Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2013, Article ID 853925, 7 pages
http://dx.doi.org/10.1155/2013/853925

Research Article

Characteristic Roots of a Class of Fractional Oscillators

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Received 9 August 2013; Accepted 13 September 2013

Academic Editor: Gongnan Xie

The fundamental theorem of algebra determines the number of characteristic roots of an ordinary differential equation of integer order. This may cease to be true for a differential equation of fractional order. The results given in this paper suggest that the number of the characteristic roots of a class of oscillators of fractional order may in general be infinitely great. Further, we infer that it may also be the case for the characteristic roots of a differential equation of fractional order greater than 1. The relationship between the range of the fractional order and the locations of characteristic roots of oscillators in the complex plane is considered.

1. Introduction

Oscillators are an essential component in devices in electron positron collider systems (see, e.g., Zhao et al. [1], Ma et al. [2], Zang et al. [3], Ding et al. [4], Marder et al. [5], Barroso [6], Miller et al. [7], and Lemke [8], just citing a few). As a matter of fact, oscillations are phenomena widely observed in sciences and engineering relating to high energy physics (see, e.g., Akhmediev et al. [9], Bachas [10], Winter et al. [11], Dodonov [12], Tan [13], Diamandis et al. [14], Greenwald et al. [15], Mathews et al. [16], Faiman [17], Cocho et al. [18], Baldiotti et al. [19], Kyu Shin [20], Kirson [21], Clement [22], Sikström et al. [23], Asghari et al. [24], Um et al. [25], Bahar and Yasuk [26], Hassanabadi et al. [27], Bhattacharya and Roy [28], and Saad et al. [29], simply mentioning a few).

There are various structures of oscillators, such as Mathieu oscillator (Floris [30]), Liénard type oscillator (Yaşar [31]), relativistic oscillator (Osborne [32]), Schrödinger equation type oscillator (Cornwall and Tiktopoulos [33]), and Duffing oscillator (Baltanás et al. [34] and Erturk and Inman [35]). In fact, oscillators play a role in various fields, ranging from experimental physics to electronics engineering (see, e.g., Riley et al. [36], Soong and Grigoriu [37], Harris [38], Papoulis [39], Bendat and Piersol [40], Devasahayam [41], Karrenberg [42], Edson [43], and Balaban et al. [44]).

This research is in the domain of fractional oscillators that attract increasing interests of physicists and engineers. More specifically, we aim at revealing specific properties of characteristic roots of a class of fractional oscillators. In doing so, we first consider an ordinary differential equation of order n given by

$$b_n \frac{d^n y(t)}{dt^n} + b_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + b_0 = x(t), \tag{1}$$

where n is a natural number and b_n is any complex number. We always assume that at least one of the higher coefficients $b_n \neq 0$ for $n > 1$. The characteristic equation of (1) is given by

$$B(\alpha) = b_n \alpha^n + b_{n-1} \alpha^{n-1} + \cdots + b_0 = 0. \tag{2}$$

The fundamental theorem of algebra says that the number of roots of (2) is n (G. A. Korn and T. M. Korn [45]). This theorem is stated in the domain of complex variables (Krantz [46]).

Suppose that the n roots of $B(\alpha)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$. For each root r of multiplicity of m, either real or complex, we always consider r the m roots in what follows unless otherwise stated. Using the partial fraction expansion, $B(\alpha)$ can be expressed by

$$B(\alpha) = b_n(\alpha-\alpha_1)(\alpha-\alpha_2)\cdots(\alpha-\alpha_n) = b_n\prod_{i=1}^{n}(\alpha-\alpha_i). \tag{3}$$

Now, we rewrite (3) by the following expression:

$$B(\alpha) = \begin{cases} K\prod_{j=1}^{n/2}\left(m_j\alpha_j^2 + c_j\alpha_j + k_j\right), & n \text{ is even}, \\ K(\alpha-\alpha_n)\prod_{j=1}^{(n-1)/2}\left(m_j\alpha_j^2 + c_j\alpha_j + k_j\right), & n \text{ is odd}, \end{cases} \tag{4}$$

where m_j, c_j, k_j, and K are constants. Without loss of generality, we can suppose that the only simple zero of $B(\alpha)$ is α_n if n is odd.

