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Research Article

$3+1D$ Massless Weyl Spinors from Bosonic Scalar-Tensor Duality

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We consider the fermionization of a bosonic-free theory characterized by the $3+1D$ scalar-tensor duality. This duality can be interpreted as the dimensional reduction, via a planar boundary, of the $4+1D$ topological BF theory. In this model, adopting the Sommerfield tomographic representation of quantized bosonic fields, we explicitly build a fermionic operator and its associated Klein factor such that it satisfies the correct anticommutation relations. Interestingly, we demonstrate that this operator satisfies the massless Dirac equation and that it can be identified with a $3+1D$ Weyl spinor. Finally, as an explicit example, we write the integrated charge density in terms of the tomographic transformed bosonic degrees of freedom.

1. Introduction

Dualities play an important role in various branches of theoretical physics since long time [1]. Perhaps, the first example is the electromagnetic duality between electric and magnetic fields in Maxwell equations. More recently, but still two decades ago, the Seiberg-Witten duality [2] relating the weak and strong coupling regime of $N = 2$ Super Yang-Mills theories induced a set of dualities between various string theories [3]. In particular, fruitful applications have been found and exploited in condensed matter physics, where the AdS/CFT correspondence [4, 5], as a realization of the gauge/gravity duality, has found recent important developments [6].

The common feature of dualities is that of relating different physics, which otherwise would not be related one with each other. The example we are dealing with in this paper is the duality which permits to build fermionic degrees of freedom (d.o.f.) out of bosonic ones.

There has been a huge activity in the opposite direction, that is, the formulation in terms of bosons of a (possibly interacting) fermionic theory. The problem has been firstly solved in $1+1$ dimensions [7–14] but is still an open question in higher dimensions and many efforts are continuously devoted toward this goal [15–17].

Less popular is the reverse process of fermionization where fermionic fields (operators) are obtained from bosonic counterparts. Fermionization has been considered in spin liquid states and heavy fermions [18, 19]. In higher dimensions, using the tomographic representation of quantized fields presented in [20], the problem was solved both in $2+1D$ [21] and $3+1D$ [22]. There, the starting point to proceed to the fermionization is the assumption of a duality relation, introduced somehow by hand, between some bosonic fields. We would like to emphasize that we do not claim that we have a biunivocal mapping from the bosonic degrees of freedom to the fermionic ones but only that, from the bosonic model, we can construct a fermionic field. The correspondence between fermionic and bosonic states has been discussed in detail in [23].

A stronger motivation for considering the fermionization recently arised again, from condensed matter, and it can

be summarized as follows. New states of matter have been predicted (and discovered), called topological insulators (TI) [24, 25], for which the low energy physics seem to be captured by a class of Schwarz-type topological quantum field theory: the BF models [26–31]. Being topological, those theories acquire local d.o.f. only on the boundary.

For what concerns the $2+1D$ TI, the abelian BF model may be easily rephrased in terms of two Chern-Simons theories with opposite coupling constants [28, 32]. On its $1+1D$ planar boundary, the action depends on two scalar fields satisfying a duality relation and it may be rewritten in terms of two counterpropagating bosonic chiral modes. The duality makes it also possible to fermionize the bosonic d.o.f. into two counterpropagating chiral electronic modes connected by time reversal symmetry: the helical Luttinger liquid. (The T symmetry required that when one of the two modes had only one spin component, the other necessarily has opposite one.)

For what concerns the $3+1D$ TI, the BF model has a $2+1D$ bosonic boundary action that depends only on a scalar and a vector field, satisfying the duality relation which allows pursuing the fermionization procedure described in [21] inferring the presence of fermionic d.o.f. on the boundary, which is what is needed to describe the surface states of a topological insulator. An alternative approach [33] for $3+1D$ TI is to consider the effective theories for the $4+1D$ TI, where a Chern-Simon satisfying T symmetry may be considered, making then a dimensional reduction (compactification) to recover the $3+1D$ or the $2+1D$ TI bulk theories [25].

In conclusion, we know that duality relations in a bosonic theory could give rise to a fermionization procedure. A systematic theoretical framework to get bosonic boundary actions with consistent duality relations relating bosonic fields exists in any space-time d-dimension. This is indeed represented by the BF topological models, which can be defined in generic d space-time dimensions, with a planar boundary. The crucial "fermionizing" duality relations are not imposed *ad hoc* but turn out to be the most general boundary conditions for the bosonic fields.

The result is that to any p-form in the bulk corresponds a $(p-1)$-form on the boundary. The $d=3+1D$ case has been treated in [34], where the duality relation involves a scalar and a vector field, and the fermionic construction has been done in [21].

Here, we pursue the program in $d=4+1D$, where, starting from the BF model, we get the duality relation on the $3+1D$ planar boundary. In this case, the resulting bosonic fields are a scalar and a tensor field, with no vector field.

It is worth to mention that our construction gives Weyl spinors in $3+1D$ and it could be relevant in the recently discussed Weyl semimetal physics where, even if the bulk is gapless, still the boundary modes are topologically protected [35, 36].

