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Research Article

Exact Solutions of the Razavy Cosine Type Potential

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We solve the quantum system with the symmetric Razavy cosine type potential and find that its exact solutions are given by the confluent Heun function. The eigenvalues are calculated numerically. The properties of the wave functions, which depend on the potential parameter a, are illustrated for a given potential parameter ξ. It is shown that the wave functions are shrunk to the origin when the potential parameter a increases. We note that the energy levels ϵ_i $(i \in [1,3])$ decrease with the increasing potential parameter a but the energy levels ϵ_i $(i \in [4,7])$ first increase and then decrease with the increasing a.

1. Introduction

As we know, the exact solutions of quantum systems have been playing an important role since the foundation of quantum mechanics. The hydrogen atom and harmonic oscillator have been taken as typical and seminal examples to explain the classic quantum phenomena in almost all quantum mechanics textbooks [1, 2]. Generally speaking, some popular methods are used to solve these quantum soluble systems. First, we call the functional analysis method, with which one solves the second-order differential equation and obtains their solutions [3] expressed by some well-known special functions. Second, it is called the algebraic method and can be realized by analyzing the Hamiltonian of quantum system. This method is relevant for the SUSYQM [4] and essentially connected to the factorization method [5]. Third, we call the exact quantization rule method [6] and further developed as the proper quantization rule method [7]. The latter approach shows more beauty and symmetry than the former one. It should be recognized that almost all soluble potentials mentioned above belong to single well potentials except for the double well potentials [8–10].

More than thirty years ago, Razavy proposed a cosine type potential [11, 12]

$$V(m,x) = \frac{\hbar^2}{2\mu}\left\{\frac{1}{8}\xi^2\left[1-\cos(2mx)\right]-(a+1)\xi\cos(mx)\right\}, \tag{1}$$

with $V(m,-x) = V(m,x)$ and $V(-m,x) = V(-m,x)$. Here the parameters a, m are positive integers and ξ is a positive real number. (The potential taken here is slightly different from original expression [11, 12], in which a proportional coefficient was included. In addition, the factor $(a+1)\xi$ is extracted from the originally proposed Razavy potential [11, 12] to incorporate the energy level E. Such a treatment does not affect the property of the quantum system.) In Figure 1, we plot it as a function of the variable x with various a, in which we take $\xi = 3$ and $m = 1$ for simplicity. We find that the minimum value of the potential $V_{\min}(m,x) = -(a+1)\xi$, which is independent of the parameter m. Razavy presented the so-called exact solutions by using the series method [11, 12]. After studying it carefully, it is found that the solutions cannot be given exactly due to the complicated three-term recurrence relation. The method used by him is nothing but the Bethe ansatz method as summarized in [13]. In this case the solutions cannot be expressed as

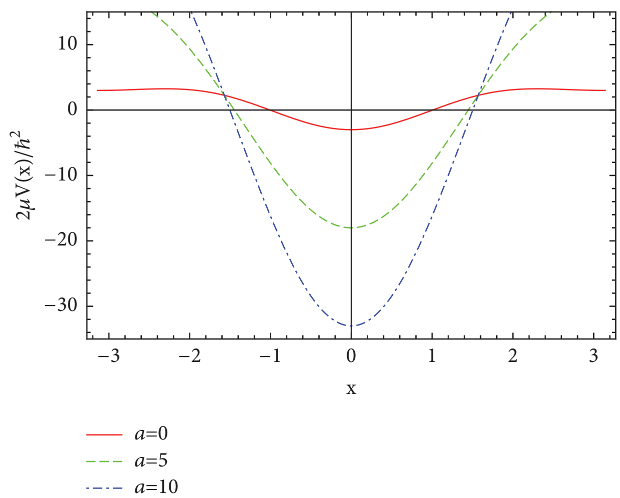

FIGURE 1: A plot of potential as function of the variables x and a.

one of the special functions due to the complicated three-term recurrence relations. One must take some constraints on the coefficients in the recurrence relations as shown in [11, 12] to obtain *quasi*-exact solutions. Recently it is found that the solutions of the hyperbolic type potentials [14–21] are given explicitly by the confluent Heun function [22]. Just recently, we have carried out the Razavy cosine hyperbolic type $V(x) = (\hbar^2\beta^2/2\mu)[(1/8)\xi^2\cosh(4\beta x) - (m+1)\xi\cosh(2\beta x) - (1/8)\xi^2]$, which was studied by Razavy in [11, 12] and found that its solutions can be written as the confluent Heun function [23]. The purpose of this work is to study the solutions of the Razavy cosine type potential (1) [11, 12] and to see whether its solutions can be written as the confluent Heun function or not. The answer is yes, but the energy spectra must be calculated numerically since the energy level term is involved inside the parameter η of the confluent Heun function $H_c(\alpha,\beta,\gamma,\delta,\eta;z)$. Even though the Heun functions have been studied well since 1889, its main topics are focused on the mathematical area. The reason why Razavy did not find its solutions related to this function is that only recent connections with the physical problems have been discovered, in particular for those hyperbolic type potentials [14–21].

