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Research Article

A Specific $\mathcal{N} = 2$ Supersymmetric Quantum Mechanical Model: Supervariable Approach

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By exploiting the supersymmetric invariant restrictions on the chiral and antichiral supervariables, we derive the off-shell nilpotent symmetry transformations for a specific $(0 + 1)$-dimensional $\mathcal{N} = 2$ supersymmetric quantum mechanical model which is considered on a $(1, 2)$-dimensional supermanifold (parametrized by a bosonic variable t and a pair of Grassmannian variables $(\theta, \bar{\theta})$). We also provide the geometrical meaning to the symmetry transformations. Finally, we show that this specific $\mathcal{N} = 2$ SUSY quantum mechanical model is a model for Hodge theory.

1. Introduction

Gauge theory is one of the most important theories of modern physics because *three* out of *four* fundamental interactions of nature are governed by the gauge theory. The Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the systematic approaches to covariantly quantize any p-form ($p = 1, 2, 3, \ldots$) gauge theories, where the local gauge symmetry of a given theory is traded with the "quantum" gauge (i.e., (anti-)BRST) symmetry transformations [1–4]. It is important to point out that the (anti-)BRST symmetries are nilpotent and absolutely anticommuting in nature. One of the unique, elegant, and geometrically rich methods to derive these (anti-)BRST transformations is the superfield formalism, where the horizontality condition (HC) plays an important role [5–12]. This HC is a useful tool to derive the BRST, as well as anti-BRST symmetry transformations for any (non-)Abelian p-form ($p = 1, 2, 3, \ldots$) gauge theory, where no interaction between the gauge and matter fields is present.

For the derivation of a full set of (anti-)BRST symmetry transformations in the case of interacting gauge theories, a powerful method known as augmented version of superfield formalism has been developed in a set of papers [13–16]. In augmented superfield formalism, some conditions named gauge invariant restrictions (GIRs), in addition to the HC, have been imposed to obtain the off-shell nilpotent and absolutely anticommuting (anti-)BRST transformations. It is worthwhile to mention here that this technique has also been applied in case of some $\mathcal{N} = 2$ supersymmetric (SUSY) quantum mechanical (QM) models to derive the off-shell nilpotent SUSY symmetry transformations [17–20]. These SUSY transformations have been derived by using the supersymmetric invariant restrictions (SUSYIRs) and it has been observed that the SUSYIRs are the generalizations of the GIRs, in case of $\mathcal{N} = 2$ SUSY QM theory.

The aim of present investigation is to explore and apply the augmented version of HC to a *new* $\mathcal{N} = 2$ SUSY QM model which is different from the earlier models present in the literature. In our present endeavor, we derive the off-shell nilpotent SUSY transformations for a specific $\mathcal{N} = 2$ SUSY QM model by exploiting the potential and power of the SUSYIRs. The additional reason behind our present investigation is to take one more step forward in the direction of the confirmation of SUSYIRs (i.e., generalization of augmented superfield formalism) as a powerful technique for the derivation of SUSY transformations for any *general* $\mathcal{N} = 2$ SUSY QM system.

One of the key differences between the (anti-)BRST and SUSY symmetry transformations is that the (anti-)BRST symmetries are nilpotent as well as absolutely anticommuting in nature, whereas SUSY transformations are *only* nilpotent and the anticommutator of fermionic transformations produces an *additional* symmetry transformation of the theory. Due to this basic reason, we are theoretically forced to use the (anti)chiral supervariables generalized on the (1, 1)-dimensional super-submanifolds of the full (1, 2)-dimensional supermanifold. The latter is parametrized by the superspace coordinate $Z^M = (t, \theta, \bar{\theta})$, where θ, $\bar{\theta}$ are the Grassmannian variables and t is the time-evolution parameter.

The contents of present investigation are organized as follows. In Section 2, we discuss the symmetry transformations associated with the specific $\mathcal{N} = 2$ SUSY QM model. It is to be noted that there are *three* continuous symmetries associated with this particular model, in which *two* of them are fermionic and *one* is bosonic in nature. Section 3 is devoted to the derivation of one of the fermionic transformations by using the antichiral supervariable approach. We derive the second SUSY fermionic symmetry by exploiting the chiral supervariable approach in Section 4. In Section 5, the Lagrangian of the model is presented in terms of the (anti)chiral supervariables and the geometrical interpretation for invariance of the Lagrangian in terms of the Grassmannian derivatives (∂_θ and $\partial_{\bar{\theta}}$) is explicated. Furthermore, we also represent the charges corresponding to the continuous symmetry transformations in terms of (anti)chiral supervariables. In Section 6, we show that the fermionic SUSY symmetry transformations satisfy the $\mathcal{N} = 2$ SUSY algebra, which is identical to the Hodge algebra obeyed by the cohomological operators of differential geometry. Thus, we show that this particular $\mathcal{N} = 2$ SUSY QM model is an example of Hodge theory. Finally, we draw conclusions in Section 7, with remarks.