The paper is organized as follows. In Section 2, we describe and motivate the duality relation we start with, identifying the bosonic d.o.f. to be fermionized. In Section 3, we implement the fermionization procedure. First, we review some basic properties of the tomographic transform, then we apply this method to our system, and, finally, starting from the fermionic variable that we have obtained, we argue how to construct a Weyl spinor. In Section 4, as an example, we relate the total charge with this expressed in terms of the tomographic transformed bosonic fields of our starting model. In Section 5, we summarize our results.

2. Duality and Bosonic Degrees of Freedom

2.1. Preliminaries. We recall some of the results obtained in [37], which represent our starting point. There, it has been shown the possibility to dimensionally reduce the abelian $4+1D$ BF model, described by the action

$$S_{\mathrm{BF}} = \frac{1}{2}\int d^5x \epsilon^{\alpha\beta\gamma\delta\eta}\partial_\alpha A_\beta B_{\gamma\delta\eta} \tag{1}$$

to a gapless bosonic theory on the $3+1D$ planar boundary $x_4=0$. In particular, it has been found that it is possible to parameterize the fields on the boundary in terms of a scalar potential Λ and an antisymmetric tensor potential $\phi_{\mu\nu} = -\phi_{\nu\mu}$, with $\mu,\nu\cdots=\{0,1,2,3\}$, as follows:

$$\begin{aligned} A_\mu &= \partial_\mu\Lambda, \\ B_{\mu\nu\rho} &= \partial_\mu\phi_{\nu\rho} + \partial_\nu\phi_{\rho\mu} + \partial_\rho\phi_{\mu\nu}. \end{aligned} \tag{2}$$

The most convenient gauge fixing term, considering the planar boundary $x_4=0$, is given by

$$S_{\mathrm{GF}} = \int_M d^5x \left(bA_4 + d^{ij}B_{ij4}\right), \tag{3}$$

where b and d^{ij} are Lagrangian multipliers. Following [37], one can conclude that the most general boundary Lagrangian compatible with locality and power counting is

$$\begin{aligned} S_{\mathrm{BD}} = \int d^4x \Big[& a_1\epsilon^{ijkl}A_iB_{jkl} + \frac{a_2}{4}F_{ij}F^{ij} \\ & + \frac{a_3}{4}G_{ij}G^{ij} - \frac{a_4}{2}m^2A_iA^i \Big], \end{aligned} \tag{4}$$

with $F_{ij}(x) = \partial_iA_j(x) - \partial_jA_i(x)$, $G_{ij}(x) = \partial_iA_j(x) + \partial_jA_i(x)$, and $a_{1,\ldots,4}$ are constant parameters.

We conclude that the dynamics on the boundary is completely determined by the $3+1D$ Lagrangian obtained in [37]

$$\begin{aligned} \mathscr{L} = & \frac{1}{2}\epsilon^{ijk}\partial_i\phi_{jk}\partial_t\Lambda + \frac{1}{2}\epsilon^{ijk}\partial_i\Lambda\partial_t\phi_{jk} \\ & - \left(\epsilon^{ijk}\partial_i\phi_{jk}\right)^2 - \frac{1}{4}\left(\partial_i\Lambda\right)^2, \end{aligned} \tag{5}$$

where $i,j\cdots=1,2,3$. The Lagrangian (5) is invariant under the following gauge symmetries:

$$\begin{aligned} \delta^{(1)}\Lambda &= c, \\ \delta^{(2)}\phi_{\mu\nu} &= \partial_{[\mu}\alpha_{\nu]}, \end{aligned} \tag{6}$$

where c is a constant and α_μ is a $3+1D$ vector gauge parameter.

The dimensional reduction on the plane $x_4 = 0$ induces a unique boundary condition which, written in terms of the boundary fields Λ and $\phi_{\mu\nu}$, reads

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu\phi_{\rho\sigma} + \partial^\mu\Lambda = 0. \tag{7}$$

This boundary condition will play the role of the fermionizing duality relation which is the starting point of [21, 22].

Summarizing, we are dealing with a nontopological $3+1D$ field theory (5) defined on a flat Minkowski space-time with metric $g_{\mu\nu} \equiv \mathrm{diag}(-1, 1, 1, 1)$. At this point, it is natural to ask which is the physics described by the $3 + 1D$ theory (5) constrained by the duality relation (7). The main purpose of this paper is to answer this question.

2.2. The Gauge Choice and the Independent Degrees of Freedom. Since the boundary model (5) is invariant under the gauge symmetries (6), it is necessary to fix a gauge for the consistency of the theory.

