

## Review Article

# Remarks on the Spin-One Duffin-Kemmer-Petiau Equation in the Presence of Nonminimal Vector Interactions in $(3 + 1)$ Dimensions

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We point out a misleading treatment in the recent literature regarding analytical solutions for nonminimal vector interaction for spin-one particles in the context of the Duffin-Kemmer-Petiau (DKP) formalism. In those papers, the authors use improperly the nonminimal vector interaction endangering in their main conclusions. We present a few properties of the nonminimal vector interactions and also present the correct equations to this problem. We show that the solution can be easily found by solving Schrödinger-like equations. As an application of this procedure, we consider spin-one particles in presence of a nonminimal vector linear potential.

## 1. Introduction

The Duffin-Kemmer-Petiau (DKP) formalism [1–4] describes spin-zero and spin-one particles and has been used to analyze relativistic interactions of spin-zero and spin-one hadrons with nuclei as an alternative to their conventional second-order Klein-Gordon (KG) and Proca counterparts. The DKP formalism proved to be better than the KG formalism in the analysis of  $K_{13}$  decays, the decay-rate ratio  $\Gamma(\eta \rightarrow \gamma\gamma)/\Gamma(\pi^0 \rightarrow \gamma\gamma)$ , and level shifts and widths in pionic atoms [5–16]. The DKP formalism enjoys a richness of couplings not capable of being expressed in the KG and Proca theories [17, 18]. Although the formalisms are equivalent in the case of minimally coupled vector interactions [19–21], the DKP formalism opens new horizons as far as it allows other kinds of couplings which are not possible in the KG and Proca theories. The nonminimal vector interaction refers to a kind of charge conjugate invariant coupling that behaves like a vector under a Lorentz transformation. The invariance of the nonminimal vector potential under charge

conjugation means that it does not distinguish particles from antiparticles. Hence, whether one considers spin-zero or spin-one bosons, this sort of interaction cannot exhibit Klein's paradox [22]. Nonminimal vector potentials, added by other kinds of Lorentz structures, have already been used in a phenomenological context for describing the scattering of mesons by nuclei [23–32], but it should be mentioned that in [23–26, 28–30, 32] the nonminimal vector couplings have been used improperly. Nonminimal vector coupling with a quadratic potential [33] and with a linear potential [34] and mixed space and time components with a step potential [35, 36], double-step potential [37], a smooth step potential [38], a linear potential [22, 39, 40], and a linear plus inversely linear potential [41] have been explored in the literature. In a recent paper published in this journal, Hassanabadi et al. [42] analyze the DKP equation in the presence of nonminimal vectorial interactions (Coulomb and harmonic oscillator potentials) in  $(3 + 1)$  dimensions for spin-one particles. In that paper, the authors used improperly the nonminimal vector interaction endangering in its main

conclusions. The same mistake is found in recent works, in [43–51], for instance. Other misconception is found in [23–26], where the space component of the nonminimal vector potential is absorbed into the spinor. As it is shown in [22], there is no chance of dissociating from this term. Furthermore, the space component of the nonminimal vector potential could be irrelevant for the formation of bound states for potentials vanishing at infinity, but its presence is an essential ingredient for confinement.

In view of the misconceptions on the nonminimal vector interaction propagated in the literature, the purpose of this review paper is to review the DKP equation in the presence of a nonminimal vectorial interaction for spin-one particles in  $(3 + 1)$  dimensions. We present a few properties of the nonminimal vector interactions and also present the correct equations to this problem. We show that the solution can be easily found by solving Schrödinger-like equations. As an application of this procedure, we consider spin-one particles in presence of a nonminimal vector linear potential. For this case in particular, the problem is mapped into the nonrelativistic three-dimensional harmonic oscillator.

