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Research Article

Relationship between Fujikawa's Method and the Background Field Method for the Scale Anomaly

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We show the equivalence between Fujikawa's method for calculating the scale anomaly and the diagrammatic approach to calculating the effective potential via the background field method, for an $O(N)$ symmetric scalar field theory. Fujikawa's method leads to a sum of terms, each one superficially in one-to-one correspondence with a vacuum diagram of the 1-loop expansion. From the viewpoint of the classical action, the anomaly results in a breakdown of the Ward identities due to scale-dependence of the couplings, whereas, in terms of the effective action, the anomaly is the result of the breakdown of Noether's theorem due to explicit symmetry breaking terms of the effective potential.

1. Introduction

Fujikawa showed that, within the path integral formalism, all anomalies are the result of noninvariance of the measure under symmetry transformations [1–3]. The resulting Jacobian then spoils the naive Ward identities. It is also known that the quantum effective action preserves the symmetries of the classical action, provided that the measure is invariant under the symmetry transformations [4]. Therefore, there should be a relationship between Fujikawa's method and the noninvariant terms of the quantum effective action. We investigate this relationship in the context of $O(N)$, $\lambda\phi^4$ theory, by comparing, term by term, the Taylor expansion of the Fujikawa determinant with all diagrams in the 1-loop expansion of the quantum effective potential.

The reason for embarking on this comparison is that a framework for applying Fujikawa's method to nonrelativistic, classically scale-invariant systems was undertaken recently [5–7]. While the quantum effective action is a standard tool in nonrelativistic physics (e.g., see [8, 9]), Fujikawa's method is not. Therefore a comparison of the two approaches, without a coupling to a gravitational background as is done for the relativisitic case, might be helpful in a first approximation as a bridge between the two methods in the context of nonrelativistic physics.

It is well known that for the chiral anomaly the choice of regulating function $f(\not{D}^2/\Lambda^2)$ one uses to regulate the Jacobian is arbitrary, except for a few conditions governing the behavior of f and its derivatives at 0 and ∞ that are quite reasonable [10]. The argument of the regulating function however is not arbitrary—one must choose the gauge invariant $\not{D}$. The anomaly calculated in this manner is both finite and exact.

For the scale anomaly things are not as clear. There is no symmetry that tells you what variable must go into the regulating function. Moreover, if one Taylor expands the anomaly as one does in the chiral case, certain terms are infinite. If one ignores those terms, then one can recover the anomaly, but it is not exact, holding only to 1-loop order. One generally chooses the quadratic part of effective action for the argument since it characterizes 1-loop effects [11].

In this paper we attempt to explore the connection between certain terms in the effective potential when it is expanded by the number of vertices and certain terms in the Jacobian of Fujikawa's method when it is Taylor expanded, thereby clarifying the statement that putting the quadratic part of the effective action in the regulating function captures the 1-loop effects. Also, we consider $O(N)$ as opposed to a single scalar field because, despite the problems of Fujikawa's method for the case of the scale anomaly compared to the chiral anomaly, such as only capturing the 1-loop result, it still retains a universal quality in that it can capture the 1-loop result for any N.

In Sections 2 and 3, we give a quick review of Fujikawa's method and the background field method for calculating the effective action. In Section 4 we apply Fujikawa's method to calculate the anomaly and the β function of N scalar fields interacting via an $O(N)$ symmetric $\lambda\phi^4$ potential. In Section 5 we use the background field method to write an expression for the effective potential, organized by the number of vertices, and compare this result with the Taylor expansion resulting from Fujikawa's method to derive conditions on the Fujikawa regulator for the two approaches to give the same result. Finally, in the sixth section, we apply Noether's theorem to the effective action and compare it to anomalous scale-breaking of the classical action.

