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Research Article

Coupling *q*-Deformed Dark Energy to Dark Matter

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We propose a novel coupled dark energy model which is assumed to occur as a q-deformed scalar field and investigate whether it will provide an expanding universe phase. We consider the q-deformed dark energy as coupled to dark matter inhomogeneities. We perform the phase-space analysis of the model by numerical methods and find the late-time accelerated attractor solutions. The attractor solutions imply that the coupled q-deformed dark energy model is consistent with the conventional dark energy models satisfying an acceleration phase of universe. At the end, we compare the cosmological parameters of deformed and standard dark energy models and interpret the implications.

1. Introduction

The standard model of cosmology states that approximately 5% of the energy content of the universe belongs to ordinary baryonic matter of standard model of particle physics. The other 95% of the energy content of the universe is made of the dark sector. Particularly, 25% of the content is an unknown form of matter having a mass but in nonbaryonic form that is called dark matter. The remaining 70% of the content consists of an unknown form of energy named dark energy. On the other hand, it is known that the universe is experiencing accelerating expansion by astrophysical observations, Supernova Ia, large-scale structure, baryon acoustic oscillations, and cosmic microwave background radiation. The dark energy is assumed to be responsible for the late-time accelerated expansion of the universe. The dark energy component of the universe is not clustered but spreads all over the universe and it generates gravitational repulsion due to its negative pressure for driving the acceleration of the expansion of the universe [1–10].

In order to explain the viable mechanism for the accelerated expansion of the universe, cosmologists have proposed various dynamical models for dark energy and dark matter, which include possible interactions between dark energy, dark matter, and other fields, such as gravitation. Particularly, the coupling between dark energy and dark matter is proposed since the energy densities of two dark components are of the same order of magnitude today [11–18].

Since there are a great number of candidates for the constitution of the dark energy, such as cosmological constant, quintessence, phantom, and tachyon fields, different interactions have been proposed between these constituents, dark matter, and gravitational field in the framework of general relativity [19–29]. However, the corresponding dynamical analyses of the interactions between different dark energy models and the dark matter and the gravity have been studied in the framework of teleparallel gravity which uses the torsion tensor instead of the curvature tensor of general relativity [30–39].

The main motivation of this study comes from the recent studies in the literature [40–45] which involves the deformation of the standard scalar field equations representing the dark energy. In this study, we propose a novel dark energy model as a q-deformed scalar field interacting with the dark matter. Since the dark energy is a negative-pressure scalar field, this scalar field can be considered as a q-deformed scalar field. The q-deformed scalar field is in fact a q-deformed boson model, and the statistical mechanical studies of q-deformed boson models have shown that the pressure of the deformed bosons is generally negative. Not only do the different deformed boson models have a negative pressure, but also the different deformed fermion models can take negative pressure values [46–50]. Here, we consider the q-deformed bosons and propose that the scalar field which is produced by these deformed bosons constitutes the dark energy in the universe. We also investigate the dynamics of the coupling q-deformed dark energy and dark matter inhomogeneities in the Friedmann-Robertson-Walker (FRW) space-time. In order to confirm our proposal, we perform the phase-space analysis of the model whether the late-time stable attractor solutions exist or not, since the stable attractor solutions imply the accelerating expansion of the universe. We finally compare the cosmological parameters of the q-deformed and standard dark energy model and interpret the implications of the comparison.

2. Dynamics of the Model: Coupling q-Deformed Dark Energy to Dark Matter

In our model, the dark energy consists of the scalar field whose field equations are defined by the q-deformed boson fields. Since the idea of q-deformation to the single particle quantum mechanics is a previous establishment in the literature [51–53], it has been natural to construct a q-deformed quantum field theory [54–56]. While the bosonic counterpart of the deformed particle fields corresponds to the deformed scalar field, the fermionic one corresponds to the deformed vector field. Here, we take into account the q-deformed boson field as the q-deformed scalar field which constitutes the dark energy under consideration. The q-deformed dark energy couples to dark matter inhomogeneities in our model. Now, we begin with defining the q-deformed dark energy in the FRW geometry.

Quantum field theory in curved space-time is important in the understanding of the scenario in the Early Universe. Quantum mechanically, constructing the coherent states for any mode of the scalar field translates the behavior of the classical scalar field around the initial singularity into quantum field regime. At the present universe, the quantum mechanical state of the scalar field around the initial singularity cannot be determined by an observer. Therefore, Hawking states that this indeterministic nature can be described by taking the random superposition of all possible states in that space-time. Berger has realized this by taking the superposition of coherent states randomly. Parker has studied the particle creation in the expanding universe with a nonquantized gravitational metric. When the evolution of the scalar field is considered in an expanding universe, Goodison and Toms stated that if the field quanta obey the Bose or Fermi statistics, then the particle creation does not occur in the vacuum state. Therefore, the scalar field dark energy must be described in terms of the deformed bosons or fermions in the coherent states or squeezed state [56–62]. Also, the q-deformed dark energy is a generalization of the standard scalar field dark energy. The free parameter q makes it possible to obtain a desired dark energy model with a preferred interaction or coupling by setting up the deformation parameter to the suitable value.

This motivates us to describe the dark energy as a q-deformed scalar field interacting with the dark matter. We can give the Dirac-Born-Infeld type action of the model as $S = \kappa^2 \int (\ell_g + \ell_{\phi_q} + \ell_m + \ell_r) d^4x$, where $\kappa^2 = 8\pi G$, $\ell_g = -(1/2\kappa) g R$, and ℓ_m and ℓ_r are the gravitational, dark matter, and radiation Lagrangian densities. Then, the deformed dark energy Lagrangian density is given for $(+,-,-,-)$ metric signature as [63]

$$\ell_{\phi_q} = g \frac{1}{2} \left[g^{\mu\nu} \left(\nabla_\mu \phi_q\right)\left(\nabla_\nu \phi_q\right) - m^2 {\phi_q}^2 \right], \tag{1}$$

where $g = \sqrt{-\det g_{\mu\nu}}$ and ϕ_q is the q-deformed scalar field operator for the dark energy and ∇_μ is the covariant derivative which is in fact the ordinary partial derivative ∂_μ for the scalar field. From the variation of the q-deformed scalar field Lagrangian density with respect to the deformed field, we obtain the deformed Klein-Gordon equation:

$$\begin{gathered} \frac{\partial \ell_{\phi_q}}{\partial \phi_q} - \partial_\mu \left(\frac{\partial \ell_{\phi_q}}{\partial \left(\partial_\mu \phi_q\right)} \right) = 0, \\ \left(\partial_\mu \partial^\mu\right) \phi_q + m^2 \phi_q = 0. \end{gathered} \tag{2}$$

Also, we get the energy-momentum tensor of the scalar field dark energy from the variation of Lagrangian ℓ_{ϕ_q} with respect to the metric tensor, such that

$$\begin{aligned} T^{\phi_q}_{\mu\nu} &= \frac{2}{g} \left(\frac{\partial \ell_{\phi_q}}{\partial g^{\mu\nu}} - g_{\mu\nu} \ell_{\phi_q} \right) \\ &= \partial_\mu \phi_q \partial_\nu \phi_q - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} \partial_\alpha \phi_q \partial_\beta \phi_q - m^2 {\phi_q}^2 \right), \end{aligned} \tag{3}$$

from which the timelike and spacelike parts of $T^{\phi_q}_{\mu\nu}$ read as follows:

$$\begin{aligned} T^{\phi_q}_{00} &= \frac{1}{2} \dot{\phi}_q^{\,2} - \frac{1}{2} g^{ii} \left(\partial_i \phi_q\right)^2 + \frac{1}{2} m^2 {\phi_q}^2, \\ T^{\phi_q}_{ii} &= -\frac{1}{2} g_{ii} \dot{\phi}_q^{\,2} + \frac{1}{2} \left(\partial_i \phi_q\right)^2 - \frac{1}{2} g_{ii} g^{jj} \left(\partial_j \phi_q\right)^2 \\ &\quad + \frac{1}{2} g_{ii} m^2 {\phi_q}^2, \end{aligned} \tag{4}$$

where $i, j = 1, 2, 3$ represent the spacelike components.

In order to calculate the deformed energy density and the pressure functions of the q-deformed energy from the energy-momentum components (4), we need to consider the quantum field theoretical description of the deformed scalar field in a FRW background with the metric

$$ds^2 = dt^2 - a^2(t) \left[dx^2 + dy^2 + dz^2 \right]. \tag{5}$$

Then, the canonically quantized q-deformed scalar field ϕ_q is introduced in terms of the Fourier expansion [57]

$$\phi_q = \int d^3k \left[a_q(k) f_k + a_q^+(k) f_k^* \right], \tag{6}$$

where $a_q(k)$ and $a_q^+(k)$ are the q-deformed boson annihilation and creation operators for the quanta of the q-deformed scalar field dark energy in the kth mode. k denotes the spatial wave vector and obeys the relativistic energy conservation law $\omega^2 = -g^{ii}k^2 + m^2$. Here, f_k is a set of orthonormal mode solutions of the deformed Klein-Gordon equation, such that

$$f_k(x^\mu) = \frac{\exp(ik_\mu x^\mu)}{\sqrt{(2\pi)^3 2\omega}}, \tag{7}$$

where $k_\mu = (\omega, -k)$ is the four-momentum vector and satisfies the relations

$$\begin{aligned}\int d^3x f_k f_{k'}^* &= \frac{\delta^3(-k+k')}{2\omega},\\ \int d^3x f_k f_{k'} &= \frac{e^{2i\omega t}\delta^3(-k-k')}{2\omega},\\ \int d^3x f_k^* f_{k'} &= \frac{\delta^3(k-k')}{2\omega},\\ \int d^3x f_k^* f_{k'}^* &= \frac{e^{-2i\omega t}\delta^3(k+k')}{2\omega}.\end{aligned} \tag{8}$$

We can give the spatial average of the q-deformed scalar field energy-momentum tensor $T_{\mu\nu}^{\phi_q}$ as [59]

$$\overline{T}_{\mu\nu}^{\phi_q} = \int d^3x T_{\mu\nu}^{\phi_q}. \tag{9}$$

By using (9), the spatial average of the timelike energy-momentum tensor component is obtained as

$$\overline{T}_{00}^{\phi_q} = \int d^3x\left[\frac{1}{2}\dot{\phi}_q^{\,2} - \frac{1}{2}g^{ii}(\partial_i\phi_q)^2 + \frac{1}{2}m^2\phi_q^{\,2}\right]. \tag{10}$$

This can be determined from (6)–(8) term by term, as follows:

$$\begin{aligned}\frac{1}{2}m^2\int d^3x\phi_q^{\,2} &= \frac{1}{2}\cdot m^2\int d^3x\iint d^3k d^3k'\left[a_q(k)f_k + a_q^+(k)f_k^*\right]\\ &\cdot\left[a_q(k')f_{k'} + a_q^+(k')f_{k'}^*\right] = \frac{1}{2}\int d^3k\frac{m^2}{2\omega}\left[a_q(k)\right.\\ &\cdot a_q(-k)e^{2i\omega t} + a_q^+(k)a_q(k) + a_q(k)a_q^+(k)\\ &\left.+ a_q^+(k)a_q^+(-k)e^{-2i\omega t}\right],\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{1}{2}\int d^3x\dot{\phi}_q^{\,2} &= \frac{1}{2}\int d^3k\frac{\omega}{2}\left[-a_q(k)a_q(-k)e^{2i\omega t} + a_q^+(k)\right.\\ &\left.\cdot a_q(k) + a_q(k)a_q^+(k) - a_q^+(k)a_q^+(-k)e^{-2i\omega t}\right],\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{1}{2}g^{ii}\int d^3x(\partial_i\phi_q)^2 &= \frac{1}{2}\int d^3k\frac{g^{ii}k^2}{2\omega}\left[a_q(k)a_q(-k)e^{2i\omega t}\right.\\ &+ a_q^+(k)a_q(k) + a_q(k)a_q^+(k) + a_q^+(k)a_q^+(-k)\\ &\left.\cdot e^{-2i\omega t}\right].\end{aligned} \tag{13}$$

