Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2014, Article ID 830109, 8 pages
http://dx.doi.org/10.1155/2014/830109

Research Article

Spherically Symmetric Solution in (1+4)-Dimensional $f(T)$ Gravity Theories

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Received 16 June 2014; Revised 1 October 2014; Accepted 5 October 2014; Published 23 October 2014

Academic Editor: Chao-Qiang Geng

A nondiagonal spherically symmetric tetrad field, involving four unknown functions of radial coordinate r plus an angle Φ, which is a generalization of the azimuthal angle ϕ, is applied to the field equations of (1+4)-dimensional $f(T)$ gravity theory. A special vacuum solution with one constant of integration is derived. The physical meaning of this constant is shown to be related to the gravitational mass of the system and the associated metric represents Schwarzschild in (1+4)-dimension. The scalar torsion related to this solution vanishes. We put the derived solution in a matrix form and rewrite it as a product of three matrices: the first represents a rotation while the second represents an inertia and the third matrix is the diagonal square root of Schwarzschild spacetime in (1+4)-dimension.

1. Introduction

Common consensus in the scientific community is that the property of the gravitational field, powered by Einstein general relativity (GR), is incorrect at scales of magnitude of Planck's length. The spacetime frame of such property should be clarified by a quantum regime. In the opposite extreme of the physical phenomena, GR also faces a fascinating problem linked to the late cosmic speedup stage of the universe. According to the previous reasons, and for other defects, GR has been the topic of many modifications which have attempted to supply a more satisfactory qualification of the gravitational field in the above aforementioned extreme regimes. Among the most important modified gravitational theories is the "$f(T)$ gravity," which is a theory constructed in a spacetime having absolute parallelism (cf. [1–12]). During the last years it has been proven that $f(T)$ gravitational theories (for a recent review, see [13]) are quite successful in description of the early-time and late-time acceleration of the universe. Calculations of the Kaluza-Klein reduction of teleparallel equivalent of general relativity (TEGR), which is an alternative gravity theory other than GR, at the low energy in the absence of the electromagnetic field have been carried out [14]. A general geometrical scenario of TEGR and investigation of the general behavior of the bulk as well as the projected effect on the brane have been studied [15].

Many of $f(T)$ gravity theories have been analyzed in [16–22]. It has been suggested that $f(T)$ gravity theory is not dynamically synonymous with the teleparallel equivalent of GR Lagrangian through conformal transformation [23]. Large-scale structure in $f(T)$ gravity theory has been analyzed in [24]. Perturbations in the area of cosmology in $f(T)$ gravity have been demonstrated [25–30]. Stationary, spherical symmetry solutions have been derived for $f(T)$ theories [31]. Relativistic stars and the cosmic expansion have been investigated [32].

$f(T)$ gravity theories have engaged many concerns: it has been indicated that the Lagrangian and the equations of motion are not invariant under local Lorentz transformations [33]. The reasons for such phenomena have been explained in [34].

The equations of motion of $f(T)$ theories differ from those of $f(R)$ theories because they are of the second order instead of the fourth order as in $f(R)$ theories. Such property has been believed as an indicator which shows that the theory might be of much interest. The nonlocality of such theories indicates that $f(T)$ seems to comprise more degrees of freedom.

Theories of $f(T)$ can be framed within the class of new gravity theories aimed at extending GR so as to solve its shortcomings at infrared and ultraviolet scales [35]. It is well known that, in extending the geometry sector, one of the goals of these theories is to solve the puzzle of dark energy and dark matter that seems to be unrivaled at basic level. In other words, $f(T)$ gravity could be a reliable tool to treat the problems of missing matter and accelerated expansion without asking for new material ingredients that have not been discovered by the experiments [36, 37]. In this work we investigate 5-dimensional $f(T)$ gravity. Exact black hole solution with vanishing scalar torsion can be derived. It is important to stress that finding analytic solutions is an essential step to set a new field theory. Exact solutions allow full control of the systems and can contribute to the well-formulation and well-position of the Cauchy problem (for a discussion on this point see [38]).

The organization of this study is as follows: brief review of $f(T)$ is presented in Section 2. In Section 3, a nondiagonal spherically symmetric tetrad field, with four unknown functions of the radial coordinate r plus a generalization to the angle ϕ, is given. The application of such tetrad field to the equations of motion of $f(T)$ is demonstrated in Section 3. Also, in Section 3, an analytic vacuum spherically symmetric solution with one constant of integration is derived. In Section 4, the physical properties of this solution are discussed. Final section is devoted to discussions and results.

