

## Research Article

# Minimal Length Effects on Tunnelling from Spherically Symmetric Black Holes

Benrong Mu,<sup>1,2</sup> Peng Wang,<sup>2</sup> and Haitang Yang<sup>2,3</sup>

<sup>1</sup>School of Physical Electronics, University of Electronic Science and Technology of China, Chengdu 610054, China

<sup>2</sup>Center for Theoretical Physics, College of Physical Science and Technology, Sichuan University, Chengdu 610064, China

<sup>3</sup>Kavli Institute for Theoretical Physics China (KITPC), Chinese Academy of Sciences, Beijing 100080, China

Correspondence should be addressed to Benrong Mu; [mubenrong@uestc.edu.cn](mailto:mubenrong@uestc.edu.cn)

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We investigate effects of the minimal length on quantum tunnelling from spherically symmetric black holes using the Hamilton-Jacobi method incorporating the minimal length. We first derive the deformed Hamilton-Jacobi equations for scalars and fermions, both of which have the same expressions. The minimal length correction to the Hawking temperature is found to depend on the black hole's mass and the mass and angular momentum of emitted particles. Finally, we calculate a Schwarzschild black hole's luminosity and find the black hole evaporates to zero mass in infinite time.

## 1. Introduction

The classical theory of black holes predicts that nothing, including light, could escape from the black holes. However, Stephen Hawking first showed that quantum effects could allow black holes to emit particles. The formula of Hawking temperature was first given in the frame of quantum field theory [1]. After that, various methods for deriving Hawking radiation have been proposed. Among them is a semiclassical method of modeling Hawking radiation as a tunneling effect proposed by Kraus and Wilczek [2, 3], which is known as the null geodesic method. Later, the tunneling behaviors of particles were investigated using the Hamilton-Jacobi method [4–6]. Using the null geodesic method and the Hamilton-Jacobi method, much fruit has been achieved [7–18]. The key point of the Hamilton-Jacobi method is using WKB approximation to calculate the imaginary part of the action for the tunneling process.

On the other hand, various theories of quantum gravity, such as string theory, loop quantum gravity, and quantum geometry, predict the existence of a minimal length [19–21]. The generalized uncertainty principle (GUP) [22] is a simple way to realize this minimal length.

An effective model of the GUP in one-dimensional quantum mechanics is given by [23, 24]

$$L_f k(p) = \tanh\left(\frac{p}{M_f}\right), \quad (1)$$

$$L_f \omega(E) = \tanh\left(\frac{E}{M_f}\right), \quad (2)$$

where the generators of the translations in space and time are the wave vector  $k$  and the frequency  $\omega$ ,  $L_f$  is the minimal length, and  $L_f M_f = \hbar$ . The quantization in position representation  $\hat{x} = x$  leads to

$$k = -i\partial_x, \quad \omega = i\partial_t. \quad (3)$$

Therefore, the low energy limit  $p \ll M_f$  including order of  $p^3/M_f^3$  gives

$$p = -i\hbar\partial_x \left(1 - \frac{\hbar^2}{M_f^2} \partial_x^2\right), \quad (4)$$

$$E = i\hbar\partial_t \left(1 - \frac{\hbar^2}{M_f^2} \partial_t^2\right), \quad (5)$$

where we neglect the factor  $1/3$ . From (1), it is noted that although one can increase  $p$  arbitrarily,  $k$  has an upper bound which is  $1/L_f$ . The upper bound on  $k$  implies that particles could not possess arbitrarily small Compton wavelengths  $\lambda = 2\pi/k$  and that there exists a minimal length  $\sim L_f$ . Furthermore, the deformed Klein-Gordon/Dirac equations incorporating (4) and (5) have already been obtained in [23], which will be briefly reviewed in Section 2. So [23] provides a way to incorporate the minimal length with special relativity, a good starting point for studying Hawking radiation as tunnelling effect.