## 2. The DKP Equation

The DKP equation for a free boson is given by [4] (with units in which  $\hbar = c = 1$ ) as follows:

$$(i\beta^\mu \partial_\mu - m)\psi = 0, \quad (1)$$

where the matrices  $\beta^\mu$  satisfy the algebra

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu \quad (2)$$

and the metric tensor is  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . The algebra expressed by (2) generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-zero particles, and a ten-dimensional representation associated with spin-one particles. The DKP spinor has an excess of components and the theory has to be supplemented by an equation which allows us to eliminate the redundant components. That constraint equation is obtained by multiplying the DKP equation by  $1 - \beta^0 \beta^0$ ; namely,

$$i\beta^i \beta^0 \beta^0 \partial_i \psi = m(1 - \beta^0 \beta^0)\psi. \quad (3)$$

This constraint equation expresses three (four) components of the spinor by the other two (six) components and their space derivatives in the scalar (vector) sector so that the superfluous components disappear and there only remain the physical components of the DKP theory. The second-order Klein-Gordon and Proca equations are obtained when one selects the spin-zero and spin-one sectors of the DKP theory.

A well-known conserved four-current is given by

$$J^\mu = \frac{1}{2} \bar{\psi} \beta^\mu \psi, \quad (4)$$

where the adjoint spinor  $\bar{\psi}$  is given by  $\bar{\psi} = \psi^\dagger \eta^0 = 2\beta^0 \beta^0 - 1$  in such a way that  $(\eta^0 \beta^\mu)^\dagger = \eta^0 \beta^\mu$  (the matrices  $\beta^\mu$

are Hermitian with respect to  $\eta^0$ ). Despite the similarity to the Dirac equation, the DKP equation involves singular matrices, the time component of  $J^\mu$  is not positive definite, and the case of massless bosons cannot be obtained by a limiting process [52]. Nevertheless, the matrices  $\beta^\mu$  plus the unit operator generate a ring consistent with integer-spin algebra and  $J^0$  may be interpreted as a charge density. The normalization condition  $\int d\tau J^0 = \pm 1$  can be expressed as

$$\int d\tau \bar{\psi} \beta^0 \psi = \pm 2, \quad (5)$$

where the plus (minus) sign must be used for a positive (negative) charge.

## 3. Interactions in the DKP Equation

With the introduction of interactions, the DKP equation can be written as

$$(i\beta^\mu \partial_\mu - m - U)\psi = 0, \quad (6)$$

where the more general potential matrix  $U$  is written in terms of 25 (100) linearly independent matrices pertinent to five- (ten-) dimensional irreducible representation associated with the scalar (vector) sector. In the presence of interaction,  $J^\mu$  satisfies the following equation:

$$\partial_\mu J^\mu + \frac{i}{2} \bar{\psi} (U - \eta^0 U^\dagger \eta^0) \psi = 0. \quad (7)$$

Thus, if  $U$  is Hermitian with respect to  $\eta^0$ , then four-current will be conserved. The potential matrix  $U$  can be written in terms of well-defined Lorentz structures. For the spin-zero sector there are two scalar, two vector, and two tensor terms [17], whereas for the spin-one sector there are two scalar, two vector, a pseudoscalar, two pseudovector, and eight tensor terms [18]. The tensor terms have been avoided in applications because they furnish noncausal effects [17, 18].

*3.1. Nonminimal Vector Couplings in the DKP Equation.* Considering only the nonminimal vector interaction, the DKP equation can be written as

$$(i\beta^\mu \partial_\mu - m - i[P, \beta^\mu] A_\mu)\psi = 0, \quad (8)$$

where  $P$  is a projection operator ( $P^2 = P$  and  $P^\dagger = P$ ) in such a way that  $\bar{\psi}[P, \beta^\mu]\psi$  behaves like a vector under a Lorentz transformation as does  $\bar{\psi}\beta^\mu\psi$ . One very important point to note is that this potential leads to a conserved four-current, but the same does not happen if instead of  $i[P, \beta^\mu]$  one uses either  $P\beta^\mu$  or  $\beta^\mu P$ , as in [23–26, 28–30, 32, 42–51]. As a matter of fact, in [23] it is mentioned that  $P\beta^\mu$  and  $\beta^\mu P$  produce identical results. Considering explicitly the condition (7) for the potential  $U = \beta^0 P V_p$  (widely used in the literature), we obtain

$$\partial_\mu J^\mu = \frac{i}{2} \bar{\psi} [P, \beta^0] V_p \psi \neq 0. \quad (9)$$

The current is not conserved and it is proportional to  $V_p$ . The fact that this current is not conserved has crucial consequences on the orthonormal condition of the DKP spinor [22, 35, 36, 38, 39].