2. Brief Review of $f(T)$

In a spacetime with absolute parallelism the parallel vector fields ${h^i}_\mu$ [39] define the nonsymmetric affine connection

$$\Gamma^{\lambda}{}_{\mu\nu} \stackrel{\text{def.}}{=} h_a{}^{\lambda} h^a{}_{\mu,\nu}, \tag{1}$$

where $h^a{}_{\mu,\nu} = \partial_\nu h_{a\mu}$ (spacetime indices $\mu, \nu, \ldots$ and indices $a, b, \ldots$ run from 0 to 4; time and space indices are indicated to $\mu = 0, i$, and $a = (0), (i)$). The curvature tensor defined by $\Gamma^{\lambda}{}_{\mu\nu}$, given by (1), is identically vanishing. The metric tensor $g_{\mu\nu}$ is defined by

$$g_{\mu\nu} \stackrel{\text{def.}}{=} \eta_{ab} h^a{}_\mu h^b{}_\nu, \tag{2}$$

with $\eta_{ab} = (-1, +1, +1, +1, +1)$ being the metric of Minkowski spacetime. Define the torsion and the contorsion components as

$$\begin{aligned} T^{\alpha}{}_{\mu\nu} &\stackrel{\text{def.}}{=} \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu} = h_a{}^{\alpha}\left(\partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu\right), \\ K^{\mu\nu}{}_{\alpha} &\stackrel{\text{def.}}{=} -\frac{1}{2}\left(T^{\mu\nu}{}_{\alpha} - T^{\nu\mu}{}_{\alpha} - T_{\alpha}{}^{\mu\nu}\right) = \Gamma^{\lambda}{}_{\mu\nu} - \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}, \end{aligned} \tag{3}$$

where contorsion equals the difference between Weitzenböck and Levi-Civita connection. The skew symmetric tensor $S_\alpha{}^{\mu\nu}$ is defined as

$$S_\alpha{}^{\mu\nu} \stackrel{\text{def.}}{=} \frac{1}{2}\left(K^{\mu\nu}{}_{\alpha} + \delta^\mu_\alpha T^{\beta\nu}{}_{\beta} - \delta^\nu_\alpha T^{\beta\mu}{}_{\beta}\right). \tag{4}$$

The torsion scalar has the form

$$T \stackrel{\text{def.}}{=} T^{\alpha}{}_{\mu\nu} S_\alpha{}^{\mu\nu}. \tag{5}$$

Similar to the $f(R)$ theory, one can define the action of $f(T)$ theory as

$$\begin{aligned} \mathscr{L}\left(h^a{}_\mu, \Phi_A\right) &= \int d^5 x h \left[\frac{1}{16\pi} f(T) + \mathscr{L}_{\text{Matter}}\left(\Phi_A\right)\right], \\ \text{where } h &= \sqrt{-g} = \det\left(h^a{}_\mu\right), \end{aligned} \tag{6}$$

and we assumed the relativistic units in which $G = c = 1$ and the Φ_A are the matter fields. Considering the action (6) as a functional of the fields $h^a{}_\mu$, Φ_A and vanishing the variation of the functional with respect to the field $h^a{}_\mu$ give the following equation of motion [40]:

$$\begin{aligned} &S_\mu{}^{\rho\nu} T_{,\rho} f(T)_{TT} + \left[h^{-1} h^a{}_\mu \partial_\rho \left(h h_a{}^{\alpha} S_\alpha{}^{\rho\nu}\right) - T^{\alpha}{}_{\lambda\mu} S_\alpha{}^{\nu\lambda}\right] f(T)_T \\ &\quad - \frac{1}{4}\delta^\nu_\mu f(T) = -4\pi \mathscr{T}^\nu_\mu, \end{aligned} \tag{7}$$

where $T_{,\rho} = \partial T/\partial x^\rho$, $f(T)_T = \partial f(T)/\partial T$, $f(T)_{TT} = \partial^2 f(T)/\partial T^2$, and $\mathscr{T}^\nu_\mu$ is the energy momentum tensor. In the next section we are going to apply the field equation (7) to a spherically symmetric spacetime and try to find new solution.

