

## Research Article

# Aharonov-Bohm Effect for Bound States on the Confinement of a Relativistic Scalar Particle to a Coulomb-Type Potential in Kaluza-Klein Theory

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Based on the Kaluza-Klein theory, we study the Aharonov-Bohm effect for bound states for a relativistic scalar particle subject to a Coulomb-type potential. We introduce this scalar potential as a modification of the mass term of the Klein-Gordon equation, and a magnetic flux through the line element of the Minkowski spacetime in five dimensions. Then, we obtain the relativistic bound states solutions and calculate the persistent currents.

## 1. Introduction

The impact of Einstein's work on the new geometric vision has changed the way of describing the possible symmetries that should lead us to the physical laws. As a consequence of this new approach, gravity was explained by a geometrical perspective. The success of this description of the gravitational interaction inspired Kaluza [1] in 1921, and thereafter Klein [2] in 1926, to create the first proposal of unifying two well-known interactions: gravitation and electromagnetism. This new proposal establishes that the electromagnetism can be introduced through an extra (compactified) dimension in the spacetime, where the spatial dimension becomes five-dimensional. In the original version of the theory there appeared a certain inconsistency that was later removed by Leibowitz and Rosen [3]. The idea behind introducing additional spacetime dimensions has found wide applications in quantum field theory, for instance, in string theory, which is consistent in a space with extra dimensions [4]. The Kaluza-Klein theory has also been investigated in the presence of torsion [5, 6], with fermions [7–9], and in studies of

the violation of the Lorentz symmetry [10–12]. In recent decades, generalizations of topological defect spacetimes have been found in the context of the Kaluza-Klein theory, for example, the magnetic cosmic string [13] and magnetic chiral cosmic string [14], both in five dimensions. Based on these generalizations of topological defect spacetimes in the Kaluza-Klein theory, the Aharonov-Bohm effect for bound states [15] has been investigated in [16] and geometric quantum phases have been discussed in graphene [17].

The aim of this work is to investigate the Aharonov-Bohm effect for bound states [15, 16] for a relativistic scalar particle subject to a Coulomb-type potential in the Kaluza-Klein theory [1, 2, 4, 7]. By using the Kaluza-Klein theory [1, 2, 4, 7], we introduce a magnetic flux through the line element of the Minkowski spacetime and thus write the Klein-Gordon equation in the five-dimensional spacetime. Besides, we introduce a Coulomb-type potential as a modification in the mass term of the Klein-Gordon equation; then, we show that the relativistic bound states solutions can be achieved, where the relativistic energy levels depend on the Aharonov-Bohm geometric quantum phase [18]. Due to this dependence of

the relativistic energy levels on the geometric quantum phase, we calculate the persistent currents [19, 20] that arise in the relativistic system.

The structure of this paper is as follows: in Section 2, we introduce the scalar potential by modifying the mass term of the Klein-Gordon equation and a magnetic flux through the Kaluza-Klein theory; thus, we analyse an Aharonov-Bohm effect for bound states and the arising of persistent currents in this relativistic quantum system; in Section 4, we present our conclusions.

## 2. Aharonov-Bohm Effect for Bound States

In this section, we analyse the Aharonov-Bohm effect for bound states [15, 16] for a relativistic scalar particle subject to a Coulomb-type potential in the Kaluza-Klein theory [1, 2, 4, 7]. It has been reported in the literature that Coulomb-type potentials have interest in several areas of physics [21–23]; for instance, in the context of condensed matter physics, studies have worked with 1-dimensional systems [24–28], molecules [29–31], pseudo-harmonic interactions [23, 32], position-dependent mass systems [33–35], the Kratzer potential [36–38], and topological defects in solids [39–43]. Other studies have dealt with Coulomb-type potential in the propagation of gravitational waves [44], quark models [45], atoms with magnetic quadrupole moment [46], neutral particle with permanent magnetic dipole moment [47], and relativistic quantum mechanics [48–51]. From this perspective, let us write the Coulomb-type potential as

$$V(\rho) = \frac{\alpha}{\rho}, \quad (1)$$

where  $\alpha$  is a constant that characterizes the scalar potential and  $\rho = \sqrt{x^2 + y^2}$  is the radial coordinate. Here, we follow [52, 53] and introduce a Coulomb-type potential by modifying the mass term in the form,  $m \rightarrow m + V(\vec{r})$ , where  $V(\vec{r})$  is the scalar potential. This procedure in modifying the mass term of the Klein-Gordon equation has been explored in studies of the quark-antiquark interaction [54], in the analysis of the behaviour of a Dirac particle in the presence of a static scalar potential and a Coulomb potential [55], in a relativistic scalar particle in the cosmic string spacetime [48], and in the Klein-Gordon oscillator [51].