To do this, we note that, differentiating (7) with respect to x_μ, we find that the scalar field Λ is massless:

$$\Box\Lambda = 0. \tag{8}$$

Then, multiplying (7) by $\epsilon_{\alpha\beta\gamma\mu}$ and differentiating with respect to x_α, we obtain

$$\Box\phi_{\beta\gamma} + \partial^\alpha\partial_\beta\phi_{\gamma\alpha} + \partial^\alpha\partial_\gamma\phi_{\alpha\beta} = 0, \tag{9}$$

which is exactly the equation of motion which must be satisfied by a free massless tensor (see the appendix of [38]). In particular, an admissible gauge choice for $\phi_{\mu\nu}$ is

$$\phi_{0i} = \partial^j\phi_{ij} = 0. \tag{10}$$

With this gauge choice, it is easy to see that $\phi_{\mu\nu}$ can be parameterized in terms of a massless vector field ξ^i:

$$\phi^{jk} = \frac{1}{2}\epsilon^{ijk}\xi_i, \tag{11}$$

with the condition that ξ_i is a longitudinal field:

$$\epsilon^{ijk}\partial_j\xi_k = 0. \tag{12}$$

Consequently, $\phi_{\mu\nu}$ has only one degree of freedom, according to the general rule of the Kalb-Ramond fields [38].

3. The Fermionization Procedure

We are dealing with a $3 + 1D$ Lagrangian (5) which involves a scalar massless field Λ and a longitudinal vector field ξ^i, related by the duality relation (7). In [27], the duality relation used in [21] between the boundary bosonic d.o.f. has been used to guess the existence of the fermionic d.o.f. of the $2+1D$ topological insulators. Inspired by this, in this section, we apply the same method to the duality relation (7), to prove that, in the low energy limit, the $3 + 1D$ boundary model described by (5) has fermionic excitations as well.

Preliminarily, we recall that the fermionization procedure relies on the tomographic transform presented in [20], which displays a known nonlocality drawback [15]. Nevertheless, this problem is not relevant in our case, since, as we said, we are dealing with a low energy effective bosonic field theory.

3.1. The Tomographic Transform. In this section, we review some basic properties of the tomographic transform, which we conveniently use in the fermionization process. For more details on the tomographic representation, we refer to [20].

The basic ingredient of the tomographic transform in three spatial dimensions is the generalized function $\delta'(y - \mathbf{n}\cdot\mathbf{r})$, defined as

$$\delta'(y - \mathbf{n}\cdot\mathbf{r}) = \int_{-\infty}^{+\infty} dk\, k e^{iky} e^{-i\mathbf{r}\cdot k\mathbf{n}}, \tag{13}$$

where $\mathbf{n}$ is an angular variable and k is a scalar. From Definition (13), it is easy to prove that $\delta'(y - \mathbf{n}\cdot\mathbf{r})$ satisfies the completeness and the orthonormality relation:

$$\frac{1}{8\pi^2}\int dy\, d^2n\, \delta'(y - \mathbf{n}\cdot\mathbf{r})\,\delta'(y - \mathbf{n}\cdot\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{14}$$

$$\begin{aligned}\frac{1}{4\pi^2}\int d^3r\,\delta'(y - \mathbf{n}\cdot\mathbf{r})\,\delta'(y' - \mathbf{n}'\cdot\mathbf{r}) \\ = \delta(y - y')\,\delta(\mathbf{n}, \mathbf{n}') - \delta(y + y')\,\delta(\mathbf{n}, -\mathbf{n}').\end{aligned} \tag{15}$$

We now review the properties of the tomographic transform of the scalar, the vector, and the fermionic field.

Following [20], the tomographic transform of the scalar field is defined as

$$\tilde{\phi}(y, \mathbf{n}) = \frac{1}{2\pi}\int d^3r\,\delta'(y - \mathbf{n}\cdot\mathbf{r})\,\phi(\mathbf{r}) = -\tilde{\phi}(-y, -\mathbf{n}), \tag{16}$$

while the inverse tomographic transformation is

$$\phi(\mathbf{r}) = \frac{1}{4\pi}\int dy\, d^2n\,\delta'(y - \mathbf{n}\cdot\mathbf{r})\,\tilde{\phi}(y, \mathbf{n}). \tag{17}$$

For what concerns the vector field A_μ, the tomographic transforms of its four space-time components are organized as

$$\begin{aligned}\widetilde{A}^S(y, \mathbf{n}) &= \frac{1}{2\pi}\int d^3r\,\delta'(y - \mathbf{n}\cdot\mathbf{r})\,A_0(\mathbf{r}), \\ \widetilde{A}^L(y, \mathbf{n}) &= \frac{1}{2\pi}\int d^3r\,\delta'(y - \mathbf{n}\cdot\mathbf{r})\,\mathbf{n}\cdot\mathbf{A}(\mathbf{r}), \\ \widetilde{A}^{Ta}(y, \mathbf{n}) &= \frac{1}{2\pi}\int d^3r\,\delta'(y - \mathbf{n}\cdot\mathbf{r})\,\boldsymbol{\varepsilon}^a(\mathbf{n})\cdot\mathbf{A}(\mathbf{r}),\end{aligned} \tag{18}$$

where $\widetilde{A}^L(y, \mathbf{n})$ and $\widetilde{A}^{Ta}(y, \mathbf{n})$ are the longitudinal and the transverse transforms, respectively, and we have introduced the polarization vectors $\boldsymbol{\varepsilon}^a(\mathbf{n})$ orthogonal to $\mathbf{n}$, with $a = 1, 2$. The spatial antitransform is defined as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int dy\, d^2n\,\delta'(y - \mathbf{n}\cdot\mathbf{r}) \\ \times\left[\mathbf{n}\widetilde{A}^L(y, \mathbf{n}) + \boldsymbol{\varepsilon}^a(\mathbf{n})\,\widetilde{A}^{Ta}(y, \mathbf{n})\right].\end{aligned} \tag{19}$$