2. Fujikawa's Method

For simplicity we will demonstrate this method for a single scalar field: the generalization to multiple fields is straightforward. With a change of variables given by $\phi'(x) = \phi(x) + \epsilon\delta\phi(x)$,

$$\begin{aligned}
\int [d\phi]\, e^{iS[\phi]} &= \int [d\phi'] \left|\frac{\delta\phi}{\delta\phi'}\right| e^{iS[\phi(\phi')]} \\
&= \int [d\phi'] \left|\delta^d(x-y) - \epsilon\frac{\delta\delta\phi'(x)}{\delta\phi'(y)}\right| e^{iS[\phi'-\epsilon\delta\phi']} \\
&= \int [d\phi] \left|\delta^d(x-y) - \epsilon\frac{\delta\delta\phi(x)}{\delta\phi(y)}\right| e^{iS[\phi-\epsilon\delta\phi]} \\
&= \int [d\phi]\, e^{-\epsilon\int d^dx(\delta\delta\phi/\delta\phi)} e^{iS[\phi]} e^{-i\epsilon\int d^dx(\delta S/\delta\phi)\delta\phi} \\
&= \int [d\phi]\, e^{iS[\phi]} \left(1 - \epsilon\int d^dx \frac{\delta\delta\phi}{\delta\phi} - i\epsilon\int d^dx\frac{\delta S}{\delta\phi}\delta\phi\right).
\end{aligned} \tag{1}$$

Since this holds for any volume V, it follows that

$$\left\langle \frac{\delta S}{\delta\phi}\delta\phi\right\rangle = i\left\langle \left.\frac{\delta\delta\phi(x)}{\delta\phi(y)}\right|_{y=x}\right\rangle. \tag{2}$$

If $\phi \rightarrow \phi + \epsilon\delta\phi$ is a symmetry transformation, then $(\delta S/\delta\phi)\delta\phi = -\partial_\mu j^\mu$, so that Fujikawa's method tells us that

$$\left\langle \partial_\mu j^\mu\right\rangle = -i\left\langle \left.\frac{\delta\delta\phi(x)}{\delta\phi(y)}\right|_{y=x}\right\rangle. \tag{3}$$

The transformations we are interested in are dilations for N scalar fields:

$$\begin{aligned}
x'^\mu &= e^{-\rho}x^\mu \\
\phi_i'(x') &= e^{\rho}\phi_i(x),
\end{aligned} \tag{4}$$

so that the Jacobian is

$$\begin{aligned}
J = \frac{\delta\delta\phi_i(x)}{\delta\phi_j(y)} &= \left(1 + x^\mu\partial_\mu\right)\delta^4(x-y)\, I_n \\
&\equiv \theta\delta^4(x-y)\, I_n,
\end{aligned} \tag{5}$$

where I_n is the N-dimensional identity matrix and $\theta = (1 + x^\mu\partial_\mu)$.

3. Background Field Method

We briefly review some facts about the effective action. The generating functional $W[J]$ for the connected correlation functions can be expressed via the path integral as

$$e^{iW[J]} = \int [d\phi]\, e^{iS[\phi]+i\int J\phi}. \tag{6}$$

The effective action is defined as the Legendre transform:

$$\begin{aligned}
\Gamma[\phi_c] &= W[J(\phi_c)] - \int J(\phi_c)\,\phi_c, \\
\phi_c &= \frac{\delta W}{\delta J} = \langle\phi\rangle_J.
\end{aligned} \tag{7}$$

$\Gamma[\phi_c]$ obeys the classical equations of motion:

$$\frac{\delta\Gamma}{\delta\phi_c} = -J, \tag{8}$$

and it can be expanded as

$$\begin{aligned}
\Gamma[\phi_c] &= \sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\cdots dx_n G_{1\mathrm{PI}}^{(n)}(x_1,\ldots,x_n) \\
&\cdot \phi_c(x_1)\cdots\phi_c(x_n) = \int dx\left(-V_{\mathrm{eff}}(\phi_c) + \frac{1}{2}Z(\phi_c)\right. \\
&\left.\cdot \partial_\mu\phi_c\partial^\mu\phi_c + \cdots\right),
\end{aligned} \tag{9}$$

which shows that $\Gamma[\phi_c]$ is the generating functional for the 1PI graphs and that the effective potential V_{eff} is the negative sum of all 1PI graphs with all external lines set to 0 momentum.