3. Spherically Symmetric Solution in 5-Dimensional $f(T)$ Gravity Theory

Assume that the manifold possesses a stationary and spherical symmetry that has the form

$$\left(h_a{}^\mu\right) = \begin{pmatrix} A_1(r) & A_2(r) & 0 & 0 & 0 \\ A_3(r)\sin\theta\cos\Phi & A_4(r)\sin\theta\cos\Phi & r\cos\theta\cos\Phi & -r\sin\theta\sin\Phi & 0 \\ A_3(r)\sin\theta\sin\Phi & A_4(r)\sin\theta\sin\Phi & r\cos\theta\sin\Phi & r\sin\theta\cos\Phi & 0 \\ A_3(r)\cos\theta & A_4(r)\cos\theta & -r\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & r\cos\theta \end{pmatrix}, \tag{8}$$

where $A_1(r)$, $A_2(r)$, $A_3(r)$, and $A_4(r)$ are four unknown functions of the radial coordinate r and the angle ϕ. $\Phi = L(\phi)$ is the generalization of the azimuthal angle ϕ.

The associated metric of the tetrad field (8) has the form

$$ds^2 = -\left[A_1{}^2(r) - A_3{}^2(r)\right]dt^2 - 2\left[A_1(r)A_2(r) - A_3(r)A_4\right]dtdr + \left[A_4{}^2(r) - A_2{}^2(r)\right]dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2. \tag{9}$$

Applying the tetrad field (8) to the field equation of $f(T)$ we get a system of nonlinear differential equation (the detailed calculations of the field equations of $f(T)$ are given in the appendix). To find an exact solution to the resulting system of nonlinear differential equation, we put the following constraints:

$$T' = 0, \qquad T_\phi = 0. \tag{10}$$

Using (10) in the resulting system of nonlinear differential equation, given in the appendix, we get

$$\begin{aligned}
4\pi\mathcal{T}_0^0 &= -\frac{f_T}{2r^2(A_1A_4 - A_2A_3)^3} \\
&\times \Big\{(A_1A_4 - A_2A_3)^2 \\
&\quad\times \left\{rA_1' + 2rA_2A_3 + 2A_1(1 - A_4)\right\}L_\phi \\
&\quad + r\left[A_4(A_4 - 3)A_1{}^2 - 2A_1A_2A_3(A_4 - 3)\right. \\
&\qquad \left. + A_3{}^2\left(A_2{}^2 - 3A_4\right)\right]A_1' \\
&\quad - 3rA_3'\left[A_3{}^2A_2 - 2A_1A_3A_4 + A_1{}^2A_2\right] \\
&\quad - 3r\left(A_3A_2' - A_1A_4'\right)\left(A_1{}^2 - A_3{}^2\right) \\
&\quad + 2(A_1A_4 - A_2A_3) \\
&\quad \times \left[A_1{}^2\left\{2A_4{}^2 + A_4 - 3\right\} - A_1A_2A_3\left\{4A_4 + 1\right\}\right] \\
&\quad + A_3{}^2\left\{2A_2{}^2 + 3\right\}\Big\} - \frac{f}{4},
\end{aligned} \tag{11}$$

$$\begin{aligned}
4\pi\mathcal{T}_1^0 &= -\frac{3f_T}{2r(A_1A_4 - A_2A_3)^3} \\
&\times (A_1A_4 - A_2A_3)\left\{A_1A_4' - A_2A_3' + A_4A_1' - A_3A_2'\right\},
\end{aligned} \tag{12}$$

$$\begin{aligned}
4\pi\mathcal{T}_1^1 &= \frac{f_T}{2r^2(A_1A_4 - A_2A_3)^2} \\
&\times \Big(L_\phi(A_1A_4 - A_2A_3)\left\{2A_1\left[1 - A_4\right] + 2A_2A_3 + rA_1'\right\} \\
&\quad + rA_1'\left[A_1(A_4 - 6) - A_2A_3\right] + 6rA_3A_3' \\
&\quad + 2A_1{}^2\left(2A_4{}^2 + A_4 - 3\right) - 2A_1A_2A_3(4A_4 + 1) \\
&\quad + 2A_3{}^2\left(3 + A_2{}^2\right)\Big) - \frac{f}{4},
\end{aligned} \tag{13}$$