The tomographic transform of the four component spinor field $\psi_\alpha(\mathbf{r})$ is as follows:

$$\tilde{\psi}^b(y, \mathbf{n}) = \frac{1}{2\pi}\int d^3r\,\delta'(y - \mathbf{n}\cdot\mathbf{r})\,u_\alpha^{\dagger b}(\mathbf{n})\,\psi_\alpha(\mathbf{r}), \tag{20}$$

$u_\alpha^{\dagger b}$ is a spinor, where $\alpha = \{1,\ldots,4\}$ is a spinor index. Introducing the usual 4×4 Dirac matrices $\boldsymbol{\alpha}$ and the spin matrices $\boldsymbol{\Sigma} = -(i/2)\boldsymbol{\alpha}\times\boldsymbol{\alpha}$, u_α^b is, by definition, an eigenspinor of $\boldsymbol{\alpha}\cdot\mathbf{n}$ with eigenvalue -1. Moreover, since $[\boldsymbol{\Sigma}\cdot\mathbf{n}, \boldsymbol{\alpha}\cdot\mathbf{n}] = 0$, $u_\alpha^b(\mathbf{n})$ is also an eigenvector of $\boldsymbol{\Sigma}\cdot\mathbf{n}$, with

$$(\boldsymbol{\Sigma}\cdot\mathbf{n})\, u_\alpha^b(\mathbf{n}) = b u_\alpha^b(\mathbf{n}), \quad b = \{1,-1\}. \tag{21}$$

The orthogonality condition $u_\alpha^{\dagger b}(\mathbf{n}) u_\alpha^c(\mathbf{n}) = \delta^{bc}$ holds, so we can write the projector as

$$\sum_b u_\alpha^b(\mathbf{n})\, u_\beta^{\dagger b}(\mathbf{n}) = \frac{1}{2}(1 - \boldsymbol{\alpha}\cdot\mathbf{n})_{\alpha\beta}. \tag{22}$$

The antitransform of a spinor field is defined as

$$\psi_\alpha(\mathbf{r}) = \frac{1}{2\pi}\int dy\, d^2n\, \delta'(y - \mathbf{n}\cdot\mathbf{r})\, u_\alpha^b(\mathbf{n})\, \widetilde{\psi}^b(y,\mathbf{n}). \tag{23}$$

Finally, by using the following identity:

$$\partial_i \delta'(y - \mathbf{n}\cdot\mathbf{r}) = -n_i \partial_y \delta'(y - \mathbf{n}\cdot\mathbf{r}), \tag{24}$$

and the completeness relation (14), it is possible to prove that the massless Dirac equation, $\gamma^\mu\partial_\mu\psi = 0$, in the tomographic representation reads:

$$\left(\partial_0 - \partial_y\right)\widetilde{\psi}^b(y,\mathbf{n}) = 0. \tag{25}$$

The last equation shows that, for a fixed value of $\mathbf{n}$, $\psi^b(y,\mathbf{n})$ is a "right moving" field propagating along the positive direction of y. It is also possible to define the tomographic transform of a spinor field as follows:

$$\widetilde{\chi}^b(y,\mathbf{n}) = \frac{1}{2\pi}\int d^3r\, \delta'(y - \mathbf{n}\cdot\mathbf{r})\, v_\alpha^{\dagger b}(\mathbf{n})\, \psi_\alpha(\mathbf{r}), \tag{26}$$

where $v_\alpha^{\dagger b}$ is an eigenspinor of $\boldsymbol{\alpha}\cdot\mathbf{n}$ with eigenvalue $+1$. For v_α^b, the following relations hold:

$$\begin{gathered}(\boldsymbol{\Sigma}\cdot\mathbf{n})\, v_\alpha^b(\mathbf{n}) = b v_\alpha^b(\mathbf{n}), \quad b = \{1,-1\}, \\ \sum_b v_\alpha^b(\mathbf{n})\, v_\beta^{\dagger b}(\mathbf{n}) = \frac{1}{2}(1 + \boldsymbol{\alpha}\cdot\mathbf{n})_{\alpha\beta}.\end{gathered} \tag{27}$$

The antitransform of the spinor field, in this case, is defined as

$$\psi_\alpha(\mathbf{r}) = \frac{1}{2\pi}\int dy\, d^2n \delta'(y - \mathbf{n}\cdot\mathbf{r})\, v_\alpha^b(\mathbf{n})\, \widetilde{\chi}^b(y,\mathbf{n}), \tag{28}$$

while the tomographic transformed Dirac equation for $\widetilde{\chi}^b(y,\mathbf{n})$ is

$$\left(\partial_0 + \partial_y\right)\widetilde{\chi}^b(y,\mathbf{n}) = 0. \tag{29}$$

Analogously, here, we have a "left moving" field. As we will see, the two previous constructions of the tomographic transform of the fermionic field are completely equivalent.