Combining (11), (12), and (13) gives

$$\overline{T}_{00}^{\phi_q} = \frac{1}{2}\int d^3k\omega\left[a_q(k)a_q^+(k) + a_q^+(k)a_q(k)\right], \tag{14}$$

where $\omega^2 = m^2 - k^2/a^2(t)$ for the FRW space-time. Correspondingly, the average of the spacelike energy-momentum tensor component can be determined, such that

$$\begin{aligned}\overline{T}_{ii}^{\phi_q} &= \int d^3x\left[-\frac{1}{2}g_{ii}\dot{\phi}_q^{\,2} + \frac{1}{2}(\partial_i\phi_q)^2 - \frac{1}{2}g_{ii}g^{jj}(\partial_j\phi_q)^2\right.\\ &\left.+ \frac{1}{2}g_{ii}m^2\phi_q^{\,2}\right] = -\frac{1}{2}\int d^3k\frac{g_{ii}\omega}{2}\\ &\cdot\left[-a_q(k)a_q(-k)e^{2i\omega t} + a_q^+(k)a_q(k)\right.\\ &\left.+ a_q(k)a_q^+(k) - a_q^+(k)a_q^+(-k)e^{-2i\omega t}\right] + \int d^3k\frac{k_i^2}{2\omega}\\ &\cdot\left[a_q(k)a_q(-k)e^{2i\omega t} + a_q^+(k)a_q(k) + a_q(k)a_q^+(k)\right.\\ &\left.+ a_q^+(k)a_q^+(-k)e^{-2i\omega t}\right] + \frac{1}{2}\int d^3k\frac{g_{ii}g^{jj}k^2}{2\omega}\\ &\cdot\left[a_q(k)a_q(-k)e^{2i\omega t} + a_q^+(k)a_q(k) + a_q(k)a_q^+(k)\right.\\ &\left.+ a_q^+(k)a_q^+(-k)e^{-2i\omega t}\right] + \frac{1}{2}\int d^3k\frac{g_{ii}m^2}{2\omega}\\ &\cdot\left[a_q(k)a_q(-k)e^{2i\omega t} + a_q^+(k)a_q(k) + a_q(k)a_q^+(k)\right.\\ &\left.+ a_q^+(k)a_q^+(-k)e^{-2i\omega t}\right].\end{aligned} \tag{15}$$

From the identities $a_q(-k)e^{2i\omega t} = a_q^+(k)$ and $a_q^+(-k)e^{-2i\omega t} = a_q(k)$, (15) turns out to be

$$\begin{aligned}\overline{T}_{ii}^{\phi_q} &= \frac{1}{2}\int d^3k\frac{1}{\omega}\left[2k_i^2 - a^2(t)\omega^2\right]\\ &\cdot\left[a_q(k)a_q^+(k) + a_q^+(k)a_q(k)\right],\end{aligned} \tag{16}$$

for the FRW geometry, and k_i is the ith spatial component of the wave vector.

The q-deformation of a quantum field theory is constructed from the standard algebra satisfied by the annihilation and creation operators introduced in the canonical quantization of the field. The deformation of a standard boson algebra satisfied by the annihilation and creation operators of a bosonic quantum field theory was firstly realized by Arik-Coon [51], and then Macfarlane and Biedenharn [52, 53] independently realized the deformation of boson algebra different from Arik-Coon. Hence, the q-deformed bosonic quantum field theory of the scalar field dark energy is constructed by the q-deformed algebra of the operators $a_q(k)$ and $a_q^+(k)$ in (6), such that

$$a_q(k)a_q^+(k') - q^2a_q^+(k')a_q(k) = \delta^3(k-k'), \tag{17}$$

$$a_q(k)a_q(k') - q^2a_q(k')a_q(k) = 0, \tag{18}$$

$$[N_k] = a_q^+(k)a_q(k). \tag{19}$$

Here, q is a real deformation parameter, and $[N_k]$ is the deformed number operator whose eigenvalue spectrum is given as [51]

$$[n_k] = \frac{1-q^{2n_k}}{1-q^2}, \tag{20}$$

where n_k is the eigenvalue of the standard number operator N_k.

The corresponding vector spaces of the annihilation and creation operators for the q-deformed scalar field dark energy are the q-deformed Fock space state vectors, which give information about the number of particles in the corresponding state. The q-deformed bosonic annihilation and creation operators $a_q(k)$ and $a_q^+(k)$ act on the Fock states $|n_k\rangle$ as follows:

$$\begin{aligned} a_q(k)|n_k\rangle &= \sqrt{[n_k]}|n_k-1\rangle, \\ a_q^+(k)|n_k\rangle &= \sqrt{[n_k+1]}|n_k+1\rangle, \\ [N_k]|n_k\rangle &= a_q^+(k)a_q(k)|n_k\rangle = [n_k]|n_k\rangle. \end{aligned} \tag{21}$$

By taking the quantum expectation values of the spatial averages of energy-momentum tensor with respect to the Fock basis $|n_k\rangle$, we obtain the energy density and the pressure of the q-deformed dark energy. Using $\rho_{\phi_q} = T_0^0$ and $p_{\phi_q} = -T_i^i$ for the energy density and pressure of the q-deformed scalar field dark energy, we obtain

$$\begin{aligned} \int \rho_{\phi_q} d^3x &= \langle n_k| \overline{T}_0^{0\phi_q} |n_k\rangle \\ &= \int d^3k \frac{\omega}{2} \langle n_k| \left[a_q(k)a_q^+(k) + a_q^+(k)a_q(k)\right] |n_k\rangle \\ &= \left(\left(1+q^2\right)[n_k]+1\right) \int d^3k_\phi \frac{\omega_\phi}{2}, \end{aligned} \tag{22}$$

where q-deformed boson algebra in (17) is used in the second line. Because the q-deformed boson algebra in (17)–(19) transforms to be the standard boson algebra and $[n_k] = n_k$ in the $q \to 1$ limit, the energy density ρ_{ϕ_q} of the q-deformed dark energy transforms into the energy density ρ_ϕ of the standard dark energy as

$$\int \rho_\phi d^3x = (2n_k+1) \int d^3k_\phi \frac{\omega_\phi}{2}. \tag{23}$$

Hence, the energy density ρ_{ϕ_q} of the q-deformed dark energy can be written in terms of the energy density ρ_ϕ of the standard dark energy by

$$\rho_{\phi_q} = \frac{\left(1+q^2\right)[n_k]+1}{2n_k+1}\rho_\phi = \Delta_q(n_k)\rho_\phi. \tag{24}$$

Accordingly, the pressure of the q-deformed scalar field dark energy can be written from (6) as

$$\begin{aligned} \int p_{\phi_q} d^3x &= \langle n_k| -\overline{T}_i^{i\phi_q} |n_k\rangle = \int d^3k \\ &\cdot \frac{1}{2\omega}\left[\frac{2k^2}{a^2(t)} - \omega^2\right]\langle n_k| \\ &\cdot \left[a_q(k)a_q^+(k) + a_q^+(k)a_q(k)\right]|n_k\rangle = \left(\left(1+q^2\right)\right. \\ &\cdot \left.[n_k]+1\right)\int d^3k \frac{1}{2\omega}\left[\frac{2k^2}{a^2(t)} - \omega^2\right], \end{aligned} \tag{25}$$

where $g^{ii}k_i^2 = (1/a^2(t))(k_1^2+k_2^2+k_3^2) = k^2/a^2(t)$ is used. In the $q \to 1$ limit, the q-deformed pressure p_{ϕ_q} of dark energy transforms to the standard pressure p_ϕ of the dark energy, such that

$$\int p_\phi d^3x = (2n_k+1)\int d^3k \frac{1}{2\omega}\left[\frac{2k^2}{a^2(t)} - \omega^2\right]. \tag{26}$$

Consequently, the q-deformed pressure p_{ϕ_q} of dark energy can be obtained in terms of the standard pressure p_ϕ of the dark energy; thus,

$$p_{\phi_q} = \frac{\left(1+q^2\right)[n_k]+1}{2n_k+1}p_\phi = \Delta_q(n_k)p_\phi. \tag{27}$$

Also, the commutation relations and plane-wave expansion of the q-deformed scalar field $\phi_q(x)$ are given by using (17)–(19) in (6), as follows:

$$\phi_q(x)\phi_q^+(x') - q^2\phi_q^+(x')\phi_q(x) = i\Delta(x-x'), \tag{28}$$

where

$$\Delta(x-x') = \frac{-1}{(2\pi)^3}\int \frac{d^3k}{w_k}\sin w_k(x-x_0). \tag{29}$$

On the other hand, the deformed and standard annihilation operators, a_q and a_s, are written as [62]

$$a_q = a_s\sqrt{\frac{[N_k]}{N_k}}. \tag{30}$$

From this, we can express the deformed bosonic scalar fields in terms of the standard one by using (20) in (30) and (6):

$$\phi_q = \sqrt{\frac{1-q^{2n_k}}{(1-q^2)n_k}}\phi = \Delta(q)\phi, \tag{31}$$

where we use the Hermiticity of the number operator N.