$$\begin{aligned}
4\pi\mathcal{T}_2^2 &= \frac{f_T}{2r^2(A_1A_4 - A_2A_3)^3} \\
&\times \Big(r^2(A_1A_4 - A_2A_3)\left[A_3A_3'' - A_1A_1''\right] + r^2A_2A_3A_1'^2 \\
&\quad - rA_1'\Big[r(A_3A_4 + A_1A_2)A_3' - (A_1A_4 - A_3A_2)^2L_\phi \\
&\qquad + rA_1A_3A_2' - rA_1{}^2A_4' + A_1{}^2A_4(5 - A_4) \\
&\qquad + A_1A_2A_3(2A_4 - 7) - A_3{}^2\left(A_2{}^2 - 2A_4\right)\Big] \\
&\quad - 2L_\phi(A_1A_4 - A_2A_3)^2\left[A_1(A_4 - 1) - A_2A_3\right] \\
&\quad + r^2A_1A_4A_3'^2 \\
&\quad + rA_3'\Big[7A_1A_3A_4 - 5A_3{}^2A_2 - 2A_1{}^2A_2 \\
&\qquad + rA_3{}^2A_2' - rA_1A_3A_4'\Big] \\
&\quad + 2rA_3\left[A_2' - A_4'\right]\left[A_3{}^2 - A_1{}^2\right] \\
&\quad + 2(A_1A_4 - A_2A_3) \\
&\quad \times \left[\left(A_4{}^2 + A_4 - 2\right)A_1{}^2 - A_1A_2A_3(2A_4 + 1)\right] \\
&\quad + A_3{}^2\left(2 + A_2{}^2\right)\Big) - \frac{f}{4}.
\end{aligned} \tag{14}$$

Equations (11)–(14) are four differential equations in five unknown functions A_1, A_2, A_3, A_4, and $L(\phi)$. Therefore, we use the system of (11)–(14) plus the constraint given by (10) to find solution of those five unknown functions. One of the solutions to these equations is the following:

$$A_1 = 1 - \frac{c_1}{2r^2}, \qquad A_2 = \frac{c_1}{2(r^2 - c_1)}, \qquad A_3 = \frac{c_1}{2r^2}, \qquad A_4 = \frac{2r^2 - c_1}{2(r^2 - c_1)}, \qquad L(\phi) = \frac{\left(3c_1 - 4r^2\right)\phi}{c_1 - 2r^2}, \tag{15}$$

where c_1 is a constant of integration. Substituting (15) into (5) we get a vanishing value of the scalar torsion. Therefore, solution (15) is an exact vacuum solution to the field equations of $f(T)$ when applied to the ansatz (8), provided that

$$f(0) = 0, \qquad f_T(0) \neq 0, \qquad f_{TT} \neq 0. \tag{16}$$

To understand the physical meaning of the integration constant that appears in solution (15) we discuss the physics related to this solution. First, the metric associated with the tetrad field given by (8) (after using solution (15) in (2)) has the form

$$ds^2 = -\alpha dt^2 + \frac{1}{\alpha}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + r^2 \cos^2\theta d\psi,$$
$$\text{where } \alpha = \frac{r^2 + c_1}{r^2}. \tag{17}$$

This is 5-dimensional Schwarzschild spacetime provided that $c_1 = -M^2$ [41].

4. Physical Properties

We rewrite the tetrad field (8) after using (15) in a way to understand its structure. To do this we use the fact that any tetrad field can be rewritten as

$$\left(h^i{}_\mu\right) \overset{\text{def.}}{=} \left(\Lambda^i{}_j\right)\left(h^j{}_\mu\right)_1, \tag{18}$$

where $(\Lambda^i{}_j)$ is a local Lorentz transformation satisfying

$$\left(\Lambda^i{}_j\right)\eta_{ik}\left(\Lambda^k{}_m\right) = \eta_{jm}, \tag{19}$$

and $(h^j{}_\mu)_1$ is a tetrad field which generates the same metric that is generated by the tetrad field $(h^i{}_\mu)$. Using (18) to rewrite the tetrad field (8) after using (15), we get

$$\left(h^i{}_\mu\right) = \left(\Lambda^i{}_j\right)\left(\Lambda^j{}_k\right)_1\left(h^k{}_\mu\right)_d, \tag{20}$$

where

$$\left(\Lambda^i{}_j\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\sin\theta\cos\Phi & \cos\theta\cos\Phi & -\sin\Phi & 0 \\ 0 & -\sin\theta\sin\Phi & \cos\theta\sin\Phi & \cos\Phi & 0 \\ 0 & -\cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

$$\left(\Lambda^j{}_k\right)_1 = \begin{pmatrix} \dfrac{2r^2 - M^2}{2r\sqrt{M^2 - r^2}} & \dfrac{M^2}{2r\sqrt{M^2 - r^2}} & 0 & 0 & 0 \\ \dfrac{-M^2}{2r\sqrt{M^2 - r^2}} & -\dfrac{2r^2 - M^2}{2r\sqrt{M^2 - r^2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

$$\left(h^k{}_\mu\right)_d = \begin{pmatrix} \sqrt{\dfrac{M^2 - r^2}{r^2}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\dfrac{r^2}{r^2 - M^2}} & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & r\sin\theta & 0 \\ 0 & 0 & 0 & 0 & r\cos\theta \end{pmatrix}. \tag{23}$$

The matrices (21) and (22) give local Lorentz transformations; that is, they satisfy (19). Equation (23) is the square root of Schwarzschild (1+4)-dimensional spacetime [41].