2. Preliminaries: A specific $\mathcal{N} = 2$ SUSY QM Model

We begin with the action of a specific (0 + 1)-dimensional $\mathcal{N} = 2$ QM model [21]:

$$S = \int dt L_0 = \int dt \left[\left(\frac{d\phi}{dt} + s \frac{\partial V}{\partial \phi} \right)^2 - i\bar{\psi} \left(\frac{d}{dt} + s \frac{\partial^2 V}{\partial \phi \partial \phi} \right) \psi \right], \tag{1}$$

where the bosonic variable ϕ and fermionic variables ψ, $\bar{\psi}$ are the functions of time-evolution parameter t, $V(\phi)$ is a general potential function, and s is an independent constant parameter. For algebraic convenience, we linearize the first term in (1) by introducing an auxiliary variable A. As a consequence, the action can be written as (henceforth, we denote $\dot{\phi} = d\phi/dt, V' = \partial V/\partial \phi, V'' = \partial^2 V/\partial \phi \partial \phi$ in the text)

$$S = \int dt L = \int dt \left[i \left(\dot{\phi} + sV' \right) A + \frac{A^2}{2} - i\bar{\psi} \left(\dot{\psi} + s\psi V'' \right) \right], \tag{2}$$

where $A = -i(\dot{\phi} + sV')$ is the equation of motion. Using this expression for A in (2), one can recover the original action.

For the present QM system, we have the following off-shell nilpotent ($s_1^2 = 0, s_2^2 = 0$) SUSY transformations (we point out that the SUSY transformations in (3) differ by an overall i-factor from [21]):

$$\begin{aligned} s_1 \phi &= i\psi, \\ s_1 \psi &= 0, \\ s_1 \bar{\psi} &= iA, \\ s_1 A &= 0, \\ s_2 \phi &= i\bar{\psi}, \\ s_2 \bar{\psi} &= 0, \\ s_2 \psi &= iA - 2sV', \\ s_2 A &= 2s\bar{\psi} V''. \end{aligned} \tag{3}$$

Under the above symmetry transformations, the Lagrangian in (2) transforms as

$$\begin{aligned} s_1 L &= 0, \\ s_2 L &= \frac{d}{dt} \left(-A\bar{\psi} \right). \end{aligned} \tag{4}$$

Thus, the action integral remains invariant (i.e., $s_1 S = 0, s_2 S = 0$). According to Noether's theorem, the continuous SUSY symmetry transformations s_1 and s_2 lead to the following conserved charges, respectively:

$$\begin{aligned} Q &= -\psi A, \\ \bar{Q} &= -\bar{\psi} \left[A + 2isV' \right]. \end{aligned} \tag{5}$$

The conservation of the SUSY charges (i.e., $\dot{Q} = 0$, $\dot{\bar{Q}} = 0$) can be proven by exploiting the following Euler-Lagrange equations of motion:

$$\begin{aligned} \dot{A} &= sAV'' - s\bar{\psi}\psi V''', \\ \dot{\bar{\psi}} &= -s\bar{\psi} V'', \\ A &= -i \left(\dot{\phi} + sV' \right), \\ \dot{\psi} &= -s\psi V''. \end{aligned} \tag{6}$$

It turns out that these charges are the generators of SUSY transformations (3). One can explicitly check that the following relations are true:

$$\begin{aligned} s_1 \Phi &= i \left[\Phi, Q \right]_{\pm}, \\ s_2 \Phi &= i \left[\Phi, \bar{Q} \right]_{\pm}, \\ \Phi &= \phi, A, \psi, \bar{\psi}, \end{aligned} \tag{7}$$

where the subscripts $(\pm)$, on the square brackets, deal with the (anti)commutator depending on the variables being fermionic/bosonic in nature.

It is to be noted that the anticommutator of the fermionic SUSY transformations (s_1 and s_2) leads to a bosonic symmetry (s_ω):

$$\begin{aligned}
s_\omega \phi &= -2\left(A + isV'\right), \\
s_\omega \psi &= -2is\psi V'', \\
s_\omega \overline{\psi} &= 2is\overline{\psi} V'', \\
s_\omega A &= 2is\left(AV'' - \overline{\psi}\psi V'''\right).
\end{aligned} \tag{8}$$

The application of bosonic symmetry (s_ω) on the Lagrangian produces total time derivative:

$$s_\omega L = \left(s_1 s_2 + s_2 s_1\right) L = \frac{d}{dt}\left(-iA^2\right). \tag{9}$$

Thus, according to Noether's theorem, the above continuous bosonic symmetry leads to a bosonic conserved charge (Q_ω) as follows:

$$\begin{aligned}
Q_\omega &= -iA^2 + 2sAV' + 2s\psi\overline{\psi}V'' \\
&= 2i\left[\frac{\Pi_\phi \Pi_\phi}{2} - s\Pi_\phi V' - s\psi\Pi_\psi V''\right] \equiv (2i)\, H,
\end{aligned} \tag{10}$$

where $\Pi_\phi = iA, \Pi_\psi = i\overline{\psi}$ are the canonical momenta corresponding to the variables ϕ, ψ, respectively. It is clear that the bosonic charge Q_ω is the Hamiltonian H (modulo a constant $2i$-factor) of our present model.

