2 describe the property of the potential well, while the parameter α is related to the range of this potential [30]. We find out that this potential has a minimum value at $r_0 = (1/\alpha)\tan^{-1}(\sqrt[4]{V_1/V_2})$. The case when $V_1 = V_2$, the minimum value is at $r_0 = \pi/4\alpha \in (0, \infty)$ for $\alpha > 0$. The second derivative which determines the force constants at $r = r_0$ is given by

$$\left.\frac{d^2V}{dr^2}\right|_{r=r_0} = \frac{8\alpha^2\left(V_2 + \sqrt{V_1V_2}\right)}{\cos^2\left[\tan^{-1}\left(\sqrt[4]{V_1/V_2}\right)\right]}, \tag{2}$$

for any α value, and thus

$$V(r_0) = \frac{\sqrt{V_1V_2} + V_2}{\cos^2\left[\tan^{-1}\left(\sqrt[4]{V_1/V_2}\right)\right]}, \tag{3}$$

which means that $V(r)$ at $r = r_0$ has a relative minimum for $\alpha > 0$. When $V_1 = V_2 = V$ then minimum value is $V(r_0) = 4V$ and $d^2V/dr^2|_{r=r_0} = 32\alpha^2V$. In Figures 1(a) and 1(b), we draw the tPT potential (1) for parameter values $V_1 = 5.0\,\mathrm{fm}^{-1}$, $V_2 = 3.0\,\mathrm{fm}^{-1}$, $\alpha = 0.02\,\mathrm{fm}^{-1}$, and $\alpha = 0.30\,\mathrm{fm}^{-1}$. Here the potential has a minimum value at $r_0 = 0.27027\pi/\alpha$. The curve is nodeless in $\alpha r \in (0, \pi/2)$. For example, with $\alpha = 0.30\,\mathrm{fm}^{-1}$, $r_0 = 2.8303\,\mathrm{fm}$ and minimum potential $V(r_0 = 2.8303\,\mathrm{fm}) = 15.746\,\mathrm{fm}^{-1}$. It is worthy to note that in the limiting case when $\alpha \to 0$, the tPT potential can be reduced to the Kratzer potential [33, 34] $V(r) = D_e((r - r_e)/r)^2 + \eta$, where r_e is the equilibrium intermolecular separation and D_e is the dissociation energy between diatomic molecules. In our case, $D_e = V_1$, $\eta = V_2$, and $r_e = 1/\alpha$. In the case of $\eta = 0$ it reduces to the molecular potential which is called the modified Kratzer potential proposed by Simons et al. [35] and Molski and Konarski [36]. In the case of $\eta = -D_e$, this potential turns into the Kratzer potential, which includes an attractive Coulomb potential and a repulsive inverse square potential introduced by Kratzer in 1920 [37].

The aim of the present work is to extend our previous work [32] to the relativistic case and $\kappa \neq \pm 1$ (rotational case) including a Coulomb-like tensor potential. We introduce a convenient approximation scheme to deal with the strong singular centrifugal term. The ansätz of this approximation possesses the same form of the potential and is singular as the centrifugal term r^{-2}.

Since the relativistic solution is indispensable, we need to solve the Dirac equation with flexible parameters tPT potential model. However, the Dirac-tPT problem can no longer be solved in a closed form due to the existence of spin-orbit coupling term $\kappa(\kappa \pm 1)r^{-2}$ and it is necessary to resort to approximation methods. Therefore, we use an approximation scheme to deal with this term and solve approximately the Dirac equation with the tPT potential for arbitrary spin-orbit quantum number κ. In the presence of spin and p-spin symmetric limitation, we obtain the approximate relativistic bound state solutions including the energy eigenvalue equations and the corresponding unnormalized upper- and lower-spinor components of the wave functions using the concepts of parametric generalization of the NU method [38] since the relativistic corrections are not neglected.

Over the past years, the Nikiforov-Uvarov (NU) method [39] has shown to be a powerful tool in solving the second-order differential equation. It was applied successfully to a large number of potential models [40–46]. This method has also been used to solve the spinless (spin-0) Schrödinger [47–51] and Klein-Gordon (KG) [52–57] equations and also relativistic spin-1/2 Dirac equation [58–63] with different potential models.

The structure of the paper is as follows. In Section 2, in the context of spin and p-spin symmetry, we briefly introduce the Dirac equation with scalar and vector tPT and also tensor potential potentials for arbitrary spin-orbit quantum number κ. The parametric generalization of the NU method is displayed in the appendix. In the presence of the spin and p-spin symmetry, the approximate energy eigenvalue equations and corresponding two-component wave functions of the Dirac-tPT problem are obtained, and effect of tensor potential is shown in this section. The nonrelativistic limit of the problem is discussed in this section too. Finally, our final concluding remarks are given in Section 3.

2. Dirac Equation including Tensor Coupling

The Dirac equation for fermionic massive spin-1/2 particles moving in an attractive scalar potential $S(r)$, a repulsive