The DKP equation is invariant under the parity operation, that is, when  $\vec{r} \rightarrow -\vec{r}$ , if  $\vec{A}$  changes sign, whereas  $A_0$  remains the same. This is because the parity operator is  $\mathcal{P} = \exp(i\delta_p)P_0\eta^0$ , where  $\delta_p$  is a constant phase and  $P_0$  changes  $\vec{r}$  into  $-\vec{r}$ . Because this unitary operator anticommutes with  $\beta^i$  and  $[P, \beta^i]$ , they change the sign under a parity transformation, whereas  $\beta^0$  and  $[P, \beta^0]$ , which commute with  $\eta^0$ , remain the same. Since  $\delta_p = 0$  or  $\delta_p = \pi$ , the spinor components have definite parities. The charge conjugation operation can be accomplished by the transformation  $\psi \rightarrow \psi_c = \mathcal{C}\psi = CK\psi$ , where  $K$  denotes the complex conjugation and  $C$  is a unitary matrix such that  $C\beta^\mu = -\beta^\mu C$ . The matrix that satisfies these relations is  $C = \exp(i\delta_C)\eta^0\eta^1$ . The phase factor  $\exp(i\delta_C)$  is equal to  $\pm 1$ , and thus  $E \rightarrow -E$ . Note also that  $J^\mu \rightarrow -J^\mu$ , as should be expected for a charge current. Meanwhile  $C$  anticommutes with  $[P, \beta^\mu]$  and the charge conjugation operation entails no change on  $A_\mu$ . The invariance of the nonminimal vector potential under charge conjugation means that it does not couple to the charge of the boson. In other words,  $A_\mu$  does not distinguish particles from antiparticles. Hence, whether one considers spin-zero or spin-one bosons, this sort of interaction cannot exhibit Klein's paradox [22].

If the potential is time-independent, one can write  $\psi(\vec{r}, t) = \varphi(\vec{r}) \exp(-iEt)$ , where  $E$  is the energy of the boson, in such a way that the time-independent DKP equation becomes

$$[\beta^0 E + i\beta^i \partial_i - (m + i[P, \beta^\mu] A_\mu)] \varphi = 0. \quad (10)$$

In this case  $J^\mu = \bar{\varphi}\beta^\mu\varphi/2$  does not depend on time, so that the spinor  $\varphi$  describes a stationary state.

**3.2. Vectorial Sector.** For the case of spin-one (vectorial sector), the  $\beta^\mu$  matrices are [53]

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \bar{0}^T & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & -is_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & -is_i & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (11)$$

where  $s_i$  are the  $3 \times 3$  spin-one matrices  $(s_i)_{jk} = -i\epsilon_{ijk}$ ,  $e_i$  are the  $1 \times 3$  matrices  $(e_i)_{1j} = \delta_{ij}$  and  $\bar{0} = (0 \ 0 \ 0)$ , while  $\mathbf{I}$  and  $\mathbf{0}$  designate the  $3 \times 3$  unit and zero matrices, respectively, while the superscript  $T$  designates matrix transposition. In this representation  $P = \beta^\mu \beta_\mu - 2 = \text{diag}(1, 1, 1, 1, 0, 0, 0, 0, 0)$ ; that is,  $P$  projects out the four upper components of the DKP spinor. The ten-component spinor can be written as

$\varphi^T = (\varphi_1, \dots, \varphi_{10})$  and partitioned as (following the notation of [42])

$$\varphi_1 = i\phi, \quad \vec{F} = \begin{pmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \quad (12)$$

$$\vec{G} = \begin{pmatrix} \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{pmatrix}.$$