Now, we will derive the Friedmann equations for our coupling q-deformed dark energy to dark matter model with a radiation field in FRW space-time by using the scale factor $a(t)$ in Einstein's equations. We can achieve this by relating the scale factor with the energy-momentum tensor of the objects in the considered universe model. It is a common fact to consider energy and matter as a perfect fluid, which will naturally be generalized to dark energy and matter. An isotropic fluid in one coordinate frame gives an isotropic metric in another frame which coincides with the first frame. This means that the fluid is at rest in commoving coordinates. Then, the four-velocity vector is given as [63]

$$U^{\mu} = (1, 0, 0, 0), \quad (32)$$

while the energy-momentum tensor reads

$$T_{\mu\nu} = (\rho + p) U_{\mu} U_{\nu} + p g_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} p & \\ 0 & & & \end{pmatrix}. \quad (33)$$

Raising one index gives a more suitable form

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p). \quad (34)$$

For a model of universe described by Dirac-Born-Infeld type action and consisting of more than one form of energy momentum, we have totally three types of energy density and pressure, such that

$$\rho_{\mathrm{tot}} = \rho_{\phi_q} + \rho_m + \rho_r, \quad (35)$$

$$p_{\mathrm{tot}} = p_{\phi_q} + p_r, \quad (36)$$

where the pressure of the dark matter p_m is explicitly zero in the total pressure p_{tot} (36). From the conservation of equation for the zero component $\nabla_{\mu} T^{\mu}_{0} = 0$, one obtains $\rho \propto a^{-3(1+w)}$. Here, w is the parameter of the equation of state $p = w\rho$ which relates the pressure and the energy density of the cosmological fluid component under consideration. Therefore, pressure is zero for the matter component and $w_m = 0$, but for the radiation component $w_r = 1/3$ due to the vanishing trace of the energy-momentum tensor of the electromagnetic field. We then express the total equation of state parameter as

$$w_{\mathrm{tot}} = \frac{p_{\mathrm{tot}}}{\rho_{\mathrm{tot}}} = w_{\phi_q} \Omega_{\phi_q} + w_r \Omega_r. \quad (37)$$

While the equations of state parameters are given as $w_{\phi_q} = p_{\phi_q}/\rho_{\phi_q}$ and $w_r = p_r/\rho_r = 1/3$, the density parameters are defined by $\Omega_{\phi_q} = \rho_{\phi_q}/\rho_{\mathrm{tot}}$, $\Omega_r = \rho_r/\rho_{\mathrm{tot}}$ for the q-deformed dark energy and the radiation fields, respectively. Since the pressure of the dark matter is $p_m = 0$, then the equation of state parameter is $w_m = p_m/\rho_m = 0$ and the density parameter is $\Omega_m = \rho_m/\rho_{\mathrm{tot}}$ for the dark matter field having no contribution to w_{tot} (37), but contributing to the total density parameter, such that

$$\Omega_{\mathrm{tot}} = \Omega_{\phi_q} + \Omega_m + \Omega_r = \frac{\kappa^2 \rho_{\mathrm{tot}}}{3H^2} = 1. \quad (38)$$

We now turn to Einstein's equations of the form $R_{\mu\nu} = \kappa^2 (T_{\mu\nu} - (1/2) g_{\mu\nu} T)$. Then, by using the components of the Ricci tensor for FRW space-time (5) and the energy-momentum tensor in (34), we rewrite Einstein's equations, for $\mu\nu = 00$ and $\mu\nu = ij$:

$$-3\frac{\ddot{a}}{a} = \frac{\kappa^2}{2}(\rho + 3p), \quad (39)$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{2}(\rho - p), \quad (40)$$

respectively. Here, the dot also represents the derivative with respect to cosmic time t. Using (39) and (40) gives the Friedmann equations for FRW metric as

$$H^2 = \frac{\kappa^2}{3}\left(\rho_{\phi_q} + \rho_m + \rho_r\right),$$
$$\dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\phi_q} + p_{\phi_q} + \rho_m + \rho_r + p_r\right), \quad (41)$$

where $H = \dot{a}/a$ is the Hubble parameter and $\rho_r = 3p_r$. From the conservation of energy, we can obtain the continuity equations for q-deformed dark energy, dark matter, and the radiation constituents in the form of evolution equations, such as

$$\dot{\rho}_{\phi_q} + 3H\left(\rho_{\phi_q} + p_{\phi_q}\right) = -Q', \quad (42)$$

$$\dot{\rho}_m + 3H\rho_m = Q, \quad (43)$$

$$\dot{\rho}_r + 3H(\rho_r + p_r) = Q' - Q, \quad (44)$$

where Q is an interaction current between the q-deformed dark energy and the dark matter which transfers the energy and momentum from the dark matter to dark energy and vice versa. Q and Q' vanish for the models having no coupling between the dark energy and the dark matter. For the models including only the interactions between dark energy and dark matter, the interaction terms become equal $Q' = Q$. The case $Q < 0$ corresponds to energy transfer from dark matter to the other two constituents, the case $Q' > 0$ corresponds to energy transfer from dark energy to the other constituents, and the case $Q' < 0$ corresponds to an energy loss from radiation. Here, we consider that the model only has interaction between dark energy and dark matter and $Q' = Q$ [64].

The energy density ρ and pressure p of this dark energy are rewritten explicitly from the energy-momentum tensor components (4) obtained by the Dirac-Born-Infeld type action of coupling q-deformed dark energy and dark matter, such that [65–68]

$$\rho_{\phi_q} = T^{0\phi_q}_{0} = \frac{1}{2}\dot{\phi}_q^{\,2} + \frac{1}{2}m^2\phi_q^{\,2},$$
$$p_{\phi_q} = -T^{i\phi_q}_{i} = \frac{1}{2}\dot{\phi}_q^{\,2} - \frac{1}{2}m^2\phi_q^{\,2}, \quad (45)$$

where the dark energy is space-independent due to the isotropy and homogeneity. Now, the equation of motion for the q-deformed dark energy can be obtained by inserting (45) into the evolution equation, such that

$$\ddot{\phi}_q + 3H\dot{\phi}_q + m^2\phi_q = -\frac{Q}{\dot{\phi}_q}. \quad (46)$$

In order to obtain the energy density and pressure and equation of motion in terms of the deformation parameter q, (31) and its time derivative will be used. Because the number of particles in each mode of the q-deformed scalar field varies in time by the particle creation and annihilation, the time derivative of $\Delta(q)$ is given as

$$\dot{\Delta}(q) = \frac{-q^{2n_k}\dot{n}_k \ln q}{\sqrt{(1-q^2)(1-q^{2n_k})n_k}} - \frac{\dot{n}_k\sqrt{1-q^{2n_k}}}{2\sqrt{(1-q^2)n_k^3}}. \quad (47)$$

Substituting (31) and (47) in (45) and (46), we obtain

$$\begin{aligned}\rho_{\phi_q} &= \frac{1}{2}\Delta^2(q)\dot{\phi}^2 + \frac{1}{2}\Delta^2(q)m^2\phi^2 + \frac{1}{2}\dot{\Delta}^2(q)\phi^2 \\ &\quad + \Delta(q)\dot{\Delta}(q)\phi\dot{\phi},\end{aligned} \quad (48)$$

$$\begin{aligned}p_{\phi_q} &= \frac{1}{2}\Delta^2(q)\dot{\phi}^2 - \frac{1}{2}\Delta^2(q)m^2\phi^2 + \frac{1}{2}\Delta^2(q)\phi^2 \\ &\quad + \Delta(q)\dot{\Delta}(q)\phi\dot{\phi},\end{aligned} \quad (49)$$

$$\begin{aligned}&\Delta(q)\ddot{\phi} + 3\Delta(q)H\dot{\phi} - \Delta(q)m^2\phi + 2\dot{\Delta}(q)\dot{\phi} \\ &\quad + \ddot{\Delta}(q)\phi + 3H\dot{\Delta}(q)\phi = -\beta\kappa\rho_m.\end{aligned} \quad (50)$$

Here, we consider the commonly used interaction current as $Q = \beta\kappa\rho_m\dot{\phi}_q$ in the literature [15], in order to obtain stationary and stable cosmological solutions in our dark model. The deformed energy density and pressure equations (24) and (27) are the same as (48) and (49), respectively. While (24) and (48) are the expression of the deformed energy density, accordingly (27) and (49) are the deformed pressure of the dark energy in terms of the deformation parameter q. The functions of the deformation parameter in (24) and (48) are

$$\begin{aligned}\Delta_q(n_k) &\approx \Delta(q), \\ \frac{(1+q^2)[n_k]+1}{2n_k+1} &\approx \sqrt{\frac{1-q^{2n_k}}{(1-q^2)n_k}},\end{aligned} \quad (51)$$

since n_k values are very large, and n_k is given as a function of time:

$$n_k \approx \frac{t}{3} + \frac{\sqrt{16t^2+16t-14}}{12} + \frac{1}{6}. \quad (52)$$

We now perform the phase-space analysis of our coupling q-deformed dark energy to dark matter model if the late-time solutions of the universe can be obtained, in order to confirm our proposal.

3. Phase-Space Analysis

We investigate the cosmological properties of the proposed q-deformed dark energy model by performing the phase-space analysis. We need to transform the equations of the dynamical system into their autonomous form [26–28, 36, 37, 69–71]. The auxiliary variables are defined to be

$$\begin{aligned}x_{\phi_q} &= \frac{\kappa(\Delta\dot{\phi} + \dot{\Delta}\phi)}{\sqrt{6}H}, \\ y_{\phi_q} &= \frac{\kappa\sqrt{e^{-\kappa\lambda\Delta\phi}}}{\sqrt{3}H}.\end{aligned} \quad (53)$$

We consider an exponential potential as $V = V_0 e^{-\kappa\lambda\phi_q}$ instead of the potential $V = (1/2)m^2{\phi_q}^2$ in Lagrangian (1), as the usual assumption in the literature, because the power-law potential does not provide a stable attractor solution [16, 18, 72–76].

We also express the density parameters for the q-deformed scalar field dark energy, dark matter, and the radiation in the autonomous system by using (35), (36), and (48) with (53):

$$\Omega_{\phi_q} = \frac{\kappa^2\rho_{\phi_q}}{3H^2} = x_{\phi_q}^2 + y_{\phi_q}^2, \quad (54)$$

$$\Omega_m = \frac{\kappa^2\rho_m}{3H^2}, \quad (55)$$

$$\Omega_r = \frac{\kappa^2\rho_r}{3H^2}, \quad (56)$$

and then the total density parameter is given by

$$\Omega_{\text{tot}} = \frac{\kappa^2\rho_{\text{tot}}}{3H^2} = x_{\phi_q}^2 + y_{\phi_q}^2 + \Omega_m + \Omega_r = 1. \quad (57)$$

Also, the equation of state parameter for the dark energy is written in the autonomous form by using (35) and (36) with (53):

$$w_{\phi_q} = \frac{p_{\phi_q}}{\rho_{\phi_q}} = \frac{x_{\phi_q}^2 - y_{\phi_q}^2}{x_{\phi_q}^2 + y_{\phi_q}^2}. \quad (58)$$

Then, the total equation of state parameter in the autonomous system from (37) and (54)–(56) and (58) is obtained as

$$w_{\text{tot}} = x_{\phi_q}^2 - y_{\phi_q}^2 + \frac{\Omega_r}{3}. \quad (59)$$

We also define $s = -\dot{H}/H^2$ in the autonomous system by using (41) and (59), such that

$$s = -\frac{\dot{H}}{H^2} = \frac{3}{2}(1 + w_{\text{tot}}) = \frac{3}{2}\left(1 + x_{\phi_q}^2 - y_{\phi_q}^2 + \frac{\Omega_r}{3}\right). \quad (60)$$

s is here only a jerk parameter which is used in other equations of cosmological parameters. However, the deceleration

Table 1: Critical points and existence conditions.