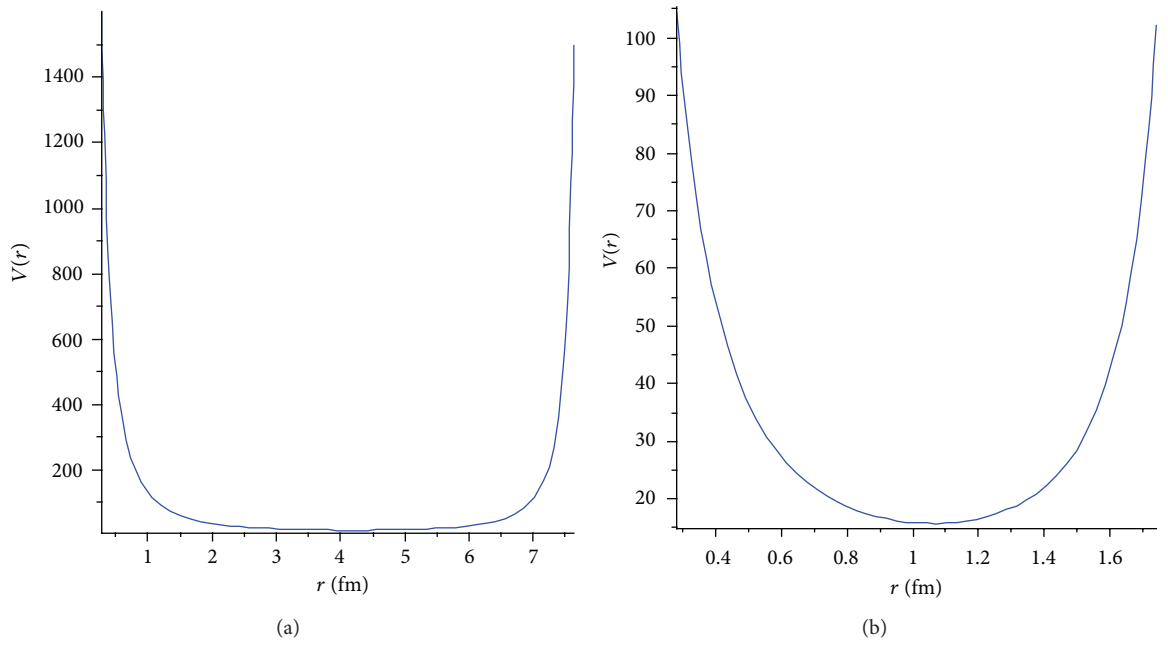

FIGURE 1: (a) A plot of the tPT potential for $\alpha = 0.2\,\mathrm{fm}^{-1}$. (b) A plot of the tPT potential for $\alpha = 0.8\,\mathrm{fm}^{-1}$.

vector potential $V(r)$, and a tensor potential $U(r)$ is [$\hbar = c = 1$]

$$\left[\vec{\alpha}\cdot\vec{p}+\beta\left(M+S\left(r\right)\right)-i\beta\vec{\alpha}\cdot\hat{r}U\left(r\right)\right]\psi\left(\vec{r}\right)=\left[E-V\left(r\right)\right]\psi\left(\vec{r}\right), \tag{4}$$

where E is the relativistic energy of the system, $\vec{p} = -i\vec{\nabla}$ is the three-dimensional momentum operator, and M is the mass of the fermionic particle [64]. $\vec{\alpha}$ and β are the 4×4 usual Dirac matrices given as

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5}$$

where I is 2×2 unitary matrix and $\vec{\sigma}$ are three-vector spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

The total angular momentum operator $\vec{J}$ and spin-orbit operator $K = (\vec{\sigma}\cdot\vec{L}+1)$, where $\vec{L}$ is orbital angular momentum operator, of the spherical nucleons commute with the Dirac Hamiltonian. The eigenvalues of spin-orbit coupling operator are $\kappa = (j + (1/2)) > 0$ and $\kappa = -(j + (1/2)) < 0$ for unaligned spin $j = l - (1/2)$ and the aligned spin $j = l + (1/2)$, respectively. (H^2, K, J^2, J_z) can be taken as the complete set of the conservative quantities. Thus, the spinor wave functions can be classified according to their angular momentum j, spin-orbit quantum number κ, and the radial quantum number n and can be written as follows:

$$\psi_{n\kappa}\left(\vec{r}\right) = \begin{pmatrix} f_{n\kappa}\left(\vec{r}\right) \\ g_{n\kappa}\left(\vec{r}\right) \end{pmatrix} = \begin{pmatrix} \dfrac{F_{n\kappa}\left(r\right)}{r} Y^{l}_{jm}\left(\theta,\varphi\right) \\ i\dfrac{G_{n\kappa}\left(r\right)}{r} Y^{\tilde{l}}_{jm}\left(\theta,\varphi\right) \end{pmatrix}, \tag{7}$$

where $f_{n\kappa}(\vec{r})$ is the upper (large) component and $g_{n\kappa}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y^{l}_{jm}(\theta,\varphi)$ and $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ are spin and pseudospin spherical harmonics, respectively, and m is the projection of the angular momentum on the z-axis. Substituting (7) into (4) and using the following relations:

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right), \tag{8a}$$

$$\left(\vec{\sigma}\cdot\vec{P}\right) = \vec{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\vec{P} + i\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \tag{8b}$$

with the following properties:

$$\begin{aligned} \left(\vec{\sigma}\cdot\vec{L}\right)Y^{\tilde{l}}_{jm}\left(\theta,\phi\right) &= \left(\kappa-1\right)Y^{\tilde{l}}_{jm}\left(\theta,\phi\right), \\ \left(\vec{\sigma}\cdot\vec{L}\right)Y^{l}_{jm}\left(\theta,\phi\right) &= -\left(\kappa-1\right)Y^{l}_{jm}\left(\theta,\phi\right), \\ \left(\vec{\sigma}\cdot\hat{r}\right)Y^{\tilde{l}}_{jm}\left(\theta,\phi\right) &= -Y^{l}_{jm}\left(\theta,\phi\right), \\ \left(\vec{\sigma}\cdot\hat{r}\right)Y^{l}_{jm}\left(\theta,\phi\right) &= -Y^{\tilde{l}}_{jm}\left(\theta,\phi\right), \end{aligned} \tag{8c}$$

one obtains two coupled differential equations for upper and lower radial wave functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)=\left(M+E_{n\kappa}-\Delta(r)\right)G_{n\kappa}(r), \tag{9a}$$

$$\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)=\left(M-E_{n\kappa}+\Sigma(r)\right)F_{n\kappa}(r), \tag{9b}$$

where

$$\Delta(r)=V(r)-S(r), \tag{10a}$$

$$\Sigma(r)=V(r)+S(r). \tag{10b}$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from (10a) and (10b), we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components, respectively:

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-\frac{dU(r)}{dr}-U^2(r)\right]F_{n\kappa}(r)+\frac{d\Delta(r)/dr}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)=\left[\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\Sigma(r)\right)\right]F_{n\kappa}(r), \tag{11}$$