In the background field method (for a review of the background field method, see [12]), we define a new generating functional $\widetilde{W}[J]$:

$$\begin{aligned}
e^{i\widetilde{W}[J]} &= \int [d\phi]\, e^{iS[\phi+\widehat{\phi}]+i\int J\phi} = \int [d\phi]\, e^{iS[\phi]+i\int J(\phi-\widehat{\phi})} \\
&= e^{iW[J]}e^{-iJ\widehat{\phi}}.
\end{aligned} \tag{10}$$

Application of (7) to $\widetilde{W}[J]$ then gives the following relationships:

$$\begin{aligned}
\widetilde{W}[J] &= W[J] - J\widehat{\phi}, \\
\widetilde{\phi}_c &= \phi_c - \widehat{\phi}, \\
\widetilde{\Gamma}\left[\widetilde{\phi}_c, \widehat{\phi}\right] &= \Gamma\left[\widetilde{\phi}_c + \widehat{\phi}\right].
\end{aligned} \tag{11}$$

Setting $\widetilde{\phi}_c = 0$ for the effective action then gives us the result we will need:

$$\Gamma\left[\widehat{\phi}\right] = \widetilde{\Gamma}\left[0, \widehat{\phi}\right], \tag{12}$$

which states that, to calculate the effective action $\Gamma[\widehat{\phi}]$ associated with the classical action $S[\widehat{\phi}]$, we need only to calculate the 1PI vacuum graphs associated with the classical action $S[\phi + \widehat{\phi}]$, that is, the original action shifted by a background $\widehat{\phi}$. In the following section we will relabel ϕ in $S[\phi + \widehat{\phi}]$ as η.

4. Fujikawa Calculation

Consider the conformally invariant Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - \frac{\lambda}{4}\left(\phi_i\phi_i\right)^2, \tag{13}$$

where repeated indices are summed and $i = 1, 2, \ldots, N$. The quadratic part of the action S expanded around the constant background fields $\widehat{\phi}_i$ ($\phi_i = \widehat{\phi}_i + \eta_i$) is given by

$$\widetilde{S}_2 = \frac{1}{2}\sum_{i,j=1}^{N} \int d^4x d^4y \frac{\delta^2 S}{\delta\phi_j(x)\,\delta\phi_i(y)} \eta_j(x)\,\eta_i(y), \tag{14}$$

which can be reexpressed in terms of the Lagrangian:

$$\begin{aligned}\widetilde{S}_2 = \frac{1}{2}\sum_{i,j=1}^{N} \int d^4x \Bigg(& \frac{\partial^2 \mathscr{L}}{\partial\phi_i\partial\phi_j}\eta_i(x)\,\eta_j(x) \\ & + 2\frac{\partial^2 \mathscr{L}}{\partial\phi_i \partial\partial_\mu\phi_j}\eta_i(x)\,\partial_\mu\eta_j(x) \\ & + \frac{\partial^2 \mathscr{L}}{\partial\partial_\nu\phi_i\partial\partial_\mu\phi_j}\partial_\nu\eta_i(x)\,\partial_\mu\eta_j(x)\Bigg).\end{aligned} \tag{15}$$

Plugging (13) into (15) gives

$$\begin{aligned}\widetilde{S}_2 &= \frac{1}{2} \\ &\cdot \sum_{i,j=1}^{N}\int d^4x \left(\left[-2\lambda\widehat{\phi}_i\widehat{\phi}_j - \lambda\left(\widehat{\phi}_k\widehat{\phi}_k\right)\delta_{ij}\right]\eta_i(x)\,\eta_j(x)\right. \\ &\left. + \partial_\mu\eta_i(x)\,\partial^\mu\eta_i(x)\right) = \frac{1}{2}\sum_{i,j=1}^{N}\int d^4x \eta_i(x)\left(B_{ij} + D_{ij}\right) \\ &\cdot \eta_j(x) = \frac{1}{2}\sum_{i,j=1}^{N}\int d^4x \eta_i(x)\, M_{ij}\eta_j(x),\end{aligned} \tag{16}$$

where

$$\begin{aligned}D_{ij} &= -\delta_{ij}\partial^2, \\ B_{ij} &= \left[-2\lambda\widehat{\phi}_i\widehat{\phi}_j - \lambda\left(\widehat{\phi}_k\widehat{\phi}_k\right)\delta_{ij}\right].\end{aligned} \tag{17}$$

We choose M_{ij} as the argument of our regulating matrix so that

$$\mathscr{A} = \operatorname{tr}\left[R\left(\frac{M}{\Lambda^2}\right)\theta\delta^4(x-y)\, I_n\right]\Bigg|_{x=y}. \tag{18}$$