One of the important features of SUSY transformations is that the application of this bosonic symmetry must produce the time translation of the variable (modulo a constant $2i$-factor), which can be checked as

$$s_\omega \Phi = \left\{s_1, s_2\right\} \Phi = (2i)\, \dot{\Phi}, \quad \Phi = \phi, A, \psi, \overline{\psi}, \tag{11}$$

where, in order to prove the sanctity of this equation, we have used the equations of motion mentioned in (6).

3. Off-Shell Nilpotent SUSY Transformations: Antichiral Supervariable Approach

In order to derive the continuous transformation s_1, we shall focus on the (1, 1)-dimensional super-submanifold (of general (1, 2)-dimensional supermanifold) parameterized by the supervariable $(t, \overline{\theta})$. For this purpose, we impose supersymmetric invariant restrictions (SUSYIRs) on the antichiral supervariables. We then generalize the basic (explicit t dependent) variables to their antichiral supervariable counterparts:

$$\begin{aligned}
\phi(t) &\longrightarrow \widetilde{\Phi}\left(t, \theta, \overline{\theta}\right)\Big|_{\theta=0} \equiv \widetilde{\Phi}\left(t, \overline{\theta}\right) = \phi(t) + \overline{\theta} f_1(t), \\
\psi(t) &\longrightarrow \widetilde{\Psi}\left(t, \theta, \overline{\theta}\right)\Big|_{\theta=0} \equiv \widetilde{\Psi}\left(t, \overline{\theta}\right) = \psi(t) + i\overline{\theta} b_1(t), \\
\overline{\psi}(t) &\longrightarrow \widetilde{\overline{\Psi}}\left(t, \theta, \overline{\theta}\right)\Big|_{\theta=0} \equiv \widetilde{\overline{\Psi}}\left(t, \overline{\theta}\right) = \psi(t) + i\overline{\theta} b_2(t), \\
A(t) &\longrightarrow \widetilde{A}\left(t, \theta, \overline{\theta}\right)\Big|_{\theta=0} \equiv \widetilde{A}\left(t, \overline{\theta}\right) = A(t) + \overline{\theta} P(t),
\end{aligned} \tag{12}$$

where $f_1(t)$, $P(t)$ and $b_1(t)$, $b_2(t)$ are the fermionic and bosonic secondary variables, respectively.

It is observed from (3) that $s_1(\psi, A) = 0$ (i.e., both ψ and A are invariant under s_1). Therefore, we demand that both variables should remain unchanged due to the presence of Grassmannian variable $\overline{\theta}$. As a result of the above restrictions, we obtain

$$\begin{aligned}
\widetilde{\Psi}\left(t, \theta, \overline{\theta}\right)\Big|_{\theta=0} &\equiv \widetilde{\Psi}\left(t, \overline{\theta}\right) = \psi(t) \Longrightarrow b_1 = 0, \\
\widetilde{A}\left(t, \theta, \overline{\theta}\right)\Big|_{\theta=0} &\equiv \widetilde{A}\left(t, \overline{\theta}\right) = A(t) \Longrightarrow P = 0.
\end{aligned} \tag{13}$$

We further point out that $s_1(\phi\psi) = 0$ and $s_1(\dot{\phi}\dot{\psi}) = 0$ due to the fermionic nature of ψ (i.e., $\psi^2 = 0$). Thus, these restrictions yield

$$\begin{aligned}
\widetilde{\Phi}\left(t, \overline{\theta}\right) \widetilde{\Psi}\left(t, \overline{\theta}\right) &= \phi(t)\, \psi(t) \Longrightarrow f_1 \psi = 0, \\
\dot{\widetilde{\Phi}}\left(t, \overline{\theta}\right) \dot{\widetilde{\Psi}}\left(t, \overline{\theta}\right) &= \dot{\phi}(t)\, \dot{\psi}(t) \Longrightarrow \dot{f}_1 \dot{\psi} = 0.
\end{aligned} \tag{14}$$

The trivial solution for the above relationships is $f_1 \propto \psi$; for algebraic convenience, we choose $f_1 = i\psi$. Here, the i-factor has been taken due to the convention we have adopted for the present SUSY QM theory. Substituting the values of the secondary variables in the expansions of antichiral supervariables (12), one obtains

$$\begin{aligned}
\widetilde{\Phi}^{(ac)}\left(t, \overline{\theta}\right) &= \phi(t) + \theta\left(i\psi\right) \equiv \phi(t) + \overline{\theta}\left(s_1 \phi(t)\right), \\
\widetilde{\Psi}^{(ac)}\left(t, \overline{\theta}\right) &= \psi(t) + \overline{\theta}(0) \equiv \psi(t) + \overline{\theta}\left(s_1 \psi(t)\right), \\
\widetilde{A}^{(ac)}\left(t, \overline{\theta}\right) &= A(t) + \theta(0) \equiv A(t) + \overline{\theta}\left(s_1 A(t)\right).
\end{aligned} \tag{15}$$