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{dU(r)}{dr}-U^2(r)\right]G_{n\kappa}(r)+\frac{d\Sigma(r)/dr}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)=\left[\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\Sigma(r)\right)\right]G_{n\kappa}(r), \tag{12}$$

where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1)=l(l+1)$. The quantum number κ is related to the quantum numbers for spin symmetry l and pseudospin symmetry $\tilde{l}$ as

$$\kappa=\begin{cases}-(l+1)=-\left(j+\frac{1}{2}\right) & (s_{1/2},p_{3/2},\text{etc.}),\\ & j=l+\frac{1}{2},\\ & \text{aligned spin}\ (\kappa<0),\\ +l=+\left(j+\frac{1}{2}\right) & (p_{1/2},d_{3/2},\text{etc.}),\\ & j=l-\frac{1}{2},\\ & \text{unaligned spin}\ (\kappa>0),\end{cases} \tag{13}$$

and the quasidegenerate doublet structure can be expressed in terms of a pseudospin angular momentum $\tilde{s}=1/2$ and pseudoorbital angular momentum $\tilde{l}$, which is defined as

$$\kappa=\begin{cases}-\tilde{l}=-\left(j+\frac{1}{2}\right) & (s_{1/2},p_{3/2},\text{etc.}),\\ & j=\tilde{l}-\frac{1}{2},\\ & \text{aligned}\\ & \text{pseudospin}\ (\kappa<0),\\ +\left(\tilde{l}+1\right)=+\left(j+\frac{1}{2}\right) & (d_{3/2},f_{5/2},\text{etc.}),\\ & j=\tilde{l}+\frac{1}{2},\\ & \text{unaligned spin}\ (\kappa>0),\end{cases} \tag{14}$$

where $\kappa=\pm1,\pm2,\ldots$. For example, $(1s_{1/2},0d_{3/2})$ and $(1p_{3/2},0f_{5/2})$ can be considered as pseudospin doublets.

2.1. Spin Symmetric Limit. In the spin symmetric limitation, $d\Delta(r)/dr=0$ or $\Delta(r)=C_s=$ constant [7, 65–67]; then, (11) with $\Sigma(r)=V_{\text{tPT}}(r)$ and tensor potential as Coulomb potential, $U(r)=-H/r$ becomes

$$\left[\frac{d^2}{dr^2}-\frac{\eta_\kappa(\eta_\kappa+1)}{r^2}-\gamma\left(\frac{V_1}{\sin^2(\alpha r)}+\frac{V_2}{\cos^2(\alpha r)}\right)-\beta^2\right]\times F_{n\kappa}(r)=0, \tag{15a}$$

$$\gamma=M+E_{n\kappa}-C_s,\qquad \beta^2=\left(M-E_{n\kappa}\right)\left(M+E_{n\kappa}-C_s\right), \tag{15b}$$

where $\eta_\kappa=\kappa+H+1$ is new spin-orbit centrifugal term and $\kappa=l$ and $\kappa=-l-1$ for $\kappa<0$ and $\kappa>0$, respectively. The Schrödinger-like equation (15a) that results from the Dirac equation is a second-order differential equation containing a spin-orbit centrifugal term $\eta_\kappa(\eta_\kappa+1)r^{-2}$ which has a strong singularity at $r=0$, and needs to be treated very carefully while performing the approximation. In absence of tensor interaction, (15a) has an exact rigorous solution only for the states with $\kappa=-1$ because of the existence of the centrifugal term $\kappa(\kappa+1)/r^2$. However, when this term taken into account, the corresponding radial Dirac equation can no longer be solved in a closed form, and it is necessary to resort to approximate methods. Over the last few decades, several schemes have been used to calculate the energy spectrum. The main idea of these schemes relies on using different approximations of the spin-orbit centrifugal coupling term $\eta_\kappa(\eta_\kappa+1)/r^2$. So, we need to perform a new approximation for the spin-orbit term as a function of the tPT potential parameters. Therefore, we resort to use an appropriate approximation scheme to deal with the centrifugal potential term as

$$\frac{1}{r^2}=\lim_{\alpha\to0}\alpha^2\left(d_0+\frac{1}{\sin^2(\alpha r)}\right),\quad 0<\alpha r<\pi/2, \tag{16}$$

where $d_0 = 1/12$ is a dimensionless shifting parameter and $\alpha r \ll 1$. The approximation (16) is done on the basis that $\sin(z) = z - z^3/3! + z^5/5! - z^7/7! + \cdots$ and in the limit when $z \to 0$, $\sin(z) \approx z$. To show the validity and accuracy of our choice to the approximation scheme (16), we plot the centrifugal potential term $1/r^2$ and its approximations: $\alpha^2/\sin^2(\alpha r)$ and $\alpha^2(d_0 + 1/\sin^2(\alpha r))$ in Figure 2. As illustrated, the three curves coincide together and show how accurate this replacement is.

Thus, employing such an approximation scheme, we can then write (15a) as

$$\left[\frac{d^2}{dr^2} - \eta_\kappa(\eta_\kappa + 1)\alpha^2\left(d_0 + \frac{1}{\sin^2(\alpha r)}\right) - \gamma\left(\frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}\right) - \beta^2\right] F_{n\kappa}(r) = 0. \tag{17}$$

Followed by making a new change of variables $s(r) = \sin^2(\alpha r)$, this allows us to decompose the spin-symmetric Dirac equation (17) into the Schrödinger-type equation satisfying the upper-spinor component $F_{n,\kappa}(s)$,

$$\begin{aligned}
&\left\{\frac{d^2}{ds^2} + \frac{(1/2) - s}{s(1-s)}\frac{d}{ds} - \frac{1}{s^2(1-s)^2} \times \left[-As^2 + Bs - C\right]\right\} F_{n,\kappa}(s) = 0,\\
&A = -\frac{1}{4\alpha^2}\left[\eta_\kappa(\eta_\kappa - 1)\alpha^2 d_0 + \beta^2\right],\\
&B = \frac{1}{4\alpha^2}\left[\eta_\kappa(\eta_\kappa - 1)\alpha^2(1 - d_0) + \gamma(V_1 - V_2) - \beta^2\right],\\
&C = \frac{1}{4\alpha^2}\left[\eta_\kappa(\eta_\kappa - 1)\alpha^2 + \gamma V_1\right],
\end{aligned} \tag{18}$$

where $F_{n\kappa}(r) \equiv F_{n,\kappa}(s)$ has been used. If the previous equation is compared with (A.2), we can obtain the specific values for constants c_i $(i = 1, 2, 3)$ as

$$c_1 = \frac{1}{2}, \qquad c_2 = 1, \qquad c_3 = 1. \tag{19}$$

In order to obtain the bound state solutions of (17), it is necessary to calculate the remaining parametric constants, that is, c_i $(i = 4, 5, \ldots, 13)$, by means of the relation (A.5). Their specific values are displayed in Table 1 for the relativistic tPT potential model. Further, using these constants along with (A.7), we can readily obtain the energy eigenvalue equation for the Dirac-tPT problem as

$$\left(n + \frac{1}{2} + \sqrt{A - B + C + \frac{1}{16}} + \sqrt{C + \frac{1}{16}}\right)^2 = A, \tag{20}$$