On the other hand, the idea behind the Kaluza-Klein theory [1, 2, 4, 7] is that the spacetime is five-dimensional with the purpose of unifying gravitation and electromagnetism. In this way, we can work with general relativity in five dimensions, that is, in  $(4 + 1)$ -dimensions. The information about the electromagnetism is given by introducing a gauge field  $A_\mu(x)$  in the line element of the spacetime as

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu - \left(dx^5 + \kappa A_\mu(x) dx^\mu\right)^2, \quad (2)$$

where  $\kappa$  is the Kaluza constant [14], the Greek indices are the spacetime indices in  $(3 + 1)$ -dimensions,  $g_{\mu\nu}(x)$  is the usual spacetime metric tensor in  $(3 + 1)$ -dimensions, and  $x^5 = y$  corresponds to the extra dimension. Here, we wish to investigate the Aharonov-Bohm effect for bound states [15, 16];

then, by following [14, 16], we can introduce a magnetic flux through the line element of the Minkowski spacetime in the form (by working with the units  $c = \hbar = 1$ )

$$ds^2 = dt^2 - d\rho^2 - \rho^2 d\varphi^2 - dz^2 - \left(dy + \frac{\phi}{2\pi} d\varphi\right)^2, \quad (3)$$

where  $\phi$  is the magnetic flux and the vector potential is given by the component  $A_\varphi = \phi/2\pi\kappa$ , which gives rise to a magnetic field  $\vec{B} = (\phi/\kappa)\delta^2(\vec{r})$  [14]. In this five-dimensional spacetime, the Klein-Gordon equation is written in the form

$$\frac{1}{\sqrt{-g}} \partial_A (g^{AB} \sqrt{-g} \partial_B) \Phi + (m + V(\rho))^2 \Phi = 0, \quad (4)$$

where  $A, B = t, \rho, \varphi, z, y$ . From (1) and (3), the Klein-Gordon equation (4) becomes

$$-\frac{\partial^2 \Phi}{\partial t^2} = \left(m + \frac{\alpha}{\rho}\right)^2 \Phi - \frac{\partial^2 \Phi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} - \frac{1}{\rho^2} \left(\frac{\partial}{\partial \varphi} - \frac{\phi}{2\pi} \frac{\partial}{\partial y}\right)^2 \Phi - \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial z^2}. \quad (5)$$

Observe that the quantum operators  $\hat{L}_z = -i\partial_\varphi$ ,  $\hat{p}_z = -i\partial_z$ , and  $\hat{p}_y = -i\partial_y$ , commute with the Hamiltonian operator given in right-hand side of (5). Thereby, a particular solution to the Klein-Gordon equation (5) can be written in terms of the eigenfunctions of these operators as

$$\Phi(t, \rho, \varphi, z, y) = e^{-i\mathcal{E}t} e^{il\varphi} e^{ikz} e^{iqy} g(\rho), \quad (6)$$

where  $l = 0, \pm 1, \pm 2, \dots, k$  and  $q$  are constants, and  $g(\rho)$  is a function of the radial coordinate. By substituting (6) into (5) we have

$$\frac{d^2 g}{d\rho^2} + \frac{1}{\rho} \frac{dg}{d\rho} - \frac{\gamma^2}{\rho^2} g - \frac{2m\alpha}{\rho} g + \beta^2 g = 0, \quad (7)$$

where we have defined the parameters

$$\beta^2 = \mathcal{E}^2 - m^2 - k^2 - q^2; \quad (8)$$

$$\gamma^2 = \left(l - \frac{\phi}{2\pi} q\right)^2 + \alpha^2.$$

Let us now discuss the asymptotic behaviour of the possible solutions to (7). For  $\rho \rightarrow \infty$ , we can obtain scattering states where the radial wave function behaves as  $g \cong e^{i\beta\rho}$ . On the other hand, bound states can be obtained if the radial wave function behaves as  $g \cong e^{-\nu\rho}$  [56]. Since our focus is on the bound state solutions, then, we consider  $\beta^2 = -\nu^2$  in (7). By replacing  $\beta^2$  with  $-\nu^2$  in (7), we also perform a change of variables given by  $\xi = 2\nu\rho$ ; thus, we have

$$\frac{d^2 g}{d\xi^2} + \frac{1}{\xi} \frac{dg}{d\xi} - \frac{\gamma^2}{\xi^2} g - \frac{m\alpha}{\nu\xi} g - \frac{1}{4} g = 0. \quad (9)$$