The black hole is a suitable venue to discuss the effects of quantum gravity. Incorporating GUP into black holes has been discussed in a lot of papers [25–30]. The thermodynamics of black holes has also been investigated in the framework of GUP [29, 30]. In [31], a new form of GUP is introduced:

$$\begin{aligned} p^0 &= k^0, \\ p^i &= k^i (1 - \alpha \mathbf{k} + 2\alpha^2 \mathbf{k}^2), \end{aligned} \quad (6)$$

where  $p^a$  is the modified four momentum,  $k^a$  is the usual four momentum and  $\alpha$  is a small parameter. The modified velocity of photons, two-dimensional Klein-Gordon equation the emission spectrum due to the Unruh effect is obtained there. Recently, the GUP deformed Hamilton-Jacobi equation for fermions in curved spacetime have been introduced and the corrected Hawking temperatures have been derived [32–36]. The authors consider the GUP of form

$$\begin{aligned} x_i &= x_{0i}, \\ p_i &= p_{0i} (1 + \beta p^2), \end{aligned} \quad (7)$$

where  $x_{0i}$ , and  $p_{0i}$  satisfy the canonical commutation relations. Fermions' tunnelling, black hole thermodynamics, and the remnants are discussed there.

In this paper, we investigate scalars and fermions tunneling across the horizons of black holes using the deformed Hamilton-Jacobi method which incorporates the minimal length via (4) and (5). Our calculation shows that the quantum gravity correction is related not only to the black hole's mass but also to the mass and angular momentum of emitted particles.

The organization of this paper is as follows. In Section 2, from the modified fundamental commutation relation, we generalize the Hamilton-Jacobi method in curved spacetime. In Section 3, incorporating GUP, we investigate the tunnelling of particles in the black holes. In Section 4, we investigate how a Schwarzschild black hole evaporates in our model. Section 5 is devoted to our conclusion. In this paper, we take Geometrized units  $c = G = 1$ , where the Planck constant  $\hbar$  is square of the Planck Mass  $m_p$ . We also assume that the emitted particles are neutral.

## 2. Deformed Hamilton-Jacobi Equations

To be generic, we will consider a spherically symmetric background metric of the form

$$ds^2 = f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (8)$$

where  $f(r)$  has a simple zero at  $r = r_h$  with  $f'(r_h)$  being finite and nonzero. The vanishing of  $f(r)$  at point  $r = r_h$  indicates the presence of an event horizon. In this section, we will first derive the deformed Klein-Gordon/Dirac equations in flat spacetime and then generalize them to the curved spacetime with the metric (8).

In the (3 + 1) dimensional flat spacetime, the relations between  $(p_i, E)$  and  $(k_i, \omega)$  can simply be generalized to

$$\begin{aligned} L_f k_i(p) &= \tanh\left(\frac{p_i}{M_f}\right), \\ L_f \omega(E) &= \tanh\left(\frac{E}{M_f}\right), \end{aligned} \quad (9)$$

where, in the spherical coordinates, one has for  $\vec{k}$

$$\vec{k} = -i \left( \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right). \quad (10)$$

Expanding (9) for small arguments to the third order gives the energy and momentum operator in position representation:

$$\begin{aligned} E &= i\hbar \partial_t \left( 1 - \frac{\hbar^2}{M_f^2} \partial_t^2 \right), \\ \vec{p} &= \frac{\hbar}{i} \left[ \hat{r} \left( \partial_r - \frac{\hbar^2 \partial_r^3}{M_f^2} \right) + \hat{\theta} \left( \frac{\partial_\theta}{r} - \frac{\hbar^2 \partial_\theta^3}{r^3 M_f^2} \right) \right. \\ &\quad \left. + \hat{\phi} \left( \frac{\partial_\phi}{r \sin \theta} - \frac{\hbar^2 \partial_\phi^3}{r^3 \sin^3 \theta M_f^2} \right) \right], \end{aligned} \quad (11)$$

where we also omit the factor  $1/3$ . Substituting the above energy and momentum operators into the energy-momentum relation, the deformed Klein-Gordon equation satisfied by the scalar field with the mass  $m$  is

$$E^2 \phi = \vec{p}^2 \phi + m^2 \phi, \quad (12)$$

where  $p^2 = \vec{p} \cdot \vec{p}$ . Making the ansatz for  $\phi$

$$\phi = \exp\left(\frac{iI}{\hbar}\right), \quad (13)$$

and substituting it into (12), one expands (12) in powers of  $\hbar$  and then finds that the lowest order gives the deformed scalar Hamilton-Jacobi equation in the flat spacetime