Table 1: The specific values of the parametric constants for the spin symmetric Dirac-tPT problem.

Constant	Analytic value
c_4	$\frac{1}{4}$
c_5	$-\frac{1}{2}$
c_6	$\frac{1}{4}(1 + 4A)$
c_7	$-\frac{1}{4}(1 + 4B)$
c_8	$\frac{1}{16}(1 + 16C)$
c_9	$A - B + C + \frac{1}{16}$
c_{10}	$\frac{1}{2}\sqrt{1 + 16C}$
c_{11}	$2\sqrt{A - B + C + \frac{1}{16}}$
c_{12}	$\frac{1}{4}\left(1 + \sqrt{1 + 16C}\right)$
c_{13}	$\frac{1}{4} + \sqrt{A - B + C + \frac{1}{16}}$

or equivalently

$$\begin{aligned}
&\left(2n + 1 + \frac{1}{2}\sqrt{1 + \frac{4V_2(M + E_{n\kappa} - C_s)}{\alpha^2}} + \frac{1}{2}\sqrt{(2\eta_\kappa - 1)^2 + \frac{4V_1(M + E_{n\kappa} - C_s)}{\alpha^2}}\right)^2\\
&\quad = \frac{1}{\alpha^2}(E_{n\kappa} - M)(M + E_{n\kappa} - C_s) - \eta_\kappa(\eta_\kappa - 1)d_0.
\end{aligned} \tag{21}$$

To show the procedure of determining the energy eigenvalues from (21), we take a set of physical parameter values, $M = 10\,\mathrm{fm}^{-1}$, $V_1 = 5.0\,\mathrm{fm}^{-1}$, $V_2 = 3.0\,\mathrm{fm}^{-1}$, $C_s = 0\,\mathrm{fm}^{-1}$, and $\alpha = 0.8, 0.6, 0.4, 0.2, 0.04$, and 0.02 [30].

In Tables 2 and 3, we present the energy spectrum for the spin symmetric case. Obviously, the pairs $(np_{1/2}, np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{9/2})$, and so on are degenerate states. Thus, each pair is considered as spin doublet and has positive energy. In Table 2, we can see that tensor interaction removes degeneracies between spin doublets. Further, when potential range parameter α approaches zero, the energy eigenvalues approaches a constant. From (21), we find that this constant is $M + V_1 + V_2 + 2\sqrt{V_1 V_2}$; that is, $\lim_{\alpha \to 0} E_{n,\kappa} = M + V_1 + V_2 + 2\sqrt{V_1 V_2}$, which can be seen from Table 3 too.

In Figure 3, we have investigated the effect of the tensor potential on the spin doublet splitting by considering the following pairs of orbital: $(1p_{3/2}, 1p_{1/2})$and $(1f_{7/2}, 1f_{5/2})$. We observe that in the case of $T = 0$ (no tensor interaction),

Table 2: The bound state energy eigenvalues in units of fm^{-1} of the spin symmetry tPT potential for several values of n and κ with $\alpha = 0.8$.

l	$n, \kappa < 0$	$(l, j = l + 1/2)$	$E_{n,\kappa<0}$ $H \neq 0$	$E_{n,\kappa<0}$ $H = 0$	$n, \kappa > 0$	$(l, j = l - 1/2)$	$E_{n,\kappa>0}$ $H \neq 0$	$E_{n,\kappa>0}$ $H = 0$
1	0, −2	$0p_{3/2}$	26.82780	26.89327	0, 1	$0p_{1/2}$	27.02322	26.89327
2	0, −3	$0d_{5/2}$	26.89327	27.02322	0, 2	$0d_{3/2}$	27.21577	27.02322
3	0, −4	$0f_{7/2}$	27.02322	27.21577	0, 3	$0f_{5/2}$	27.46825	27.21577
4	0, −5	$0g_{9/2}$	27.21577	27.46825	0, 4	$0g_{7/2}$	27.77739	27.46825
1	1, −2	$1p_{3/2}$	28.96349	29.02725	1, 1	$1p_{1/2}$	29.15386	29.02725
2	1, −3	$1d_{5/2}$	29.02725	29.15386	1, 2	$1d_{3/2}$	29.34155	29.15386
3	1, −4	$1f_{7/2}$	29.15386	29.34155	1, 3	$1f_{5/2}$	29.58785	29.34155
4	1, −5	$1g_{9/2}$	29.34155	29.58785	1, 4	$1g_{7/2}$	29.88968	29.58785

Table 3: The spin symmetric bound state energy eigenvalues in units of fm^{-1} of the tPT potential for several values of n and κ with $M = 10.0, V_1 = 5.0, V_2 = 3.0, c_s = 0$, and $H = 0$.