3.2. Tomographic Duality. The duality relation (7) can be written in terms of the tomographic transformed fields $\widetilde{\Lambda}$ and $\widetilde{\xi}^L$ as follows:

$$\begin{gathered}\partial_0\widetilde{\Lambda}(y,\mathbf{n}) = \partial_y\widetilde{\xi}^L(y,\mathbf{n}), \\ \partial_y\widetilde{\Lambda}(y,\mathbf{n}) = \partial_0\widetilde{\xi}^L(y,\mathbf{n}).\end{gathered} \tag{30}$$

On the other hand, the longitudinal condition (12) requires that the transverse components vanish, as well as their tomographic counterparts $\widetilde{\xi}^{Ta}(y,\mathbf{n}) \equiv 0$. Consequently, we find that $\widetilde{\xi}^L(y,\mathbf{n})$ is the tomographic transform of the unique d.o.f. of the massless tensor $\phi_{\mu\nu}$, and from now on $\widetilde{\xi}^L(y,\mathbf{n}) \equiv \widetilde{\xi}(y,\mathbf{n})$.

3.3. Fermionization. In the previous sections, we defined the properties of the tomographic transform and we wrote the duality relation (7) in terms of tomographic variables (30). Now, we implement the fermionization process following the steps described in [21, 22].

Identifying $\widetilde{\Lambda}(y,\mathbf{n})$ and $\widetilde{\xi}(y,\mathbf{n})$ as the tomographic transforms of the bosonic d.o.f. on which the $3+1D$ theory described by the Lagrangian (5) depends, we define a fermionic field as

$$\widetilde{\Psi}(y,\mathbf{n}) = \frac{1}{\sqrt{2\pi\alpha}} :e^{i\sqrt{\pi}[\widetilde{\xi}(y,\mathbf{n})+\widetilde{\Lambda}(y,\mathbf{n})]}:, \tag{31}$$

where α is a regularizing constant. The normal ordering prescription appearing in (31) is $\{\widetilde{\xi}^{(-)}, \widetilde{\Lambda}^{(-)}, \widetilde{\Lambda}^{(+)}, \widetilde{\xi}^{(+)}\}$. The ordinary time evolution of the free fields Λ and ξ allows us to write down the Heisenberg operators of the positive and negative frequency parts of $\widetilde{\xi}$ and $\widetilde{\Lambda}$ as follows:

$$\begin{gathered}\widetilde{\xi}^{(\pm)}(y,\mathbf{n}) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{1}{\sqrt{2|p|}}a^{(\pm)}(p)\, e^{\mp(ipy - i|p|t)}e^{-(\alpha/2)|p|}, \\ \widetilde{\Lambda}^{(\pm)}(y,\mathbf{n}) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{1}{\sqrt{2|p|}}b^{(\pm)}(p)\, e^{\mp(ipy - i|p|t)}e^{-(\alpha/2)|p|},\end{gathered} \tag{32}$$

and $a^{(+)}(p)$ ($a^{(-)}(p)$) and $b^{(+)}(p)$ ($b^{(-)}(p)$) are the creation (annihilation) operators for $\widetilde{\xi}(y,\mathbf{n})$ and $\widetilde{\Lambda}(y,\mathbf{n})$, respectively. Because of the duality relations (30), they are related by

$$b^{(\pm)}(p) = -\operatorname{sign}(p)\, a^{(\pm)}(p) \tag{33}$$

with $\operatorname{sign}(x)$ denoting the sign of x. Let us define $\phi(y,\mathbf{n})$ as

$$\phi(y,\mathbf{n}) = \widetilde{\xi}(y,\mathbf{n}) + \widetilde{\Lambda}(y,\mathbf{n}), \tag{34}$$

and, accordingly,

$$\phi^{(\pm)}(y,\mathbf{n}) = \int_0^{\infty}\frac{dp}{\pi}\frac{1}{\sqrt{2|p|}}a^{(\pm)}(p)\, e^{\mp(ipy - i|p|t)}e^{-(\alpha/2)|p|}. \tag{35}$$

The presence of only positive p momenta reminds what happens in the $1+1D$ bosonization, where only the right moving components are involved [13]. The minus sign in (34) would correspond to terms with only negative momenta components, in analogy with the left movers in $1+1D$.