The factor $(m_j\alpha_j^2 + c_j\alpha_j + k_j)$ in (4) corresponds to the oscillator equation in the form

$$m_j\frac{d^2y(t)}{dt^2} + c_j\frac{dy(t)}{dt} + k_jy(t) = x(t). \tag{5}$$

The characteristic equation of (5) is in the form

$$\left(m_j\alpha_j^2 + c_j\alpha_j + k_j\right) = 0. \tag{6}$$

There are two classes of fractional oscillators. One is in the form (Ryabov and Puzenko [47, Eq. (1)], Ahmad et al. [48], Radwan et al. [49], Drozdov [50, Eq. (9)], Tofighi and Pour [51], Tofighi [52, Eq. (2)], Blaszczyk et al. [53, Eq. (10)], and Narahari Achar et al. [54, 55])

$$m_j\frac{d^{2\pm\varepsilon}y(t)}{dt^{2\pm\varepsilon}} + c_j\frac{d^{\pm\varepsilon}y(t)}{dt^{\pm\varepsilon}} + k_jy(t) = x(t) \quad \text{for } 0 \le \varepsilon < 1. \tag{7}$$

The other is expressed with the form (Lim et al. [56–58], Muniandy and Lim [59], Eab and Lim [60], and Li et al. [61])

$$\left(m_jD^2y(t) + c_jDy(t) + k_jy(t)\right)^\beta = x(t) \quad \text{for } \beta > 0. \tag{8}$$

This research utilizes the form of (8).

According to the fundamental theorem of algebra, there are only two characteristic roots with respect to the oscillator equation (5). They are

$$\alpha_{j,12} = \frac{-c_j \pm \sqrt{c_j^2 - 4m_jk_j}}{2m_j}. \tag{9}$$

One might be carelessly misled to consider that there exist only two characteristic roots regarding the fractional oscillator equation (8) because

$$\left(m_j\alpha_{j,12}^2 + c_j\alpha_{j,12} + k_j\right)^\beta = 0. \tag{10}$$

However, we shall show that the number of the roots in the above expression dramatically differs from what in the following expression:

$$\left(m_j\alpha_{j,12}^2 + c_j\alpha_{j,12} + k_j\right) = 0. \tag{11}$$

The contributions of this paper are twofold. One is to exhibit that the number of the characteristic roots of (8) is in general infinitely great. The other is to reveal the relationship between the range of β and the locations of the characteristic roots of (8) in a complex plane. In addition, if all α_j ($j = 1,\ldots,n$) are simple complex pair of roots, the ordinary differential equation of order n (1) and its generalization given by

$$\left(b_nD^ny(t) + b_{n-1}D^{n-1}y(t) + \cdots + b_0\right)^\beta = x(t), \quad \beta > 0 \tag{12}$$

may be taken as the product of oscillators of integer order and fractional order in series in the wide sense for n being even, respectively.

The rest of the paper is organized as follows. We shall give the results in Section 2, including the proof that there are infinite characteristic roots regarding (8), and the explanation that (1) and (12) may be taken as oscillators in series in the wide sense. Discussions are given in Section 3, which is followed by Conclusions.

2. Results

2.1. Result 1. The number of the characteristic roots of (8) may be infinitely great.

Denote by $\mathbf{C}$ the set of complex numbers. Let $z \in \mathbf{C}$. Suppose that a power function is given by

$$w = z^b = e^{b\ln(z)}. \tag{13}$$

Then, the number of different values of w relies on the value of b for a given z. More precisely, we express that by the following lemmas, which can be found in the literature, such as [45] or Yu [62].

Lemma 1. *If b is a rational number expressed by the irreducible fraction l/m, where $m \ge 1$, the number of values of z^b is m.*

Lemma 2. *If b is an irrational number or imaginary number, the number of values of z^b is infinitely great.*

The general expression of w is in the form

$$w = z^b = e^{b\ln(z)} = e^{b[\ln|z| + i(\arg z + 2m\pi)]}, \quad i = \sqrt{-1}. \tag{14}$$

Therefore, from Lemma 2, we have the theorem below.

Theorem 3. *The number of the characteristic roots of the fractional oscillator (8) is infinitely great if $\beta \ne 1$.*

Proof. Let $c_j = 0$. Then, the characteristic roots $\alpha_{j,12}$ in (9) become the imaginary numbers expressed by

$$\alpha_{j,12} = \frac{\pm i\sqrt{m_j k_j}}{m_j}. \tag{15}$$

□

According to Lemma 2, the number of the roots of either $(\alpha - \alpha_{j,1})^\beta$ or $(\alpha - \alpha_{j,2})^\beta$ is infinitely great. Thus, Theorem 3 results.

2.2. Result 2. Equations (1) and (12) may be taken as oscillators in series.

Denote $(m_j\alpha_j^2 + c_j\alpha_j + k_j)$ in (4) by $B_j(\alpha)$:

$$B_j(\alpha) = \left(m_j\alpha_j^2 + c_j\alpha_j + k_j\right). \tag{16}$$

Without loss of generality, n is assumed to be even. In addition, we suppose that all α_j ($j = 1, \dots, n$) are simple complex pair of roots. Then, we have the theorem below.

