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Research Article

Holographic Brownian Motion in Three-Dimensional Gödel Black Hole

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By using the AdS/CFT correspondence and Gödel black hole background, we study the dynamics of heavy quark under a rotating plasma. In that case we follow Atmaja (2013) about Brownian motion in BTZ black hole. In this paper we receive some new results for the case of $\alpha^2 l^2 \neq 1$. In this case, we must redefine the angular velocity of string fluctuation. We obtain the time evolution of displacement square and angular velocity and show that it behaves as a Brownian particle in non relativistic limit. In this plasma, it seems that relating the Brownian motion to physical observables is rather a difficult work. But our results match with Atmaja work in the limit $\alpha^2 l^2 \to 1$.

1. Introduction

In the last several years, the holographic AdS/CFT [1–4] has been exploited to study strongly coupled systems, in particular quark gluon plasmas [5–7]. The quark gluon plasma (QGP) is produced, when two heavy ions collide with each other at very high temperature. A relatively heavy particle, for example, a heavy quark, immerses in a soup of quarks and gluons with small fluctuations due to its interaction with constituent of QGP. The random motion of this particle is well known as Brownian motion [8–10]. The Brownian motion is a universal phenomenon in finite temperature systems and any particle immersed in a fluid at finite temperature undergoes Brownian motion. The Brownian motion opens a wide view from microscopic nature. It offers a better understanding of the microscopic origin of thermodynamics of black holes. Therefore, it is a natural step to study Brownian motion using the AdS/CFT correspondence. Particularly, the AdS/CFT correspondence can be utilized to investigate the Brownian motion for a quark in the quark gluon plasma.

In the field theory or boundary side of AdS/CFT story, a mathematical description of Brownian motion is given by the Langevin equation which phenomenologically describes the force acting on Brownian particles [10–12] which is given by

$$\dot{p}(t) = -\gamma_0 p(t) + R(t), \tag{1}$$

where p is momentum of Brownian particle and γ_0 is the friction coefficient. These forces originate from losing energy to medium due to friction term (first term) and getting a random kick from the thermal bath (second term). One can learn about the microscopic interaction between the Brownian particle and the fluid constituents, if these forces be clear. By assuming $\langle m\dot{x}^2\rangle = T$, the time evolution of displacement square is given as follows [10]:

$$\left\langle s(t)^2 \right\rangle = \left\langle [x(t) - x(0)]^2 \right\rangle \approx \begin{cases} \dfrac{T}{m}t^2, & \left(t \ll \dfrac{1}{\gamma_0}\right), \\ 2Dt, & \left(t \gg \dfrac{1}{\gamma_0}\right), \end{cases} \tag{2}$$

where $D = T/\gamma_0 m$ is diffusion constant, T is the temperature, and m is the mass of Brownian particle. At early time, $t \ll 1/\gamma_0$ (ballistic regime), the Brownian particle moves with constant velocity $\dot{x} \sim \sqrt{T/m}$, while at the late time, $t \gg 1/\gamma_0$ (diffusive regime), the particle undergoes a random walk.

In the gravity or bulk side of AdS/CFT version for Brownian motion, we need a gravitational analog of a quark immersed in QGP. This is achieved by introducing a bulk fundamental string stretching between the boundary at infinity and event horizon of an asymptotically AdS black hole background [13–17]. The dual statement of a quark in QGP on the boundary corresponds to the black hole environment that excites the modes of string. In the context of this duality, the end of string at the boundary corresponds to the quark which shows Brownian motion and its dynamics is formulated by Langevin equation. In the formulation of AdS/CFT correspondence, fields of gravitational theory would be related to the corresponding boundary theory operators [3, 4]. In this way, instead of using the boundary field theory to obtain the correlation function of quantum operators, we can determine these correlators by the thermal physics of black holes and use them to compute the correlation functions. In [14–17], the Brownian motion has been studied in holographic setting and the time evolution of displacement square. If we consider different gravity theories, we know that strings live in a black hole background and excitation of the modes is done by Hawking radiation of black hole. So, different theories of gravity can be associated with various plasma in the boundary. In this paper we follow different works to investigate Brownian motion of a particle in rotating plasmas. We try to consider the motion of a particle in two-dimensional rotating plasma whose gravity dual is described by three-dimensional Gödel metric background. In this case, we will see that, for the α parameter in the Gödel metric, new conditions for Brownian motion will be provided. As we know, the Brownian motion of a particle in two-dimensional rotating plasma with the corresponding gravity of BTZ black hole has been studied in [16]. The Gödel metric background in the special case receives to the BTZ black hole [18], so the comparison of our results with [16] gives us motivation to understanding the Brownian motion in rotating plasmas in general form of background as Gödel black hole.

This paper is arranged as follows. In Section 2, we give some review of three-dimensional Gödel black hole and derive string action from this metric background. Section 3 is devoted to investigate a holographic realization of Brownian motion and obtain the solution for equation of motion of string in Gödel black hole geometry. We study the Hawking radiation of the transverse modes near the outer horizon of Gödel black hole to describe the random motion of the external quark in Section 4. In Section 5, we make some summery about our results.