5. Main Results and Discussion

In this study we have addressed the $f(T)$ gravitational theories in higher dimension, for example, (1+4)-dimensional spacetime. We have considered a solution in the vacuum case. The field equations of $f(T)$ have been applied to a nondiagonal spherically symmetric tetrad field having four unknown functions of the redial coordinate besides a generalization of the angle ϕ. Nine nonlinear differential equations have been obtained. To solve these differential equations, we have imposed two constraints related to the derivative of the scalar torsion T. Then the number of the nonlinear differential equations reduced to four nonlinear differential equations. We have solved these four differential equations in addition to the constraints $T_\phi = 0$, which are the derivative of the scalar torsion with respect to the azimuthal angle ϕ. We have derived a solution which contains one constant of integration. This solution represents an exact vacuum spherically symmetric solution to the field equations of $f(T)$ gravitational theory in higher dimensional. This solution has a property that its scalar torsion is vanishing and satisfying the field equations of $f(T)$, provided that (16) holds. To understand the physical meaning of the constant of integration, which exists in the solution, we have calculated its associated metric. It has been shown that this constant is related to the gravitational mass to reproduced Schwarzschild spacetime in (1+4)-dimension.

To understand the properties of the derived solution, we have rewritten it as three matrices. Two of these matrices represent local Lorentz transformations while the third represents the square root of the Schwarzschild spacetime in (1 + 4)-dimension. The first local Lorentz transformation (local Lorentz transformation means that it satisfies (A.7)) represents a rotation matrix which is some kind of Euler's angle while the second one represents some kind of inertia.

Appendix

Calculations of Torsion Scalar, Its Derivative, and the Field Equations of $f(T)$ Using Tetrad (8)

By the use of (5) and (8), we obtain the torsion scalar and its derivatives in terms of r and ϕ as

$$\begin{aligned} T(r,\phi) &= \frac{1}{r^2\left(A_1A_4 - A_2A_3\right)^2} \\ &\quad \times \left(2\left(\left[2 - 2A_4\right]A_1 + 2A_2A_3 + rA_1'\right)\left(A_1A_4 - A_2A_3\right)L_\phi\right. \\ &\qquad + 2rA_1'\left[\left[A_4 - 3\right]A_1 - A_2A_3\right] \\ &\qquad + 6rA_3A_3' + 2{A_1}^2\left({A_4}^2 + 2A_4 - 3\right) \end{aligned}$$

$$-4A_2A_3(A_4+1)+2{A_3}^2\left(3+{A_2}^2\right)\Big), \quad \text{where}$$

$$A_1' = \frac{\partial A_1(r)}{\partial r}, \qquad A_2' = \frac{\partial A_2(r)}{\partial r}, \qquad A_3' = \frac{\partial A_3(r)}{\partial r},$$

$$A_4' = \frac{\partial A_4(r)}{\partial r}, \qquad L_\phi = \frac{\partial L(\phi)}{\partial \phi},$$