Going into Fourier space,

$$\begin{aligned}\mathscr{A} &= \operatorname{tr}\int\frac{d^4k}{(2\pi)^4}\left[R\left(\frac{M}{\Lambda^2}\right)\theta e^{ik\cdot(x-y)}I_n\right]\Bigg|_{x=y} \\ &= \operatorname{tr}\int\frac{d^4k}{(2\pi)^4}\left[R\left(\frac{M}{\Lambda^2}\right)\left(1 + x_\mu k_\mu\right)I_n\right] \\ &= \Lambda^4\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}\left[R\left(D + \frac{B}{\Lambda^2}\right)I_n\right],\end{aligned} \tag{19}$$

where in the second line y has been set equal to x and $D_{ij} = -\delta_{ij}\partial^2 \rightarrow \delta_{ij}k^2$. D_{ij} is even in k^2; therefore the $x_\mu k_\mu$ term vanishes upon integration. Since $[D, B] = 0$, $R(D + B/\Lambda^2)$ admits a power series expansion about D:

$$\begin{aligned}\mathscr{A} = \Lambda^4\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}\Bigg[& R(D) + R'(D)\frac{B}{\Lambda^2} \\ & + \frac{1}{2!}R''(D)\left(\frac{B}{\Lambda^2}\right)^2 + \cdots\Bigg].\end{aligned} \tag{20}$$

D is diagonal; hence we can write $R^{(n)}(D) = f^{(n)}(k^2)I_n$ for some scalar function $f(k^2)$, so that (20) becomes

$$\begin{aligned}\mathscr{A} &= \Lambda^4 N\int\frac{d^4k}{(2\pi)^4}f\left(k^2\right) + \Lambda^2(\operatorname{tr}B)\int\frac{d^4k}{(2\pi)^4}f'\left(k^2\right) \\ &\quad + \frac{1}{2!}\left(\operatorname{tr}B^2\right)\int\frac{d^4k}{(2\pi)^4}f''\left(k^2\right) + \cdots \\ &= \Lambda^4 N\int\frac{d^4k}{(2\pi)^4}f\left(k^2\right) \\ &\quad + \Lambda^2(\operatorname{tr}B)\int\frac{\Omega_3 dk^2}{2(2\pi)^4}k^2 f'\left(k^2\right) \\ &\quad + \frac{1}{2!}\left(\operatorname{tr}B^2\right)\int\frac{\Omega_3 dk^2}{2(2\pi)^4}k^2 f''\left(k^2\right) \\ &\quad + \sum_{n=3}^{\infty}\frac{1}{\Lambda^{(2n-4)}}\frac{1}{n!}\left(\operatorname{tr}B^n\right)\int\frac{\Omega_3 dk^2}{2(2\pi)^4}k^2 f^{(n)}\left(k^2\right),\end{aligned} \tag{21}$$

where $\Omega_3 = 2\pi^2$ is the solid angle. The minimum conditions on $f(k^2)$ required to produce the anomaly are

$$\begin{aligned}f(0) &= 1, \\ f(\infty) &= 0, \\ \left[k^2 f'\left(k^2\right)\right]\Big|_0^\infty &= 0,\end{aligned} \tag{22}$$

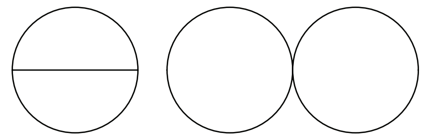

FIGURE 1: Lowest-loop 1PI vacuum graphs with 3 and 4 vertices.

which are the same conditions for the chiral anomaly [10]. However, for simplicity we will specialize to $f(k^2) = e^{-k^2}$, which satisfies (22) but, in addition, has the nice property that

$$\int dk^2 k^2 f^{(n)}\left(k^2\right) = (-1)^n, \tag{23}$$

so that plugging this regulator into (21) gives us

$$\begin{aligned}\mathscr{A} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Lambda^{(2n-4)}} \frac{1}{n!} \left(\operatorname{tr} B^n\right) \frac{\Omega_3}{2(2\pi)^4} \\ &= \Lambda^4 \left(\operatorname{tr} B^0\right) \frac{\Omega_3}{2(2\pi)^4} - \Lambda^2 (\operatorname{tr} B) \frac{\Omega_3}{2(2\pi)^4} \\ &\quad + \frac{1}{2!} \left(\operatorname{tr} B^2\right) \frac{\Omega_3}{2(2\pi)^4} \\ &\quad + \sum_{n=3}^{\infty} \frac{(-1)^n}{\Lambda^{(2n-4)}} \frac{1}{n!} \left(\operatorname{tr} B^n\right) \frac{\Omega_3}{2(2\pi)^4}.\end{aligned} \tag{24}$$