2. Background and String Action

2.1. Gödel Black Hole. Three-dimensional Gödel spacetime is an exact solution of Einstein-Maxwell theory with a negative cosmological constant and a Chern-Simons term [19]. When the electromagnetic field acquires a topological mass α Maxwell equation will be modified by an additional term. In that case, we receive to Einstein-Maxwell-Chern-Simons system, and the geometry is the Gödel space time [20]. This theory can be viewed as a lower dimensional toy model for the bosonic part of five-dimensional supergravity theory, so it can be an advantage in development of string theory. Three-dimensional Gödel black holes are like their higher dimensional counterparts in special properties. The action of Einstein-Maxwell-Chern-Simons theory in three dimensions is given by [21]

$$I = \frac{1}{16\pi G}\int d^3x\left[\sqrt{-g}\left(R + \frac{2}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) - \frac{\alpha}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\right]. \tag{3}$$

A general spherically symmetric static solution to the above action in various cases for the α parameter can be written by [22]

$$ds^2 = \frac{dr^2}{h^2 - pq} + pdt^2 + 2hdtd\phi + qd\phi^2, \tag{4}$$

where p, q, and h are functions of r as

$$\begin{aligned} p(r) &= 8G\mu, \\ q(r) &= \frac{-4GJ}{\alpha} + 2r - 2\frac{\gamma^2}{l^2}r^2, \\ h(r) &= -2\alpha r, \end{aligned} \tag{5}$$

with

$$\gamma = \sqrt{\frac{1-\alpha^2 l^2}{8G\mu}}. \tag{6}$$

The gauge potential is given by

$$A = A_t(r)\,dt + A_\phi(r)\,d\phi, \tag{7}$$

with

$$A_t(r) = \frac{\alpha^2 l^2 - 1}{\gamma \alpha l} + \varepsilon, \qquad A_\varphi(r) = \frac{-4GQ}{\alpha} + 2\frac{\gamma}{l}r. \tag{8}$$

The parameters μ and J are mass and angular momentum. The arbitrary constant ε is a pure gauge. We can rewrite metric (4) in the ADM form as follows:

$$ds^2 = -\frac{\Delta}{q}dt^2 + \frac{dr^2}{\Delta} + q\left(d\phi + \frac{h}{q}dt\right)^2, \tag{9}$$

where

$$\begin{gathered} \Delta = h^2 - pq = \lambda(r - r_+)(r - r_-), \qquad \lambda = \frac{2(1+\alpha^2 l^2)}{l^2}, \\ r_\pm = \frac{8G}{\lambda}\left[\mu \pm \sqrt{\mu^2 - \frac{\mu J\lambda}{2\alpha}}\right]. \end{gathered} \tag{10}$$

The Hawking temperature that gives the temperature of the plasma is [23]

$$T_H = \frac{\lambda(r_+ - r_-)}{8\pi\alpha r_+}. \tag{11}$$

Here r_- is the inner horizon and r_+ is the outer horizon. In the sector $\alpha^2 l^2 > 1$, we have real solution only for μ negative. In this regime, there are Gödel particles and theory supports time-like constants fields. When $\alpha^2 l^2 < 1$, μ has positive values. In this case black hole will be will be constructed and theory supports space-like constants fields. For $\alpha^2 l^2 = 1$, metric (4) reduces to BTZ metric as can be explicitly seen by transforming to the standard frame that is nonrotating at infinity with respect to anti-de Sitter space:

$$\phi \longrightarrow \phi + \alpha t, \qquad r \longrightarrow \frac{r^2}{2} + \frac{2GJ}{\alpha}. \tag{12}$$

In the standard frame, energy and angular momentum become $M = \mu - \alpha J$ and J, instead of μ and J in rotating frame.

2.2. String Action. In the general case for a $d+2$-dimensional black hole metric background is

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu + G_{IJ}(x)\, dx^I dx^J. \tag{13}$$

Here $x^\mu = r, t$ stands for the string worldsheet coordinates and $X^I = X^I(x)$ $(I, J = 1, \ldots, d-2)$ for the spacetime coordinates. If we stretch a string along the r direction and consider small fluctuation in the transverse direction X^I, the dynamics of this string follows from the Nambu-Goto action [14]:

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\acute{\alpha}} \int dx^2 \sqrt{\left(\dot{X}X'\right)^2 - \dot{X}^2 X'^2}. \tag{14}$$

If the scalars X^I do not fluctuate too far from their equilibrium values ($X^I = 0$), we can expand the above action up to quadratic order in X^I:

$$S_{\mathrm{NG}} \approx -\frac{1}{4\pi\acute{\alpha}} \int dx^2 \sqrt{-g(x)} g^{\mu\nu} G_{IJ} \frac{\partial X^I}{\partial x^\mu} \frac{\partial X^J}{\partial x^\nu}. \tag{15}$$

In fact, this quadratic fluctuation Lagrangian can be interpreted as taking the nonrelativistic limit, so we must use the dual Langevin dynamics on boundary in the nonrelativistic case.

3. Strings in Gödel Black Hole

As we said in the Introduction, an external quark is dual to an open string that extends from the boundary to the horizon of the black hole [24]. We can obtain the dynamics of this string in a threedimensional Gödel black hole with the metric background (9) by the Nambu-Goto action (14) in the following form:

$$\begin{aligned} S_{\mathrm{NG}} = &-\frac{1}{2\pi\acute{\alpha}} \\ &\times \int dx^2 \left(\frac{r^2}{q(r^2)} + \Delta\left(r^2\right)\phi'^2 - \frac{q\left(r^2\right)r^2}{\Delta\left(r^2\right)} \right. \\ &\left. \times \left(\dot{\phi} + \alpha + \frac{h\left(r^2\right)}{q\left(r^2\right)} \right)^2 \right)^{1/2}. \end{aligned} \tag{16}$$

We have obtained the above relation in the standard frame. The equation of motion for ϕ derived from (16) is

$$-\frac{\partial}{\partial t}\left[\frac{r^2 q}{\Delta\sqrt{-g}}\left(\dot{\phi} + \alpha + \frac{h}{q}\right)\right] + \frac{\partial}{\partial r}\left[\frac{\Delta\phi'}{\sqrt{-g}}\right] = 0. \tag{17}$$