Label	x_{ϕ_q}	y_{ϕ_q}	Ω_m	Ω_r	Ω_{ϕ_q}	ω_{ϕ_q}	ω_{tot}
A	$\dfrac{3}{\sqrt{6}\,(\lambda+\beta)}$	$\dfrac{\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)}$	$\dfrac{\lambda(\lambda+\beta)-3}{(\lambda+\beta)^2}$	0	$\dfrac{\beta(\lambda+\beta)+3}{(\lambda+\beta)^2}$	$\dfrac{-\beta(\lambda+\beta)}{\beta(\lambda+\beta)+3}$	$\dfrac{-\beta}{(\lambda+\beta)}$
B	$\dfrac{3}{\sqrt{6}\,(\lambda+\beta)}$	$\dfrac{-\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)}$	$\dfrac{\lambda(\lambda+\beta)-3}{(\lambda+\beta)^2}$	0	$\dfrac{\beta(\lambda+\beta)+3}{(\lambda+\beta)^2}$	$\dfrac{-\beta(\lambda+\beta)}{\beta(\lambda+\beta)+3}$	$\dfrac{-\beta}{(\lambda+\beta)}$
C	$\dfrac{\sqrt{6}\lambda}{6}$	$\sqrt{1-\dfrac{\lambda^2}{6}}$	0	0	1	$\dfrac{\lambda^2}{3}-1$	$\dfrac{\lambda^2}{3}-1$
D	$\dfrac{4}{\sqrt{6}\lambda}$	$\dfrac{2}{\sqrt{3}\lambda}$	0	$1-\dfrac{4}{\lambda^2}$	$\dfrac{4}{\lambda^2}$	$\dfrac{1}{3}$	$\dfrac{1}{3}$
E	$\dfrac{-1}{\sqrt{6}\beta}$	0	$\dfrac{1}{3\beta^2}$	$1-\dfrac{1}{2\beta^2}$	$\dfrac{1}{6\beta^2}$	1	$\dfrac{1}{3}$

parameter q_d which is not used in the equations but is not also a jerk parameter is defined as

$$q_d = -1 - \frac{\dot{H}}{H^2}. \tag{61}$$

Now, we convert the Friedmann equation (41), the continuity equations (43) and (44), and the equation of motion (50) into the autonomous system by using the auxiliary variables in (53)–(56) and their derivatives with respect to $N = \ln a$, for which the time derivative of any quantity F is $\dot{F} = H(dF/dN)$. Thus, we will obtain $X' = f(X)$, where X is the column vector including the auxiliary variables and $f(X)$ is the column vector of the autonomous equations. We then find the critical points X_c of X, by setting $X' = 0$. We then expand $X' = f(X)$ around $X = X_c + U$, where U is the column vector of perturbations of the auxiliary variables, such as δx_{ϕ_q}, δy_{ϕ_q}, $\delta\Omega_m$, and $\delta\Omega_r$ for each constituent in our model. Thus, we expand the perturbation equations up to the first order for each critical point as $U' = MU$, where M is the matrix of perturbation equations. The eigenvalues of perturbation matrix M determine the type and stability of the critical points for each critical point [77–79].

With the definitions for the interaction current and the potential, the autonomous form of the cosmological system reads [80–89]

$$\begin{aligned} x'_{\phi_q} &= -3x_{\phi_q} + s x_{\phi_q} + \frac{\sqrt{6}}{2}\lambda y^2_{\phi_q} - \frac{\sqrt{6}}{2}\beta\Omega_m, \\ y'_{\phi_q} &= s y_{\phi_q} - \frac{\sqrt{6}}{2}\lambda y_{\phi_q} x_{\phi_q}, \\ \Omega'_m &= \Omega_m\left[-3 + \sqrt{6}\beta x_{\phi_q} + 2s\right], \\ \Omega'_r &= \Omega_r\left[-4 + 2s\right]. \end{aligned} \tag{62}$$

In order to perform the phase-space analysis of the model, we obtain the critical points of the autonomous system in (62). We will obtain these points by equating the left hand sides of (62) to zero for stationary solutions, by using the condition $\Omega_{\text{tot}} = 1$. After some calculations, five critical points are found as listed in Table 1 with the existence conditions.

Now, we will get the perturbations $\delta x'_{\phi_q}$, $\delta y'_{\phi_q}$, $\delta\Omega'_m$, and $\delta\Omega'_r$ for each constituent in our model by using the variations of (62), such as

$$\begin{aligned} \delta x'_{\phi_q} &= \left[-\frac{3}{2} + \frac{9}{2}x^2_{\phi_q} - \frac{3}{2}y^2_{\phi_q} + \frac{\Omega_r}{2}\right]\delta x_{\phi_q} \\ &\quad + \left[\sqrt{6}\lambda - 3x_{\phi_q}y_{\phi_q}\right]\delta y_{\phi_q} + \frac{\sqrt{6}}{2}\beta\delta\Omega_m \\ &\quad + \frac{x_{\phi_q}}{2}\delta\Omega_r, \end{aligned} \tag{63}$$

$$\begin{aligned} \delta y'_{\phi_q} &= \left[3x_{\phi_q}y_{\phi_q} - \frac{\sqrt{6}}{2}\lambda y_{\phi_q}\right]\delta x_{\phi_q} \\ &\quad + \left[\frac{3}{2} + \frac{\sqrt{6}}{2}\lambda x_{\phi_q} + \frac{3}{2}x^2_{\phi_q} - \frac{9}{2}y^2_{\phi_q} + \frac{\Omega_r}{2}\right]\delta y_{\phi_q} \\ &\quad + \frac{y_{\phi_q}}{2}\delta\Omega_r, \end{aligned} \tag{64}$$

$$\begin{aligned} \delta\Omega'_m &= \left[6x_{\phi_q}\Omega_m + \sqrt{6}\beta\Omega_m\right]\delta x_{\phi_q} - 6y_{\phi_q}\Omega_m\delta y_{\phi_q} \\ &\quad + \left[\sqrt{6}\beta x_{\phi_q} + 3x^2_{\phi_q} - 3y^2_{\phi_q} + \Omega_r\right]\delta\Omega_m \\ &\quad + \Omega_m\delta\Omega_r, \end{aligned} \tag{65}$$

$$\begin{aligned} \delta\Omega'_r &= 6x_{\phi_q}\Omega_r\delta x_{\phi_q} - 6y_{\phi_q}\Omega_r\delta y_{\phi_q} \\ &\quad + \left[-1 + 3x^2_{\phi_q} - 3y^2_{\phi_q} + 2\Omega_r\right]\delta\Omega_r. \end{aligned} \tag{66}$$

Thus, we obtain a 4×4 perturbation matrix M whose nonzero elements are given as

$$M_{11} = -\frac{3}{2} + \frac{9}{2}x_{\phi_q}^2 - \frac{3}{2}y_{\phi_q}^2 + \frac{\Omega_r}{2},$$
$$M_{12} = \sqrt{6}\lambda y_{\phi_q} - 3x_{\phi_q}y_{\phi_q},$$
$$M_{13} = \frac{\sqrt{6}}{2}\beta,$$
$$M_{14} = \frac{x_{\phi_q}}{2},$$
$$M_{21} = 3x_{\phi_q}y_{\phi_q} - \frac{\sqrt{6}}{2}\lambda y_{\phi_q},$$
$$M_{22} = \frac{3}{2} + \frac{\sqrt{6}}{2}\lambda x_{\phi_q} + \frac{3}{2}x_{\phi_q}^2 - \frac{9}{2}y_{\phi_q}^2 + \frac{\Omega_r}{2},$$
$$M_{24} = \frac{y_{\phi_q}}{2},$$
$$M_{31} = 6x_{\phi_q}\Omega_m + \sqrt{6}\beta\Omega_m,$$
$$M_{32} = -6y_{\phi_q}\Omega_m,$$
$$M_{33} = \sqrt{6}\beta x_{\phi_q} + 3x_{\phi_q}^2 - 3y_{\phi_q}^2 + \Omega_r,$$
$$M_{34} = \Omega_m,$$
$$M_{41} = 6x_{\phi_q}\Omega_r,$$
$$M_{42} = -6y_{\phi_q}\Omega_r,$$
$$M_{44} = -1 + 3x_{\phi_q}^2 - 3y_{\phi_q}^2 + 2\Omega_r. \tag{67}$$

Then, we insert linear perturbations $x_{\phi_q} \to x_{\phi_q,c} + \delta x_{\phi_q}$, $y_{\phi_q} \to y_{\phi_q,c} + \delta y_{\phi_q}$, $\Omega_m \to \Omega_{m,c} + \delta\Omega_m$, and $\Omega_r \to \Omega_{r,c} + \delta\Omega_r$ about the critical points for three constituents in the autonomous system (62). So, we can calculate the eigenvalues of perturbation matrix M for five critical points given in Table 1, with the corresponding existing conditions.

In what follows, we find and represent five perturbation matrices for each of the five critical points. We obtain five sets of eigenvalues. In order to determine the type and stability of critical points, we investigate the sign of the real parts of eigenvalues. A critical point is stable if all the real part of eigenvalues is negative. The physical meaning of the negative eigenvalue is always stable attractor; namely, if the universe is in this state, it keeps its state forever and thus it can attract the universe at a late time. There can occur accelerated expansion only for $w_{\text{tot}} < -1/3$.

A:

$$M = \begin{pmatrix} \frac{9}{2(\lambda+\beta)^2} + \frac{3\lambda}{2(\lambda+\beta)} - 3 & \left[\frac{-9}{6(\lambda+\beta)} + \sqrt{6}\lambda\right]\frac{\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)} & \frac{-\sqrt{6}\beta}{2} & \frac{\sqrt{6}}{4(\lambda+\beta)} \\ \left[\frac{9}{6(\lambda+\beta)} - \frac{\sqrt{6}\lambda}{2}\right]\frac{\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)} & \frac{-3(2\beta(\lambda+\beta)+3)}{2(\lambda+\beta)^2} & 0 & \frac{\sqrt{2\beta(\lambda+\beta)+3}}{2\sqrt{2}(\lambda+\beta)} \\ \left[\frac{18}{\sqrt{6}(\lambda+\beta)} + \sqrt{6}\beta\right]\frac{\lambda(\lambda+\beta)-3}{(\lambda+\beta)^2} & \frac{-6(\lambda(\lambda+\beta)-3)\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)^3} & 0 & \frac{\lambda(\lambda+\beta)-3}{(\lambda+\beta)^2} \\ 0 & 0 & 0 & -4 + \frac{3\lambda}{\lambda+\beta} \end{pmatrix} \tag{68}$$

B:

$$M = \begin{pmatrix} \frac{9}{2(\lambda+\beta)^2} + \frac{3\lambda}{2(\lambda+\beta)} - 3 & \left[\frac{9}{6(\lambda+\beta)} - \sqrt{6}\lambda\right]\frac{\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)} & \frac{-\sqrt{6}\beta}{2} & \frac{\sqrt{6}}{4(\lambda+\beta)} \\ \left[\frac{-9}{6(\lambda+\beta)} + \frac{\sqrt{6}\lambda}{2}\right]\frac{\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)} & \frac{-3(2\beta(\lambda+\beta)+3)}{2(\lambda+\beta)^2} & 0 & \frac{-\sqrt{2\beta(\lambda+\beta)+3}}{2\sqrt{2}(\lambda+\beta)} \\ \left[\frac{18}{\sqrt{6}(\lambda+\beta)} + \sqrt{6}\beta\right]\frac{\lambda(\lambda+\beta)-3}{(\lambda+\beta)^2} & \frac{6(\lambda(\lambda+\beta)-3)\sqrt{2\beta(\lambda+\beta)+3}}{\sqrt{2}(\lambda+\beta)^3} & 0 & \frac{\lambda(\lambda+\beta)-3}{(\lambda+\beta)^2} \\ 0 & 0 & 0 & -4 + \frac{3\lambda}{\lambda+\beta} \end{pmatrix} \tag{69}$$