The first term in (24) is independent of the coupling λ so it would be present even in the free theory. Since the free theory is taken to be nonanomalous, we ignore this term [13]. The second term, proportional to Λ^2, is removed by mass renormalization: the precise meaning of this is discussed in the next section. The third term is the only remaining nonvanishing term in the $\Lambda \to \infty$ limit and is independent of Λ. Evaluating $(\operatorname{tr} B^2) = B_{ij} B_{ji}$ by substituting in B_{ij} from (17) gives

$$\begin{aligned}\mathscr{A} &= \frac{1}{2!} \left[\lambda^2 (N+8) \left(\hat{\phi}_k \hat{\phi}_k\right)^2\right] \frac{\Omega_3}{2(2\pi)^4} \\ &= \frac{\lambda^2 (N+8)}{32\pi^2} \left(\hat{\phi}_k \hat{\phi}_k\right)^2 = \beta(\lambda) \frac{\left(\hat{\phi}_k \hat{\phi}_k\right)^2}{4} \\ &= \beta(\lambda) \frac{\partial \mathscr{H}_I}{\partial \lambda},\end{aligned} \tag{25}$$

where $\beta(\lambda) = \lambda^2 (N+8)/8\pi^2$ and $\mathscr{H}_I$ is the interacting Hamiltonian.

5. Equivalence of Fujikawa with Background Field Calculation

We now apply the background field method to the Lagrangian in (13). We make the shift $\phi_i(x) = \hat{\phi}_i + \eta_i(x)$ so that the $O(N)$ Lagrangian becomes

$$\begin{aligned}\widetilde{\mathscr{L}} &= \frac{1}{2} \sum_{i,j=1}^{N} \int d^4 x \eta_i(x) \left(D_{ij} + B_{ij}\right) \eta_j(x) \\ &\quad + \mathscr{L}\left(\hat{\phi}_i, \partial_\mu \hat{\phi}_i\right) + \mathscr{L}_T + \mathscr{L}_I.\end{aligned} \tag{26}$$

In the above expression, $\mathscr{L}(\hat{\phi}_i, \partial_\mu \hat{\phi}_i)$ is the original $O(N)$ Lagrangian with the background field substituted for ϕ. This term has no dependence on η and contributes to the 1PI vacuum graphs at tree-level (i.e., with respect to the η field, this term is like a cosmological constant). $\mathscr{L}_T$ are terms that contain only one η field: these produce tadpole diagrams which are reducible, so $\mathscr{L}_T$ can be neglected in calculation of 1PI graphs. L_I are terms involving η^3 and η^4 interactions. For 1PI vacuum graphs, these interactions contribute beginning at the 2-loop level and hence can be ignored for a 1-loop calculation (see Figure 1).

So the Lagrangian we will use to calculate the 1PI vacuum graphs at 1-loop is

$$\begin{aligned}\widetilde{\mathscr{L}} &= \frac{1}{2} \sum_{i,j=1}^{N} \int d^4 x \eta_i(x) D_{ij} \eta_j(x) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{N} \int d^4 x \eta_i(x) B_{ij} \eta_j(x).\end{aligned} \tag{27}$$

Since the background field $\hat{\phi}_i$ (contained in B_{ij} of (17)) is constant and the Lagrangian is only quadratic in η, we could sum all the 1-loop vacuum graphs at once by calculating the determinant $D_{ij} + B_{ij}$ [14]. However, instead we choose as the propagator D_{ij}^{-1} and treat interaction B_{ij} as an interaction vertex that joins two propagators and categorize the loops by the number of vertices B_{ij} which corresponds to twice the number of background fields $\hat{\phi}$ (see Figure 2). We do this to match the result of (24) from Fujikawa's method, which is an expansion in powers of B_{ij}.

The Feynman rules are straightforward. For each vertex we write iB_{ij}, as the $1/2$ in (27) accounts for swapping connections of the two propagators to which each vertex connects. For each propagator we write iD_{ij}^{-1}, where the $1/2$ takes care of which end of the propagator connects to a vertex.