$$\begin{aligned} (\partial_t I)^2 \left( 1 + \frac{2(\partial_t I)^2}{M_f^2} \right) - (\partial_r I)^2 \left( 1 + \frac{2(\partial_r I)^2}{M_f^2} \right) \\ - \frac{(\partial_\theta I)^2}{r^2} \left( 1 + \frac{2(\partial_\theta I)^2}{r^2 M_f^2} \right) \\ - \frac{(\partial_\phi I)^2}{r^2 \sin^2 \theta} \left( 1 + \frac{2(\partial_\phi I)^2}{r^2 \sin^2 \theta M_f^2} \right) = m^2, \end{aligned} \quad (14)$$

which is truncated at  $\mathcal{O}(1/M_f^2)$ .

Similarly, the deformed Dirac equation for a spin-1/2 fermion with the mass  $m$  takes the form of

$$(\gamma_0 E + \vec{\gamma} \cdot \vec{p} - m) \psi = 0, \quad (15)$$

where  $\{\gamma_0, \gamma_0\} = 2$ ,  $\{\gamma_a, \gamma_b\} = -2\delta_{ab}$ , and  $\{\gamma_0, \gamma_a\} = 0$  with the Latin index  $a$  running over  $r, \theta$ , and  $\phi$ . Multiplying  $(\gamma_0 E + \vec{\gamma} \cdot \vec{p} + m)$  by (15) and using the gamma matrices anticommutation relations, the deformed Dirac equation can be written as

$$E^2 \psi = (p^2 + m^2) \psi - \frac{[\gamma_a, \gamma_b]}{2} p_a p_b \psi. \quad (16)$$

To obtain the Hamilton-Jacobi equation for the fermion, the ansatz for  $\psi$  takes the form of

$$\psi = \exp\left(\frac{iI}{\hbar}\right) \nu, \quad (17)$$

where  $\nu$  is a vector function of the spacetime. Substituting (17) into (16) and noting that the second term on RHS of (16) does not contribute to the lowest order of  $\hbar$ , we find that the deformed Hamilton-Jacobi equation for a fermion is the same as the deformed one for a scalar with the same mass, namely, (14). Note that one can use the deformed Maxwell's equations obtained in [23] to get the deformed Hamilton-Jacobi equation for a vector boson. However, for simplicity we just stop here.

In order to generalize the deformed Hamilton-Jacobi equation, (14), to the curved spacetime with the metric (8), we first consider the Hamilton-Jacobi equation without GUP modifications. In [37], we show that the unmodified Hamilton-Jacobi equation in curved spacetime with  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$  is

$$g^{\mu\nu} \partial_\mu I \partial_\nu I - m^2 = 0. \quad (18)$$

Therefore, the unmodified Hamilton-Jacobi equation in the metric (8) becomes

$$\frac{(\partial_t I)^2}{f(r)} - f(r) (\partial_r I)^2 - \frac{(\partial_\theta I)^2}{r^2} - \frac{(\partial_\phi I)^2}{r^2 \sin^2 \theta} = m^2. \quad (19)$$

On the other hand, the unmodified Hamilton-Jacobi equation in flat spacetime can be obtained from (14) by taking  $M_f \rightarrow \infty$ ,

$$(\partial_t I)^2 - (\partial_r I)^2 - \frac{(\partial_\theta I)^2}{r^2} - \frac{(\partial_\phi I)^2}{r^2 \sin^2 \theta} = m^2. \quad (20)$$