3.1. Trivial Solution. The Nambu-Goto action up to quadratic terms after subsisting the small fluctuation $\phi \rightarrow C + \phi$ under Gödel metric background in the standard frame is given by

$$\begin{aligned} S^{(2)}_{\mathrm{NG}} = &-\frac{1}{4\pi\acute{\alpha}} \int dt\, dr \frac{\Delta^{3/2}\phi'^2}{r^3\left[\Delta/q - q(h/q+\alpha)^2\right]^{1/2}} \\ &- \frac{\Delta^{1/2}\dot{\phi}^2}{r\left[\Delta/q - q(h/q+\alpha)^2\right]^{3/2}}. \end{aligned} \tag{18}$$

By changing coordinate to $s = x - x_+$ (where $x = r^2/2 + 2GJ/\alpha$) and defining $\phi(t, s) = e^{-i\omega t} f_\omega(s)$, one can write the equation of motion as follows:

$$\begin{aligned} &W(s)\, Z(s)\, \partial_s^2 f_\omega + \frac{1}{2}\left[3\partial Z(s)\, W(s) - \partial W(s)\, Z(s)\right]\partial_s f_\omega \\ &\quad + \omega^2 f_\omega = 0, \end{aligned} \tag{19}$$

where

$$W(s) = \frac{\Delta(s)}{q(s)} - q(s)\left(\frac{h(s)}{q(s)} + \alpha\right)^2, \qquad Z(s) = \Delta(s). \tag{20}$$

The solution for this equation of motion is the trivial solution for the relation (17). In general, solution of this equation is very complicated. However, for the extremal case $\mu = J(1 + \alpha^2 l^2)/\alpha l^2$, we can find an analytical solution as

$$\begin{aligned} &f_\omega^\pm = s^{-1\pm D} Y(s), \\ &\text{where } D = \sqrt{1 + \frac{\omega^2 l^2}{16GM}} \text{ with } M = \frac{\mu}{1+\alpha^2 l^2}, \end{aligned} \tag{21}$$

where $Y(s)$ is obtained from the following relation:

$$\begin{aligned} & s\left(cs^2 + 4z\left(s - \frac{1}{4}\frac{z}{\alpha^2}\right)\right)\frac{d^2}{ds^2}Y(s) \\ & \quad + \left(2\chi\left(cs^2 + 4z\left(s - \frac{1}{4}\frac{z}{\alpha^2}\right)\right)\right. \\ & \qquad \left. + 2cs^2 + 10sz - 3\frac{z^2}{\alpha^2}\right)\frac{d}{ds}Y(s) \\ & \quad + \chi\left((\chi - 1)(cs + 4z) + 2cs + 10z\right)Y(s) = 0, \end{aligned} \tag{22}$$

where $\chi = -1 \pm D$ and $z = 2x_+/l^2$. The complete solution of $Y(s)$ is derived in the Appendix. With similar argument as in [16], this solution is not acceptable, because it does not have oscillatory modes in radial coordinates and also, for this trivial constant solution, one can investigate that the square root determinant of the worldsheet metric is not real everywhere; therefore, it is not a physical solution. Due to nonphysical motivation about the mentioned solution, we have to consider another approach, which is linear solution.

3.2. Linear Solution. We can take the linear ansatz for the small fluctuation in the transverse direction ϕ to achieve a nontrivial solution, so we expand it as

$$\phi(t, r) = wt + \eta(r), \tag{23}$$

where w is a constant angular velocity. By replacing this relation into (17), the solution for η is obtained as follows:

$$\eta'(r) = -\frac{\pi_\phi}{\Delta}\sqrt{\frac{\left(r^2/q\right)\left(\Delta - q^2(h/q + \alpha + w)^2\right)}{\Delta - \pi_\phi^2}}, \tag{24}$$

where π_ϕ is a constant which has a concept as the total force to keep string moving with linear angular velocity w and also is related to momentum conjugate of ϕ in r direction. At $r = r_{\mathrm{NH}}$, the numerator becomes zero, so the denominator should also vanish there, because the string solution (23) must be real everywhere along the worldsheet. For $w = 0$ and $\alpha^2 l^2 \neq 1$, r_{NH} is given by

$$r_{\mathrm{NH}}^2 = -\frac{l^2}{\gamma^2}\left[1 \pm \sqrt{1 + \frac{8G\gamma^2}{l^2\alpha^2}(2\mu - J\alpha)}\right] - \frac{4GJ}{\alpha}. \tag{25}$$

When $w \neq 0$ and $\alpha^2 l^2 = 1$, we receive the excepted relation for BTZ black hole [16]. For $w \neq 0$ and $\alpha^2 l^2 \neq 1$, we obtain

$$\begin{aligned} r_{\mathrm{NH}}^2 = {} & \frac{l^2}{\gamma^2(\alpha + w)} \\ & \times\left[(w - \alpha) \pm \sqrt{(w - \alpha)^2 + \frac{16G\gamma^2}{l^2} > \left(\mu - \frac{J(\alpha + w)^2}{2\alpha}\right)}\right] \\ & - \frac{4GJ}{\alpha}. \end{aligned} \tag{26}$$

According to [16] we set dominator to zero, so we have

$$\pi_\phi^2 = \Delta = \left(\frac{h\alpha + p}{\alpha}\right)^2, \qquad h = h\left(r_{\mathrm{NH}}^2\big|_{w=0}\right). \tag{27}$$

The external force F_{ext} can be obtained by considering the rotation and the topological mass of black hole which is given by

$$F_{\mathrm{ext}} = \frac{\pi_\phi}{2\pi\acute{\alpha}} = \frac{h\alpha + p}{2\pi\acute{\alpha}\alpha}. \tag{28}$$

After extracting this external force, we can derive the friction coefficient γ_0 for nonzero w, by considering the relation $p_\phi = m_0 w$, as

$$\gamma_0 m_0 = \frac{q_{\mathrm{NH}}}{2\pi\acute{\alpha}} = \frac{r_{\mathrm{NH}}^2 - \left(2\gamma^2/l^2\right)\left(r_{\mathrm{NH}}^2/2 + 2GJ/\alpha\right)^2}{2\pi\acute{\alpha}}. \tag{29}$$

With $\alpha^2 l^2 = 1$ this coefficient reduces to the excepted value $\gamma_0 = r_{\mathrm{NH}}^2/2\pi\acute{\alpha}m_0$ for BTZ black hole [16].