Theorem 4. *The ordinary differential equation* (1) *may be taken as an oscillator (i.e., product of oscillators) in the wide sense if n is even and all α_j ($j = 1, \dots, n$) are simple complex pair of roots. By wide sense, one means that it is a system consisting of the product of a series of conventional 2-order oscillators.*

Proof. On the one hand, $B_j(\alpha)$ stands for the characteristic equation of the jth oscillator of order 2 since n is even and all α_j ($j = 1, \dots, n$) are simple complex pair of roots. On the other hand, the characteristic equation of (1) can be expressed by

$$B(\alpha) = \begin{cases} K\prod_{j=1}^{n/2} B_j(\alpha), & n \text{ is even}, \\ K(\alpha - \alpha_n)\prod_{j=1}^{(n-1)/2} B_j(\alpha), & n \text{ is odd}. \end{cases} \tag{17}$$

□

Based on the theory of filter design (Mitra and Kaiser [63]), the system (1) in the case of n being even may be expressed by Figure 1.

Therefore, the system (1) may be expressed by the product of a series of 2-order oscillators.

Denote by $B^\beta(\alpha)$ the characteristic equation of (12). Then,

$$B^\beta(\alpha) = \begin{cases} K\prod_{j=1}^{n/2} B_j^\beta(\alpha), & n \text{ is even}, \\ K(\alpha - \alpha_n)\prod_{j=1}^{(n-1)/2} B_j^\beta(\alpha), & n \text{ is odd}, \end{cases} \tag{18}$$

where

$$B_j^\beta(\alpha) = \left(m_j\alpha_j^2 + c_j\alpha_j + k_j\right)^\beta. \tag{19}$$

Thus, the system of fractional order expressed by (12) may be the product of a series of fractional oscillators of (8).

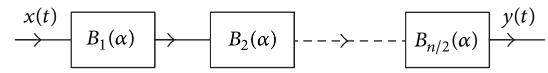

FIGURE 1: Oscillators of 2-order in concatenation.

3. Discussions

The previous section says that there are infinite roots in $B_j^\beta(\alpha)$ if $\beta \neq 1$. In the case of $\beta = 1$, $B_j^\beta(\alpha)$ reduces to the characteristic equation of the conventional oscillator (5) with two roots only. Thus, the fraction $\beta \neq 1$ dramatically alters the behavior of characteristic roots of oscillators. For facilitating our discussions, we omit the subscript j in what follows if not confused. More precisely, we specifically consider the fractional oscillator in the form

$$\begin{gathered}\left(mD^2y(t) + cDy(t) + ky(t)\right)^\beta \\ = x(t) \quad \text{for } \beta > 0.\end{gathered} \tag{20}$$

Figure 2 shows an RLC resonance circuit in series, where R, L, and C represent resistor, inductor, and capacity, respectively. In Figure 2, $I(t)$ is the electronic current and $v(t)$ the power source. According to the Kirchhoff voltage law, one has

$$D^2I(t) + \frac{R}{L}DI(t) + \frac{1}{LC}I(t) = \frac{1}{L}\frac{dv(t)}{dt}. \tag{21}$$

Let $\omega = \sqrt{1/LC}$ and $R/L = 2b$. Denote $(1/L)(dv(t)/dt)$ by $e(t)$. Then, (21) becomes the form

$$D^2I(t) + 2bDI(t) + \omega^2I(t) = e(t). \tag{22}$$

Generalizing (22) to the fractional order β yields

$$\left(D^2I(t) + 2bDI(t) + \omega^2I(t)\right)^\beta = e(t). \tag{23}$$

Below, we specifically study the circuit in Figure 2 with $R = 0$, as indicated in Figure 3.

In the case of Figure 3, (23) becomes the form

$$\left(D^2I(t) + \omega^2I(t)\right)^\beta = e(t). \tag{24}$$

Denote by $h(t)$ the impulse response function of (24). Then, using the techniques in fractional calculus and differential equations [64–81], we have (see [61] for details)

$$\begin{gathered}h(t) = \frac{\sqrt{\pi}}{\Gamma(\beta)(2\omega)^{\beta-1/2}}t^{\beta-1/2}J_{\beta-1/2}(\omega t), \\ \beta > 0,\ t \geq 0,\end{gathered} \tag{25}$$

where $J_{\beta-1/2}(\omega t)$ is the Bessel function of the first kind of order $\beta - 1/2$.

The following theorems reflect the particularity of roots of $B^\beta(\alpha)$.

Theorem 5. *If $0 < \beta < 1$, all roots of $B^\beta(\alpha)$ are located in the left side of the complex plane.*

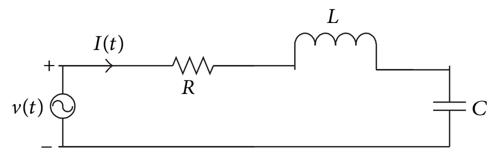

FIGURE 2: Illustration of RLC resonance circuit in series.