The superscript (ac) in the above represents the antichiral supervariables, obtained after the application of SUSIRs. Furthermore, we note that the 1D potential function $V(\phi)$ can be generalized to $\widetilde{V}(\widetilde{\Phi}^{(ac)})$ onto (1, 1)-dimensional super-submanifold as

$$V(\phi) \longrightarrow \widetilde{V}\left(\widetilde{\Phi}^{(ac)}\right) = \widetilde{V}^{(ac)}\left(\phi + \overline{\theta}\left(i\psi\right)\right) = V(\phi) + \overline{\theta}\left(i\psi V'\right) \equiv V(\phi) + \overline{\theta}\left(s_1 V(\phi)\right), \tag{16}$$

where we have used the expression of antichiral supervariable $\widetilde{\Phi}^{(ac)}(t,\bar{\theta})$ as given in (15).

In order to find out the SUSY transformation for $\bar{\psi}$, it can be checked that the application of s_1 on the following vanishes:

$$s_1\left[i\left(\dot{\phi}+sV'\right)A-i\bar{\psi}\left(\dot{\psi}+s\psi V''\right)\right]=0. \tag{17}$$

As a consequence, the above can be used as a SUSYIR and we replace the ordinary variables by their antichiral supervariables as

$$\begin{aligned}&\left[i\left(\dot{\widetilde{\Phi}}^{(ac)}+s\widetilde{V}'^{(ac)}\right)\widetilde{A}^{(ac)}\right.\\&\left.-i\widetilde{\overline{\Psi}}\left(\dot{\widetilde{\Psi}}^{(ac)}+s\widetilde{\Psi}^{(ac)}\widetilde{V}''^{(ac)}\right)\right]=\left[i\left(\dot{\phi}+sV'\right)A\right.\\&\left.-i\bar{\psi}\left(\dot{\psi}+s\psi V''\right)\right].\end{aligned} \tag{18}$$

After doing some trivial computations, we obtain $b_2 = A$. Recollecting all the value of secondary variables and substituting them into (12), we finally obtain the following antichiral supervariable expansions:

$$\begin{aligned}\widetilde{\Phi}^{(ac)}\left(t,\bar{\theta}\right)&=\phi(t)+\bar{\theta}(i\psi)\equiv\phi(t)+\bar{\theta}\left(s_1\phi(t)\right),\\\widetilde{\Psi}^{(ac)}\left(t,\bar{\theta}\right)&=\psi(t)+\bar{\theta}(0)\equiv\psi(t)+\bar{\theta}\left(s_1\psi(t)\right),\\\widetilde{\overline{\Psi}}^{(ac)}\left(t,\bar{\theta}\right)&=\bar{\psi}(t)+\bar{\theta}(iA)\equiv\bar{\psi}(t)+\bar{\theta}\left(s_1\bar{\psi}(t)\right),\\\widetilde{A}^{(ac)}\left(t,\bar{\theta}\right)&=A(t)+\bar{\theta}(0)\equiv A(t)+\bar{\theta}\left(s_1A(t)\right).\end{aligned} \tag{19}$$

Finally, we have derived explicitly the SUSY transformation s_1 for all the variables by exploiting SUSY invariant restrictions on the antichiral supervariables. These symmetry transformations are

$$\begin{aligned}s_1\phi&=i\psi,\\s_1\psi&=0,\\s_1\bar{\psi}&=iA,\\s_1A&=0,\\s_1V&=i\psi V'.\end{aligned} \tag{20}$$

It is worthwhile to mention here that, for the antichiral supervariable expansions given in (11), we have the following relationship between the Grassmannian derivative $\partial_{\bar{\theta}}$ and SUSY transformations s_1:

$$\frac{\partial}{\partial\bar{\theta}}\widetilde{\Omega}^{(ac)}\left(t,\theta,\bar{\theta}\right)\Big|_{\theta=0}\equiv\frac{\partial}{\partial\bar{\theta}}\widetilde{\Omega}^{(ac)}\left(t,\bar{\theta}\right)=s_1\Omega(t), \tag{21}$$

where $\Omega^{(ac)}(t,\bar{\theta})$ is the generic supervariable obtained by exploiting the SUSY invariant restriction on the antichiral supervariables. It is easy to check from the above equation that the symmetry transformation (s_1) for any generic variable $\Omega(t)$ is equal to the translation along the $\bar{\theta}$-direction of the antichiral supervariable. Furthermore, it can also be checked that nilpotency of the Grassmannian derivative $\partial_{\bar{\theta}}$ (i.e., $\partial_{\bar{\theta}}^2=0$) implies $s_1^2=0$.