C:

$$M = \begin{pmatrix} \lambda^2 - 3 & \frac{\sqrt{6}\lambda}{2}\sqrt{1-\frac{\lambda^2}{6}} & \frac{-\sqrt{6}\beta}{2} & \frac{\sqrt{6}\lambda}{12} \\ 0 & \frac{\lambda^2}{2} - 3 & 0 & \frac{1}{2}\sqrt{1-\frac{\lambda^2}{6}} \\ 0 & 0 & \lambda^2 + \lambda\beta - 3 & 0 \\ 0 & 0 & 0 & \lambda^2 - 4 \end{pmatrix} \quad (70)$$

D:

$$M = \begin{pmatrix} \frac{8}{\lambda^2} - 1 & 2\sqrt{2} - \frac{4\sqrt{2}}{\lambda^2} & \frac{-\sqrt{6}\beta}{2} & \frac{2}{\sqrt{6}\lambda} \\ -\sqrt{2} + \frac{4\sqrt{2}}{\lambda^2} & \frac{4}{\lambda^2} & 0 & \frac{1}{\sqrt{3}\lambda} \\ 0 & 0 & \frac{4\beta}{\lambda} + 1 & 0 \\ \frac{24}{\sqrt{6}\lambda}\left(1 - \frac{4}{\lambda^2}\right) & \frac{-12}{\sqrt{3}\lambda}\left(1 - \frac{4}{\lambda^2}\right) & 0 & 1 - \frac{4}{\lambda^2} \end{pmatrix} \quad (71)$$

E:

$$M = \begin{pmatrix} \frac{1}{2\beta^2} - 1 & 0 & \frac{-\sqrt{6}\beta}{2} & \frac{-1}{2\sqrt{6}\beta} \\ 0 & 2 + \frac{\lambda}{\beta} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3\beta^2} \\ \frac{\sqrt{6}}{\beta}\left(1 - \frac{1}{2\beta^2}\right) & 0 & 0 & 1 - \frac{1}{2\beta^2} \end{pmatrix} \quad (72)$$

Eigenvalues of the five M matrices with the existence conditions, stability conditions, and acceleration condition are represented in Table 2, for each of the critical points A, B, C, D, and E. As seen in Table 2, the first two critical points A and B have the same eigenvalues. Here, the eigenvalues and the stability conditions of the perturbation matrices for critical points A, B, D, and E have been obtained by the numerical methods, due to the complexity of the matrices. The stability conditions of each critical point are listed in Table 2, according to the sign of the eigenvalues.

We now analyze the cosmological behavior of each critical point by noting the attractor solutions in scalar field cosmology [90]. From the theoretical cosmology studies, we know that the energy density of a scalar field has an effect on the determination of the evolution of the universe. Cosmological attractors provide the understanding of the evolution and the affecting factors on this evolution; for example, from the dynamical conditions, the scalar field evolution approaches a certain kind of behavior without initial fine tuning conditions [91–101]. Attractor behavior is known as a situation in which a collection of phase-space points evolve into a particular region and never leave from there.

Critical Point A. This point exists for $\beta(\lambda + \beta) > -3/2$ and $\lambda(\lambda + \beta) > 3$. Because of $w_{\text{tot}} < -1/3$, acceleration occurs at this point if $\lambda < 2\beta$ and it is an expansion phase since y_{ϕ_q} is positive, so H is positive, too. Point A is stable, meaning that the universe keeps its further evolution, if λ and β take the values for the negative eigenvalues given in the second column of Table 2. In Figure 1, we also represent the 2-dimensional and 3-dimensional projections of 4-dimensional phase-space trajectories for $\beta = 2.51$, $\lambda = 3.1$, $\lambda = 3.6$, and $\lambda = 4.1$. This state corresponds to a stable attractor starting from the critical point $A = (0.21, 0.70, 0.45, 0)$, as seen from the plots in Figure 1. Also, zero value of critical point Ω_r cancels the total behavior Ω_r' in (66).

Critical Point B. Point B also exists for $\beta(\lambda + \beta) > -3/2$ and $\lambda(\lambda + \beta) > 3$. Acceleration phase is again valid here if $\lambda < 2\beta$ leading $w_{\text{tot}} < -1/3$, but this point refers to contraction phase because y_{ϕ_q} is negative here. Stability of the point B is again satisfied for λ and β values given in the second column of Table 2. Therefore, the stable attractor behavior is represented starting from the critical point $B = (0.21, -0.70, 0.45, 0)$ for $\beta = 2.51$, $\lambda = 3.1$, $\lambda = 3.6$, and $\lambda = 4.1$ values, in Figure 2. The zero value of critical point Ω_r again cancels the total behavior Ω_r' in (66).

Critical Point C. Critical point C occurs for all values of β, while $\lambda < \sqrt{6}$. The cosmological behavior is again an acceleration phase that occurs if $\lambda < \sqrt{2}$ providing $w_{\text{tot}} < -1/3$ and an expansion phase since y_{ϕ_q} is positive. Point C is stable if $\beta < (3 - \lambda^2)/\lambda$ and $\lambda < \sqrt{3}$. Two-dimensional projection of phase-space is represented in Figure 3, for $\beta = 1.5$, $\lambda = 0.001$, $\lambda = 0.5$, and $\lambda = 1.1$. The stable attractor starting from the critical point $C = (0.48, 0.87, 0, 0)$ can be inferred from Figure 3. We again find zero plots containing zero values Ω_m and Ω_r, since they cancel the total behaviors Ω_m' and Ω_r' in (65) and (66).

Critical Point D. This point exists for any values of β, while $\lambda > 2$. Acceleration phase never occurs due to $w_{\text{tot}} = 1/3$. Point D is always unstable for any values of β and λ. This state corresponds to an unstable saddle point starting from the point $D = (0.54, 0.38, 0, 0.55)$ for $\beta = 1.5$, $\lambda = 3$, $\lambda = 4$, and $\lambda = 6$, as seen from the plots in Figure 4. Zero plots containing the axis Ω_m lead to the cancellation of the total behavior Ω_m' in (65), since $\Omega_m = 0$, so they are not represented in Figure 4.

Critical Point E. This point exists for any values of λ, while $\beta > 1/\sqrt{2}$. Acceleration phase never occurs due to $w_{\text{tot}} = 1/3$. Point E is always unstable for any values of β and λ. This state corresponds to an unstable saddle point starting from the point $= (-0.24, 0, 0.11, 0.82)$ for $\lambda = 1$, $\beta = 1.7$, $\beta = 2.6$, and $\beta = 3.5$, as seen from the plots in Figure 5. Zero plots containing the axis y_{ϕ_q} lead to the cancellation of the total behavior y_{ϕ_q}' in (64), since $y_{\phi_q} = 0$, so they are not represented in Figure 5.

All the plots in Figures 1–3 have the structure of stable attractor, since each of them evolves to a single point which is in fact one of the critical points in Table 1. These evolutions

Table 2: Eigenvalues and stability of critical points.

	Eigenvalues				λ	β	
A and B	−0.7516	−0.7516	−0.0065	−1.0065	4.6000	0.0100	Existing condition is $\beta(\lambda+\beta) > -3/2$ and $\lambda(\lambda+\beta) > 3$. Stable point if λ and β are the given values for the negative eigenvalues in the second column. Acceleration phase occurs if $\lambda < 2\beta$.
	−0.7518	−0.7518	−0.0073	−1.0073	4.1000	0.0100	
	−0.7521	−0.7521	−0.0083	−1.0083	3.6000	0.0100	
	−0.7524	−0.7524	−0.0096	−1.0096	3.1000	0.0100	
	−0.8249	−0.8249	−0.2994	−1.2994	4.6000	0.5100	
	−0.8330	−0.8330	−0.3319	−1.3319	4.1000	0.5100	
	−0.8431	−0.8431	−0.3723	−1.3723	3.6000	0.5100	
	−0.8560	−0.8560	−0.4238	−1.4238	3.1000	0.5100	
	−0.8850	−0.8850	−0.5401	−1.5401	4.6000	1.0100	
	−0.8982	−0.8982	−0.5930	−1.5930	4.1000	1.0100	
	−0.9143	−0.9143	−0.6573	−1.6573	3.6000	1.0100	
	−0.9343	−0.9343	−0.7372	−1.7372	3.1000	1.0100	
	−0.9354	−0.9354	−0.7414	−1.7414	4.6000	1.5100	
	−0.9519	−0.9519	−0.8075	−1.8075	4.1000	1.5100	
	−0.9716	−0.9716	−0.8865	−1.8865	3.6000	1.5100	
	−0.9781	−0.9781	−0.9123	−1.9123	4.6000	2.0100	
	−0.9957	−0.9957	−0.9826	−1.9826	3.1000	1.5100	
	−0.9967	−0.9967	−0.9869	−1.9869	4.1000	2.0100	
	−1.0148	−1.0148	−1.0591	−2.0591	4.6000	2.5100	
	−1.0187	−1.0187	−1.0749	−2.0749	3.6000	2.0100	
	−1.0348	−1.0348	−1.1392	−2.1392	4.1000	2.5100	
	−1.0450	−1.0450	−1.1800	−2.1800	3.1000	2.0100	
	−1.0581	−1.0581	−1.2324	−2.2324	3.6000	2.5100	
	−1.0856	−1.0856	−1.3422	−2.3422	3.1000	2.5100	
	Eigenvalues						
C	$\lambda^2+\lambda\beta-3$, λ^2-3, λ^2-4, $\frac{\lambda^2}{2}-3$,						Existing condition is $\lambda < \sqrt{6}$. Stable point if $\beta < (3-\lambda^2)/\lambda$ and $\lambda < \sqrt{3}$. Acceleration phase occurs if $\lambda < \sqrt{2}$.

Table 2: Continued.

	Eigenvalues				λ	β	
D	−0.5000	−0.5000	1.0000	0.2000	2.5000	−0.5000	Existing condition is $\lambda > 2$ and $\forall\beta$. Unstable point. Acceleration phase never occurs.
	−0.5000	−0.5000	1.0000	0.3333	3.0000	−0.5000	
	−0.5000	−0.5000	1.0000	0.4286	3.5000	−0.5000	
	−0.5000	−0.5000	1.0000	0.5000	4.0000	−0.5000	
	1.0000	−1.0000	0	0	2.0000	−0.5000	
	1.0000	−1.0000	2.0000	0	2.0000	0.5000	
	1.0000	−1.0000	4.0000	0	2.0000	1.5000	
	1.0000	−1.0000	−2.0000	0	2.0000	−1.5000	
	−0.5000	−0.5000	1.0000	0	4.0000	−1.0000	
	−0.5000	−0.5000	1.0000	1.0000	2.5000	0	
	−0.5000	−0.5000	1.0000	1.0000	3.5000	0	
	−0.5000	−0.5000	1.0000	1.0000	4.0000	0	
	−0.5000	−0.5000	1.0000	1.5000	4.0000	0.5000	
	−0.5000	−0.5000	1.0000	1.5714	3.5000	0.5000	
	−0.5000	−0.5000	1.0000	1.6667	3.0000	0.5000	
	−0.5000	−0.5000	1.0000	1.8000	2.5000	0.5000	
	−0.5000	−0.5000	1.0000	2.0000	4.0000	1.0000	
	−0.5000	−0.5000	1.0000	2.1429	3.5000	1.0000	
	−0.5000	−0.5000	1.0000	2.3333	3.0000	1.0000	
	−0.5000	−0.5000	1.0000	2.5000	4.0000	1.5000	
	−0.5000	−0.5000	1.0000	2.6000	2.5000	1.0000	
	−0.5000	−0.5000	1.0000	2.7143	3.5000	1.5000	
	−0.5000	−0.5000	1.0000	3.0000	3.0000	1.5000	
	−0.5000	−0.5000	1.0000	3.0000	4.0000	2.0000	
	−0.5000	−0.5000	1.0000	3.2857	3.5000	2.0000	
	−0.5000	−0.5000	1.0000	3.4000	2.5000	1.5000	
	−0.5000	−0.5000	1.0000	3.6667	3.0000	2.0000	
	−0.5000	−0.5000	1.0000	4.2000	2.5000	2.0000	
	−0.5000	−0.5000	1.0000	−0.1429	3.5000	−1.0000	
	−0.5000	−0.5000	1.0000	−0.3333	3.0000	−1.0000	
	−0.5000	−0.5000	1.0000	−0.5000	4.0000	−1.5000	
	−0.5000	−0.5000	1.0000	−0.6000	2.5000	−1.0000	
	−0.5000	−0.5000	1.0000	−0.7143	3.5000	−1.5000	
	−0.5000	−0.5000	1.0000	−1.0000	3.0000	−1.5000	
	−0.5000	−0.5000	1.0000	−1.0000	4.0000	−2.0000	
	−0.5000	−0.5000	1.0000	−1.2857	3.5000	−2.0000	
	−0.5000	−0.5000	1.0000	−1.4000	2.5000	−1.5000	
	−0.5000	−0.5000	1.0000	−1.6667	3.0000	−2.0000	
	−0.5000	−0.5000	1.0000	−2.2000	2.5000	−2.0000	

Table 2: Continued.