The DKP equation in (3 + 1) dimensions can be expressed in the following compact forms:

$$i\vec{\nabla} \times \vec{F} - i\vec{A} \times \vec{F} = m\vec{H}, \quad (13)$$

$$\vec{\nabla} \cdot \vec{G} + \vec{A} \cdot \vec{G} = m\phi, \quad (14)$$

$$i\vec{\nabla} \times \vec{H} + i\vec{A} \times \vec{H} = m\vec{F} - (E - iA_0)\vec{G}, \quad (15)$$

$$\vec{\nabla}\phi - \vec{A}\phi = m\vec{G} - (E + iA_0)\vec{F}. \quad (16)$$

At this stage it is worthwhile to mention that (13)–(16) are completely different from those given in [42] and this fact is due to using improperly the nonminimal vector coupling. These facts should be enough to jeopardize the results presented in [42–51].

Using the standard procedure developed in [54], we put

$$\phi = \frac{\phi_{n,j}(r)}{r} Y_j^{m_j}(\theta, \varphi), \quad (17)$$

$$\vec{F} = \sum_l \frac{F_{n,j,l}(r)}{r} \vec{Y}_{j,l,m_j}(\theta, \varphi), \quad (18)$$

$$\vec{G} = \sum_l \frac{G_{n,j,l}(r)}{r} \vec{Y}_{j,l,m_j}(\theta, \varphi), \quad (19)$$

$$\vec{H} = \sum_l \frac{H_{n,j,l}(r)}{r} \vec{Y}_{j,l,m_j}(\theta, \varphi), \quad (20)$$

where  $\phi_{n,j}$ ,  $F_{n,j,l}$ ,  $G_{n,j,l}$ , and  $H_{n,j,l}$  are radial wave functions, while  $Y_j^{m_j}(\theta, \varphi)$  are the usual spherical harmonics of order  $j$  and  $\vec{Y}_{j,l,m_j}(\theta, \varphi)$  are the vector spherical harmonics. Then, using the notation

$$F_{n,j,j} = F_0, \quad F_{n,j,j\pm 1} = F_{\pm 1} \quad (21)$$

and similar definitions for  $G_0$ ,  $G_{\pm 1}$ ,  $H_0$ , and  $H_{\pm 1}$  together with the properties of vector spherical harmonics (see the Appendix), we can get a set of first-order coupled differential radial equations. Substituting (18) and (20) in (13) and if we consider spherically symmetric potentials  $A_0 = A_0(r)$  and

$\vec{A} = A_r(r)\vec{r}$ , the radial differential equations obtained from (13) are

$$\left(\frac{dF_0}{dr} - \frac{j+1}{r}F_0 - A_rF_0\right) = -\frac{1}{\zeta_j}mH_{+1}, \quad (22)$$

$$\left(\frac{dF_0}{dr} + \frac{j}{r}F_0 - A_rF_0\right) = -\frac{1}{\alpha_j}mH_{-1}, \quad (23)$$

$$-\zeta_j\left(\frac{dF_{+1}}{dr} + \frac{j+1}{r}F_{+1} - A_rF_{+1}\right) \quad (24)$$

$$-\alpha_j\left(\frac{dF_{-1}}{dr} - \frac{j}{r}F_{-1} - A_rF_{-1}\right) = mH_0,$$

where  $\alpha_j = \sqrt{(j+1)/(2j+1)}$  and  $\zeta_j = \sqrt{j/(2j+1)}$ .

Similarly, substituting (17) and (19) in (14), we obtain

$$-\alpha_j\left(\frac{dG_{+1}}{dr} + \frac{j+1}{r}G_{+1} + A_rG_{+1}\right) \quad (25)$$

$$+\zeta_j\left(\frac{dG_{-1}}{dr} - \frac{j}{r}G_{-1} + A_rG_{-1}\right) = m\phi.$$