Comparing (19) with (20), one finds that the Hamilton-Jacobi equation in the metric (8) can be obtained from the one in flat spacetime by making replacements  $\partial_r I \rightarrow \sqrt{f(r)} \partial_r I$  and  $\partial_t I \rightarrow \partial_t I / \sqrt{f(r)}$  in the no GUP modifications scenario. Therefore, by making replacements  $\partial_r I \rightarrow \sqrt{f(r)} \partial_r I$  and  $\partial_t I \rightarrow \partial_t I / \sqrt{f(r)}$ , the deformed Hamilton-Jacobi equation

in flat spacetime, (14), leads to the deformed Hamilton-Jacobi equation in the metric (8), which is to  $\mathcal{O}(1/M_f^2)$ ,

$$\begin{aligned} & \frac{(\partial_t I)^2}{f(r)} \left(1 + \frac{2(\partial_t I)^2}{f(r) M_f^2}\right) - f(r) (\partial_r I)^2 \left(1 + \frac{2f(r) (\partial_r I)^2}{M_f^2}\right) \\ & - \frac{(\partial_\theta I)^2}{r^2} \left(1 + \frac{2(\partial_\theta I)^2}{r^2 M_f^2}\right) \\ & - \frac{(\partial_\phi I)^2}{r^2 \sin^2 \theta} \left(1 + \frac{2(\partial_\phi I)^2}{r^2 \sin^2 \theta M_f^2}\right) = m^2. \end{aligned} \quad (21)$$

### 3. Quantum Tunnelling

In this section, we investigate the particles' tunneling at the event horizon  $r = r_h$  of the metric (8) where GUP is taken into account. Since metric (8) does not depend on  $t$  and  $\phi$ ,  $\partial_t$  and  $\partial_\phi$  are killing vectors. Taking into account the Killing vectors of the background spacetime, we can employ the following ansatz for the action:

$$I = -\omega t + W(r, \theta) + p_\phi \phi, \quad (22)$$

where  $\omega$  and  $p_\phi$  are constants and they are the energy and the  $z$ -component of angular momentum of emitted particles, respectively. Inserting (22) into (21), we find that the deformed Hamilton-Jacobi equation becomes

$$\begin{aligned} & p_r^2 \left(1 + \frac{2f(r) p_r^2}{M_f^2}\right) \\ & = \frac{1}{f^2(r)} \left[ \omega^2 \left(1 + \frac{2\omega^2}{f(r) M_f^2}\right) - f(r) (m^2 + \lambda) \right], \end{aligned} \quad (23)$$

where  $p_r = \partial_r W$ ,  $p_\theta = \partial_\theta W$ , and

$$\lambda = \frac{p_\theta^2}{r^2} \left(1 + \frac{2p_\theta^2}{r^2 M_f^2}\right) + \frac{p_\phi^2}{r^2 \sin^2 \theta} \left(1 + \frac{2p_\phi^2}{r^2 \sin^2 \theta M_f^2}\right). \quad (24)$$

Since the magnitude of the angular momentum of the particle  $L$  can be expressed in terms of  $p_\theta$  and  $p_\phi$ ,

$$L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}, \quad (25)$$

one can rewrite  $\lambda$  as

$$\lambda = \frac{L^2}{r^2} + \mathcal{O}\left(\frac{1}{M_f^2}\right). \quad (26)$$

Solving (23) for  $p_r$  to  $\mathcal{O}(1/M_f^2)$  gives

$$\begin{aligned} \partial_r W_{\mp} &= \pm \frac{1}{f(r)} \sqrt{\omega^2 \left( 1 + \frac{2\omega^2}{f(r)M_f^2} \right) - f(r)(m^2 + \lambda)} \\ &\times \sqrt{1 - \frac{2}{M_f^2 f(r)} \left[ \omega^2 \left( 1 + \frac{2\omega^2}{f(r)M_f^2} \right) - f(r)(m^2 + \lambda) \right]}, \end{aligned} \quad (27)$$

where  $+/-$  represent the outgoing/ingoing solutions. In order to get the imaginary part of  $W_{\pm}$ , we need to find residue of the RHS of (27) at  $r = r_h$  by expanding the RHS in a Laurent series with respect to  $r$  at  $r = r_h$ . We then rewrite (27) as

$$\begin{aligned} \partial_r W_{\mp} &= \pm \frac{1}{f(r)^{5/2}} \sqrt{\omega^2 \left( f(r) + \frac{2\omega^2}{M_f^2} \right) - f^2(r)(m^2 + \lambda)} \\ &\times \sqrt{f^2(r) - \frac{2}{M_f^2} \left[ \omega^2 \left( f(r) + \frac{2\omega^2}{M_f^2} \right) - f^2(r)(m^2 + \lambda) \right]}. \end{aligned} \quad (28)$$