The crucial observation is that the Lorentz scalar defined in (31) is a fermionic anticommuting variable. To see this, we compute the anticommutator

$$\begin{aligned}&\left\{\widetilde{\Psi}(y,\mathbf{n}),\widetilde{\Psi}^{\dagger}(y',\mathbf{n})\right\}\\&=\frac{1}{2\pi\alpha}:e^{i\sqrt{\pi}[\phi^{(+)}(y,\mathbf{n})+\phi^{(-)}(y,\mathbf{n})]}::e^{-i\sqrt{\pi}[\phi^{(+)}(y',\mathbf{n})+\phi^{(-)}(y',\mathbf{n})]}:\\&\quad+\frac{1}{2\pi\alpha}:e^{-i\sqrt{\pi}[\phi^{(+)}(y',\mathbf{n})+\phi^{(-)}(y',\mathbf{n})]}::e^{i\sqrt{\pi}[\phi^{(+)}(y,\mathbf{n})+\phi^{(-)}(y,\mathbf{n})]}:.\end{aligned}\tag{36}$$

We observe that

$$\begin{aligned}&:e^{i\sqrt{\pi}[\phi^{(+)}(y,\mathbf{n})+\phi^{(-)}(y,\mathbf{n})]}::e^{-i\sqrt{\pi}[\phi^{(+)}(y',\mathbf{n})+\phi^{(-)}(y',\mathbf{n})]}\\&:=e^{\pi\langle\phi(y,\mathbf{n})\phi(y',\mathbf{n})-(\phi^2(y,\mathbf{n})+\phi^2(y',\mathbf{n}))/2\rangle}=\frac{\alpha}{\alpha-i(y-y')}.\end{aligned}\tag{37}$$

where we have used [13] $:e^{A}e^{B}=:e^{A+B}:e^{\langle AB+(A^2+B^2)/2\rangle}$. The second term in the right-hand side of (36) can be treated in the same way, with the outcome that

$$\begin{aligned}&\left\{\widetilde{\Psi}(y,\mathbf{n}),\widetilde{\Psi}^{\dagger}(y',\mathbf{n})\right\}\\&=\left[\frac{\alpha}{\alpha-i(y-y')}+\frac{\alpha}{\alpha+i(y-y')}\right]\frac{1}{2\pi\alpha}\\&=\frac{\alpha}{\pi\left(\alpha^2+(y-y')^2\right)}\underset{\alpha\to0}{\longrightarrow}\delta(y-y').\end{aligned}\tag{38}$$

Therefore, we verified that the scalar field defined in (31) satisfies the anticommutation relations for $\mathbf{n}=\mathbf{n}'$, but the commutation relation is still bosonic for $\mathbf{n}\neq\mathbf{n}'$. The complete Fermi statistics can be achieved introducing the Klein factors [15]:

$$O_{\mathbf{n}}=e^{(i\sqrt{\pi}/2)\int^{(\theta,\varphi)}d^2n'[\alpha(\mathbf{n}')+\beta(\mathbf{n}')]},\tag{39}$$

where the operators

$$\begin{aligned}\alpha(\mathbf{n})&=\int_{-\infty}^{\infty}dy\,\partial_0\widetilde{\xi}(y,\mathbf{n}),\\\beta(\mathbf{n})&=\int_{-\infty}^{\infty}dy\,\partial_0\widetilde{\Lambda}(y,\mathbf{n})\end{aligned}\tag{40}$$

take the role of generalized charges [21, 22]. The angle parameterization (θ',ϕ') defined the versor direction $\mathbf{n}'$ and the integration domain $\int^{(\theta,\varphi)}$ is $0<\varphi'\leq2\pi\cup0<\theta'<\theta$ if $\theta'\neq\theta$ and $0<\varphi'<\varphi$ if $\theta'=\theta$. Using the definitions (39)-(40), the following rule can be verified:

$$O_{\mathbf{n}}\widetilde{\Psi}(y,\mathbf{m})=\begin{cases}-\widetilde{\Psi}(y,\mathbf{m})O_{\mathbf{n}}, & \mathbf{m}<\mathbf{n},\\\widetilde{\Psi}(y,\mathbf{m})O_{\mathbf{n}}, & \mathbf{m}\geq\mathbf{n},\end{cases}\tag{41}$$

where $\mathbf{m}<\mathbf{n}$ means $\theta_{\mathbf{m}}<\theta_{\mathbf{n}}$ or $\theta_{\mathbf{m}}=\theta_{\mathbf{n}}$ and $\varphi_{\mathbf{m}}<\varphi_{\mathbf{n}}$.

At this point, it is straightforward to check that the operator

$$\widetilde{\psi}(y,\mathbf{n})\equiv\widetilde{\Psi}(y,\mathbf{n})O_{\mathbf{n}}\tag{42}$$

satisfies the anticommutation relations

$$\begin{aligned}\left\{\widetilde{\psi}(x,\mathbf{n}),\widetilde{\psi}^{\dagger}(y,\mathbf{m})\right\}&=\delta(\mathbf{n},\mathbf{m})\,\delta(x-y),\\\left\{\widetilde{\psi}(x,\mathbf{n}),\widetilde{\psi}(y,\mathbf{m})\right\}&=0,\end{aligned}\tag{43}$$

which allow us to conclude that $\widetilde{\psi}(y,\mathbf{n})$ is a well-defined fermionic field in the tomographic representation.