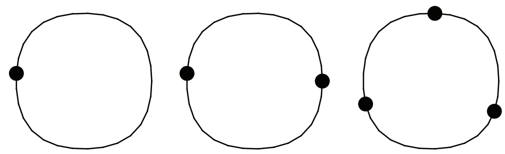

FIGURE 2: 1-loop 1PI vacuum graphs with 1, 2, and 3 vertices.

An overall symmetry factor is required that depends on the number of vertices B_{ij}. This symmetry factor is $1/2n$ where n is the number of vertices: the 2 is due to reflection symmetry and n to cyclic permutation of the vertices.

For an n-vertex diagram,

$$-iV_{\text{eff}}^{n} = \frac{1}{2n}\int \frac{id^4k}{(2\pi)^4}\left(\frac{i}{-k^2}\right)^n \operatorname{tr}\left[(iB)^n\right] = \frac{i}{2n}\frac{\Omega_3}{(2\pi)^4}\operatorname{tr} B^n\left(\int_0^\Lambda dk\frac{k^3}{k^{2n}}\right), \tag{28}$$

where a Wick rotation was performed. The anomaly in Fujikawa's method was given in (24) as $\mathscr{A} = \sum_{n=0}^{\infty}((-1)^n/2n!)(\Omega_3/(2\pi)^4)(\operatorname{tr} B^n)\Lambda^{4-2n}$. Following the renormalization group analysis of [15], we apply the operator $\partial/\partial\ln\Lambda = \Lambda(\partial/\partial\Lambda)$ to (28). Then, from the fundamental theorem of calculus $\Lambda(\partial/\partial\Lambda)\int_0^\Lambda(k^3/k^{2n}) = \Lambda^{4-2n}$, we get the following result:

$$-\frac{\partial}{\partial\ln\Lambda}V_{\text{eff}} = \sum_{n=1}^{\infty}\frac{1}{2n}\frac{\Omega_3}{(2\pi)^4}\left(\operatorname{tr} B^n\right)\Lambda^{4-2n}. \tag{29}$$

Only for $n = 2$ does this match the anomaly given by Fujikawa's method. Indeed, it is impossible to construct a regulator in Fujikawa's method that exactly produces (29). However, the terms for $n \geq 3$ vanish in the limit $\Lambda \to \infty$. Since diagrams for which $n \geq 3$ are convergent, they do not contribute to the anomaly, and in Fujikawa's method they correspond to the vanishing $n \geq 3$ terms in the Taylor expansion. The anomaly is contained entirely in Figure 2(b). The quadratic divergence in Figure 2(a) is a well-known artifact of cutoff regularization and can be avoided by dimensional regularization, where the loop integral is zero [16]. However, Fujikawa's method does not work with dimensional regularization since, in $d - 2\epsilon$ dimensions, the δ-function is zero [17] (however, in the nonrelativistic context, this need not be the case [18]). Within the context of dimensional regularization, the anomaly arises from the fact that $\lambda\phi^4$ in $d-2\epsilon$ dimensions is not conformally invariant [19] rather than through the noninvariance of the path integral measure.

This can readily be seen by calculating the effective potential. The effective potential is given by summing across all n of (28):

$$V_{\text{eff}} = -\sum_{n=1}^{\infty}\frac{1}{2n}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2}\right)^n \operatorname{tr} B^n. \tag{30}$$

One can swap the integral with the summation: this avoids the need for an IR regulator, as the summation results in a log which is IR-free. However, we are interested in the contribution of each n-vertex diagram—therefore we introduce a fictitious mass m to regulate the theory in the IR and a cutoff Λ to regulate the theory in the UV:

$$\begin{aligned}-V_{\text{eff}} &= \sum_{n=1}^{\infty}\frac{1}{2n}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right)^n \operatorname{tr} B^n \\ &= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right)\operatorname{tr} B \\ &\quad + \frac{1}{4}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right)^2\operatorname{tr} B^2 \\ &\quad + \sum_{n=3}^{\infty}\frac{1}{2n}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right)^n \operatorname{tr} B^n.\end{aligned} \tag{31}$$