l	n, κ	$(l, j = l \pm 1/2)$	$\alpha = 0.8$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.04$	$\alpha = 0.02$
1	1, −2,1	$1p_{3/2}$, $1p_{1/2}$	29.02725	27.36375	26.54876	25.90551	25.82567
2	1, −3,2	$1d_{5/2}$, $1d_{3/2}$	29.15386	27.39629	26.55699	25.90584	25.82575
3	1, −4,3	$1f_{7/2}$, $1f_{5/2}$	29.34155	27.44495	26.56933	25.90634	25.82588
4	1, −5,4	$1g_{9/2}$, $1g_{7/2}$	29.58785	27.50956	26.585773	25.90701	25.82605
1	2, −2,1	$2p_{3/2}$, $2p_{1/2}$	31.16696	28.43165	27.08151	26.01179	25.87879
2	2, −3,2	$2d_{5/2}$, $2d_{3/2}$	31.29020	28.46376	27.08969	26.01212	25.87887
3	2, −4,3	$2f_{7/2}$, $2f_{5/2}$	31.47301	28.51179	27.10195	26.01262	25.878100
4	2, −5,4	$2g_{9/2}$, $2g_{7/2}$	31.71308	28.57555	27.11828	26.01328	25.87916

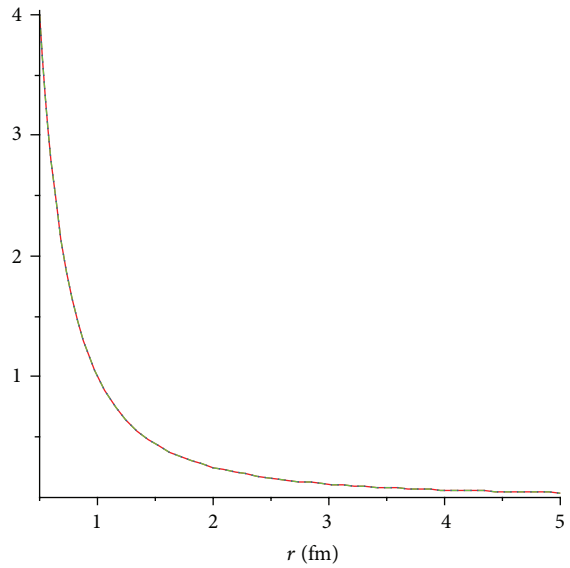

Figure 2: The centrifugal term $1/r^2$ (red line) and its approximations (16) with $d_0 = 0$ (green dash line) and $d_0 = 1/12$ (blue dot line), while $\alpha = 0.002$ fm^{-1}.

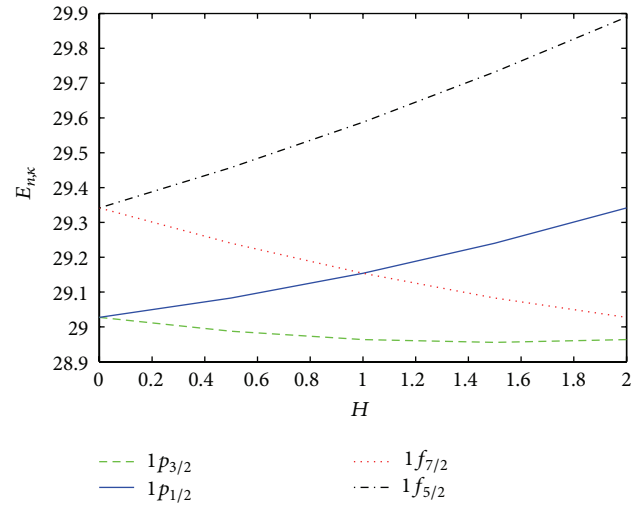

Figure 3: Effect of tensor potential on spin doublets.

members of spin doublets have same energy. However, in the presence of the tensor potential $T \neq 0$, these degeneracies are removed. We can also see that spin doublet splitting increases with increasing T. The reason is that term $2\kappa T$ gives different contributions to each level in the spin doublet because T takes different values for each state in the spin doublet.

On the other hand, in order to establish the upper-spinor component of the wave functions $F_{n,\kappa}(r)$, namely, (15a), the relations (A.8), (A.9), (A.10), and (A.11) are used. Firstly, we find the first part of the wave function as

$$\phi(s) = s^{(1/4)(1+\sqrt{1+16C})}(1-s)^{(1/4)+\sqrt{A-B+C+(1/16)}}. \quad (22)$$

Secondly, we calculate the weight function as

$$\rho(s) = s^{(1/2)\sqrt{1+16C}}(1-s)^{2\sqrt{A-B+C+(1/16)}}, \quad (23)$$

which gives the second part of the wave function as

$$y_n(s) = P_n^{\left((1/2)\sqrt{1+16C},\, 2\sqrt{A-B+C+(1/16)}\right)}(1-2s), \tag{24}$$

where $P_n^{(a,b)}(y)$ are the orthogonal Jacobi polynomials. Finally the upper spinor component for arbitrary κ can be found through the relation (A.11)

$$\begin{aligned} F_{n\kappa}(s) = {} & N_{n\kappa} s^{(1/4)(1+\sqrt{1+16C})}(1-s)^{(1/4)+\sqrt{A-B+C+(1/16)}} \\ & \times P_n^{\left((1/2)\sqrt{1+16C},\, 2\sqrt{A-B+C+(1/16)}\right)}(1-2s), \end{aligned} \tag{25}$$

or

$$\begin{aligned} F_{n\kappa}(r) = {} & N_{n\kappa}(\sin(\alpha r))^{(1/2)(1+\xi_\kappa)}(\cos(\alpha r))^{(1/2)(1+\delta)} \\ & \times P_n^{((1/2)\xi_\kappa,(1/2)\delta)}(\cos(2\alpha r)), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \xi_\kappa &= \sqrt{1+\frac{4}{\alpha^2}\left[\eta_\kappa(\eta_\kappa-1)\alpha^2+\gamma V_1\right]}, \\ \delta &= \sqrt{1+\frac{4\gamma V_2}{\alpha^2}}, \end{aligned} \tag{27}$$

and $N_{n\kappa}$ is the normalization constant. Further, the lower-spinor component of the wave function can be calculated by using

$$G_{n\kappa}(r) = \frac{1}{M+E_{n\kappa}-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r), \tag{28}$$

where $E \neq -M + C_s$ and in the presence of the exact spin symmetry ($C_s = 0$), only positive energy states do exist.