In [20], it was shown how the tomographic transformed fields behave under space-time rotations. We have verified that definition (31) is consistent with those properties. Moreover, by using the definition (42) and the duality relations (30), it is easy to see that $\widetilde{\psi}$ must satisfy

$$\left(\partial_0-\partial_y\right)\widetilde{\psi}(y,\mathbf{n})=0,\tag{44}$$

which is the tomographic version of the massless Dirac equation (25).

An additional fermionic field can be introduced:

$$\widetilde{\chi}(y,\mathbf{n})=\frac{1}{\sqrt{2\pi\alpha}}:e^{i\sqrt{\pi}[\widetilde{\xi}(y,\mathbf{n})-\widetilde{\Lambda}(y,\mathbf{n})]}:P_{\mathbf{n}},\tag{45}$$

where

$$P_{\mathbf{n}}=e^{(i\sqrt{\pi}/2)\int^{(\theta,\varphi)}d^2n'[\alpha(\mathbf{n}')-\beta(\mathbf{n}')]},\tag{46}$$

which satisfies the Fermi statistics:

$$\begin{aligned}\left\{\widetilde{\chi}(x,\mathbf{n}),\widetilde{\chi}^{\dagger}(y,\mathbf{m})\right\}&=\delta(\mathbf{n},\mathbf{m})\,\delta(x-y),\\\left\{\widetilde{\chi}(x,\mathbf{n}),\widetilde{\chi}(y,\mathbf{m})\right\}&=0,\end{aligned}\tag{47}$$

together with the massless Dirac equation (29):

$$\left(\partial_0+\partial_y\right)\widetilde{\chi}(y,\mathbf{n})=0.\tag{48}$$

Consequently, $\widetilde{\chi}(y,\mathbf{n})$ obeys the tomographic construction (26).

As we have anticipated, it is possible to reconstruct the spinor $\psi_\alpha(r)$ both from $\widetilde{\psi}(y,\mathbf{n})$ and $\widetilde{\chi}(y,\mathbf{n})$ in a completely equivalent way. In fact, choosing a fixed eigenvalue b of $\mathbf{\Sigma}\cdot\mathbf{n}$ associated with $\widetilde{\chi}(y,\mathbf{n})$ (we are not able to determine the value of b a priori), we have from (26)

$$\begin{aligned}\psi_\alpha(r)&=\int dy\,d^2n\delta'(y-\mathbf{n}\cdot\mathbf{r})\,v_\alpha^b(\mathbf{n})\,\widetilde{\chi}(y,\mathbf{n})\\&=(y\longrightarrow-y,\mathbf{n}\longrightarrow-\mathbf{n})\\&=\int dy\,d^2n\delta'(-y+\mathbf{n}\cdot\mathbf{r})\,v_\alpha^b(-\mathbf{n})\,\widetilde{\chi}(-y,-\mathbf{n}).\end{aligned}\tag{49}$$

But, keeping in mind that $\delta'(y-\mathbf{n}\cdot\mathbf{r})$ is an odd function under the transformation ($y \to -y, \mathbf{n} \to -\mathbf{n}$) and the well-known relation $v^b(-\mathbf{n}) = u^{-b}(\mathbf{n})$ (remember that $\boldsymbol{\alpha}\cdot\mathbf{n}u^b(\mathbf{n}) = -u^b(\mathbf{n})$ and $\boldsymbol{\alpha}\cdot\mathbf{n}v^b(\mathbf{n}) = v^b(\mathbf{n})$), we obtain that (49) is equal to

$$-\int dy\, d^2n\delta'(y-\mathbf{n}\cdot\mathbf{r})\, u_\alpha^{-b}(\mathbf{n})\, \widetilde{\chi}(-y,-\mathbf{n}). \tag{50}$$

Finally, since, by construction, $\widetilde{\xi}(-y,-\mathbf{n}) = \widetilde{\xi}(y,\mathbf{n})$ and $\widetilde{\Lambda}(-y,-\mathbf{n}) = -\widetilde{\Lambda}(y,\mathbf{n})$, we obtain from the definition of $\widetilde{\chi}(y,\mathbf{n})$ (45) that

$$\widetilde{\chi}(-y,-\mathbf{n}) = \widetilde{\psi}(y,\mathbf{n}), \tag{51}$$

And, consequently, (50) is equal to

$$-\int dy\, d^2n\delta'(y-\mathbf{n}\cdot\mathbf{r})\, u_\alpha^{-b}(\mathbf{n})\, \widetilde{\psi}(y,\mathbf{n}), \tag{52}$$

which proves that the construction (20) and construction (26) are completely equivalent.