4. Off-Shell Nilpotent SUSY Transformations: Chiral Supervariable Approach

For the derivation of second fermionic SUSY transformation s_2, we concentrate on the chiral super-submanifold parametrized by the supervariables (t,θ). Now, all the ordinary variables (depending explicitly on t) are generalized to a (1, 1)-dimensional chiral super-submanifold as

$$\begin{aligned}\phi(t)&\longrightarrow\widetilde{\Phi}\left(t,\theta,\bar{\theta}\right)\Big|_{\bar{\theta}=0}\equiv\widetilde{\Phi}(t,\theta)=\phi(t)+\theta\bar{f}_1(t),\\\psi(t)&\longrightarrow\widetilde{\Psi}\left(t,\theta,\bar{\theta}\right)\Big|_{\bar{\theta}=0}\equiv\widetilde{\Psi}(t,\theta)=\psi(t)+i\theta\bar{b}_1(t),\\\bar{\psi}(t)&\longrightarrow\widetilde{\overline{\Psi}}\left(t,\theta,\bar{\theta}\right)\Big|_{\bar{\theta}=0}\equiv\widetilde{\overline{\Psi}}(t,\theta)=\psi(t)+i\theta\bar{b}_2(t),\\A(t)&\longrightarrow\widetilde{A}\left(t,\theta,\bar{\theta}\right)\Big|_{\bar{\theta}=0}\equiv A(t,\theta)=A(t)+\theta\bar{P}(t).\end{aligned} \tag{22}$$

In the above, secondary variables $\bar{f}_1(t)$, $\bar{P}(t)$ and $\bar{b}_1(t)$, $\bar{b}_2(t)$ are fermionic and bosonic variables, respectively. We can derive the values of these secondary variables in terms of the basic variables, by exploiting the power and potential of SUSY invariant restrictions.

It is to be noted from (3) that $\bar{\psi}$ does not transform under SUSY transformations s_2 (i.e., $s_2\bar{\psi}=0$) so the variable $\bar{\psi}$ would remain unaffected by the presence of Grassmannian variable θ. As a consequence, we have the following:

$$\widetilde{\overline{\Psi}}\left(t,\theta,\bar{\theta}\right)\Big|_{\bar{\theta}=0}\equiv\widetilde{\overline{\Psi}}(t,\theta)=\bar{\psi}(t)\Longrightarrow\bar{b}_2=0. \tag{23}$$

Furthermore, we observe that $s_2(\phi\bar{\psi})=0$ and $s_2(\dot{\phi}\dot{\bar{\psi}})=0$ due to the fermionic nature of $\bar{\psi}$. Generalizing these invariant restrictions to the chiral supersubmanifold, we have the following SUSYIRs in the following forms; namely,

$$\begin{aligned}\widetilde{\Phi}(t,\theta)\widetilde{\overline{\Psi}}(t,\theta)&=\phi(t)\bar{\psi}(t),\\\dot{\widetilde{\Phi}}(t,\theta)\dot{\overline{\Psi}}(t,\theta)&=\dot{\phi}(t)\dot{\bar{\psi}}(t).\end{aligned} \tag{24}$$

After putting the expansions for supervariable (22) in the above, we get

$$\begin{aligned}\bar{f}_1\bar{\psi}&=0,\\\dot{\bar{f}}_1\dot{\bar{\psi}}&=0.\end{aligned} \tag{25}$$

The solution for the above relationship is $\bar{f}_1=i\bar{\psi}$. Substituting the value of secondary variables in the chiral supervariable expansions (21), we obtain the following expressions:

$$\begin{aligned}\widetilde{\Phi}^{(c)}(t,\theta)&=\phi(t)+\theta\left(i\bar{\psi}\right)\equiv\phi(t)+\theta\left(s_2\phi(t)\right),\\\widetilde{\overline{\Psi}}^{(c)}(t,\theta)&=\bar{\psi}(t)+i\theta(0)\equiv\psi(t)+\theta\left(s_2\bar{\psi}(t)\right),\end{aligned} \tag{26}$$

where superscript (c) represents the chiral supervariables obtained after the applicationof SUSYIRs. Using (26), one can generalize $V(\phi)$ to $\widetilde{V}(\widetilde{\Phi}^{(c)})$ onto the (1, 1)-dimensional chiral super-submanifold as

$$V(\phi) \longrightarrow \widetilde{V}\left(\widetilde{\Phi}^{(c)}\right) = \widetilde{V}^{(c)}(\phi + \theta(i\bar{\psi})) = V(\phi) + \theta\left(i\bar{\psi}V'(\phi)\right) \equiv V(\phi) + s_2(V(\phi)), \tag{27}$$

where we have used the expression of chiral supervariable $\widetilde{\Phi}^{(c)}(t,\theta)$ given in (26).

We note that $s_2[iA - 2s\bar{\psi}V'] = 0$ because of the nilpotency of s_2 [cf. (3)]. Thus, we have the following SUSYIR in our present theory:

$$i\widetilde{A}^{(c)} - 2s\widetilde{\bar{\Psi}}\widetilde{V}'^{(c)} = \left[iA - 2s\bar{\psi}V'\right]. \tag{28}$$

This restriction serves our purpose for the derivation of SUSY transformation of $A(t)$. Exploiting the above restriction, we get the value of secondary variable $\bar{P}(t)$ in terms of basic variables as $\bar{P} = 2s\bar{\psi}V''$.