The Nambu-Goto action, with the small fluctuation, $\phi \to wt + \eta(r) + \phi$, under the Gödel background becomes

$$\begin{aligned} S_{\mathrm{NG}}^{(2)} = -\frac{1}{4\pi\acute{\alpha}}\int dt dr & \frac{\Delta^{3/2}\phi'^2}{r^3\left[\Delta/q - q(h/q + \alpha + w)^2\right]^{1/2}} \\ & - \frac{\Delta^{1/2}\dot{\phi}^2}{r\left[\Delta/q - q(h/q + \alpha + w)^2\right]^{3/2}}. \end{aligned} \tag{30}$$

The equation of motion from the above Nambu-Goto action is given by

$$\begin{aligned} & \frac{-r\Delta^{1/2}\ddot{\phi}}{\left[\Delta/q - q(h/q + \alpha + w)^2\right]^{3/2}} \\ & \quad + \frac{\partial}{\partial r}\frac{\Delta^{3/2}\phi'^2}{r\left[\Delta/q - q(h/q + \alpha + w)^2\right]^{1/2}} = 0. \end{aligned} \tag{31}$$

Solving this equation is quite complicated for more values of w. However, one can find that there are some values like

$$\begin{aligned} w & = \frac{x_- + (x_+ - x_-)\left(\left(1 - \alpha^2 l^2\right)/2\right)}{\alpha x_+ l^2} \\ & = \frac{r_- + (r_+ - r_-)\left(\left(1 - \alpha^2 l^2\right)/2\right)}{\alpha r_+ l^2}, \end{aligned} \tag{32}$$

where this makes it possible to solve the equation of motion. In derivation of the right-hand side of the above relation we use $r_- r_+ = 4GJ/\alpha$. The special radius r_{NH} approaches the outer horizon of the Gödel black hole for this value of angular velocity w, where $\Delta(r_+) = 0$ (and $\pi_\phi = 0$); then the steady state solution is the case that $r_{\mathrm{NH}} = r_+$. From relation (32) for the angular velocity, it is evident that we can receive to $w = r_-/r_+$ for BTZ black hole ($\alpha^2 l^2 = 1$). Furthermore, for

($\alpha^2 l^2 > 1$), we can check that $w^2 < \alpha^2$, but there must be some condition on μ and J to have $w^2 < \alpha^2$ for ($\alpha^2 l^2 < 1$). We can write the equation of motion for this terminal angular velocity with changing coordinate to $s = x - x_+$ as

$$W(s)Z(s)\phi_s'' + \frac{1}{2}\left[3\partial Z(s)W(s) - \partial W(s)Z(s)\right]\phi_s' - \ddot{\phi}_s = 0, \tag{33}$$

where

$$W(s) = \frac{p}{(2\alpha x_+)^2}\left[s(cs+\lambda\zeta)\right], \qquad Z(s) = \lambda s(s+\zeta),$$
$$c = \lambda - 4\alpha^2, \qquad \zeta = x_+ - x_-. \tag{34}$$

As before, we take $\phi(t,s) = e^{-i\omega t} f_\omega(s)$, so (33) reduce to

$$W(s)Z(s)f_\omega'' + \frac{1}{2}\left[3Z'(s)W(s) - W'(s)Z(s)\right]f_\omega' + \omega^2 f_\omega = 0. \tag{35}$$

Consequently, the independent linear solutions to the above equation are obtained as below:

$$f_\omega^\pm(s) = (\lambda\zeta s)^{\pm i\vartheta}\,{}_2F_1\left(\pm i\vartheta, \frac{3}{2} \pm i\vartheta; 1 \pm 2i\vartheta, \frac{(-\lambda + c)s}{cs+\lambda\zeta}\right) \times (cs+\lambda\zeta)^{\mp i\vartheta}, \tag{36}$$

where

$$\vartheta = \frac{2\alpha\omega x_+}{\lambda\zeta\sqrt{p}}, \tag{37}$$

or with $\xi = 2\zeta = r_+^2 - r_-^2$ and $x_+/\sqrt{p} = 4GJ/(\alpha r_-\sqrt{\lambda})$, we have $\vartheta = 16GJ\omega/\lambda^{3/2}\xi r_-$. By considering the following relation for hypergeometric functions,

$${}_2F_1\left(\kappa, \kappa + \frac{3}{2}, 2\kappa + 1, z\right) = \frac{2^{2\kappa}}{\kappa + 1/2}\left[1 + (1-z)^{1/2}\right]^{-2\kappa}\left[\frac{1}{2} + \kappa(1-z)^{-1/2}\right], \tag{38}$$

(36) reduces as

$$f_w^\pm(s) = \frac{(4\lambda\zeta)^{\pm i\vartheta}}{1 \pm 2i\vartheta}\frac{\left[1 \pm 2i\vartheta/\sqrt{1 + (\lambda - c)s/(cs+\lambda\zeta)}\right]}{\left[1 + \sqrt{1 + (\lambda - c)s/(cs+\lambda\zeta)}\right]^{\pm 2i\vartheta}} \times (cs+\lambda\zeta)^{\mp i\vartheta} s^{\pm i\vartheta}, \tag{39}$$

which gives oscillation modes. We have the following asymptotic behavior from the solutions near the outer horizon ($s \to 0$) and the boundary ($s \to \infty$):

$$f_w^\pm(s) \sim \begin{cases} e^{\pm i\omega s_\star} & (s \longrightarrow 0) \\ \dfrac{(4\lambda\zeta)^{\pm i\vartheta}\left(1 \pm 2i\vartheta/\sqrt{\lambda/c}\right)}{(1 \pm 2i\vartheta)\left(1 + \sqrt{\lambda/c}\right)^{\pm 2i\vartheta}} & (s \longrightarrow \infty), \end{cases} \tag{40}$$

with $s_\star = (\vartheta/\omega)\ln(s) = (16GJ/\lambda^{3/2}\xi r_-)\ln(s)$.