Using  $f(r) = f'(r_h)(r - r_h) + (f''(r_h)/2)(r - r_h)^2 + \mathcal{O}((r - r_h)^3)$ , one can single out the  $1/(r - r_h)$  term of the Laurent series

$$\partial_r W_{\mp} \sim \frac{\pm a_{-1}}{r - r_h}, \quad (29)$$

where we have

$$a_{-1} = \frac{\omega}{f'(r_h)} \left[ 1 + \frac{2}{M_f^2} \left( m^2 + \frac{L^2}{r_h^2} \right) \right] + \mathcal{O}\left(\frac{1}{M_f^4}\right). \quad (30)$$

Using the residue theory for semicircles, we obtain for the imaginary part of  $W_{\pm}$  to  $\mathcal{O}(1/M_f^2)$

$$\text{Im } W_{\pm} = \pm \frac{\pi\omega}{f'(r_h)} \left[ 1 + \frac{2}{M_f^2} \left( m^2 + \frac{L^2}{r_h^2} \right) \right]. \quad (31)$$

However, when one tries to calculate the tunneling rate  $\Gamma$  from  $\text{Im } W_{\pm}$ , there is so called ‘‘factor-two problem’’ [38]. One way to solve the ‘‘factor-two problem’’ is introducing a ‘‘temporal contribution’’ [14–16, 39, 40]. To consider an invariance under canonical transformations, we also follow the recent work [14–16, 39, 40] and adopt the expression  $\oint p_r dr = \int p_r^+ dr - \int p_r^- dr$  for the spatial contribution to

$\Gamma$ . The spatial and temporal contributions to  $\Gamma$  are given as follows.

*Spatial Contribution.* The spatial part contribution comes from the imaginary part of  $W(r)$ . Thus, the spatial part contribution is proportional to

$$\begin{aligned} &\exp \left[ -\frac{1}{\hbar} \text{Im} \oint p_r dr \right] \\ &= \exp \left[ -\frac{1}{\hbar} \text{Im} \left( \int p_r^+ dr - \int p_r^- dr \right) \right] \\ &= \exp \left\{ -\frac{2\pi\omega}{f'(r_h)} \left[ 1 + \frac{2}{M_f^2} \left( m^2 + \frac{L^2}{r_h^2} \right) \right] \right\}. \end{aligned} \quad (32)$$

*Temporal Contribution.* As pointed in [15, 38–40], the temporal part contribution comes from the ‘‘rotation’’ which connects the interior region and the exterior region of the black hole. Thus, the imaginary contribution from the temporal part when crossing the horizon is  $\text{Im}(\omega\Delta t^{\text{out,in}}) = \omega(\pi/2\kappa)$ , where  $\kappa = f'(r_h)/2$  is the surface gravity at the event horizon. Then the total temporal contribution for a round trip is

$$\text{Im}(\omega\Delta t) = \frac{2\pi\omega}{f'(r_h)}. \quad (33)$$

Therefore, the tunnelling rate of the particle crossing the horizon is

$$\begin{aligned} \Gamma &\propto \exp \left[ -\frac{1}{\hbar} \left( \text{Im}(\omega\Delta t) + \text{Im} \oint p_r dr \right) \right] \\ &= \exp \left\{ -\frac{4\pi\omega}{f'(r_h)} \left[ 1 + \frac{1}{M_f^2} \left( m^2 + \frac{L^2}{r_h^2} \right) \right] \right\}. \end{aligned} \quad (34)$$

This is the expression of Boltzmann factor with an effective temperature

$$T = \frac{f'(r_h)}{4\pi} \frac{\hbar}{1 + \left(1/M_f^2\right) \left(m^2 + (L^2/r_h^2)\right)}, \quad (35)$$

where  $T_0 = \hbar f'(r_h)/4\pi$  is the original Hawking temperature.