The radial equations obtained from (15) are

$$\left(\frac{dH_0}{dr} - \frac{j+1}{r}H_0 + A_rH_0\right) \quad (26)$$

$$= -\frac{1}{\zeta_j}(mF_{+1} - (E - iA_0)G_{+1}),$$

$$\left(\frac{dH_0}{dr} + \frac{j}{r}H_0 + A_rH_0\right) \quad (27)$$

$$= -\frac{1}{\alpha_j}(mF_{-1} - (E - iA_0)G_{-1}),$$

$$-\zeta_j\left(\frac{dH_{+1}}{dr} + \frac{j+1}{r}H_{+1} + A_rH_{+1}\right)$$

$$-\alpha_j\left(\frac{dH_{-1}}{dr} - \frac{j}{r}H_{-1} + A_rH_{-1}\right) = (mF_0 - (E - iA_0)G_0). \quad (28)$$

Finally, from (16) we get

$$(E + iA_0)F_0 = mG_0, \quad (29)$$

$$\left(\frac{d\phi}{dr} - \frac{j+1}{r}\phi - A_r\phi\right) = -\frac{1}{\alpha_j}(mG_{+1} - (E + iA_0)F_{+1}), \quad (30)$$

$$\left(\frac{d\phi}{dr} + \frac{j}{r}\phi - A_r\phi\right) = \frac{1}{\zeta_j}(mG_{-1} - (E + iA_0)F_{-1}). \quad (31)$$

Now, following the procedure used in [54], the ten coupled differential radial equations obtained above (equations (22)–(31)) can be decoupled into two classes of radial equations associated with a specific parity. For states of  $(-1)^j$  parity, the relevant differential equations are (22), (23), (28),

and (29). The remaining six radial wave functions are zero. On the other hand, for states of  $(-1)^{j+1}$  parity, the relevant differential equations are (24), (25), (26), (27), (30), and (31). Similarly to the previous case, the other four radial wave functions are zero.

**3.2.1.  $(-1)^j$  Parity States.** Using (22), (23), and (29), the components  $H_{+1}$ ,  $H_{-1}$  and  $G_0$  can be eliminated in favor of  $F_0$  then by inserting them in (28), the radial function  $F_0(r)$  obeys the second-order differential equation as follows:

$$\frac{d^2F_0(r)}{dr^2} + \left[\kappa^2 - \frac{dA_r}{dr} - \frac{j(j+1)}{r^2} - A_r^2\right]F_0(r) = 0, \quad (32)$$

where  $\kappa^2 = E^2 - m^2 + A_0^2$  and because  $\nabla^2(1/r) = -4\pi\delta(\vec{r})$ , unless the potentials contain a delta function at the origin, one must impose the homogeneous Dirichlet condition  $F_0(0) = 0$ . At this stage it is worthwhile to mention that (32) is very similar to DKP equation for spin-zero particles in (3+1) dimensions except for the term  $-2A_r/r$  [40]. Therefore, for motion in a central field, the solution of the three-dimensional DKP equation with nonminimal vectorial interaction can be found by solving a Schrödinger-like equation for states of  $(-1)^j$  parity. The other components are obtained through (22), (23), and (29).

**3.2.2.  $(-1)^{j+1}$  Parity States.** Using (26) and (30), we obtain

$$\begin{pmatrix} F_{+1} \\ G_{+1} \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (E - iA_0)\alpha_j\Delta_- & m\zeta_j\Delta_+ \\ m\alpha_j\Delta_- & (E + iA_0)\zeta_j\Delta_+ \end{pmatrix} \begin{pmatrix} \phi \\ H_0 \end{pmatrix}, \quad (33)$$

where  $\Delta_{\pm} = d/dr - (j+1)/r \pm A_r$ . Similarly, using (27) and (31), we get

$$\begin{pmatrix} F_{-1} \\ G_{-1} \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} -(E - iA_0)\zeta_j\Xi_- & m\alpha_j\Xi_+ \\ -m\zeta_j\Xi_- & (E + iA_0)\alpha_j\Xi_+ \end{pmatrix} \begin{pmatrix} \phi \\ H_0 \end{pmatrix}, \quad (34)$$

where  $\Xi_{\pm} = d/dr + j/r \pm A_r$ . In this general case, we are not able to obtain analytical solutions to this kind of parity states, because we cannot decouple the differential equations for the components  $H_0$  and  $\phi$ . An alternative to overcome this disadvantage is to restrict our analysis for  $j = 0$ . Considering  $j = 0$ , the differential equations are decoupled for the components  $H_0$  and  $\phi$ , but those differential equations are very complicated and do not furnish exact solutions.