2.2. p-Spin Symmetric Limit. Ginocchio showed that there is p-spin symmetry in case when the relationship between the vector potential and the scalar potential is given by $V(r) = -S(r)$ [7]. Further, Meng et al. showed that if $d[V(r)+S(r)]/dr = d\Sigma(r)/dr = 0$, then $\Sigma(r) = C_{ps}$ = constant, for which the p-spin symmetry is exact in the Dirac equation [65–67]. Thus, choosing the $\Delta(r)$ as tPT potential and $U(r)$ as Coulomb potential, (12) under this symmetry becomes

$$\begin{aligned} & \left[\frac{d^2}{dr^2}-\frac{\lambda_\kappa(\lambda_\kappa-1)}{r^2}-\widetilde{\gamma}\left(\frac{V_1}{\sin^2(\alpha r)}+\frac{V_2}{\cos^2(\alpha r)}\right)-\widetilde{\beta}^2\right] \\ & \quad \times G_{n\kappa}(r) = 0, \end{aligned} \tag{29a}$$

$$\widetilde{\gamma} = E_{n\kappa}-M-C_{ps}, \qquad \widetilde{\beta}^2 = \left(M+E_{n\kappa}\right)\left(M-E_{n\kappa}+C_{ps}\right), \tag{29b}$$

where $\lambda_\kappa = \kappa + H$, $\kappa = -\widetilde{l}$ and $\kappa = \widetilde{l}+1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Employing the new approximation for the spin-orbit pseudocentrifugal term, $\lambda_\kappa(\lambda_\kappa-1)/r^2$, that is, (16), the p-spin Dirac equation (29a) can be written as

$$\begin{aligned} & \left[\frac{d^2}{dr^2}-\lambda_\kappa(\lambda_\kappa-1)\alpha^2\left(d_0+\frac{1}{\sin^2(\alpha r)}\right)\right. \\ & \left.-\widetilde{\gamma}\left(\frac{V_1}{\sin^2(\alpha r)}+\frac{V_2}{\cos^2(\alpha r)}\right)-\widetilde{\beta}^2\right]G_{n\kappa}(r) = 0. \end{aligned} \tag{30}$$

To avoid repetition of the negative energy solution of (30), the p-spin symmetric case can be readily obtained directly via the spin symmetric solution throughout the following parametric mappings:

$$\begin{aligned} & F_{n\kappa}(r) \longleftrightarrow G_{n\kappa}(r), \qquad \kappa \longrightarrow \kappa-1, \qquad V(r) \longrightarrow -V(r) \\ & \qquad \left(\text{i.e.}, V_1 \longrightarrow -V_1, V_2 \longrightarrow -V_2\right), \\ & \mathrm{E}_{n\kappa} \longrightarrow -E_{n\kappa}, \qquad C_s \longrightarrow -C_{ps}. \end{aligned} \tag{31}$$

Following the previous procedure, one can obtain the p-spin symmetric energy equation as

$$\begin{aligned} & -\left(2n+1+\frac{1}{2}\sqrt{1+\frac{4V_2\left(E_{n\kappa}-M-C_{ps}\right)}{\alpha^2}}\right. \\ & \left.+\frac{1}{2}\sqrt{(2\lambda_\kappa-1)^2+\frac{4V_1\left(E_{n\kappa}-M-C_{ps}\right)}{\alpha^2}}\right)^2 \\ & = \frac{1}{\alpha^2}\left(M+E_{n\kappa}\right)\left(M-E_{n\kappa}+C_{ps}\right)+\lambda_\kappa(\lambda_\kappa-1)d_0. \end{aligned} \tag{32}$$

Furthermore, the lower-spinor component of the wave functions is found as

$$\begin{aligned} G_{n\kappa}(r) = {} & \widetilde{N}_{n,\kappa}(\sin(\alpha r))^{(1/2)(1+\widetilde{\xi}_\kappa)} \\ & \times(\cos(\alpha r))^{(1/2)(1+\widetilde{\delta})}P_n^{((1/2)\widetilde{\xi},(1/2)\widetilde{\delta})}(\cos(2\alpha r)), \end{aligned} \tag{33}$$

with

$$\begin{aligned} \widetilde{\xi}_\kappa &= \sqrt{(2\lambda_\kappa-1)^2+\frac{4V_1\left(E_{n\kappa}-M-C_{ps}\right)}{\alpha^2}}, \\ \widetilde{\delta} &= \sqrt{1+\frac{4V_2\left(E_{n\kappa}-M-C_{ps}\right)}{\alpha^2}}, \end{aligned} \tag{34}$$

where $\widetilde{N}_{n\kappa}$ is the normalization constant.

In Tables 4 and 5, we give the numerical results for the p-spin symmetric case. In this case, we take the set of parameter values, $M = 10\,\mathrm{fm}^{-1}$, $V_1 = -5.0\,\mathrm{fm}^{-1}$, $V_2 = -3.0\,\mathrm{fm}^{-1}$, $c_{ps} = 0\,\mathrm{fm}^{-1}$, and $\alpha = 0.8, 0.4, 0.2, 0.04,$

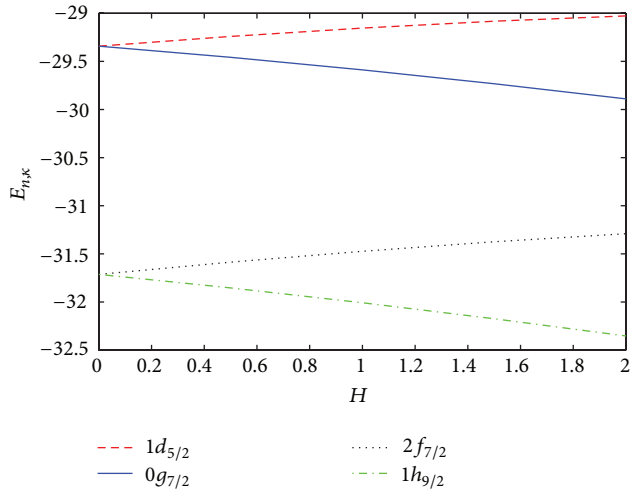

FIGURE 4: Effect of tensor potential on p-spin doublets.

and 0.02 [30]. We observe the degeneracy in the following doublets $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$, and so on. Thus, each pair is considered as p-spin doublet and has negative energy. In Table 4, we can see that tensor interaction removes degeneracies between spin doublets.