$$\begin{aligned}
&T'(r,\phi)\\
&= \frac{\partial T(r)}{\partial r} = \frac{1}{r^3(A_1A_4-A_2A_3)^3}\\
&\times\Big(2\left(\left[A_1A_4-A_2A_3\right]L_\phi+\left[A_4-3\right]A_1-A_3A_2\right)\\
&\quad\times(A_1A_4-A_2A_3)r^2A_1''\\
&\quad+6r^2A_3A_3''(A_1A_4-A_2A_3)\\
&\quad-2r^2{A_1'}^2\left[A_4(A_1A_4-A_2A_3)A_1L_\phi\right.\\
&\qquad\left.+A_4(A_4-3)A_1-A_2A_3(3+A_4)\right]\\
&\quad+2L_\phi\Big[(A_1A_4-A_2A_3)\left[rA_3A_2'-rA_1A_4'-A_1A_4\right.\\
&\qquad\left.+rA_2A_3'-A_2A_3\right]\\
&\qquad+rA_3'\left(A_1A_2\left[A_4-6\right]-A_3\left(6A_4+{A_2}^2\right)\right)\\
&\qquad+rA_3A_2'\left(A_1\left[A_4-6\right]-A_2A_3\right)\\
&\qquad-rA_1A_4'\left(A_1\left[A_4-6\right]-A_2A_3\right)\\
&\qquad+{A_1}^2A_4(3-A_4)+3A_1A_2A_3\\
&\qquad+r{A_3}^2A_1'\left({A_2}^2-6A_4\right)\Big]\\
&\quad+4L_\phi(A_1A_4-A_2A_3)\\
&\quad\times\left[rA_1A_2A_3'+rA_1A_3A_2'-r{A_1}^2A_4'\right.\\
&\qquad\left.+2(A_1A_4-A_2A_3)(\left[A_1-1\right]A_1-A_2A_3)\right]\\
&\quad+6r^2{A_3'}^2(A_2A_3+A_1A_4)\\
&\quad+12rA_3'\left[r{A_3}^2A_2'-rA_1A_3A_4'+\frac{1}{3}A_2(A_4-3){A_1}^2\right.\\
&\qquad\left.-\frac{1}{3}\left({A_2}^2-\frac{3}{2}A_4\right)A_1A_3+\frac{1}{2}A_2{A_3}^2\right]\\
&\quad+4rA_3A_2'\left(\left[A_4-3\right]{A_1}^2-A_1A_2A_3+3{A_3}^2\right)\\
&\quad-4rA_1A_4'\left[\left[A_4-3\right]{A_1}^2-A_1A_2A_3+3{A_3}^2\right]\\
&\quad-4(A_1A_4-A_2A_3)\\
&\quad\times\Big(\left[2A_4+{A_4}^2-3\right]{A_1}^2\\
&\qquad-2A_2A_3(A_4+1)A_1+{A_3}^2\left(3+{A_2}^2\right)\Big)\Big),
\end{aligned}$$

$$\begin{aligned}
T(r,\phi)_\phi &= \frac{\partial T(r,\phi)}{\partial \phi}\\
&= \frac{2L_{\phi\phi}\left(rA_1'+2A_2A_3+A_1\left[2-2A_4\right]\right)}{r^2(A_1A_4-A_2A_3)^3}.
\end{aligned} \tag{A.1}$$

Applying the field equation (7) to the tetrad field (8) and using the above equation we get

$$\begin{aligned}
&4\pi\mathcal{T}_0^{\,0}\\
&= -\Big(f_{TT}T'\left[A_4{A_1}^2\left(1+L_\phi\right)-3{A_1}^2\right.\\
&\qquad\left.-A_1A_2A_3\left(1+L_\phi\right)+3{A_3}^2\right]\Big)\\
&\quad\times\left(r\left({A_1}^2{A_4}^2+{A_2}^2{A_3}^2-2A_1A_2A_3A_4\right)\right)^{-1}\\
&\quad-\frac{f_T}{2r^2(A_1A_4-A_2A_3)^3}\\
&\quad\times\Big\{(A_1A_4-A_2A_3)^2\\
&\qquad\times\left\{rA_1'+2rA_2A_3+2A_1(1-A_4)\right\}L_\phi\\
&\qquad+r\Big[A_4(A_4-3){A_1}^2\\
&\qquad\quad-2A_1A_2A_3(A_4-3)+{A_3}^2\left({A_2}^2-3A_4\right)\Big]A_1'\\
&\qquad-3rA_3'\left[{A_3}^2A_2-2A_1A_3A_4+{A_1}^2A_2\right]\\
&\qquad-3r\left(A_3A_2'-A_1A_4'\right)\left({A_1}^2-{A_3}^2\right)\\
&\qquad+2(A_1A_4-A_2A_3)\\
&\qquad\times\left[{A_1}^2\left\{2{A_4}^2+A_4-3\right\}-A_1A_2A_3\left\{4A_4+1\right\}\right]\\
&\qquad+{A_3}^2\left\{2{A_2}^2+3\right\}\Big\}-\frac{f}{4},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&4\pi\mathcal{T}_1^{\,0}\\
&= \Big(f_{TT}T'\left[A_4A_1A_2\left(1+L_\phi\right)+3A_3A_4\right.\\
&\qquad\left.-3A_1A_2-A_3\left(1+L_\phi\right){A_2}^2\right]\Big)\\
&\quad\times\left(r\left({A_1}^2{A_4}^2+{A_2}^2{A_3}^2-2A_1A_2A_3A_4\right)\right)^{-1}\\
&\quad-\frac{3f_T}{2r(A_1A_4-A_2A_3)^3}\\
&\quad\times(A_1A_4-A_2A_3)\left\{A_1A_4'-A_2A_3'+A_4A_1'-A_3A_2'\right\},
\end{aligned} \tag{A.3}$$