Finally, we are dealing with only one independent tomographic fermionic field, from which we can only construct a Weyl spinor since, as it is well known, for massless fermion in $3+1D$, $\boldsymbol{\Sigma}\cdot\mathbf{n}$ is equivalent to γ^5 and, consequently, our tomographic transformed spinor field is an eigenvalue of γ^5 by construction. Then, we can use it only to construct a Weyl spinor.

4. The Integrated Charge Density

In this section, as an example, we compute the integrated charge density expressed in terms of the tomographic transformed bosonic fields of our starting model.

The total charge expressed in terms of the tomographic fermionic variables is

$$\rho_F = \int d^3r\psi_\alpha^\dagger(\mathbf{r})\,\psi_\alpha(\mathbf{r}) = \int dy\, d^2n\, \widetilde{\psi}^\dagger(y,\mathbf{n})\,\widetilde{\psi}(y,\mathbf{n}), \tag{53}$$

where we have used the generalization of the orthonormality relation (15) for the fermionic field:

$$\begin{aligned}\frac{1}{4\pi^2}\int d^3r\delta'(y-\mathbf{n}\cdot\mathbf{r})\,\delta'(y'-\mathbf{n}'\cdot\mathbf{r})\,u_\alpha^{\dagger b}(\mathbf{n})\,u_\alpha^c(\mathbf{n}')\\ = \delta^{bc}\delta(y-y')\,\delta(\mathbf{n},\mathbf{n}').\end{aligned} \tag{54}$$

We evaluate the right-hand side of (53) with the point-splitting regularization technique [13]:

$$\begin{aligned}\rho_\psi &= \lim_{\varepsilon\to 0}\left[\widetilde{\psi}^\dagger(y+\varepsilon,\mathbf{n})\,\widetilde{\psi}(y,\mathbf{n})\right]\\ &= \lim_{\varepsilon\to 0}\left[\frac{1}{2\pi}\left(\frac{1}{i\varepsilon}-\sqrt{\pi}\partial_y\left[\widetilde{\xi}(y,\mathbf{n})+\widetilde{\Lambda}(y,\mathbf{n})\right]\right)\right].\end{aligned} \tag{55}$$

Here, we have used the usual point-splitting assumption, with the limit $\alpha \to 0$ taken before that $\epsilon \to 0$.

The total charge is obtained by subtracting from (55) the vacuum average charge (actually, the total charge is an integral of a total derivative and, as usual, is equal to zero if specific boundary conditions are not imposed on the fields of the theory):

$$\begin{aligned}\overline{\rho}_F &= \rho_F - \langle\rho_F\rangle\\ &= \lim_{\varepsilon\to 0}\int dy\, d^2n\\ &\qquad\times\left[\frac{1}{2\pi}\left(\frac{1}{i\varepsilon}-\sqrt{\pi}\partial_y\left(\widetilde{\xi}(y,\mathbf{n})+\widetilde{\Lambda}(y,\mathbf{n})\right)-\frac{1}{i\varepsilon}\right)\right]\\ &= -\int dy\, d^2n\,\frac{1}{2\sqrt{\pi}}\partial_y\left(\widetilde{\xi}(y,\mathbf{n})+\widetilde{\Lambda}(y,\mathbf{n})\right).\end{aligned} \tag{56}$$

This result is exactly what we have expected, since $\widetilde{\psi}(y,\mathbf{n})$ is a tomographic "right moving" fermion because, in the representation of (20), it satisfies the tomographic Dirac equation with the minus sign (25). Analogously, the current associated with this spinor must also be "right moving" and, indeed, it only depends on the "right moving" combination of fields $\widetilde{\xi}(y,\mathbf{n})+\widetilde{\Lambda}(y,\mathbf{n})$. (Equivalently, in the representation (28), only the negative p components need to be considered and, in such case, we will obtain a minus sign between the two bosonic fields.)

5. Summary of Results

In this paper, we explicitly constructed the fermionic d.o.f. for a $3+1D$ bosonic theory, where a scalar field and a tensor field are related by a duality relation. The most natural interpretation of this duality relation comes from the dimensional reduction (on a planar boundary) of the $4+1D$ BF theory. This is done in complete analogy with the 3 + 1 TI, where the $3+1D$ BF theory for the bulk, if restricted to the $2+1D$ boundary, naturally displays fermionic d.o.f. [27]. From a more field theoretical point of view, we stress that the duality relation is not imposed by hand but emerges as the unique boundary condition for the fields of the topological $4+1D$ BF model with a planar boundary [37]. Following the tomographic representation of quantized fields presented in [20], we have given the tomographic representation of a fermionic field corresponding to the bosonic original d.o.f. We have shown that this fermionic field satisfies the correct anticommutation relations and the massless Dirac equations. In addition, we have shown that it is only possible, with our tomographic transformed spinor field, to construct a Weyl spinor. Finally, as an explicit example, the fermionic integrated charge density has been considered, and we showed that its bosonized tomographic counterpart coincides, indeed, with what we expected for the right mover tomographic fermion we are dealing with.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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