For  $A_0 = 0$ , (33) and (34) reduce to

$$\begin{pmatrix} F_{+1} \\ G_{+1} \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} E\alpha_j\Delta_- & m\zeta_j\Delta_+ \\ m\alpha_j\Delta_- & E\zeta_j\Delta_+ \end{pmatrix} \begin{pmatrix} \phi \\ H_0 \end{pmatrix}, \quad (35)$$

$$\begin{pmatrix} F_{-1} \\ G_{-1} \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} -E\zeta_j\Xi_- & m\alpha_j\Xi_+ \\ -m\zeta_j\Xi_- & E\alpha_j\Xi_+ \end{pmatrix} \begin{pmatrix} \phi \\ H_0 \end{pmatrix},$$

where  $\bar{\kappa}^2 = E^2 - m^2$ . In this case, we obtain that the radial functions  $H_0(r)$  and  $\phi(r)$  obey the second-order differential equations as follows:

$$\begin{aligned} \frac{d^2 H_0}{dr^2} + \left[ \bar{\kappa}^2 + \frac{dA_r}{dr} - \frac{j(j+1)}{r^2} - A_r^2 \right] H_0 &= 0, \\ \frac{d^2 \phi}{dr^2} + \left[ \bar{\kappa}^2 - \frac{dA_r}{dr} - \frac{j(j+1)}{r^2} - \frac{A_r}{r} - A_r^2 \right] \phi &= 0. \end{aligned} \quad (36)$$

Therefore, for the particular case  $A_0(r) = 0$ , the solution of the three-dimensional DKP equation with nonminimal vectorial interaction can be found by solving two Schrödinger-like equations for states of  $(-1)^{j+1}$  parity. The other components are obtained through (35). It should not be forgotten, though, that the equations for  $H_0$  and  $\phi$  are not indeed independent because the energy  $E$  appears in both equations. Therefore, one has to search for bound-state solutions for  $H_0$  and  $\phi$  with a common energy.

**3.3. Nonminimal Vector Linear Potential.** Having set up the spin-one equations for nonminimal vector interaction, we are now in a position to use the machinery developed above in order to solve the DKP equation for some specific form of the nonminimal interaction. As an application of this procedure, let us consider a nonminimal vector linear potential in the following form:

$$A_0 = m^2 \lambda_0 r, \quad A_r = m^2 \lambda_r r, \quad (37)$$

where  $\lambda_0$  and  $\lambda_r$  are dimensionless quantities.

**3.3.1.  $(-1)^j$  Parity States.** Substituting (37) in (32), one finds that  $F_0(r)$  obeys the second-order differential equation

$$\frac{d^2 F_0}{dr^2} + \left[ K^2 - \lambda^2 r^2 - \frac{j(j+1)}{r^2} \right] F_0 = 0, \quad (38)$$

where

$$K = \sqrt{E^2 - m^2 (1 + \lambda_r)}, \quad \lambda = m^2 \sqrt{\lambda_r^2 - \lambda_0^2}. \quad (39)$$

Considering  $F_0(0) = 0$  and  $\int_0^\infty dr |F_0|^2 < \infty$ , the solution for (38) with  $K$  and  $\lambda$  real is the well-known solution of the Schrödinger equation for the three-dimensional harmonic oscillator. Note that the condition  $\lambda$  real implies that  $|\lambda_r| > |\lambda_0|$ , meaning that the radial component of the nonminimal vectorial potential must be stronger than its time component in order for the effective potential to be a true confining potential. On the other hand, if  $\lambda_r = 0$  or  $|\lambda_r| < |\lambda_0|$ , we obtain  $\lambda = i|\lambda|$ , the effective potential in this case will be an inverted harmonic oscillator, and the energy spectrum will consist of a continuum corresponding to unbound states. Therefore, the presence of radial component of the nonminimal vector potential is an essential ingredient for confinement.