4. Displacement Square

So far, we have succeeded to drive oscillation modes for a string moving in the Gödel black hole background. In the following, we follow the same procedure as in [14, 16] to compute the displacement square for Brownian motion. In order to achieve this, we write the solutions for bulk equation of motion as a linear combination of $f_\omega^\pm$:

$$f_\omega(s) = A\left[f_\omega^+(s) + Bf_\omega^-(s)\right]e^{-i\omega t}, \tag{41}$$

where A and B are constants. By exerting the Neumann boundary condition near the boundary, $\partial_s f_s(\omega) = 0$ with $s = s_c \gg 0$, to put the UV-cutoff, we obtain

$$B = \frac{(4\lambda\zeta)^{2i\vartheta}(cs_c+\lambda\zeta)^{-2i\vartheta}s_c^{2i\vartheta}(1-2i\vartheta)}{\left[1 + \sqrt{1 + (\lambda - c)s_c/(cs_c+\lambda\zeta)}\right]^{4i\vartheta}(1+2i\vartheta)} \times \frac{\left[1 + 2i\vartheta\sqrt{1 + (\lambda - c)s_c/(cs_c+\lambda\zeta)}\right]}{\left[1 - 2i\vartheta\sqrt{1 + (\lambda - c)s_c/(cs_c+\lambda\zeta)}\right]} \equiv e^{i\theta_\omega}. \tag{42}$$

Note that the constant B is a pure phase, so by using (40) in the near horizon we can write

$$\Phi(t,s) = f_\omega(s)e^{i\omega t} \sim e^{-i\omega(t-s_\star)} + e^{i\theta_\omega}e^{-i\omega(t+s_\star)}. \tag{43}$$

To regulate the theory, we implement another cutoff near the outer horizon at $s_h = \epsilon$, $\epsilon \ll 1$, which is called IR-cutoff; we obtain

$$B \approx \epsilon^{2i\vartheta} = e^{-2i\vartheta\ln(1/\epsilon)}. \tag{44}$$

If we take B in the terms of ω by relation (41) only, then B has continuous values, since the ω can have any value. Using relation (43) for B will satisfy our requirements to have discrete values in $\epsilon \ll 1$. In this case, the discreteness is [14, 16]

$$\Delta\vartheta = \frac{\pi}{\ln(1/\epsilon)}, \tag{45}$$

where, in terms of ω, it is given by

$$\Delta\omega = \frac{\lambda^{3/2}\pi r_-\xi}{16GJ\ln(1/\epsilon)}. \tag{46}$$

Following the above processes and using IR-cutoff to discrete the continuous spectrum makes it easy to find normalized bases of modes and to quantize $\phi(t,r)$ by extending in these modes.

4.1. Brownian Particle Location. In this section we are going to use quantized modes of the string near the outer horizon of Gödel black hole to describe the Brownian motion of an external quark. Therefore we consider the Nambu-Goto action for certain amount of terminal angular velocity, near the outer horizon ($s \to 0$):

$$S_{\mathrm{NG}}^2 \sim \frac{1}{2}\int dt ds_\star\left(\dot{\Phi}^2 - \Phi'^2\right), \tag{47}$$

where $\Phi \equiv (8GJ/(r_- \sqrt{2\pi\lambda\acute{\alpha}}))\phi$. Thus, according to the same procedure for standard scalar fields, we introduce the following mode expansions:

$$\Phi(t,s) = \sum_{\omega>0} \left[a_\omega u_\omega(t,s) + a_\omega^\dagger u_\omega(t,s)^* \right], \tag{48}$$

with

$$u_\omega(t,s) = \sqrt{\frac{\lambda^{3/2}\xi r_-}{32GJ\omega \ln(1/\epsilon)}} \left[f_\omega^+(s) + B f_\omega^-(s) \right] e^{-i\omega t}, \qquad \left[a_\omega, a_{\acute{\omega}}^\dagger \right] = \delta_{\omega\acute{\omega}}. \tag{49}$$

Now, by considering the above quantum modes on the probe string in the bulk, we want to work out the dynamics of the endpoint which corresponds to an external quark. We investigate the wave-functions of the world-sheet fields in the two interesting regions: (i) near the black hole horizon and (ii) close to the boundary. From (40), near the horizon ($S \sim 0$), expansion (48) becomes

$$\phi(t, S \longrightarrow 0) = \frac{r_-^{3/2}\sqrt{2\pi\acute{\alpha}\lambda^{5/2}}\,\xi}{(16GJ)^{3/2}\sqrt{\ln(1/\varepsilon)}} \times \sum_{\omega=-\infty}^{\infty} \frac{1}{\sqrt{\omega}} \left(e^{-i\omega(t-S_*)} + e^{i\theta_\omega} e^{-i\omega(t+S_*)} \right) a_\omega. \tag{50}$$