	Eigenvalues				λ	β	
E	0.2634	−0.1317	−0.1317	2.0704	0.1000	0.7100	Existing condition is $\beta > 1/\sqrt{2}$ and $\forall\lambda$. Unstable point. Acceleration phase never occurs.
	0.2634	−0.1317	−0.1317	2.4225	0.6000	0.7100	
	0.2634	−0.1317	−0.1317	2.7746	1.1000	0.7100	
	0.2634	−0.1317	−0.1317	3.1268	1.6000	0.7100	
	0.2634	−0.1317	−0.1317	3.4789	2.1000	0.7100	
	0.2634	−0.1317	−0.1317	3.8310	2.6000	0.7100	
	0.2634	−0.1317	−0.1317	4.1831	3.1000	0.7100	
	0.2634	−0.1317	−0.1317	4.5352	3.6000	0.7100	
	1.0273	−0.8883	−0.1390	2.0185	0.1000	2.7100	
	1.0273	−0.8883	−0.1390	2.1107	0.6000	2.7100	
	1.0273	−0.8883	−0.1390	2.2030	1.1000	2.7100	
	1.0273	−0.8883	−0.1390	2.2952	1.6000	2.7100	
	1.0273	−0.8883	−0.1390	2.3875	2.1000	2.7100	
	1.0273	−0.8883	−0.1390	2.4797	2.6000	2.7100	
	1.0273	−0.8883	−0.1390	2.5720	3.1000	2.7100	
	1.0273	−0.8883	−0.1390	2.6642	3.6000	2.7100	
	1.0368	−0.8208	−0.2160	2.0226	0.1000	2.2100	
	1.0368	−0.8208	−0.2160	2.1357	0.6000	2.2100	
	1.0368	−0.8208	−0.2160	2.2489	1.1000	2.2100	
	1.0368	−0.8208	−0.2160	2.3620	1.6000	2.2100	
	1.0368	−0.8208	−0.2160	2.4751	2.1000	2.2100	
	1.0368	−0.8208	−0.2160	2.5882	2.6000	2.2100	
	1.0368	−0.8208	−0.2160	2.7014	3.1000	2.2100	
	1.0368	−0.8208	−0.2160	2.8145	3.6000	2.2100	
	1.0437	−0.5219	−0.5219	2.0413	0.1000	1.2100	
	1.0437	−0.5219	−0.5219	2.2479	0.6000	1.2100	
	1.0437	−0.5219	−0.5219	2.4545	1.1000	1.2100	
	1.0437	−0.5219	−0.5219	2.6612	1.6000	1.2100	
	1.0437	−0.5219	−0.5219	2.8678	2.1000	1.2100	
	1.0437	−0.5219	−0.5219	3.0744	2.6000	1.2100	
	1.0437	−0.5219	−0.5219	3.2810	3.1000	1.2100	
	1.0437	−0.5219	−0.5219	3.4876	3.6000	1.2100	
	1.0485	−0.5910	−0.4575	2.0292	0.1000	1.7100	
	1.0485	−0.5910	−0.4575	2.1754	0.6000	1.7100	
	1.0485	−0.5910	−0.4575	2.3216	1.1000	1.7100	
	1.0485	−0.5910	−0.4575	2.4678	1.6000	1.7100	
	1.0485	−0.5910	−0.4575	2.6140	2.1000	1.7100	
	1.0485	−0.5910	−0.4575	2.7602	2.6000	1.7100	
	1.0485	−0.5910	−0.4575	2.9064	3.1000	1.7100	

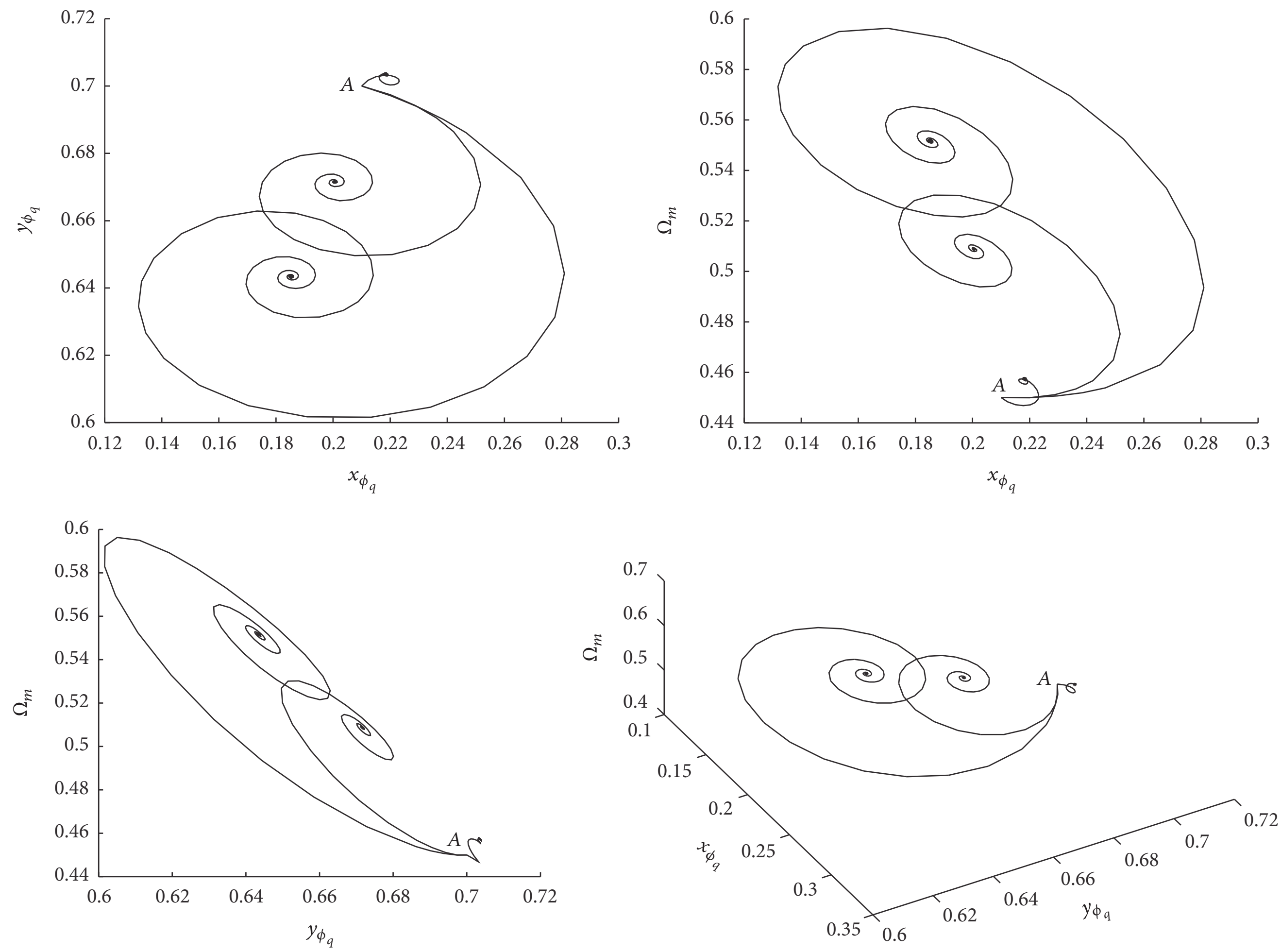


Figure 1: Two- and three-dimensional projections of the phase-space trajectories for $\beta = 2.51$, $\lambda = 3.1$, $\lambda = 3.6$, and $\lambda = 4.1$. All plots begin from the critical point $A = (0.21, 0.70, 0.45, 0)$ being a stable attractor.

to the critical points are the attractor solutions in coupling q-deformed dark energy to dark matter cosmology of our model, which imply an expanding universe. Therefore, we confirm that the dark energy in our model can be defined in terms of the q-deformed scalar fields obeying the q-deformed boson algebra in (17)–(20). According to the stable attractor behaviors, it makes sense to consider the dark energy as a scalar field defined by the q-deformed scalar field, due to the negative pressure of q-deformed boson field, as dark energy field.

Finally, we can investigate the relation between q-deformed and standard dark energy density, pressure, and scalar field equations in (24), (27), and (31). We illustrate the behavior of q-deformed energy density and pressure in terms of the standard ones with respect to the total number of particles and the deformation parameter q in Figures 6 and 7, respectively. We observe that for a large particle number the q-deformed energy density and pressure function decrease with the decrease in deformation parameter q. On the contrary, if the particle number is small, the deformed energy density and pressure increase with the decrease in deformation parameter. Note that when the deformation parameter decreases from 1, this increases the deformation of the model, since the deformation vanishes by approaching 1. The deformation parameter significantly affects the value of the deformed energy density and pressure. In the $q \to 1$ limit, deformed energy density and pressure function became identical to the standard values, as expected.

In Figure 8, we represent the q-deformed scalar field behavior in terms of the standard one. It is observed that while the deformation parameter $q \to 1$, q-deformed scalar field becomes identical to the standard one. However, it asymptotically approaches lower values, while q decreases with large number of particles. Since the square of a quantum mechanical field means the probability density, q-deformed probability density decreases when the deformation increases, and in the $q \to 0$ limit it approaches zero.