For the standard Hawking radiation, all particles very close to the horizon are effectively massless on account of infinite blueshift. Thus, the conformal invariance of the horizon make Hawking temperatures of all particles the same. The mass, angular momentum, and identity of the particles are only relevant when they escape the potential barrier. However, if quantum gravity effects are considered, behaviors of particles near the horizon could be different. For example, if we send a wave packet which is governed by a subluminal dispersion relation backwards in time toward the horizon, it reaches a minimum distance of approach and then reverses direction and propagate back away from the horizon, instead of getting unlimited blueshift toward the horizon [41, 42]. Thus, quantum gravity effects might make fermions and scalars experience different (effective)

Hawking temperatures. However, our result shows that in our model, the tunnelling rates of fermions and scalars depend on their masses and angular momentums, but independent of the identities of the particles, to  $\mathcal{O}(1/M_f^2)$ . In other words, effective Hawking temperatures of fermions and scalars are the same to  $\mathcal{O}(1/M_f^2)$  in our model as long as their masses, energies, and angular momentums are the same.

#### 4. Thermodynamics of Black Holes

For simplicity, we consider the Schwarzschild metric with  $f(r) = 1 - 2M/r$  with the black hole's mass,  $M$ . The event horizon of the Schwarzschild black hole is  $r_h = 2M$ . In this section, we work with massless particles. Near the horizon of the the black hole, angular momentum of the particle  $L \sim pr_h \sim \omega r_h$ . Thus, one can rewrite  $T$

$$T \sim \frac{T_0}{1 + 2\omega^2/M_f^2}, \quad (36)$$

where  $T_0 = \hbar/8\pi M$  for the Schwarzschild black hole. As reported in [43], the authors obtained the relation  $\omega \geq \hbar/\delta x$  between the energy of a particle and its position uncertainty in the framework of GUP. Near the horizon of the the Schwarzschild black hole, the position uncertainty of a particle will be of the order of the Schwarzschild radius of the black hole [44]  $\delta x \sim r_h$ . Thus, one finds for  $T$

$$T \sim \frac{T_0}{1 + m_p^4/2M^2M_f^2}, \quad (37)$$

where we use  $\hbar = m_p^2$ . Using the first law of the black hole thermodynamics, we find that the corrected black hole entropy is

$$\begin{aligned} S &= \int \frac{dM}{T} \\ &\sim \frac{A}{4m_p^2} + \frac{4\pi m_p^2}{M_f^2} \ln\left(\frac{A}{16\pi}\right), \end{aligned} \quad (38)$$

where  $A = 4\pi r_h^2 = 16\pi M^2$  is the area of the horizon. The logarithmic term in (38) is the well known correction from quantum gravity to the classical Bekenstein-Hawking entropy, which has appeared in different studies of GUP modified thermodynamics of black holes [27–29, 45–51]. In general, the entropy for the Schwarzschild black hole of mass  $M$  in four spacetime dimensions can be written in form of

$$S = \frac{A}{4} + \sigma \ln\left(\frac{A}{16\pi}\right) + \mathcal{O}\left(\frac{M_f^2}{A}\right), \quad (39)$$

where  $\sigma = 2M_f^2/M^2$  in our paper. Neglecting the terms  $\mathcal{O}(M_f^2/A)$  in (39), there could be three scenarios depending on the sign of  $\sigma$ .

(1)  $\sigma = 0$ . This case is just the standard Hawking radiation. The black holes evaporate completely in finite time.

(2)  $\sigma < 0$ . The entropy  $S$  as function of mass develops a minimum at some value of  $M_{\min}$ . This predicts the existence of black hole remnants. Furthermore, the black holes stop evaporating in finite time. This is what happened in [27–29, 45–51], which is consistent with the existence of a minimal length.

(3)  $\sigma > 0$ . This is a subtle case. In the remainder of the section, we will investigate how the black holes evaporate in our model.