$$
\begin{aligned}
&4\pi\mathcal{T}^{1}_{\ 1}\\
&=\frac{f_T}{2r^2(A_1A_4-A_2A_3)^2}\\
&\quad\times\Big(L_\phi\left(A_1A_4-A_2A_3\right)\left\{2A_1\left[1-A_4\right]+2A_2A_3+rA_1'\right\}\\
&\qquad+rA_1'\left[A_1\left(A_4-6\right)-A_2A_3\right]\\
&\qquad+6rA_3A_3'+2{A_1}^2\left(2{A_4}^2+A_4-3\right)\\
&\qquad-2A_1A_2A_3\left(4A_4+1\right)+2{A_3}^2\left(3+{A_2}^2\right)\Big)-\frac{f}{4},
\end{aligned}
\tag{A.4}
$$

$$
4\pi\mathcal{T}^{2}_{\ 1}=-\frac{f_{TT}T'\left[1-\cos^2\theta\left(2-L_\phi\right)\right]}{r^2\sin2\theta},
\tag{A.5}
$$

$$
\begin{aligned}
&4\pi\mathcal{T}^{2}_{\ 2}\\
&=\Big(f_{TT}T'\Big[A_1\left(A_1A_4-A_2A_3\right)L_\phi\\
&\qquad\quad-2{A_1}^2+2{A_3}^2-r\left(A_1A_1'-A_3A_3'\right)\Big]\Big)\\
&\quad\times\left(2r(A_1A_4-A_2A_3)^2\right)^{-1}\\
&\quad+\frac{f_T}{2r^2(A_1A_4-A_2A_3)^3}\\
&\quad\times\Big(r^2\left(A_1A_4-A_2A_3\right)\left[A_3A_3''-A_1A_1''\right]+r^2A_2A_3{A_1'}^2\\
&\qquad-rA_1'\Big[r\left(A_3A_4+A_1A_2\right)A_3'-\left(A_1A_4-A_3A_2\right)^2L_\phi\\
&\qquad\qquad+rA_1A_3A_2'-r{A_1}^2A_4'+{A_1}^2A_4\left(5-A_4\right)\\
&\qquad\qquad+A_1A_2A_3\left(2A_4-7\right)-{A_3}^2\left({A_2}^2-2A_4\right)\Big]\\
&\qquad-2L_\phi(A_1A_4-A_2A_3)^2\left[A_1\left(A_4-1\right)-A_2A_3\right]\\
&\qquad+r^2A_1A_4{A_3'}^2\\
&\qquad+rA_3'\Big[7A_1A_3A_4-5{A_3}^2A_2-2{A_1}^2A_2\\
&\qquad\qquad+r{A_3}^2A_2'-rA_1A_3A_4'\Big]\\
&\qquad+2rA_3\left[A_2'-A_4'\right]\left[{A_3}^2-{A_1}^2\right]\\
&\qquad+2\left(A_1A_4-A_2A_3\right)\Big[\left({A_4}^2+A_4-2\right){A_1}^2\\
&\qquad-A_1A_2A_3\left(2A_4+1\right)\Big]+{A_3}^2\left(2+{A_2}^2\right)\Big)-\frac{f}{4},
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
&4\pi\mathcal{T}^{0}_{\ 3}\\
&=\Big(f_{TT}T_\phi\Big[A_1A_2A_4+rA_4A_3'+2A_3A_4\\
&\qquad-2A_1A_2-{A_2}^2A_3-rA_2A_1'\Big]\Big)\\
&\quad\times\left(2r(A_1A_4-A_2A_3)^2\right)^{-1},
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
&4\pi\mathcal{T}^{1}_{\ 3}\\
&=\Big(f_{TT}T_\phi\Big[A_1A_2A_3+rA_1A_1'-{A_1}^2A_4\\
&\qquad+2{A_1}^2-2{A_3}^2-rA_3A_3'\Big]\Big)\\
&\quad\times\left(2r(A_1A_4-A_2A_3)^2\right)^{-1},
\end{aligned}
\tag{A.8}
$$