In Figure 4, we have investigated the effect of the tensor potential on the p-spin doublet splitting by considering the following pairs of orbital: $(1d_{5/2}, 0g_{7/2})$ and $(2f_{7/2}, 1h_{9/2})$, and one can observe that the results obtained in the p-spin symmetric limit resemble the ones observed in the spin symmetric limit.

On the other hand, the upper-spinor component of the Dirac wave function can be calculated by

$$F_{n\kappa}(r) = \frac{1}{M - E_{n\kappa} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}(r), \quad (35)$$

where $E \neq M + C_{ps}$, and in the presence of the exact p-spin symmetry ($C_{ps} = 0$), only negative energy states do exist.

2.3. The Nonrelativistic Limiting Case. In this section, we study the energy eigenvalue equation (21) and upper-spinor component of wave function (26) of the Dirac-tPT problem under the nonrelativistic limits: $C_s = 0$, $\kappa \to l$, $E_{n\kappa} - M \simeq E_{nl}$, and $M + E_{n\kappa} \simeq 2m$. Thereby, we obtain the energy equation of the Schrödinger equation with any arbitrary orbital state for the tPT potential as

$$E_{nl} = \frac{\hbar^2\alpha^2 l(l+1) d_0}{2m} + \frac{2\hbar^2\alpha^2}{m} \times \left[n + \frac{1}{2} + \frac{1}{4} \left(\sqrt{(2l+1)^2 + \frac{8mV_1}{\hbar^2\alpha^2}} + \sqrt{1 + \frac{8mV_2}{\hbar^2\alpha^2}} \right) \right]^2. \quad (36)$$

In the limit when $\alpha \to 0$, the vibration-rotation energy formula (36) reduces into a constant value:

$$\lim_{\alpha \to 0} E_{nl} = \left(\sqrt{V_1} + \sqrt{V_2} \right)^2. \quad (37)$$

Further, there is no loss of generality if $d_0 = 0$; then, (36) becomes

$$E_{nl} = \frac{2\hbar^2\alpha^2}{m} \times \left[n + \frac{1}{2} + \frac{1}{4} \left(\sqrt{(2l+1)^2 + \frac{8mV_1}{\hbar^2\alpha^2}} + \sqrt{1 + \frac{8mV_2}{\hbar^2\alpha^2}} \right) \right]^2, \quad (38)$$

where $n = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$ are the vibration and rotation quantum numbers, respectively. To obtain numerical energy eigenvalues for the present potential model, we take the following set of parameter values; namely, $M = 10\,\mathrm{fm}^{-1}$, $V_1 = 5.0\,\mathrm{fm}^{-1}$, $V_2 = 3.0\,\mathrm{fm}^{-1}$, and $\alpha = 1.2, 0.8, 0.4, 0.2, 0.02$, and 0.002 [30]. As seen from Table 6, in the limit when potential range parameter α approaches zero, the energy eigenvalues approach a constant value given by (37).

Also, we can get the radial wave functions of the Schrödinger equation with tPT potential as

$$R_{n,l}(s) = s^{(1/4)[1+\sqrt{(2l+1)^2+(8mV_1/\hbar^2\alpha^2)}]} \times (1-s)^{(1/4)(1+\sqrt{1+(8mV_2/\hbar^2\alpha^2)})} \times P_n^{((1/2)\sqrt{(2l+1)^2+(8mV_1/\hbar^2\alpha^2)}, (1/2)\sqrt{1+(8mV_2/\hbar^2\alpha^2)})} \times (1-2s). \quad (39)$$

Inserting $s = \sin^2(\alpha r)$ in the previous equation, we can obtain

$$R_{n,l}(r) = N_{nl} (\sin(\alpha r))^{(1+\eta_l)/2} \times (\cos(\alpha r))^{(1+\delta)/2} P_n^{(\eta_l/2, \delta/2)} (\cos(2\alpha r)), \quad (40a)$$

$$\eta_l = \sqrt{(2l+1)^2 + \frac{8mV_1}{\hbar^2\alpha^2}}, \qquad \delta = \sqrt{1 + \frac{8mV_2}{\hbar^2\alpha^2}}, \quad (40b)$$

where N_{nl} is a normalization factor to be calculated from the normalization conditions.

3. Concluding Remarks

In this work, we have studied the bound state solutions of the Dirac equation with trigonometric Pöschl-Teller and Coulomb-like tensor potentials for any spin-orbit quantum number κ. By making an appropriate approximation to deal with the spin-orbit centrifugal (pseudo-centrifugal) coupling term, we have obtained the approximate energy eigenvalue equation and the unnormalized two components of the radial wave functions expressed in terms of the Jacobi polynomials using the NU method. It is found that tensor interaction removes degeneracies between each pair of pseudospin or spin doublets.

Table 4: The bound state energy eigenvalues in units of fm^{-1} of the p-spin symmetry tPT potential for several values of n and κ with $\alpha = 0.8$.

$\widetilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}\ H \neq 0$	$E_{n,\kappa<0}\ H = 0$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}\ H \neq 0$	$E_{n-1,\kappa>0}\ H = 0$
1	1, −1	$1s_{1/2}$	−28.96349	−29.02725	0, 2	$0d_{3/2}$	−29.15386	−29.02725
2	1, −2	$1p_{3/2}$	−29.02725	−29.15386	0, 3	$0f_{5/2}$	−29.34155	−29.15386
3	1, −3	$1d_{5/2}$	−29.15386	−29.34155	0, 4	$0g_{7/2}$	−29.58785	−29.34155
4	1, −4	$1f_{7/2}$	−29.34155	−29.58785	0, 5	$0h_{9/2}$	−29.88968	−29.58785
1	2, −1	$2s_{1/2}$	−31.10492	−31.16696	1, 2	$1d_{3/2}$	−31.29020	−31.16696
2	2, −2	$2p_{3/2}$	−31.16696	−31.29020	1, 3	$1f_{5/2}$	−31.47301	−31.29020
3	2, −3	$2d_{5/2}$	−31.29020	−31.47301	1, 4	$1g_{7/2}$	−31.71308	−31.47301
4	2, −4	$2f_{7/2}$	−31.47301	−31.71308	1, 5	$1h_{9/2}$	−32.00754	−31.71308

Table 5: The bound state energy eigenvalues in units of fm^{-1} of the p-spin symmetry tPT potential for several values of n and κ with $H = 0$.