This paper is organized as follows. In Section 2, we show how to obtain the solutions of the Schrödinger equation with the Razavy cosine type potential. This is realized by transforming the Schrödinger equation into a confluent Heun differential equation through taking some variable transformations. In Section 3, some fundamental properties of the solutions are studied and illustrated graphically. The energy levels for different parameter values a are calculated numerically. We summarize our results and conclusions in Section 4.

2. Exact Solutions

Let us consider the one-dimensional Schrödinger equation,

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x). \tag{2}$$

Substituting potential (1) into (2), we have

$$\begin{aligned}&\frac{d^2}{dx^2}\psi(x) + \Big\{\varepsilon \\ &\quad - \left[\frac{1}{8}\xi^2(1-\cos(2mx)) - (a+1)\xi\cos(mx)\right]\Big\} \\ &\quad \cdot\psi(x) = 0, \\ &\epsilon = \frac{2\mu E}{\hbar^2} - (a+1)\xi.\end{aligned} \tag{3}$$

Take the wave functions of the form

$$\psi(x) = \exp\left[\frac{\xi\cos(mx)}{2m}\right]\phi(x). \tag{4}$$

Substituting this into (3) yields

$$\begin{aligned}&\phi''(x) - \xi\sin(mx)\phi'(x) \\ &\quad + \frac{1}{2}\left[\xi(2a-m+2)\cos(mx) + 2\epsilon\right]\phi(x) = 0.\end{aligned} \tag{5}$$

Choose a new variable $z = \cos^2(mx/2)$. The above equation becomes

$$\begin{aligned}&(z-1)z\phi''(z) + \frac{1}{2}\left(\frac{4\xi(z-1)z}{m} + 2z - 1\right)\phi'(z) \\ &\quad - \frac{\phi(z)\left[\xi(2z-1)(2a-m+2) + 2\epsilon\right]}{2m^2} = 0,\end{aligned} \tag{6}$$

which can be rearranged as

$$\begin{aligned}&\phi''(z) + \left[\frac{2\xi}{m} + \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z}\right)\right]\phi'(z) \\ &\quad - \frac{\xi(2z-1)(2a-m+2) + 2\epsilon}{2m^2(z-1)z}\phi(z) = 0.\end{aligned} \tag{7}$$

Compared this with the confluent Heun differential equation in the simplest uniform form [22]

$$\begin{aligned}&\frac{d^2H(z)}{dz^2} + \left(\alpha + \frac{1+\beta}{z} + \frac{1+\gamma}{z-1}\right)\frac{dH(z)}{dz} \\ &\quad + \left(\frac{\mu}{z} + \frac{\nu}{z-1}\right)H(z) = 0,\end{aligned} \tag{8}$$

we find the solution to (7) is given by the acceptable confluent Heun function $H_c(\alpha,\beta,\gamma,\delta,\eta;z)$ with the following parameters:

$$\begin{aligned}\alpha &= \frac{2\xi}{m}, \\ \beta &= -\frac{1}{2}, \\ \gamma &= -\frac{1}{2}, \\ \mu_\pm &= \frac{\xi(-2a+m-2)\pm 2\epsilon}{2m^2}, \\ \nu &= \mu_-\end{aligned} \tag{9}$$

Table 1: Spectra of the Schrödinger equation with potential (1).

a	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
$a = 0$	-1.25000	1.79206	3.88549	5.52599	7.571	10.2587	13.4768
$a = 1$	-3.85555	0.105572	3.35555	5.88783	8.10716	10.6369	13.7448
$a = 2$	-6.5289	-1.86011	2.22966	5.68013	8.54924	11.2207	14.2107
$a = 3$	-9.2426	-3.97095	0.793349	5.00238	8.62128	11.7636	14.8280
$a = 4$	-11.9842	-6.17622	-0.831098	4.01497	8.31808	12.0716	15.4571
$a = 5$	-14.7467	-8.44991	-2.58550	2.81773	7.72170	12.0880	15.9545
$a = 6$	-17.5255	-10.7764	-4.43719	1.46786	6.90679	11.8388	16.2407
$a = 7$	-20.3178	-13.1452	-6.36553	0.00034	5.92502	11.3718	16.3004
$a = 8$	-23.1212	-15.5492	-8.35647	-1.56169	4.81114	10.7299	16.1532
$a = 9$	-25.9342	-17.9829	-10.3999	-3.20197	3.58925	9.94554	15.8289
$a = 10$	-28.7554	-20.4422	-12.4881	-4.90854	2.27678	9.04264	15.3551