Also, since the dark matter pressure is taken to be zero, $\omega_m \approx 0$ and $\omega_{\text{tot}} \approx \omega_{\phi_q}$. For the stable accelerated expansion condition $\omega_{\text{tot}} \approx \omega_{\phi_q} < -1/3$, solutions require the scalar field dark energy pressure to be negative $p_{\phi_q} < 0$. From the relation $p_{\phi_q} = \Delta_q(n_k) p_\phi$ in (27), we finally represent the effect of q on the deformed dark energy pressure p_{ϕ_q}, namely, on the

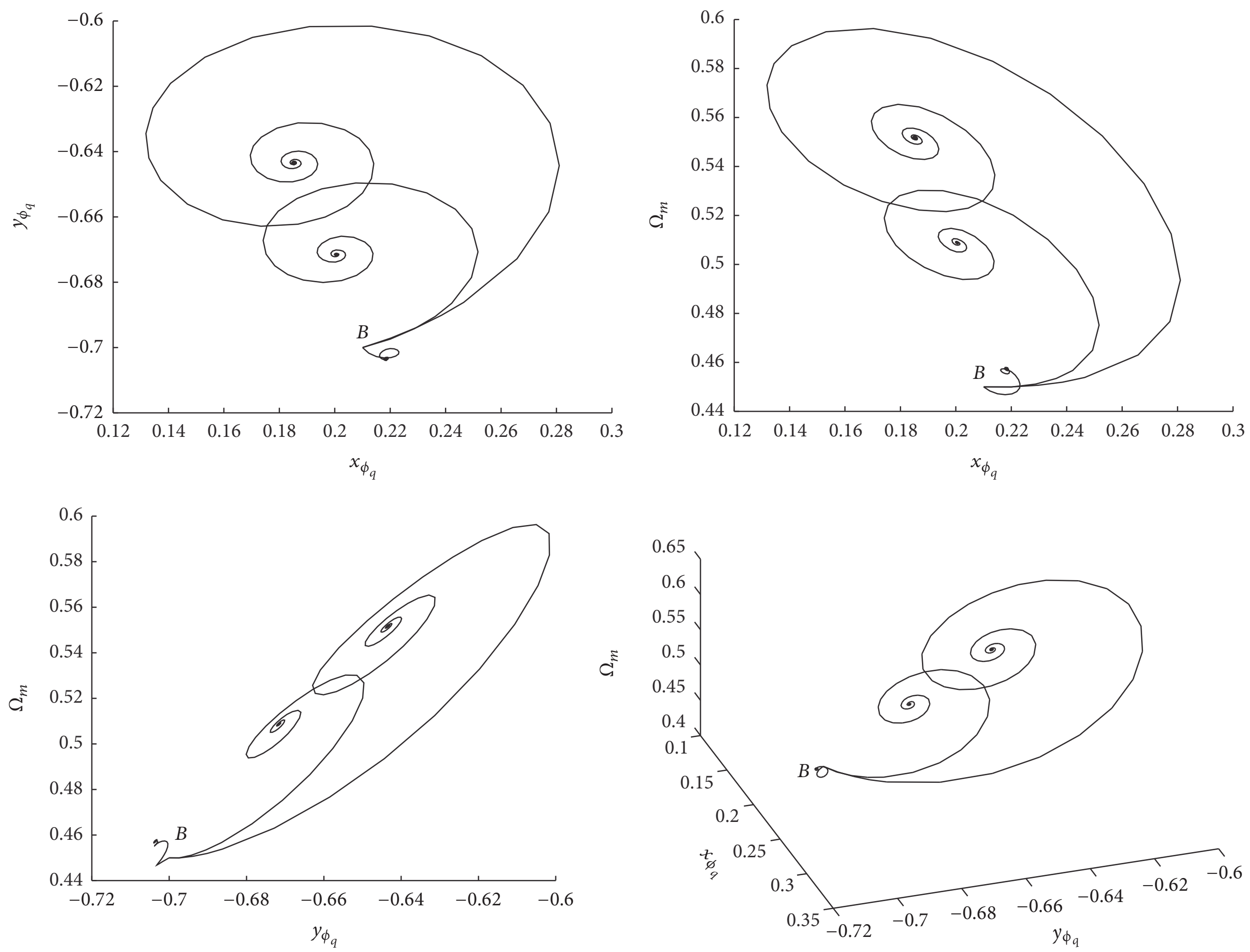


FIGURE 2: Two- and three-dimensional projections of the phase-space trajectories for $\beta = 2.51$, $\lambda = 3.1$, $\lambda = 3.6$, and $\lambda = 4.1$. All plots begin from the critical point $B = (0.21, -0.70, 0.45, 0)$ being a stable attractor.

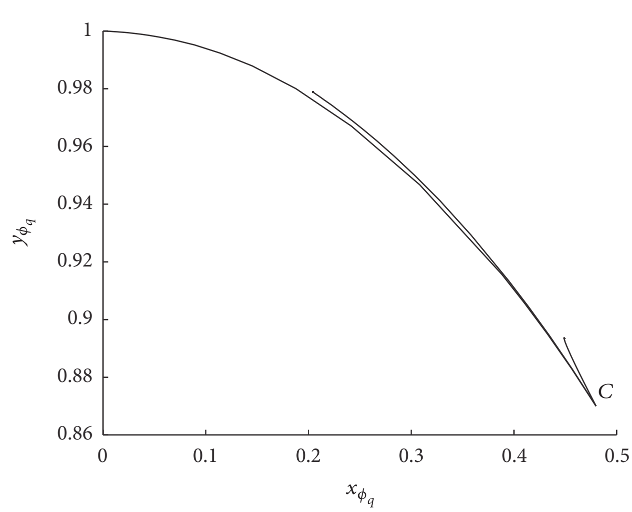


FIGURE 3: Two-dimensional projections of the phase-space trajectories for $\beta = 1.5$, $\lambda = 0.001$, $\lambda = 0.5$, and $\lambda = 1.1$. All plots begin from the critical point $C = (0.48, 0.87, 0, 0)$ being a stable attractor.

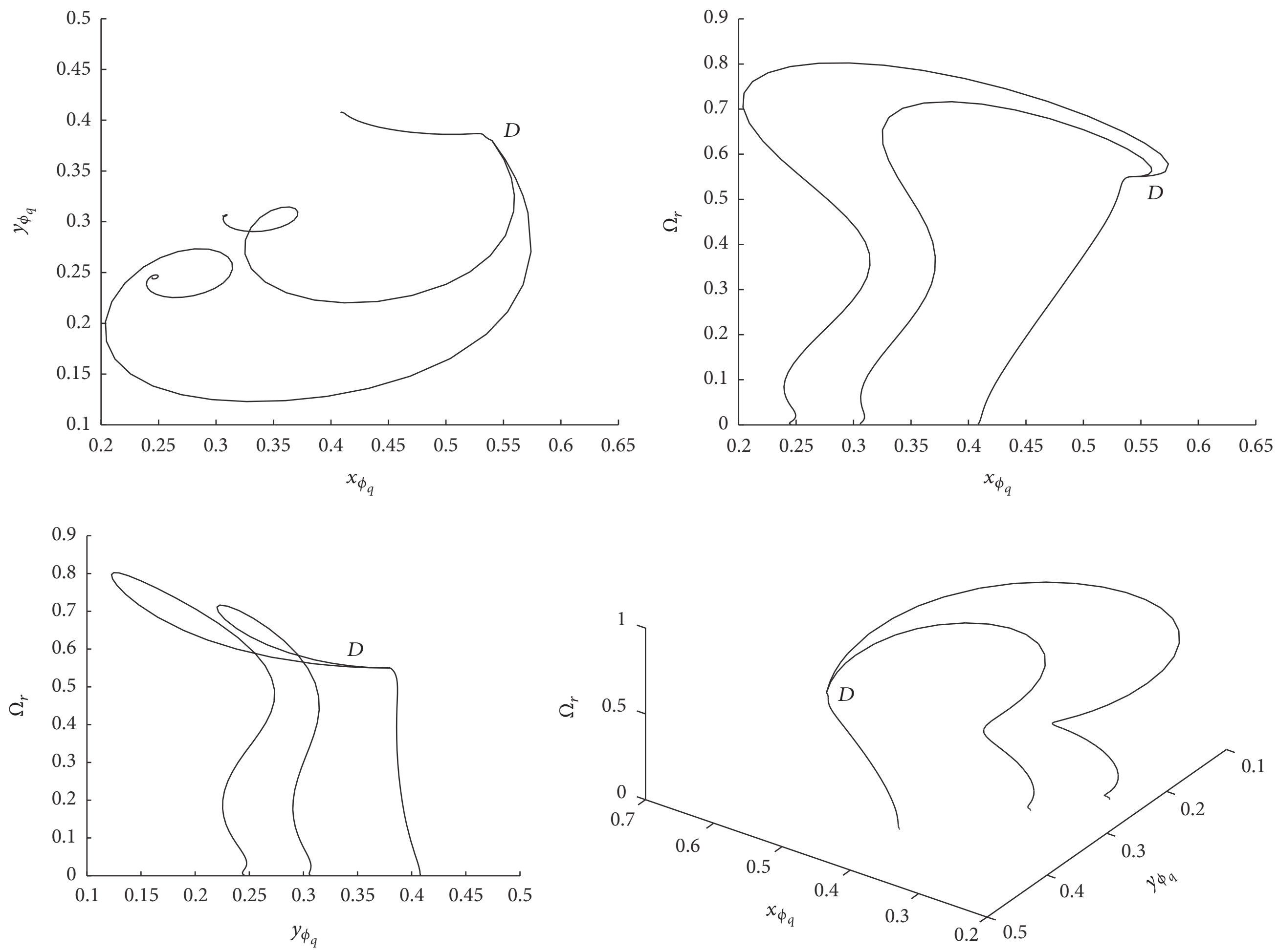


Figure 4: Two- and three-dimensional projections of the phase-space trajectories for $\beta = 1.5$, $\lambda = 3$, $\lambda = 4$, and $\lambda = 6$. All plots begin from the critical point $D = (0.54, 0.38, 0, 0.55)$ being an unstable solution.

accelerated expansion behavior in Figure 9. From the figure, we deduce that, for any values of q, the deformed dark energy shows the accelerated expansion behavior with the negative deformed dark energy pressure.

4. Conclusion

Since it is known that the dark energy has a negative pressure acting as gravitational repulsion to drive the accelerated expansion of the universe, we are motivated to propose that the dark energy consists of negative-pressure q-deformed scalar field whose field equation is defined by the q annihilation and creation operators satisfying the q-deformed boson algebra in (17)–(20). In order to confirm our proposal, we consider q-deformed dark energy coupling to the dark matter inhomogeneities and then investigate the dynamics of the model. Later on, we perform the phase-space analysis, whether it will give stable attractor solutions or not, which refers to the accelerating expansion phase of the universe. Therefore, the action integral of coupling q-deformed dark energy model is set up to study its dynamics, and the Hubble parameter and Friedmann equations of the model are obtained in spatially flat FRW geometry. Later on, we find the energy density and pressure values with the evolution equations for q-deformed dark energy, dark matter, and the radiation fields from the variation of the action and the Lagrangian of the model. After that, we translate these dynamical equations into the autonomous form by introducing suitable auxiliary variables, in order to perform the phase-space analysis of the model. Then, the critical points of the autonomous system are obtained by setting each autonomous equation to zero and four perturbation matrices can be written for each critical point by constructing the perturbation equations. We determine the eigenvalues of four perturbation matrices to examine the stability of critical points. There are also some important calculated cosmological parameters, such as the total equation of state parameter and the deceleration parameter to check whether the critical points satisfy an accelerating universe. We obtain four stable attractors for the model depending on the coupling parameter β. An accelerating universe exists for all stable solutions due to $w_{\text{tot}} < -1/3$. The critical points A and B are late-time stable attractors for the given λ and β values for the negative eigenvalues in the second column of Table 2.