$\widetilde{l}$	n, κ	(l, j)	$\alpha = 0.8$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.04$	$\alpha = 0.02$
1	1, −1,2	$1s_{1/2}, 0d_{3/2}$	−29.02725	−27.36375	−26.54876	−25.90551	−25.82567
2	1, −2,3	$1p_{3/2}, 0f_{5/2}$	−29.15386	−27.39629	−26.55699	−25.90584	−25.82575
3	1, −3,4	$1d_{5/2}, 0g_{7/2}$	−29.34155	−27.44495	−26.56933	−25.90634	−25.82588
4	1, −4,5	$1f_{7/2}, 0h_{9/2}$	−29.58785	−27.50956	−26.58577	−25.90701	−25.82605
1	2, −1,2	$2s_{1/2}, 1d_{3/2}$	−31.16696	−28.43165	−27.08151	−26.01179	−25.87879
2	2, −2,3	$2p_{3/2}, 1f_{5/2}$	−31.29020	−28.46376	−27.08969	−26.01212	−25.87887
3	2, −3,4	$2d_{5/2}, 1g_{7/2}$	−31.47301	−28.51179	−27.10195	−26.01262	−25.87810
4	2, −4,5	$2f_{7/2}, 1h_{9/2}$	−31.71308	−28.57555	−27.11828	−26.01328	−25.87916

Table 6: The bound state energy levels E_{nl} of the Schrödinger equation for the tPT potential.

State (n, l)	E_{nl} $M = 10.0\ \mathrm{fm}^{-1}, V_1 = 5\ \mathrm{fm}^{-1}, V_2 = 3\ \mathrm{fm}^{-1}$ [30]					
$\alpha = 1.2$	$\alpha = 0.8$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.02$	$\alpha = 0.002$	
$1s$ [32]	22.87051710	20.32991862	17.95616357	16.83082621	15.85264289	15.75661628
$2s$ [32]	28.29143398	23.68420415	19.50420742	17.57271070	15.92394680	15.76371786
$2p$	28.64395419	23.82847894	19.53712286	17.58054181	15.92402153	15.76371860
$3s$ [32]	34.28835086	27.2944896	21.11625126	18.33059518	15.99541071	15.77082105
$3p$	34.67512504	27.44896381	21.15044543	18.33858626	15.99548560	15.77082179
$3d$	35.43921159	27.75631556	21.21875330	18.35456399	15.99563534	15.77082328
$4s$ [32]	40.86126774	31.16077522	22.79229510	19.10447967	16.06703463	15.77792584
$4p$	41.28229584	31.32544868	22.82776800	19.11263070	16.06710967	15.77792658
$4d$	42.11348590	31.65300783	22.89862721	19.12892817	16.06725974	15.77792806
$4f$	43.33519178	32.14003977	23.00470171	19.15336297	16.06748485	15.77793030

Appendix

Parametric Generalization of the NU Method

The NU method is used to solve second-order differential equations with an appropriate coordinate transformation $s = s(r)$ [39]

$$\psi_n''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{A.1}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second degree, and $\widetilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution, we present a shortcut for the method. So, at first, we write the general form of the Schrödinger-like equation (A.1) in a more general form applicable to any potential as follows [38]:

$$\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1 - c_3 s)}\right)\psi_n'(s) + \left(\frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1 - c_3 s)^2}\right)\psi_n(s) = 0, \tag{A.2}$$

satisfying the wave functions

$$\psi_n(s) = \phi(s)\, y_n(s). \tag{A.3}$$

Comparing (A.2) with its counterpart (A.1), we obtain the following identifications:

$$\widetilde{\tau}(s) = c_1 - c_2 s, \qquad \sigma(s) = s(1 - c_3 s), \\ \widetilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{A.4}$$

Following the NU method [39], we obtain the following [38]:

(i) the relevant constants:

$$c_4 = \frac{1}{2}(1 - c_1), \qquad c_5 = \frac{1}{2}(c_2 - 2c_3), \\ c_6 = c_5^2 + A, \qquad c_7 = 2c_4 c_5 - B, \\ c_8 = c_4^2 + C, \qquad c_9 = c_3(c_7 + c_3 c_8) + c_6, \\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, \\ c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \quad c_3 \neq 0, \\ c_{12} = c_4 + \sqrt{c_8} > 0, \\ c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) > 0, \quad c_3 \neq 0; \tag{A.5}$$

(ii) the essential polynomial functions:

$$\pi(s) = c_4 + c_5 s - \left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right], \\ k = -\left(c_7 + 2c_3 c_8\right) - 2\sqrt{c_8 c_9}, \\ \tau(s) = c_1 + 2c_4 - \left(c_2 - 2c_5\right)s - 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right], \\ \tau'(s) = -2c_3 - 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) < 0; \tag{A.6}$$

(iii) the energy equation:

$$\left(c_2 - c_3\right)n + c_3 n^2 - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) \\ + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0; \tag{A.7}$$

(iv) the wave functions:

$$\rho(s) = s^{c_{10}}(1 - c_3 s)^{c_{11}}, \tag{A.8}$$

$$\phi(s) = s^{c_{12}}(1 - c_3 s)^{c_{13}}, \quad c_{12} > 0,\ c_{13} > 0, \tag{A.9}$$

$$y_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 s), \quad c_{10} > -1,\ c_{11} > -1, \tag{A.10}$$

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}}(1 - c_3 s)^{c_{13}} P_n^{(c_{10}, c_{11})}(1 - 2c_3 s), \tag{A.11}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1, 1]$ are Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1 - 2s) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n, 1 + \alpha + \beta + n; \alpha + 1; s), \tag{A.12}$$

and $N_{n\kappa}$ is a normalization constant. Also, the previous wave functions can be expressed in terms of the hypergeometric function as

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}}(1 - c_3 s)^{c_{13}} \\ \times {}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3 s), \tag{A.13}$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

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