We used $S = 2s = r^2 - r_+^2$. On the other hand, expansion (48) at $S = R$ (the location of the regulated boundary) is given by

$$\begin{aligned}
\phi(t, S = R) &= \frac{r_-^{3/2}\sqrt{2\pi\acute{\alpha}\lambda^{5/2}}\,\xi}{(16GJ)^{3/2}\sqrt{\ln(1/\epsilon)}} \times \sum_{\omega>0} \frac{1}{\sqrt{\omega}} \\
&\quad \times \Bigg[\Bigg(2^{1+2i\vartheta} (\lambda\xi R)^{i\vartheta} (cR + \lambda\xi)^{-i\vartheta} (1 - 2i\vartheta) \\
&\quad \times \Bigg(\left[1 + \sqrt{1 + \frac{(\lambda - c) R}{cR + \lambda\xi}} \right]^{2i\vartheta} \times \left[1 - 2i\vartheta \sqrt{1 + \frac{(\lambda - c) R}{cR + \lambda\xi}} \right] \Bigg)^{-1} \Bigg) \\
&\quad \times e^{-i\omega t} a_\omega + h.c. \Bigg].
\end{aligned} \tag{51}$$

One can see that there are two modes in the solutions. The outgoing modes ($\omega > 0$) that are excited because of Hawking radiation [25, 26] and incoming modes ($\omega < 0$) which fall into black hole. The outgoing mode correlators are determined by the thermal density matrix:

$$\rho_0 = \frac{e^{-\beta H}}{Tr(e^{-\beta H})}, \qquad H = \sum_{\omega>0} \omega a_\omega^\dagger a_\omega, \tag{52}$$

and the expectation value of occupation number is given by the Bose-Einstein distribution:

$$\left\langle a_\omega^\dagger a_{\acute{\omega}} \right\rangle = \frac{\delta\omega\acute{\omega}}{e^{\beta\omega} - 1}, \tag{53}$$

with $\beta = 1/T$. Using the knowledge of relation (52) about outgoing modes correlators in the bulk, we can investigate the motion of the endpoint of the string at $S = R \gg 1$. We can also determine the behavior of the Brownian motion, by computing displacement square, as came in (2). So we can predict the nature of Brownian motion of external particle on the boundary. For this purpose, we compute the modes correlators at $S = R \gg 1$ as

$$\left\langle \phi_R(t)\,\phi_R(0) \right\rangle = \sum_{\omega>0} \frac{r_-^3 \pi\acute{\alpha}\lambda^{5/2}\xi}{(16GJ)^3 \omega \ln(1/\epsilon)} \times \frac{\left[1 + 4\vartheta(\omega)^2 \right]}{\left[1 + 4\vartheta(\omega)^2 (\lambda(R+\xi)/(cR + \lambda\zeta)) \right]} \times \left[\frac{2\cos\omega t}{e^{\beta\omega} - 1} + e^{-i\omega t} \right]. \tag{54}$$

By utilizing (46), we can write the above relation in the integral form. We see that the integral is diverging. So we regularize it by normally ordering the $a, a^\dagger$ oscillators: $a_\omega a_\omega^\dagger \equiv a_\omega^\dagger a_\omega$; then we have

$$\begin{aligned}
\left\langle : \phi_R(t)\,\phi_R(0) : \right\rangle &= \frac{8\lambda r_-^2 \acute{\alpha}}{(16GJ)^2} \times \int_0^\infty \frac{d\omega}{\omega} \left(1 + 4\frac{(16GJ)^2\omega^2}{\lambda^3\xi^2 r_-^2} \right) \\
&\quad \times \left(1 + 4\frac{(16GJ)^2\omega^2}{\lambda^3\xi^2 r_-^2} \left(\frac{\lambda(R+\xi)}{cR + \lambda\xi} \right) \right)^{-1} \times \left[\frac{2\cos(\omega t)}{e^{\beta\omega} - 1} \right],
\end{aligned} \tag{55}$$

and the displacement square becomes

$$\begin{aligned}
S_{\text{reg}}(t)^2 &\equiv \left\langle : \left[\phi_R(t) - \phi_R(0) \right]^2 : \right\rangle \\
&= \frac{16\lambda r_-^2 \acute{\alpha}}{(16GJ)^2} \left[\frac{(\lambda - c) R}{\lambda(R+\xi)} I_1 + \frac{cR + \lambda\xi}{\lambda(R+\xi)} I_2 \right],
\end{aligned} \tag{56}$$

with

$$I_1 = 4\int_0^\infty \frac{dy}{y\left(1+a^2y^2\right)}\frac{\sin^2\left(ky/2\right)}{e^y-1},$$
$$I_2 = 4\int_0^\infty \frac{dy}{y}\frac{\sin^2\left(ky/2\right)}{e^y-1}, \tag{57}$$

and we have defined

$$y=\beta\omega,\qquad k=\frac{t}{\beta},\qquad a^2=4\left(\frac{16GJ}{\lambda\xi r_-\beta}\right)^2\left(\frac{R+\xi}{cR+\lambda\xi}\right). \tag{58}$$

The evaluation of these integrals and their behavior for $R\gg 1$ and $a\gg 1$ can be found in Appendix B of [14]. From relation (58), we can see that when $R\gg 1$, we have $a\propto 1/c$. Thus in general case for a, we use the following relations for integrals (57):

$$\begin{aligned}I_1 &= \frac{1}{2}\left[e^{k/a}Ei\left(-\frac{k}{a}\right)+e^{-k/a}Ei\left(\frac{k}{a}\right)\right]\\&\quad+\frac{1}{2}\left[\psi\left(1+\frac{1}{2\pi a}\right)+\psi\left(1-\frac{1}{2\pi a}\right)\right]\\&\quad-\frac{\pi}{2}\left(1-e^{|k|/a}\right)\cot\frac{1}{2a}+\log\left(\frac{2a\sinh\pi k}{k}\right)+\frac{e^{-2\pi|k|}}{2}\\&\quad\times\left[\frac{{}_2F_1\left(1,1+1/2\pi a,2+1/2\pi a;e^{-2\pi|k|}\right)}{1+1/2\pi a}\right.\\&\quad\left.+\frac{{}_2F_1\left(1,1-1/2\pi a,2-1/2\pi a;e^{-2\pi|k|}\right)}{1-1/2\pi a}\right],\\I_2 &= \log\left(\frac{\sinh\pi k}{\pi k}\right).\end{aligned} \tag{59}$$