The integrals are standard, and the result in the $m^2 \to 0$ limit is

$$\begin{aligned}&\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right)\operatorname{tr} B = -\frac{\Lambda^2}{32\pi^2}\operatorname{tr} B, \\ &\frac{1}{4}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right)^2\operatorname{tr} B^2 \\ &\quad = \frac{1}{64\pi^2}\left[1-\log\left(\frac{\Lambda^2}{m^2}\right)\right]\operatorname{tr} B^2, \\ &\sum_{n=3}^{\infty}\frac{1}{2n}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right)^n\operatorname{tr} B^n \\ &\quad = \frac{1}{128\pi^2}\operatorname{tr}\left[-3B^2+2B^2\log\left(\frac{-B}{m^2}\right)\right].\end{aligned} \tag{32}$$

One can see that diagrams with $n \geq 3$ are independent of Λ and that $-\partial/\partial\ln\Lambda$ acting on $n = 2$ produces the anomaly. Both $\operatorname{tr} B = -\lambda(N+2)\phi_k\phi_k$ and $\operatorname{tr} B^2 = \lambda^2(N+8)(\phi_k\phi_k)^2$ are of the form of the original Lagrangian, so they can be cancelled by counterterms. Adding all the terms in (32) gives

$$V_{\text{eff}} = -\frac{\Lambda^2}{32\pi^2}\operatorname{tr} B - \frac{\operatorname{tr} B^2}{128\pi^2} + \frac{1}{64\pi^2}\operatorname{tr}\left[B^2\log\left(\frac{-B}{\Lambda^2}\right)\right]. \tag{33}$$

The result is independent of m^2 as it should be. The $n \geq 3$ terms have produced a nonpolynomial log interaction, and the $n = 2$ term has provided the scale for this interaction.

6. Noether's Theorem and Dimensional Transmutation

The field ϕ_c obeys the classical equations of motion (8), with the effective action $\Gamma[\phi_c]$ replacing the classical one $S[\phi_c]$. Therefore, Noether's theorem, which is based on the classical EOM, would apply if $\Gamma[\phi_c]$ retains the symmetry. In general the quantum corrections will create terms in $\Gamma[\phi_c]$ that explicitly break scale symmetry. The measure of symmetry-breaking is $\sum_{i=1}^{N}(\partial V_{\text{eff}}/\partial\phi_{ic})\phi_{ic} - 4V_{\text{eff}}$, which gives zero for the classically scale-invariant tree-level contribution $V = (\lambda/4)(\phi_{ic}\phi_{ic})^2$ to the effective potential. Specializing to $N = 1$, effective potential (33) reads

$$V_{\text{eff}} = \frac{\lambda\phi_c^4}{4} + \frac{9\lambda^2\phi_c^4}{64\pi^2}\left(\ln\left(\frac{3\lambda\phi_c^2}{\Lambda^2}\right) - \frac{1}{2}\right). \tag{34}$$

Applying $\sum_{i=1}^{N}(\partial V_{\text{eff}}/\partial\phi_{ic})\phi_{ic} - 4V_{\text{eff}}$ to (34), we get the scale anomaly:

$$\mathscr{A} = \frac{9\lambda^2\phi_c^4}{32\pi^2}, \tag{35}$$

in agreement with (25). From the viewpoint of classical physics, a term like $\phi_c^4 \ln M^2$ is scale-invariant, acting like a ϕ_c^4 potential. It is $\phi_c^4 \ln\phi_c^2$ term that breaks scale-invariance. Both terms are related since dimensional transmutation of the $n = 2$ graph provides the scale for the $n \geq 3$ graphs which generate nonpolynomial interactions.

7. Conclusion

The scale anomaly and anomalies in general are the result of the failure to maintain classical symmetry upon quantization. One cannot regularize the system in a way to preserve all the symmetries of the theory. The absence of dimensionful parameters in the action is sufficient for the classical theory to be scale-invariant. However, the introduction of a dimensionful parameter through regularization can provide a scale to support noninvariant ϕ^{2n} interactions with $n \geq 3$ in the $O(N)$ quantum theory. Fujikawa's method is equivalent to the 1-loop calculation of the anomaly in the effective potential.

We plan to investigate these connections and apply the insights gained to the nonrelativistic case in order to study questions of interest in atomic and molecular physics, in particular in the field of ultracold atoms where, unlike the situation in particle physics, the manifestations of the scale anomaly in these systems have only now been accessible to experimentalists in this decade.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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