from which we are able to calculate the parameters δ and η as

$$\begin{aligned}\delta &= \mu_+ + \mu_- - \frac{1}{2}\alpha(\beta+\gamma+2) = -\frac{2(a+1)\xi}{m^2},\\ \eta &= \frac{1}{2}\alpha(\beta+1) - \mu_+ - \frac{1}{2}(\beta+\gamma+\beta\gamma)\\ &= \frac{8(a+1)\xi + 3m^2 - 8\epsilon}{8m^2},\end{aligned} \tag{10}$$

which implies the parameter η involved in the confluent Heun function is related to energy levels. The wave function given by this Heun function seems to be analytical, but the key issue is how to first get the energy levels. Otherwise, the solution becomes unsolvable. Generally, the confluent Heun function can be expressed as a series expansion

$$H_C(\alpha,\beta,\gamma,\delta,\eta;z) = \sum_{n=0}^{\infty} v_n(\alpha,\beta,\gamma,\delta,\eta,\xi)\, z^n, \quad |z| < 1. \tag{11}$$

The coefficients v_n are given by a three-term recurrence relation

$$\begin{aligned}A_n v_n - B_n v_{n-1} - C_n v_{n-2} &= 0,\\ v_{-1} &= 0,\\ v_0 &= 1,\end{aligned} \tag{12}$$

with

$$\begin{aligned}A_n &= 1 + \frac{\beta}{n} = 1 - \frac{1}{2n},\\ B_n &= 1 + \frac{1}{n}(\beta+\gamma-\alpha-1)\\ &\quad + \frac{1}{n^2}\left\{\eta - \frac{1}{2}(\beta+\gamma-\alpha) - \frac{\alpha\beta}{2} + \frac{\beta\gamma}{2}\right\},\\ &= \frac{2(a\xi+\xi-\epsilon) + 2m^2(n-1)^2 + m(3-4n)\xi}{2m^2n^2}\\ C_n &= \frac{\alpha}{n^2}\left(\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + n - 1\right)\\ &= \frac{\xi(-2a + m(2n-3) - 2)}{m^2n^2}\end{aligned} \tag{13}$$

To make the confluent Heun function reduce to polynomials, two termination conditions have to be satisfied [22]

$$\begin{aligned}\mu_+ + \mu_- + N\alpha &= 0,\\ \Delta_{N+1}(\mu_+) &= 0.\end{aligned} \tag{14}$$

The second condition is a tridiagonal determinant and can be constructed by the matrix elements

$$\begin{aligned}a_{ii} &= \mu_+ - s_i + (i-1)\alpha,\\ a_{ii+1} &= i(i+\beta),\\ a_{i+1i} &= (N-i+1)\alpha,\\ s_i &= (i-1)(i+\beta+\gamma),\\ & i = 1,2,\ldots,N,N+1.\end{aligned} \tag{15}$$

The explicit expression of this determinant can refer to [16–19] for some detail.

For present case, there is a problem for the first condition. That is, $\mu_+ + \mu_- + \alpha = 0$ when $N = 1$. From this we have $m = 2(1+a)/(1+4\xi)$. This is contrary to the assumption m is positive integer. Therefore, how to obtain the eigenvalues becomes a challenging task. Due to $z \in [0,1]$ we would like to solve this problem via series expansion method as shown in [15]. Unfortunately, the calculation results are not ideal. We have to solve it in another way as shown in [14].

3. Fundamental Properties

Now, let us study some basic properties of the solutions as shown in Figures 2 and 3. We find that the wave functions are shrunk to the origin when the potential parameter a increases. This makes the amplitude of the wave function be increased. We list the energy levels ϵ_i $(i \in [1,7])$ in Table 1 and illustrate them in Figure 3. We notice that the energy levels ϵ_i