A detailed study of this effective potential is done in [40]. Using the results of [40], the solution is expressed as

$$|E| = m \sqrt{1 + \lambda_r + (2n + 3) \sqrt{\lambda_r^2 - \lambda_0^2}}, \quad n = 0, 1, 2, \dots, \quad (40)$$

$$F_0(r) = N_{nj} r^j e^{-\lambda r^2/2} L_{(n-j)/2}^{(j+1/2)}(\lambda r^2), \quad (41)$$

where  $N_{nj}$  is a normalization constant and  $n = 2N + j$  with  $N$  a nonnegative integer. Note that  $j$  can take values  $0, 2, \dots, n$  when  $n$  is an even number and  $1, 3, \dots, n$  when  $n$  is an odd number and also that for each value of  $j$  there are  $2j + 1$  different values of  $m_j$ . All the energy levels are degenerate with the exception of  $n = 0$ . The degeneracy of the level of energy for a given principal quantum number  $n$  is given by  $(n + 1)(n + 2)/2$  as a consequence of the presence of essential and accidental degeneracy.

From (40), we can see that there is an infinite set of discrete energies (symmetrical about  $E = 0$ ) irrespective to sign of  $\lambda_0$  and although positive- and negative-energy levels do not touch, they can be very close to each other for moderately strong coupling constants without any danger of reaching the conditions for Klein's paradox. The absence of Klein's paradox for this kind of interaction is attributed to fact that the nonminimal vectorial interaction does not distinguish particles from antiparticles [36].

For the case of  $A_0 = 0$  ( $\lambda_0 = 0$ ), the solution is expressed as

$$|E| = m \sqrt{1 + \lambda_r + (2n + 3) |\lambda_r|}, \quad n = 0, 1, 2, \dots, \quad (42)$$

$$F_0(r) = N_{nj} r^j e^{-\Omega r^2/2} L_{(n-j)/2}^{(j+1/2)}(\Omega r^2),$$

where  $\Omega = m^2 |\lambda_r|$ .

**3.3.2.  $(-1)^{j+1}$  Parity States.** As mentioned in Section 3.2.2, we cannot consider the general case (37), because we are not able to obtain analytical solutions to this kind of parity states.

Otherwise, considering (37) with  $A_0 = 0$  ( $\lambda_0 = 0$ ) and using the notations,  $\Phi_+ = H_0$  and  $\Phi_- = \phi$ , (36) reduces to

$$\frac{d^2 \Phi_\pm}{dr^2} + \left[ K_\pm^2 - \Omega^2 r^2 - \frac{j(j+1)}{r^2} \right] \Phi_\pm = 0, \quad (43)$$

where

$$\begin{aligned} K_+ &= \sqrt{E^2 - m^2 (1 - \lambda_r)}, \\ K_- &= \sqrt{E^2 - m^2 (1 - 2\lambda_r)}. \end{aligned} \quad (44)$$

The solution for (43) with  $K_\pm$  and  $\Omega$  real is the solution of the Schrödinger equation for the three-dimensional harmonic oscillator, as in the case of states of  $(-1)^j$  parity.

The energy can be obtained from the relation

$$K_{\pm}^2 = (2n_{\pm} + 3) \Omega. \quad (45)$$

Now we move on to match a common energy to spin-one particles problem for states of  $(-1)^{j+1}$  parity. The compatibility of the solutions for  $\Phi_+$  and  $\Phi_-$  demands that the quantum number  $n_+$  and  $n_-$  must satisfy the relation