It is important to note that the following sum of the composite variables is invariant under s_2; namely,

$$s_2\left[isV'A + \frac{1}{2}A^2 - is\bar{\psi}\psi V''\right] = 0. \tag{29}$$

In order to calculate the fermionic symmetry transformation corresponding to variable ψ, we use the above relationship as a SUSYIR:

$$\begin{aligned} &is\widetilde{V}'^{(c)}\widetilde{A}^{(c)} + \frac{1}{2}\widetilde{A}^{2(c)} - is\widetilde{\bar{\Psi}}^{(c)}\widetilde{\Psi}\widetilde{V}''^{(c)} \\ &\quad = isV'A + \frac{1}{2}A^2 - is\bar{\psi}\psi V'', \end{aligned} \tag{30}$$

and after some computations, one gets $\bar{b}_1 = A + 2isV'$.

Recollecting all the values of secondary variables and substituting them into (21), we have the expansions of the chiral supervariables as

$$\begin{aligned} \widetilde{\Phi}^{(c)}(t,\theta) &= \phi(t) + \theta(i\bar{\psi}) \equiv \phi(t) + \theta(s_2\phi(t)), \\ \widetilde{\Psi}^{(c)}(t,\theta) &= \psi(t) + \theta\left(iA - 2sV'\right) \\ &\equiv \psi(t) + \theta(s_2\psi(t)), \\ \widetilde{\bar{\Psi}}^{(c)}(t,\theta) &= \bar{\psi}(t) + i\theta(0) \equiv \psi(t) + \theta(s_2\bar{\psi}(t)), \\ \widetilde{A}^{(c)}\left(t,\bar{\theta}\right) &= A(t) + \theta\left(2s\bar{\psi}V''\right) \equiv A(t) + \theta(s_2 A(t)). \end{aligned} \tag{31}$$

Finally, the supersymmetric transformations (s_2) for all the basic and auxiliary variables are listed as

$$\begin{aligned} s_2\phi &= i\bar{\psi}, \\ s_2\bar{\psi} &= 0, \\ s_2\psi &= iA - 2sV', \\ s_2V &= i\bar{\psi}V', \\ s_2A &= 2s\bar{\psi}V''. \end{aligned} \tag{32}$$

It is important to point out here that we have the following mapping between the Grassmannian derivative (∂_θ) and the symmetry transformation s_2:

$$\frac{\partial}{\partial\theta}\widetilde{\Omega}^{(c)}\left(t,\theta,\bar{\theta}\right)\Big|_{\bar{\theta}=0} = \frac{\partial}{\partial\theta}\widetilde{\Omega}^{(c)}(t,\theta) = s_2\Omega(t), \tag{33}$$

where $\widetilde{\Omega}^{(c)}(t,\theta)$ is the generic chiral supervariables obtained after the application of supersymmetric invariant restrictions and $\Omega(t)$ denotes the basic variables of our present QM theory. The above equation captures the geometrical interpretation of transformation (s_2) in terms of the Grassmannian derivative (∂_θ) because of the fact that the translation along θ-direction of chiral supervariable is equivalent to the symmetry transformation (s_2) of the same basic variable. We observe from (33) that the nilpotency of SUSY transformation s_2 (i.e., $s_2^2 = 0$) can be generalized in terms of Grassmannian derivative $\partial_\theta^2 = 0$.

5. Invariance and Off-Shell Nilpotency: Supervariable Approach

It is interesting to note that, by exploiting the expansions of supervariables (11), the Lagrangian in (2) can be expressed in terms of the antichiral supervariables as

$$L \Longrightarrow L^{(ac)} = i\left(\dot{\widetilde{\Phi}}^{(ac)} + s\widetilde{V}'^{(ac)}\right)\widetilde{A}^{(ac)} + \frac{1}{2}\widetilde{A}^{(ac)}\widetilde{A}^{(ac)} - i\widetilde{\bar{\Psi}}^{(ac)}\left(\dot{\widetilde{\Psi}}^{(ac)} + s\widetilde{\Psi}^{(ac)}\widetilde{V}''^{(ac)}\right). \tag{34}$$

In the earlier section, we have shown that SUSY transformation (s_1) and translational generator ($\partial_{\bar{\theta}}$) are geometrically related to each other (i.e., $s_1 \leftrightarrow \partial_{\bar{\theta}}$). As a consequence, one can also capture the invariance of the Lagrangian in the following fashion:

$$\frac{\partial}{\partial\bar{\theta}}L^{(ac)} = 0 \Longleftrightarrow s_1 L = 0. \tag{35}$$

Similarly, Lagrangian (2) can also be written in terms of the chiral supervariables as

$$L \Longrightarrow L^{(c)} = i\left(\dot{\widetilde{\Phi}}^{(c)} + s\widetilde{V}'^{(c)}\right)\widetilde{A}^{(c)} + \frac{1}{2}\widetilde{A}^{(c)}\widetilde{A}^{(c)} - i\widetilde{\overline{\Psi}}^{(c)}\left(\dot{\widetilde{\Psi}}^{(c)} + s\widetilde{\Psi}^{(c)}\widetilde{V}''^{(c)}\right). \quad (36)$$

Since the fermionic symmetry (s_2) is geometrically connected with the translational generator (∂_θ), therefore, the invariance of the Lagrangian can be geometrically interpreted as follows:

$$\frac{\partial}{\partial\theta}L^{(c)} = \frac{d}{dt}\left[-\psi A\right] \Longleftrightarrow s_2 L = \frac{d}{dt}\left[-\psi A\right]. \quad (37)$$

As a result, the action integral $S = \int dt L^{(c)}|_{\theta=0}$ remains invariant.