For particles emitted in a wave mode labelled by energy  $\omega$  and  $L$ , we find from (34) that [52]

(Probability for a black hole to emit

a particle in this mode)

$$= \exp\left(-\frac{\omega}{T}\right) \times (\text{Probability for a black hole to absorb$$

a particle in the same mode),

(40)

where  $T$  is given by (35). Neglecting back-reaction, detailed balance condition requires that the ratio of the probability of having  $N$  particles in a particular mode with  $\omega$  and  $L$  to the probability of having  $N - 1$  particles in the same mode is  $\exp(-\omega/T)$ . One then follows the standard textbook procedure to get the average number  $n_{\omega,L}$  in the mode

$$n_{\omega,L} = n\left(\frac{\omega}{T}\right), \quad (41)$$

where we define

$$n(x) = \frac{1}{\exp x - (-1)^\epsilon}, \quad (42)$$

and  $\epsilon = 0$  for bosons and  $\epsilon = 1$  for fermions. In [53], counting the number of modes per frequency interval with periodic boundary conditions in a large container around the black hole, Page related the expected number emitted per mode  $n_{\omega,L}$  to the average emission rate per frequency interval  $dn_{\omega,L}/dt$  by

$$\frac{dn_{\omega,L}}{dt} = n_{\omega,L} \frac{d\omega}{2\pi\hbar}. \quad (43)$$

Following Page's argument, we find that in our model

$$\frac{dn_{\omega,L}}{dt} = n_{\omega,L} \frac{\partial\omega}{\partial p_r} \frac{dp_r}{2\pi\hbar} = n_{\omega,L} \frac{d\omega}{2\pi\hbar}, \quad (44)$$

where  $\partial\omega/\partial p_r$  is the radial velocity of the particle and the number of modes between the wavevector interval  $(p_r, p_r + dp_r)$  is  $dp_r/2\pi\hbar$ , where  $p_r = \partial_r I$  is the radial wavevector. Since each particle carries off the energy  $\omega$ , the total luminosity is obtained from  $dn_{\omega,L}/dt$  by multiplying by the energy  $\omega$  and summing up over all energy  $\omega$  and  $L$ ,

$$L = \sum_{l=0} (2l+1) \int \omega n_{\omega,l} \frac{d\omega}{2\pi\hbar}, \quad (45)$$

where  $L^2 = (l+1)\hbar^2$  and the degeneracy for  $l$  is  $(2l+1)$ . However, some of the radiation emitted by the horizon might not be able to reach the asymptotic region. We need to consider the greybody factor  $|T_i(\omega)|^2$ , where  $T_i(\omega)$  represents the transmission coefficient of the black hole barrier which in general can depend on the energy  $\omega$  and angular momentum  $l$  of the particle. Taking the greybody factor into account, we find for the total luminosity

$$L = \sum_{l=0}^{\infty} (2l+1) \int |T_i(\omega)|^2 \omega n_{\omega,l} \frac{d\omega}{2\pi\hbar}. \quad (46)$$

Usually, one needs to solve the exact wave equations for  $|T_i(\omega)|^2$ , which is very complicated. On the other hand, one can use the geometric optics approximation to estimate  $|T_i(\omega)|^2$ . In the geometric optics approximation, we assume  $\omega \gg M$  and high energy waves will be absorbed unless they are aimed away from the black hole. Hence we have  $|T_i(\omega)|^2 = 1$  for all the classically allowed energy  $\omega$  and angular momentum  $l$  of the particle. In this approximation, the Schwarzschild black hole is just like a black sphere of radius  $R = 3^{3/2}M$  [54], which puts an upper bound on  $l(l+1)\hbar^2$ ,

$$l(l+1)\hbar^2 \leq 27M^2\omega^2. \quad (47)$$

Note that we neglect possible modifications from GUP to the bound since we are interested in the GUP effects near the horizon. Thus, the luminosity is

$$L = \int_0^{\infty} \frac{\omega d\omega}{2\pi\hbar^3} \int_0^{27M^2\omega^2} n \left[ \frac{\omega}{T_0} \left( 1 + \frac{1}{M_f^2} \frac{l(l+1)\hbar^2}{2M^2} \right) \right] \cdot d[l(l+1)\hbar^2] \quad (48)$$

$$= \frac{T_0^4 M^2}{2\pi\hbar^3} \int_0^{\infty} u^3 du \int_0^{27} n[u(1+a^2ux)] dx,$$