However, for $R\gg 1$ and $\ll 1(\alpha^2l^2\to 1)$, then $a\gg 1$, one can utilize the following relation for I_1 and I_2:

$$I_1=\begin{cases}\dfrac{\pi k^2}{2a}+O\left(a^{-2}\right)\\ \pi k+O\left(\log k\right),\end{cases}$$
$$I_2=\begin{cases}O\left(a^0\right), & (k\ll a)\\ \pi k+O\left(\log k\right), & (k\gg a).\end{cases} \tag{60}$$

Therefore, $S_{\mathrm{reg}}(t)^2$ has the following form:

$$\left\langle S_{\mathrm{reg}}(t)^2\right\rangle=\begin{cases}\dfrac{16r_-^3\lambda\pi\acute{\alpha}}{(16GJ)^3\beta}\left[\dfrac{\xi(\lambda-c)(cR+\lambda\xi)^{1/2}}{4(R+\xi)^{1/2}}\right]t^2\\ \quad+O\left(\dfrac{cR+\lambda\xi}{R+\xi}\right), & (t\ll\beta),\\ \dfrac{16r_-^2\lambda\pi\acute{\alpha}}{(16GJ)^2\beta}t+O\left(\log\dfrac{t}{\beta}\right), & (t\gg\beta).\end{cases} \tag{61}$$

One can check that the displacement square (61) is consistent with BTZ black hole in [16] by setting $c=0$ or $\alpha^2l^2=1$. In that case, the w vanishes for $J=0$ (or $r_-=0$), but when $\alpha^2l^2\neq 1$, the w will have zero value only for $J/\alpha l^2=(\alpha^2l^2-1)\mu$ (see relation (32)). Then our static solution is achieved by this condition. The diffusion constant from (61) is given by

$$D=\frac{\lambda\pi\acute{\alpha}\alpha^2}{2r_+^2}T. \tag{62}$$

So, the relaxation time of Brownian particle is as follows:

$$t_c=\frac{1}{\gamma_0}=\frac{\lambda m_0\pi\acute{\alpha}\alpha^2}{2r_+^2}. \tag{63}$$

The mass of external particle, m_0, can be computed by using the total energy and momentum of string [27] under the metric background (4):

$$E=\frac{1}{2\pi\acute{\alpha}}\int dr\pi_t^0,\qquad p_\phi=\frac{1}{2\pi\acute{\alpha}}\int dr\pi_\phi^0, \tag{64}$$

with

$$\begin{aligned}\pi_t^0&=\frac{\phi'^2}{\sqrt{-g}}\left(g_{t\phi}^2-g_{tt}g_{\phi\phi}\right)-\frac{g_{rr}}{\sqrt{-g}}\left(g_{tt}+g_{t\phi}\dot{\phi}\right),\\ \pi_\phi^0&=\frac{g_{rr}}{\sqrt{-g}}\left(g_{t\phi}+g_{\phi\phi}\dot{\phi}\right).\end{aligned} \tag{65}$$

Then we have

$$\begin{aligned}E&=\frac{\alpha}{2\pi\acute{\alpha}}\int ds\frac{c\left(s+x_+\right)+\lambda x_+}{\sqrt{\lambda p\left(s+\zeta\right)\left(cs+\lambda\zeta\right)}},\\ p_\phi&=\frac{1}{2\pi\acute{\alpha}}\int ds\frac{-c\left(s+x_+\right)+\lambda x_-}{\sqrt{\lambda p\left(s+\zeta\right)\left(cs+\lambda\zeta\right)}}.\end{aligned} \tag{66}$$

One can check that, after putting $c=0$ in the above relation, the result of integral is as excepted for BTZ black hole. However, for $c\neq 0$ we obtain

$$\begin{aligned}E&=\alpha\frac{\sqrt{(cR+\lambda\xi)(R+\xi)}-\sqrt{\lambda\xi^2}}{\sqrt{\lambda p}}\\&\quad+2\alpha\frac{\lambda+c}{\lambda}\sqrt{\frac{p}{\lambda c}}\ln\left[\frac{\sqrt{cR+\lambda\xi}+\sqrt{c(R+\xi)}}{\sqrt{c\xi}+\sqrt{\lambda\xi}}\right],\\ p_\phi&=-\frac{\sqrt{(cR+\lambda\xi)(R+\xi)}-\sqrt{\lambda\xi^2}}{\sqrt{\lambda p}}\\&\quad+2\frac{\lambda-c}{\lambda}\sqrt{\frac{p}{\lambda c}}\ln\left[\frac{\sqrt{cR+\lambda\xi}+\sqrt{c(R+\xi)}}{\sqrt{c\xi}+\sqrt{\lambda\xi}}\right],\end{aligned} \tag{67}$$

where $\sqrt{p/\lambda}=(r_++r_-)/2$. Then the mass is defined as

$$m_0^2=E^2-p_\phi^2. \tag{68}$$

From the above relations, we see that relating the physical mass to displacement square is difficult.