We point out that the conserved charges Q and $\overline{Q}$ corresponding to the continuous symmetry transformations s_1 and s_2 can also be expressed as

$$\begin{aligned} Q &= s_1\left[-i\psi\overline{\psi}\right] = -\psi A \Longleftrightarrow Q = \frac{\partial}{\partial\bar{\theta}}\left[-i\widetilde{\Psi}^{(ac)}\widetilde{\overline{\Psi}}^{(ac)}\right], \\ \overline{Q} &= s_2\left[i\psi\overline{\psi}\right] = -\overline{\psi}\left[A + isV'\right] \Longleftrightarrow \overline{Q} = \frac{\partial}{\partial\theta}\left[i\widetilde{\Psi}^{(c)}\widetilde{\overline{\Psi}}^{(c)}\right]. \end{aligned} \quad (38)$$

The nilpotency properties of the above charges can be shown in a straightforward manner with the help of symmetry properties:

$$\begin{aligned} s_1 Q &= +i\{Q, Q\} = 0 \Longrightarrow Q^2 = 0, \\ s_2\overline{Q} &= +i\left\{\overline{Q}, \overline{Q}\right\} = 0 \Longrightarrow \overline{Q}^2 = 0. \end{aligned} \quad (39)$$

In the language of translational generators, these properties can be written as $\partial_{\bar{\theta}}Q = 0 \Rightarrow \overline{Q}^2 = 0$ and $\partial_\theta\overline{Q} = 0 \Rightarrow Q^2 = 0$. These relations hold due to the nilpotency of the Grassmannian derivatives (i.e., $\partial_{\bar{\theta}}^2 = 0,\ \partial_\theta^2 = 0$).

6. $\mathcal{N} = 2$ SUSY Algebra and Its Interpretation

We observe that, under the discrete symmetry

$$\begin{aligned} t &\longrightarrow t, \\ \phi &\longrightarrow \phi, \\ \psi &\longrightarrow \overline{\psi}, \\ \overline{\psi} &\longrightarrow \psi, \\ s &\longrightarrow -s, \\ A &\longrightarrow A + 2isV', \end{aligned} \quad (40)$$

the Lagrangian (L) transforms as $L \to L + (d/dt)[i\overline{\psi}\psi - 2sV]$. Hence, action integral (2) of the SUSY QM system remains invariant. It is to be noted that the above discrete symmetry transformations are important because they relate the two SUSY transformations (s_1, s_2):

$$s_2\Omega = \pm * s_1 * \Omega, \quad (41)$$

where Ω is the generic variables present in the model. It is to be noted that, generally, the ($\pm$) signs are governed by the two successive operations of the discrete symmetry on the variables as

$$*(*\Omega) = \pm\Omega. \quad (42)$$

In the present case, *only* the (+) sign will occur for all the variables (i.e., $\Omega = \phi, \psi, \overline{\psi}, A$). It can be easily seen that relationship (41) is analogous to the relationship $\delta = \pm * d *$ of differential geometry (where d and δ are the exterior and coexterior derivative, resp., and ($*$) is the Hodge duality operation).

We now focus on the physical identifications of the de Rham cohomological operators of differential geometry in terms of the symmetry transformations. It can be explicitly checked that the continuous symmetry transformations, together with discrete symmetry for our SUSY QM model, satisfy the following algebra [17–20, 22–24]:

$$\begin{aligned} s_1^2 &= 0, \\ s_2^2 &= 0, \\ \{s_1, s_2\} &= (s_1 + s_2)^2 = s_\omega, \\ [s_1, s_\omega] &= 0, \\ [s_2, s_\omega] &= 0, \\ s_2 &= \pm * s_1 *, \end{aligned} \quad (43)$$

which is identical to the algebra obeyed by the de Rham cohomological operators (d, δ, Δ) [25–29],

$$\begin{aligned} d^2 &= 0, \\ \delta^2 &= 0, \\ \{d, \delta\} &= (d + \delta)^2 = \Delta, \\ [d, \Delta] &= 0, \\ [\delta, \Delta] &= 0, \\ \delta &= \pm \star d \star. \end{aligned} \quad (44)$$