$$n_- - n_+ = \frac{1}{2} \frac{\lambda_r}{|\lambda_r|}. \quad (46)$$

#### 4. Final Remarks

In this review paper, we showed the correct use and also presented a few properties of the nonminimal vector interactions in the Duffin-Kemmer-Petiau (DKP) formalism. A relativistic wave equation must carry a conserved four-current to exhibit symmetries in physical problems. In this spirit, we showed that the four-current is not conserved when one uses either the matrix potential  $P\beta^\mu$  or  $\beta^\mu P$  (widely used in the literature), even though the linear forms constructed from those matrices potentials behave as true Lorentz vectors. Also, we presented the correct equations for the problem addressed in [42]. In this case, we found an equation very similar to DKP equation for spin-zero particles in  $(3+1)$  dimensions, except for some additional terms. Therefore, the solution of the three-dimensional DKP equation with nonminimal vectorial interactions can be found by solving Schrödinger-like equations. As an application of the procedure developed, we considered the problem of spin-one particles in the presence of a nonminimal linear vector potential and discussed the necessary conditions in order for the effective potential to be true confining potential. The absence of Klein's paradox is attributed to the fact that the nonminimal vectorial interaction does not distinguish particles from antiparticles [36].

Our results are definitely useful because they shed some light on the understanding of the nonminimal vector interactions. Furthermore, the correct use of the nonminimal vectorial interaction may be useful due to wide applications in the description of elastic meson-nucleus scattering.

#### Appendix

##### The Vector Spherical Harmonics

The properties of the vector spherical harmonics used in this work are obtained from [55]. The list of properties is the following:

$$\begin{aligned} \hat{r} Y_j^{m_j} &= -\alpha_j \vec{Y}_{j,j+1,m_j} + \zeta_j \vec{Y}_{j,j-1,m_j}, \\ \vec{\nabla} Y_j^{m_j} &= \alpha_j \frac{j}{r} \vec{Y}_{j,j+1,m_j} + \zeta_j \frac{j+1}{r} \vec{Y}_{j,j-1,m_j}, \\ \vec{\nabla} \cdot (f(r) \vec{Y}_{j,j+1,m_j}) &= -\alpha_j \left( \frac{df}{dr} + \frac{j+2}{r} f \right) Y_j^{m_j}, \\ \vec{\nabla} \cdot (f(r) \vec{Y}_{j,j,m_j}) &= 0, \end{aligned}$$

$$\begin{aligned} \vec{\nabla} \cdot (f(r) \vec{Y}_{j,j-1,m_j}) &= \zeta_j \left( \frac{df}{dr} - \frac{j-1}{r} f \right) Y_j^{m_j}, \\ \vec{\nabla} \times (f(r) \vec{Y}_{j,j+1,m_j}) &= i\zeta_j \left( \frac{df}{dr} + \frac{j+2}{r} f \right) \vec{Y}_{j,j,m_j}, \\ \vec{\nabla} \times (f(r) \vec{Y}_{j,j,m_j}) &= i\zeta_j \left( \frac{df}{dr} - \frac{j}{r} f \right) \vec{Y}_{j,j+1,m_j} \\ &\quad + i\alpha_j \left( \frac{df}{dr} + \frac{j+1}{r} f \right) \vec{Y}_{j,j-1,m_j}, \\ \vec{\nabla} \times (f(r) \vec{Y}_{j,j-1,m_j}) &= i\alpha_j \left( \frac{df}{dr} - \frac{j-1}{r} f \right) \vec{Y}_{j,j,m_j}, \\ \hat{r} \times \vec{Y}_{j,j+1,m_j} &= i\zeta_j \vec{Y}_{j,j,m_j}, \\ \hat{r} \times \vec{Y}_{j,j,m_j} &= i\zeta_j \vec{Y}_{j,j+1,m_j} + i\alpha_j \vec{Y}_{j,j-1,m_j}, \\ \hat{r} \times \vec{Y}_{j,j-1,m_j} &= i\alpha_j \vec{Y}_{j,j,m_j}, \\ \hat{r} \cdot \vec{Y}_{j,j+1,m_j} &= -\alpha_j Y_j^{m_j}, \\ \hat{r} \cdot \vec{Y}_{j,j,m_j} &= 0, \\ \hat{r} \cdot \vec{Y}_{j,j-1,m_j} &= \zeta_j Y_j^{m_j}. \end{aligned} \quad (A.1)$$

#### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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