where we define  $u = \omega/T_0$ ,  $y = l(l+1)\hbar^2/M^2\omega^2$ , and  $a = T_0/\sqrt{2}M_f = m_p^2/8\sqrt{2}\pi M_f M$ . For  $M \gg m_p^2/8\sqrt{2}\pi M_f$ , we have  $a \ll 1$  and hence the luminosity is

$$L \approx \frac{27}{32\pi^2\hbar^3} T_0^4 A \int_0^{\infty} u^3 n(u) du, \quad (49)$$

which is just Stefan's law for black holes. Therefore for large black holes, they evaporate in almost the same way as in Case 1 until  $M \sim m_p^2/8\sqrt{2}\pi M_f$ , when the term  $a^2ux$  starts to dominate in (48). Then the luminosity is approximated by

$$\begin{aligned} L &\sim \frac{T_0^4 M^2}{2\pi\hbar^3} \int_0^{\infty} u^3 du \int_0^{27} n(a^2u^2x) dx \\ &= \frac{2M^2 M_f^4}{\pi m_p^6} \int_0^{\infty} v^3 dv \int_0^{27} n(v^2x) dx, \end{aligned} \quad (50)$$

where  $v = au$ . Not worrying about exact numerical factors, one has for the evaporation rate

$$\frac{dM}{dt} = -L \sim -A \frac{M^2}{M_f^2}, \quad (51)$$

where  $A > 0$  is a constant. Solving (51) for  $M$  gives  $M \sim M_f^2/At$ . The evaporation rate considerably slows down when black holes' mass  $M \sim m_p^2/8\sqrt{2}\pi M_f$ . The black hole then evaporates to zero mass in infinite time. However, the GUP predicts the existence of a minimal length. It would make much more sense if there are black hole remnants in the GUP models. How can we reconcile the contradiction? When we write down the deformed Hamilton-Jacobi equation, (21), we neglect terms higher than  $\mathcal{O}(1/M_f^2)$ . However, when  $M \sim m_p^2/8\sqrt{2}\pi M_f$ , our effective approach starts breaking down since contributions from higher order terms become the same important as these from terms  $\mathcal{O}(1/M_f^2)$ . Thus, one then has to include these higher order contributions. For example, if there are higher order corrections to (51) as in

$$\frac{dM}{dt} \sim -A \frac{M^2}{M_f^2} + B \frac{M^3}{M_f^3}, \quad (52)$$

where  $B > 0$ , one can easily see that there exists a minimum mass  $M_{\min} \sim M_f \sqrt{A/B}$  for black holes. In another word, the  $\mathcal{O}(1/M_f^2)$  terms used in our paper are not sufficient enough to produce black hole remnants predicted by the GUP. To do so, one needs to resort to higher order terms if the full theory is not available. It is also noted that when at late stage of a black hole with  $a \geq 1$ , (36) becomes

$$T \sim \frac{T_0}{m_p^4/2M^2 M_f^2} \sim M, \quad (53)$$

which means that the temperature of the black hole goes to zero as the mass goes to zero.

The GUP is closely related to noncommutative geometry. In fact, when the GUP is investigated in more than one dimension, a noncommutative geometric generalization of position space always appears naturally [22]. On the other hand, quantum black hole physics has been studied in the noncommutative geometry [55]. In [56], a noncommutative black hole's entropy received a logarithmic correction with  $\sigma < 0$ . However, [57, 58] showed that the corrections to a noncommutative schwarzschild black hole's entropy might not involve any logarithmic terms. In either case, the temperature of the noncommutative schwarzschild black hole reaches zero in finite time with remnants left.

## 5. Conclusion

In this paper, incorporating effects of the minimal length, we derived the deformed Hamilton-Jacobi equations for both scalars and fermions in curved spacetime based on the modified fundamental commutation relations. We investigated the particles' tunneling in the background of a spherically symmetric black hole. In this spacetime configurations, we showed that the corrected Hawking temperature is not only determined by the properties of the black holes, but also dependent on the angular momentum and mass of the emitted particles. Finally, we studied how a Schwarzschild black hole evaporates in our model. We found that the black hole evaporates to zero mass in infinite time.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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