5. Summary

In this paper, by using AdS/CFT correspondence, we studied the Brownian motion of an external quark in plasma. It is corresponded to a string stretched from horizon of AdS to boundary. By using the Nambu-Goto action, we obtained the equation of motion for this string in the Gödel background. For an acceptable solution with oscillatory modes, we had to redefine the terminal angular velocity. We found that turning on a finite density for a conserved $U(1)$ charge (reflected by a CS term in the bulk) and the rotation of black hole influence oscillatory modes. For realization of the Brownian motion, we derived the time evolution of the displacement square from the modes correlators. We showed that in general case ($\alpha^2 l^2 \neq 1$ Gödel black hole), our results for displacement square are different in comparison with [16]. However, in $\alpha^2 l^2 = 1$ limit (BTZ black hole), we confirmed that our results agree with the work of Atmaja [16]. We derived the physical mass, but we found that relating the displacement square to physical observables is a difficult work. This is the problem that we would like to consider in future work. Also we would like to investigate the Brownian motion of external quarks in different environments, in particular plasmas which correspond to metric backgrounds as Lifshitz geometry [28] and metric backgrounds with hyperscaling violation [29, 30].

Appendix

The solution to the following differential equation

$$
\begin{aligned}
& s\left(cs^2 + 4z\left(s - \frac{1}{4}\frac{z}{\alpha^2}\right)\right)\frac{d^2}{ds^2}Y(s) \\
&\quad + \left(2\chi\left(cs^2 + 4z\left(s - \frac{1}{4}\frac{z}{\alpha^2}\right)\right)\right. \\
&\qquad\left. + 2cs^2 + 10sz - 3\frac{z^2}{\alpha^2}\right)\frac{d}{ds}Y(s) \\
&\quad + \chi\left((\chi - 1)(cs + 4z) + 2cs + 10z\right)Y(s) \\
& = 0
\end{aligned}
\tag{A.1}
$$

can be obtained analytically as

$$
\begin{aligned}
Y(s) = {} & C_1 s^{-3/2-\chi}\sqrt[4]{s(cs+4z)\alpha^2 - z^2} \\
& \times \left(\frac{s}{-c\alpha s - 2z\alpha + z\sqrt{\lambda}}\right)^{(1/2)(1-p(\alpha)/r(\alpha)q(\alpha))} \\
& \times \left(\frac{-\alpha\sqrt{\lambda}s + 2s\alpha^2 - z}{-z\sqrt{\lambda} + \alpha(cs+2z)}\right)^{(1/2)(1+3\sqrt{-c^2}/4r(\alpha)q(\alpha))} \\
& \times \left(-c\alpha s - 2z\alpha + z\sqrt{\lambda}\right) \\
& \times {}_2F_1\left(E(\alpha)(T(\alpha)+U(\alpha)),\right. \\
& \qquad E(\alpha)(T(\alpha)+U(\alpha)), \\
& \qquad \left(1 - \frac{p(\alpha)}{q(\alpha)r(\alpha)}\right); \\
& \qquad \left.\frac{c\alpha\sqrt{\lambda}s}{\left(-z\sqrt{\lambda} + \alpha(cs+2z)\right)r(\alpha)}\right) \\
& + C_2 s^{-3/2-\chi}\sqrt[4]{s(cs+4z)\alpha^2 - z^2} \\
& \times \left(\frac{s}{-c\alpha s - 2z\alpha + z\sqrt{\lambda}}\right)^{(1/2)(1+p(\alpha)/r(\alpha)q(\alpha))} \\
& \times \left(\frac{-\alpha\sqrt{\lambda}s + 2s\alpha^2 - z}{-z\sqrt{\lambda} + \alpha(cs+2z)}\right)^{(1/2)(1+3\sqrt{-c^2}/4r(\alpha)q(\alpha))} \\
& \times \left(-c\alpha s - 2z\alpha + z\sqrt{\lambda}\right) \\
& \times {}_2F_1\left(E(\alpha)(T(\alpha)+R(\alpha)),\right. \\
& \qquad E(\alpha)(T(\alpha)+R(\alpha)), \\
& \qquad \left(1 + \frac{p(\alpha)}{q(\alpha)r(\alpha)}\right); \\
& \qquad \left.\frac{c\alpha\sqrt{\lambda}s}{\left(-z\sqrt{\lambda} + \alpha(cs+2z)\right)r(\alpha)}\right),
\end{aligned}
\tag{A.2}
$$

where E, R, T, U, r, q, and $p(\alpha)$ are given by relations (A.2)–(A.6):

$$
\begin{aligned}
E(\alpha) &= \frac{1}{2\sqrt{-c - 8\alpha^2 + 4\alpha\sqrt{\lambda}}\left(\alpha + 1/2\sqrt{\lambda}\right)\left(2\alpha\sqrt{\lambda} + \lambda\right)} \\
&= \frac{1}{4\sqrt{\lambda}r^2(\alpha)q(\alpha)},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
r(\alpha) &= \sqrt{-c - 8\alpha^2 + 4\alpha\sqrt{\lambda}}, \\
q(\alpha) &= \left(\alpha + \frac{1}{2}\sqrt{\lambda}\right), \\
p(\alpha) &= \sqrt{-(\chi+1)^2 c^2},
\end{aligned}
\tag{A.4}
$$

$$
T(\alpha) = \left(\left(2\alpha^2 + \frac{1}{4}c\right)\sqrt{\lambda} + \alpha\lambda\right)\sqrt{-c - 8\alpha^2 + 4\alpha\sqrt{\lambda}}, \tag{A.5}
$$

$$U(\alpha) = \left(\frac{3}{2}\alpha + \frac{3}{4}\sqrt{\lambda}\right)\sqrt{-c^2\lambda} - \sqrt{-(\chi+1)^2 c^2}\left(2\alpha\sqrt{\lambda}+\lambda\right), \tag{A.6}$$

$$R(\alpha) = \left(\frac{3}{2}\alpha + \frac{3}{4}\sqrt{\lambda}\right)\sqrt{-c^2\lambda} + \sqrt{-(\chi+1)^2 c^2}\left(2\alpha\sqrt{\lambda}+\lambda\right). \tag{A.7}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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