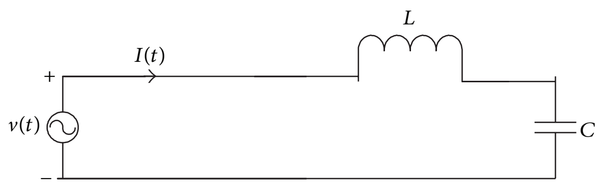

FIGURE 3: Illustration of LC resonance circuit in series with damping $R = 0$.

Proof. Note that

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{t}{2}\right)^{2m+\nu}. \tag{26}$$

□

From the above, we have the asymptotic expression in the form

$$J_\nu(t) \sim \frac{1}{\sqrt{t}} \quad \text{for } t \longrightarrow \infty. \tag{27}$$

Applying (27) to (25) produces

$$\begin{aligned} \lim_{t\to\infty} h(t) &= \lim_{t\to\infty} \frac{\sqrt{\pi}}{\Gamma(\beta)(2\omega)^{\beta-1/2}} t^{\beta-1/2} J_{\beta-1/2}(\omega t) \\ &= 0, \quad 0 < \beta < 1. \end{aligned} \tag{28}$$

Denote by $H(s)$ the Laplace transform of $h(t)$. Then, according to the final-value theorem, we have

$$\lim_{s\to 0} sH(s) = 0, \quad 0 < \beta < 1. \tag{29}$$

The above implies that all poles of $H(s)$ except the origin are strictly in the left side of s plane. In the right of the s plane, $H(s)$ is analytic. This completes the proof.

Theorem 6. *If $\beta > 1$, at least, parts of roots of $B^\beta(\alpha)$ are located in the right side of the complex plane.*

Proof. Note that

$$\begin{aligned} J_\nu(t) = &\frac{(t/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \\ &\times \int_{-1}^{1} \left(1-u^2\right)^{\nu-1/2} \cos(tu)\, du, \quad \operatorname{Re}\nu > -\frac{1}{2}. \end{aligned} \tag{30}$$

From the above, we have the following:

$$\begin{aligned} &\frac{-(t/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^{1} \left(1-u^2\right)^{\nu-1/2} du \\ &\quad \le J_\nu(t) \le \frac{(t/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^{1} \left(1-u^2\right)^{\nu-1/2} du. \end{aligned} \tag{31}$$

□

Since $\beta > 1$ implies $\nu > 1/2$, we immediately see that both the right side and the left one on the above expression are respectively unbounded when $t \to \infty$. Thus, for $\beta > 1$, the fractional oscillator (24) is nonstable according to the theory of systems (Gabel and Roberts [77], Dorf and Bishop [78]). Consequently, at least, some of poles of $H(s)$ are in the right of the s plane. Therefore, at least, parts of roots of $B^\beta(\alpha)$ are located in the right side of the complex plane.

Most of previous discussions take oscillators of fractional order (24) as a specific object. Note that the number of the characteristic roots of differential equation in general in the form of (12) may also be infinitely great. Hence, comes the following theorem in passing.

Theorem 7. *Fractional-order differential equation expressed by* (12) *has infinite characteristic roots if $n > 1$ and if there is at least a pair of roots that are simple complex.*

Proof. The characteristic equation of (12) may be decomposed in the form of (18) due to $n > 1$. Because there is at least a pair of roots that are simple complex, the number of the characteristic roots of (19) is infinitely great. Thus, the number of characteristic roots of (12) is infinitely great. This completes the proof. □

The previous discussions exhibit interesting phenomena of the characteristic roots of the oscillators of the fractional type of (24). In the future, we will work on exploring the answers of the questions described below.

(i) Are all poles of $H(s)$ with respect to (24) in the right of the s plane when $\beta > 1$?

(ii) Might there be interesting oscillation behavior of (12) if all $c_j = 0$ in (18) and if n is even?

4. Conclusions

We have explained that the number of the characteristic roots of fractional-order oscillators of (24) is usually infinitely great. This conclusion has been further inferred to the case of fractional-order differential equation of (12). We have exhibited that all characteristic roots of (24) are strictly located in the left side of the complex plane if $0 < \beta < 1$ and at least some of characteristic roots of (24) are in the right side of the complex plane if $\beta > 1$. In the case of $\beta = 1$, (24) reduce to an ordinary damping-free oscillator.

Acknowledgments

This work was partly supported by the National Natural Science Foundation of China (NSFC) under the project Grant nos. 61272402, 61070214, 60873264, and the 973 plan under the project no. 2011CB302800.

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