$$
4\pi\mathcal{T}^{2}_{\ 3}=-\frac{f_{TT}T_\phi\cot\theta}{2r^2},
\tag{A.9}
$$

$$
\begin{aligned}
&4\pi\mathcal{T}^{3}_{\ 3}\\
&=\Big(f_{TT}T'\Big[A_1\left(A_1A_4-2A_1-A_2A_3\right)\\
&\qquad+2{A_3}^2-r\left(A_1A_1'-A_3A_3'\right)\Big]\Big)\\
&\quad\times\left(2r(A_1A_4-A_2A_3)^2\right)^{-1}\\
&\quad+\frac{f_T}{2r^2(A_1A_4-A_2A_3)^3}\\
&\quad\times\Big(r^2\left(A_1A_4-A_2A_3\right)\left[A_3A_3''-A_1A_1''\right]+r^2A_2A_3{A_1'}^2\\
&\qquad-rA_1'\Big[r\left(A_3A_4+A_1A_2\right)A_3'\\
&\qquad\qquad-\left(A_1A_4-A_3A_2\right)^2L_\phi+rA_1A_3A_2'-r{A_1}^2A_4'\\
&\qquad\qquad+{A_1}^2A_4\left(5-A_4\right)+A_1A_2A_3\left(2A_4-7\right)\\
&\qquad\qquad-{A_3}^2\left({A_2}^2-2A_4\right)\Big]\\
&\qquad-2L_\phi(A_1A_4-A_2A_3)^2\left[A_1\left(A_4-1\right)-A_2A_3\right]\\
&\qquad+r^2A_1A_4{A_3'}^2\\
&\qquad+rA_3'\Big[7A_1A_3A_4-5{A_3}^2A_2-2{A_1}^2A_2+r{A_3}^2A_2'\\
&\qquad\qquad-rA_1A_3A_4'\Big]+2rA_3\left[A_2'-A_4'\right]\left[{A_3}^2-{A_1}^2\right]\\
&\qquad+2\left(A_1A_4-A_2A_3\right)\\
&\qquad\times\left[\left({A_4}^2+A_4-2\right){A_1}^2-A_1A_2A_3\left(2A_4+1\right)\right]\\
&\qquad+{A_3}^2\left(2+{A_2}^2\right)\Big)-\frac{f}{4},
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
&4\pi\mathcal{T}^{4}_{\ 4}\\
&=\Big(f_{TT}T'\Big[A_1\left(A_1A_4-2A_1-A_2A_3\right)+2{A_3}^2\\
&\qquad-r\left(A_1A_1'-A_3A_3'\right)+A_1L_\phi\left(A_1A_4-A_2A_3\right)\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(2r\left(A_1A_4-A_2A_3\right)^2\right)^{-1}\\
&+\frac{f_T}{2r^2\left(A_1A_4-A_2A_3\right)^3}\\
&\times\Big(r^2\left(A_1A_4-A_2A_3\right)\left[A_3A_3''-A1A_1''\right]+r^2A_2A_3{A_1'}^2\\
&\quad-rA_1'\Big[r\left(A_3A_4+A_1A_2\right)A_3'\\
&\qquad-\left(A_1A_4-A_3A_2\right)^2L_\phi+rA_1A_3A_2'-r{A_1}^2A_4'\\
&\qquad+{A_1}^2A_4\left(5-A_4\right)+A_1A_2A_3\left(2A_4-7\right)\\
&\qquad-{A_3}^2\left({A_2}^2-2A_4\right)\Big]\\
&\quad-2L_\phi\left(A_1A_4-A_2A_3\right)^2\left[A_1\left(A_4-1\right)-A_2A_3\right]\\
&\quad+r^2A_1A_4{A_3'}^2\\
&\quad+rA_3'\Big[7A_1A_3A_4-5{A_3}^2A_2-2{A_1}^2A_2+r{A_3}^2A_2'\\
&\qquad-rA_1A_3A_4'\Big]+2rA_3\left[A_2'-A_4'\right]\left[{A_3}^2-{A_1}^2\right]\\
&\quad+2\left(A_1A_4-A_2A_3\right)\\
&\quad\times\left[\left({A_4}^2+A_4-2\right){A_1}^2-A_1A_2A_3\left(2A_4+1\right)\right]\\
&\quad+{A_3}^2\left(2+{A_2}^2\right)\Big)-\frac{f}{4}.
\end{aligned}
\tag{A.11}
$$

Using (6) and (8), one can obtain $h = \det({h^\mu}_a) = r^3\sin\theta\cos\theta(A_1A_4 - A_2A_3)$ (for brevity, we write $A_1(r) \equiv A_1$, $A_2(r) \equiv A_2$, $A_3(r) \equiv A_3$, $A_4(r) \equiv A_4$, and $L(\phi) = L$). From (A.1)–(A.11), it is clear that $A_1A_4 \neq A_3A_2$. When $f(T) = T$ and $\Phi = L(\phi) = \phi$, then (A.1)–(A.11) will be identical with the TEGR [42]. Equations (A.1)–(A.11) cannot be easily solved. This is because of the existence of the term f_{TT}. If this term is vanishing, then (A.1)–(A.11) can be easily solved.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is partially supported by the Egyptian Ministry of Scientific Research under Project no. 24-2-12.

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