Here, Δ is the Laplacian operator. From (43) and (44), we can identify the exterior derivative with s_1 and coexterior derivative δ with s_2. The discrete symmetry (40) provides the *analogue* of Hodge duality ($\star$) operation of differential geometry. In fact, there is a one-to-one mapping between the symmetry transformations and the de Rham cohomological operators. It is also clear from (43) and (44) that the bosonic symmetry (s_ω) and Laplacian operator (Δ) are the Casimir operators of the algebra given in (43) and (44), respectively. Thus, our present $\mathcal{N} = 2$ SUSY model provides a model for Hodge theory. Furthermore, a similar algebra given in (44) is also satisfied by the conserved charges Q, $\overline{Q}$, and Q_ω:

$$\begin{aligned} Q^2 &= 0, \\ \overline{Q}^2 &= 0, \\ \left\{Q, \overline{Q}\right\} &= Q_\omega = 2iH, \\ [Q, H] &= 0, \\ \left[\overline{Q}, H\right] &= 0. \end{aligned} \tag{45}$$

In the above, we have used the canonical quantum (anti)commutation relations $[\phi, A] = 1$ and $\{\psi, \overline{\psi}\} = 1$. It is important to mention here that the bosonic charge (i.e., the Hamiltonian of the theory modulo a $2i$-factor) is the Casimir operator in algebra (45).

Some crucial properties related to the de Rham cohomological operators (d, δ, Δ) can be captured by these charges. For instance, we observe from (45) that the Hamiltonian is the Casimir operator of the algebra. Thus, it can be easily seen that $HQ = QH$ implies that $QH^{-1} = H^{-1}Q$, if the inverse of the Hamiltonian exists. Since we are dealing with the nosingular Hamiltonian, we presume that the Casimir operator has its well-defined inverse value. By exploiting (45), it can be seen that

$$\begin{aligned} \left[\frac{Q\overline{Q}}{H}, Q\right] &= Q, \\ \left[\frac{Q\overline{Q}}{H}, \overline{Q}\right] &= -\overline{Q}, \\ \left[\frac{\overline{Q}Q}{H}, Q\right] &= -Q, \\ \left[\frac{\overline{Q}Q}{H}, \overline{Q}\right] &= \overline{Q}. \end{aligned} \tag{46}$$

Let us define an eigenvalue equation $(Q\overline{Q}/H)|\chi\rangle_p = p|\chi\rangle_p$, where $|\chi\rangle_p$ is the quantum Hilbert state with eigenvalue p. By using algebra (46), one can verify the following:

$$\begin{aligned} \left(\frac{Q\overline{Q}}{H}\right) Q\,|\chi\rangle_p &= (p+1)\,|\chi\rangle_p, \\ \left(\frac{Q\overline{Q}}{H}\right) \overline{Q}\,|\chi\rangle_p &= (p-1)\,|\chi\rangle_p, \\ \left(\frac{Q\overline{Q}}{H}\right) H\,|\chi\rangle_p &= p\,|\chi\rangle_p. \end{aligned} \tag{47}$$

As a consequence of (47), it is evident that $Q|\chi\rangle_p$, $\overline{Q}|\chi\rangle_p$, and $H|\chi\rangle_p$ have the eigenvalues $(p+1)$, $(p-1)$, and p, respectively, with respect to to the operator $Q\overline{Q}/H$.

The above equation provides a connection between the conserved charges $(Q, \overline{Q}, H)$ and de Rham cohomological operators (d, δ, Δ) because as we know the action of d on a given form increases the degree of the form by one, whereas application of δ decreases the degree by one unit and operator Δ keeps the degree of a form intact. These important properties can be realized by the charges $(Q, \overline{Q}, Q_\omega)$, where the eigenvalues and eigenfunctions play the key role [22].

7. Conclusions

In summary, exploiting the supervariable approach, we have derived the off-shell nilpotent symmetry transformations for the $\mathcal{N} = 2$ SUSY QM system. This has been explicated through the 1D SUSY invariant quantities, which remain unaffected due to the presence of the Grassmannian variables θ and $\overline{\theta}$. Furthermore, we have provided the geometrical interpretation of the SUSY transformations (s_1 and s_2) in terms of the translational generators ($\partial_{\overline{\theta}}$ and ∂_θ) along the Grassmannian directions $\overline{\theta}$ and θ, respectively. Further, we have expressed the Lagrangian in terms of the (anti)chiral supervariables and the invariance of the Lagrangian under continuous transformations (s_1, s_2) has been shown within the translations generators along $(\theta, \overline{\theta})$-directions. The conserved SUSY charges corresponding to the fermionic symmetry transformations have been expressed in terms of (anti)chiral supervariables and the Grassmannian derivatives. The nilpotency of fermionic charges has been captured geometrically, within the framework of supervariable approach by the Grassmannian derivatives.

Finally, we have shown that the algebra satisfied by the continuous symmetry transformations s_1, s_2, and s_ω (and corresponding charges) is exactly analogous to the Hodge algebra obeyed by the de Rham cohomological operators (d, δ, and Δ) of differential geometry. The discrete symmetry of the theory provides physical realization of the Hodge duality ($*$) operation. Thus, the present $\mathcal{N} = 2$ SUSY QM model provides a model for Hodge theory.

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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