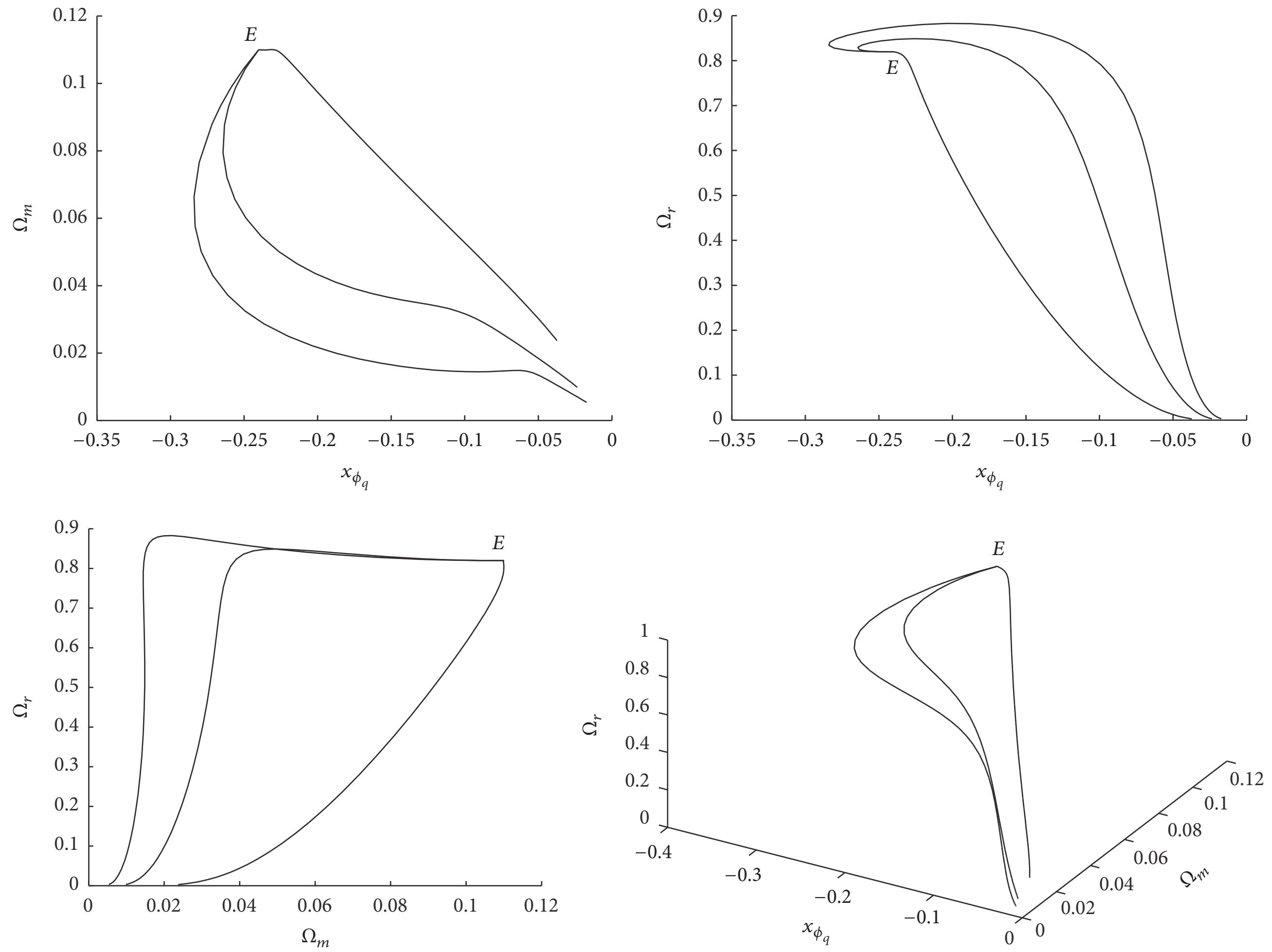


FIGURE 5: Two- and three-dimensional projections of the phase-space trajectories for $\lambda = 1$, $\beta = 1.7$, $\beta = 2.6$, and $\beta = 3.5$. All plots begin from the critical point $E = (-0.24, 0, 0.11, 0.82)$ being an unstable solution.

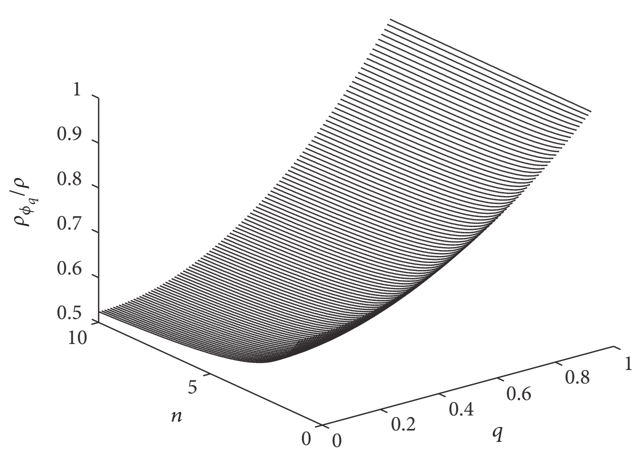


FIGURE 6: q-deformed energy density for various values of n and q, in terms of standard energy density.

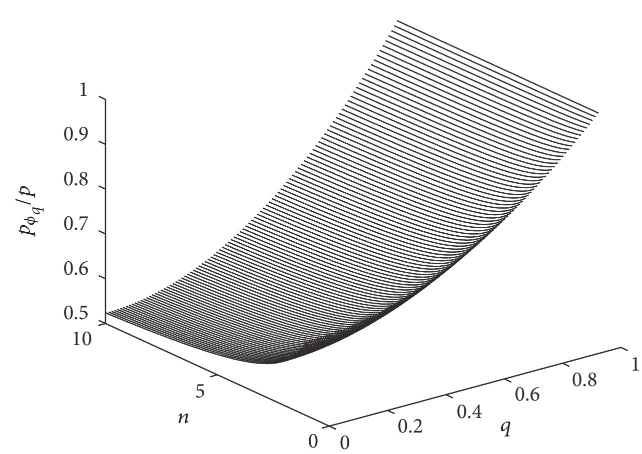


FIGURE 7: q-deformed pressure for various values of n and q, in terms of standard pressure.

The point A refers to expansion, while the point B refers to contraction with stable acceleration for $\lambda < 2\beta$. However, the critical point C is late-time stable attractor for $\beta < (3 - \lambda^2)/\lambda$ and $\lambda < \sqrt{3}$ with expansion. The stable attractor behavior of the model at each critical point is demonstrated in Figures 1–3. In order to solve the differential equation system (62)

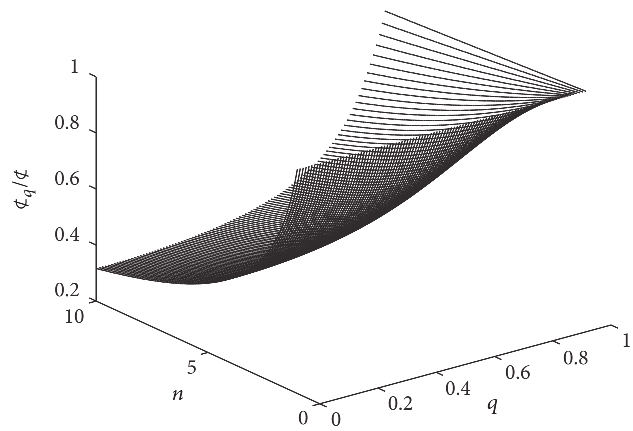
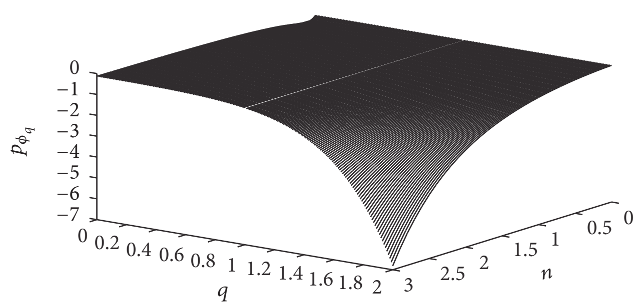


FIGURE 8: q-deformed scalar field for various values of n and q, in terms of standard scalar field.



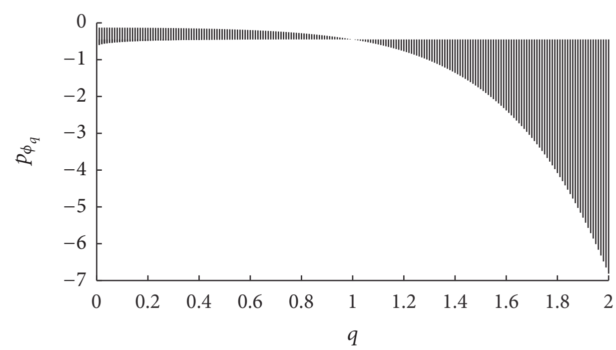


FIGURE 9: Effect of q on the accelerated expansion behavior with negative dark energy pressure.

with the critical points and plot the graphs in Figures 1–5, we use adaptive Runge-Kutta method of 4th and 5th order, in Matlab programming. Then, the solutions with the stability conditions of critical points are plotted for each pair of the solution sets being the auxiliary variables in (53), (55), and (56).

These figures represent the notion that, by choosing the suitable parameters of the model, we obtain the stable and unstable attractors as A, B, C, D, and E, depending on the existence conditions of critical points A, B, C, D, and E. Also, the suitable parameters with the stability conditions give the stable accelerated behavior for A, B, and C attractor models.

The q-deformed dark energy is a generalization of the standard scalar field dark energy. As seen from the behavior of the deformed energy density, pressure, and scalar field functions with respect to the standard functions, in the $q \to 1$ limit, they all approach the standard corresponding function values. However, in the $q \to 0$ limit, the deformed energy density and the pressure functions decrease to smaller values of the standard energy density and the pressure function values, respectively. This implies that the energy momentum of the scalar field decreases when the deformation becomes more apparent, since q reaches 1 which gives the nondeformed state. Also, when $q \to 0$ for large n values, the deformed scalar field approaches zero value meaning a decrease in the probability density of the scalar field. This state is expected to represent an energy-momentum decrease leading to a decrease in the probability of finding the particles of the field. Consequently, q deformation of the scalar field dark energy gives a self-consistent model due to the existence of standard case parameters of the dark energy in the $q \to 1$ limit and the existence of the stable attractor behavior of the accelerated expansion phase of the universe for the considered coupling dark energy and dark matter model.

The results confirm that the proposed q-deformed scalar field dark energy model is consistent since it gives the expected behavior of the universe. The idea to consider the dark energy as a q-deformed scalar field is a very recent approach. There are more deformed particle algebras in the literature which can be considered as other and maybe more suitable candidates for the dark energy. As a further study on the purpose of confirming whether the dark energy can be considered as a general deformed scalar field, the other couplings between dark energy and dark matter and also in the other framework of gravity, such as teleparallel or maybe $f(R)$ gravity, can be investigated.

Competing Interests

The author declares that they there are no competing interests regarding the publication of this paper.

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