Moreover, a particular solution to (9) which is regular at the origin can be written as  $g(\xi) = e^{-\xi/2} \xi^{|\gamma|} f(\xi)$ , where  $f(\xi)$  is

an unknown function. Thereby, by substituting this solution into (9), we obtain a second-order differential equation given by

$$\xi \frac{d^2 f}{d\xi^2} + [2|\gamma| + 1 - \xi] \frac{df}{d\xi} + \left(-|\gamma| - \frac{1}{2} - \frac{m\alpha}{\nu}\right) f = 0, \quad (10)$$

which corresponds to the Kummer equation or the confluent hypergeometric equation [57, 58]. Hence, the function  $f(\xi)$  is the confluent hypergeometric function:

$$f(\xi) = {}_1F_1\left(|\gamma| + \frac{1}{2} + \frac{m\alpha}{\nu}, 2|\gamma| + 1, \xi\right). \quad (11)$$

Besides, by analyzing the asymptotic behavior for  $\xi \rightarrow \infty$ , we have that the confluent hypergeometric function diverges for large values of  $\xi$  [57, 58]. In order that the wave function can vanish at  $\xi \rightarrow \infty$ , we must impose the condition that the confluent hypergeometric series becomes a polynomial of degree  $n$  ( $n = 0, 1, 2, \dots$ ). This occurs when  $|\gamma| + 1/2 + m\eta/\nu = -n$ . In this way, by using (8), we have

$$\mathcal{E}_{n,l} = \pm m \sqrt{1 - \frac{\alpha^2}{[n + |\gamma| + 1/2]^2} + \frac{k^2}{m^2} + \frac{q^2}{m^2}}, \quad (12)$$

where  $n = 0, 1, 2, 3, \dots$  is the quantum number associated with the radial modes,  $l = 0, \pm 1, \pm 2, \dots$  is the quantum number associated with the angular momentum, and  $k$  and  $q$  are constants. Equation (12) corresponds to the relativistic energy levels for a scalar particle subject to a Coulomb-type potential in the Kaluza-Klein theory. Observe that, due to the presence of the parameter  $\gamma$  in (12), we have that the energy levels depend on the Aharonov-Bohm geometric quantum phase [18], where it has a periodicity  $\phi_0 = -2\pi/q$ ; thus, we have that  $\mathcal{E}_{n,l}(\phi + \phi_0) = \mathcal{E}_{n,l+1}(\phi)$ . For  $\phi_0 = +2\pi/q$ , we have that  $\mathcal{E}_{n,l}(\phi + \phi_0) = \mathcal{E}_{n,l-1}(\phi)$ . This dependence of the relativistic energy levels on the geometric quantum phase  $\phi$  corresponds to an Aharonov-Bohm effect for bound states [15, 16].

Furthermore, it is well-known in condensed matter systems [19, 20, 59–61] and in relativistic quantum systems [62, 63] that when there exists dependence of the energy levels on the geometric quantum phase, then, persistent current can arise in the system. Studies of persistent currents have explored systems that deal with the Berry phase [64, 65], the Aharonov-Anandan quantum phase [66, 67], and the Aharonov-Casher geometric quantum phase [68–71]. By following [19, 20, 59], the expression for the total persistent currents is given by  $\mathcal{I} = \sum_{n,l} \mathcal{I}_{n,l}$ , where  $\mathcal{I}_{n,l} = -\partial \mathcal{E}_{n,l} / \partial \phi$  is called the Byers-Yang relation [19]. Therefore, the persistent current that arises in this relativistic system is given by

$$\mathcal{I}_{n,l} = -\frac{\partial \mathcal{E}_{n,l}}{\partial \phi} = \mp \frac{q m \alpha^2}{\pi |\gamma|} \frac{(l - (\phi/2\pi)q)}{[n + |\gamma| + 1/2]^3} \cdot \frac{1}{\sqrt{1 - \alpha^2/[n + |\gamma| + 1/2]^2 + k^2/m^2 + q^2/m^2}}. \quad (13)$$

Hence, we have that relativistic bound states solutions to the Klein-Gordon equation can be obtained for a scalar particle confined to a Coulomb-type potential in the Kaluza-Klein theory. By introducing a magnetic flux through the line element of the Minkowski spacetime in five dimensions, we have seen that the relativistic energy levels (12) depend on the geometric quantum phase which gives rise to an Aharonov-Bohm effect for bound states in Kaluza-Klein theory [15, 16]. Moreover, this dependence of the relativistic energy levels on the geometric quantum phase has yielded persistent currents in the relativistic quantum system.