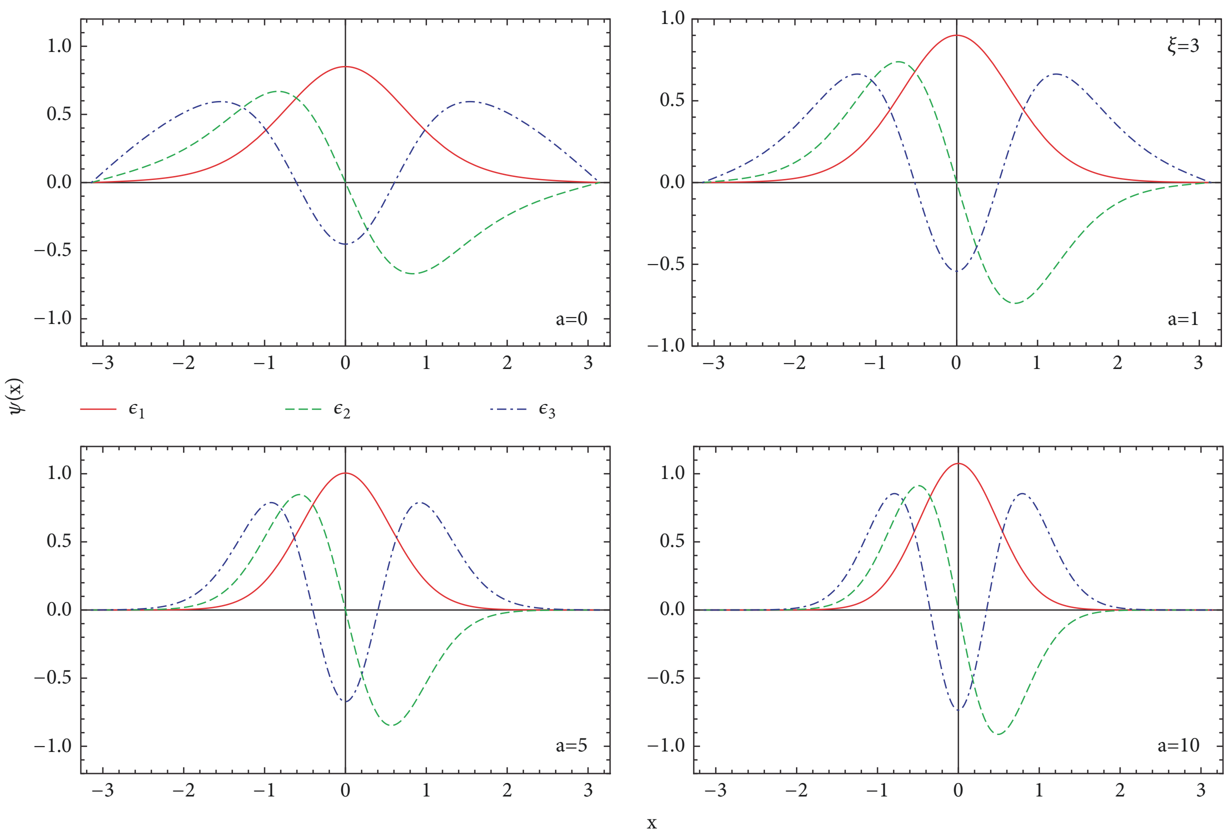

FIGURE 2: The characteristics of wave functions as a function of the position x. We take $\xi = 3$.

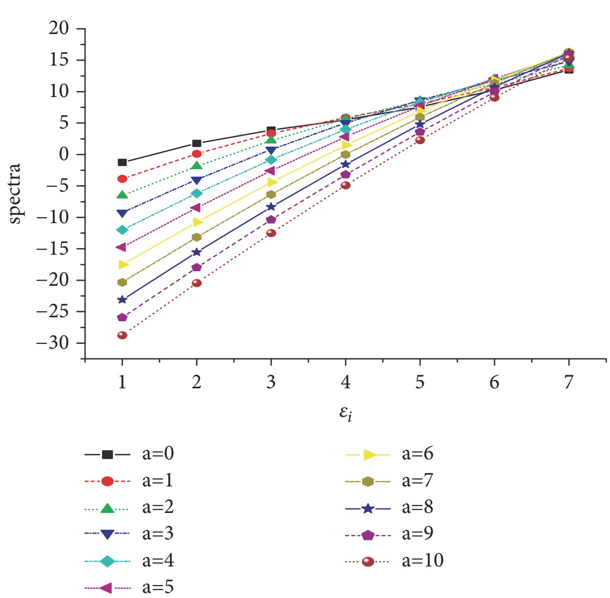

FIGURE 3: The variation of the energy spectra ϵ_i and $\xi = 3$.

$(i \in [1,3])$ decrease with the increasing potential parameter a but ϵ_i $(i \in [4,7])$ first increase and then decrease with the increasing potential parameter a.

4. Conclusions

In this work we have studied the quantum system with the Razavy cosine type potential and found that its exact solutions are given by confluent Heun function $\psi(z) = \exp[(2z-1)\xi/2]H_c(\alpha,\beta,\gamma,\delta,\eta;z)$ by transforming the original differential equation into a confluent type Heun differential equation. The fact that the energy levels are involved inside the parameter η makes us calculate the eigenvalues numerically. The properties of the wave functions depending on the potential parameter a have been illustrated graphically for a given potential parameter ξ. We have also noticed that the energy levels ϵ_i $(i \in [1,3])$ decrease with the increasing potential parameter a but ϵ_i $(i \in [4,7])$ first increase and then decrease with the increasing a.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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