### 3. Nonrelativistic Limit

Let us now discuss the nonrelativistic limit of the Klein-Gordon equation (5). The motivation for discussing this nonrelativistic limit comes from the studies of the violation of the Lorentz symmetry through the Kaluza-Klein theory [10–12]. A great deal of work has investigated and established bounds for the parameters associated with the violation of the Lorentz symmetry at low energy scenarios in recent decades [72]. Interesting examples are the hydrogen atom [73], Weyl semimetals [74] and on the Rashba coupling [75], geometric quantum phases [76], and the quantum Hall effect [77]. Hence, investigating Aharonov-Bohm-type effects [15, 18, 72] through Kaluza-Klein theories at low energy scenarios can be important for studies of the violation of the Lorentz symmetry since it could be a way of establishing bounds for the parameters associated with the Lorentz symmetry breaking effects. Thereby, let us analyse the behaviour of the present system at a low energy regime. By following [14, 52], the wave function can be written in the form  $\Phi(t, \rho, \varphi, z, y) = e^{-imt} \psi(t, \rho, \varphi, z, y)$ ; thus, by assuming that  $|i(\partial\psi/\partial t)| \ll m$  and by substituting this ansatz into (5), we obtain

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \left[ \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \left( \frac{\partial}{\partial \varphi} + \frac{\phi}{2\pi} \frac{\partial}{\partial y} \right)^2 \psi + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right] + \frac{\alpha}{\rho} \psi + \frac{\alpha^2}{2m\rho^2} \psi. \quad (14)$$

Performing the steps from (6) to (11), we have

$$\mathcal{E}_{n,l}^{\text{NR}} = -\frac{m\alpha^2}{2[n + |\gamma| + 1/2]^2} + \frac{k^2}{2m} + \frac{q^2}{2m}. \quad (15)$$

Equation (15) corresponds to the nonrelativistic energy levels for a spinless particle confined to the Coulomb-type potential in Kaluza-Klein theory. Note that the nonrelativistic energy levels (15) depend on the Aharonov-Bohm geometric quantum phase, whose periodicity is  $\phi_0 = \mp 2\pi/q$ ; thus, we have that  $\mathcal{E}_{n,l}(\phi \pm \phi_0) = \mathcal{E}_{n,l \pm 1}(\phi)$ , and the persistent currents are given by

$$\mathcal{I}_{n,l}^{\text{NR}} = -\frac{q}{\pi} \frac{\gamma}{|\gamma|} \left( l - \frac{q}{2\pi} \right) \frac{m\alpha^2}{[n + |\gamma| + 1/2]^3}. \quad (16)$$

In view of the introduction of the scalar potential by modifying the mass term of the Klein-Gordon equation,

then, the mass term can be considered to be an effective position-dependent mass [52]:  $m(\rho) = m + V(\rho)$ . Position-dependent mass systems, Coulomb-type potentials, and the Aharonov-Bohm effect for bound states have a great interest in condensed matter physics [15, 29–31, 78–81]; thereby the analysis of the behaviour of the present system at a low energy regime can open new discussions about quantum effects on nonrelativistic systems by analogy, for instance, with position-dependent mass systems.

#### 4. Conclusions

We have investigated relativistic quantum effects on a scalar particle subject to a Coulomb-type potential due to the presence of the Aharonov-Bohm geometric quantum [18] which is introduced in the system through an extra dimension of the spacetime. Thereby, we have shown that relativistic bound state solutions can be achieved, where the relativistic energy levels depend on the Aharonov-Bohm quantum phase. This dependence of the relativistic energy levels on the geometric quantum phase corresponds to an Aharonov-Bohm effect for bound states [15, 16] and gives rise to the appearance of persistent currents in this relativistic quantum system.

Another interesting context in which the Aharonov-Bohm effect for bound states in Kaluza-Klein theory can be explored is based on topological defect spacetime. In [13] it is shown that the cosmic string spacetime and the magnetic cosmic string spacetime can have analogues in five dimensions. Besides, in [14] a five-dimensional chiral cosmic string is shown. From this perspective, interesting discussions about Aharonov-Bohm effects for bound states and persistent currents can be made from the confinement of a relativistic scalar particle to a Coulomb-type potential with the background of the Kaluza-Klein magnetic cosmic string spacetime [13] and the Kaluza-Klein magnetic chiral cosmic string [14]. We hope to present these discussions in the near future.

#### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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