

Research Article

The Quantum Effect on Friedmann Equation in FRW Universe

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We study the modified Friedmann equation in the Friedmann-Robertson-Walker universe with quantum effect. Our modified results mainly stem from the new entropy-area relation and the novel idea of Padmanabhan, who considers the cosmic space to be emerging as the cosmic time progresses, so that the expansion rate of the universe is determined by the difference of degrees of freedom between the holographic surface and the bulk inside. We also discuss the possibility of having bounce cosmological solution from the modified Friedmann equation in spatially flat geometry.

1. Introduction

In the 1970s, the thermodynamic property of black holes has been proposed [1–3], and it reveals that the gravitational dynamics is entwined with thermodynamics. Inspired by Bekenstein's entropy-area theorem [1], Bardeen et al. put forward the four thermodynamical laws of black hole systems [2]. In 1995, Jacobson considered Einstein's field equation as an equation of state. Afterwards, he reproduced Einstein's field equation by demanding that the fundamental relation $\delta Q = TdS$ holds for all local Rindler causal horizons through each space-time point, and δQ and T are treated as the energy flux and Unruh temperature, respectively, felt by an accelerated observer inside the horizon [4]. In 2010, Verlinde defined gravity as an entropic force due to the changes of the information related to the positions of the materials, and the space is emergent based on the holographic principle in his discussions [5]. Moreover, Verlinde's proposal has been applied to reproduce the Friedmann equation into brane cosmology [6] and Friedmann-Robertson-Walker (FRW) universe [7], respectively.

On the other hand, it was addressed in [8–12] that the Friedmann equation can be modified by a bounce solution of the universe as

$$H^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_c}\right) \quad (1)$$

in loop quantum cosmology (LQC). Then the authors of [13] attempted to derive the Friedmann equation by borrowing the Clausius relation, that is, $\delta Q = TdS$, and an entropy-area relation with quantum correction; however, they failed to reproduce the same modified Friedmann equation as that in LQC with bounce solution. Luckily, the difficulty was overcome by the authors of [14], where they proposed a modified dispersion relation at quantum phenomenological level and then obtained the modified Friedmann equation for a bounce solution to the flat FRW universe in LQC. It was found that the role of their modified dispersion relation is to explicitly modify the Clausius relation. This proposal has also been extended into the spatially curved cases and the corresponding modified entropy-area relations have been derived [15].

However, the above studies are only involved in the gravity as an emergent phenomenon rather than the space-time itself as an emergent structure. This situation has been improved by Padmanabhan. In detail, he proposed in [16] that the accelerated expansion of the universe is related to the difference between the surface degrees of freedom (N_{sur}) and the bulk degrees of freedom (N_{bulk}) in a region of space; that is, $\Delta V = \Delta t(N_{\text{sur}} - N_{\text{bulk}})$, where V is the Hubble volume and t is the cosmic time in Planck units. Moreover, the standard evolution of the universe was also reproduced directly from the proposed relation. This proposal inspired

plenty of related studies and remarkable progress [17–33]. However, in the framework of Padmanabhan’s conjecture, the study of quantum effect is missing. So, it is interesting to introduce the quantum effect to Padmanabhan’s conjecture and study the related cosmology.

Thus, in this paper, we will introduce the modified dispersion relation in the framework of Padmanabhan’s conjecture and derive the modified Friedmann equation. Then we will analyze whether the quantum effect in Padmanabhan’s conjecture will bring in modified Friedmann equation (1) with bounce solution. It is notable that starting from the Clausius relation to the apparent horizon along with the modified dispersion relation, one can easily get the modified Friedmann equation with bounce solution to the FRW universe [14, 15], but the answer is not direct in Padmanabhan’s conjecture in the emergent universe. Our study will give an insight into the answer.

Our paper is organized as follows. In next section we briefly review Padmanabhan’s idea that the cosmic space is emergent as cosmic time progresses and give the standard Friedmann equation governing the dynamical evolution of the FRW universe. Then, in Section 3, we will analyze the modified Friedmann equation from Padmanabhan’s conjecture based on modified entropy-area relation. Finally, we will give our summary and discussions in Section 4. In this paper, we use the natural units with $c = \hbar = k_B = 1$.

2. The Emergence of Cosmic Space of the FRW Universe

In this section, we will give a brief review on the process of obtaining standard Friedmann equation in the emergent universe, which was addressed by Padmanabhan in [16]. The main idea is that the expansion of the universe (or the emergence of space) tends to fulfill the holographic equipartition condition, which stated that the number of degrees of freedom (N_{bulk}) inside the Hubble volume is equal to the number of degrees of freedom (N_{sur}) on the spherical surface of Hubble radius; that is, $N_{\text{bulk}} = N_{\text{sur}}$. So in our asymptotic de Sitter universe, the natural law governing the emergence of space in an infinitesimal interval dt is

$$\frac{dV}{dt} = G(N_{\text{sur}} - N_{\text{bulk}}), \quad (2)$$

where $V = 4\pi/3H^3$ is the Hubble volume and t is the cosmic time.

For a spatially flat FRW universe with Hubble constant H and apparent horizon $r_A = 1/H$, we have

$$N_{\text{sur}} = 4S = \frac{4\pi}{GH^2}, \quad (3)$$

where $S = A/4G = \pi/GH^2$ is the entropy of the apparent horizon, and

$$N_{\text{bulk}} = \frac{2|E|}{T} = -\frac{2(\rho + 3p)V}{T}, \quad (4)$$

where in the second equality we recalled the horizon temperature $T = H/2\pi$ and Komar energy $|E| = -(\rho + 3p)V$ for

accelerating part with dark energy having $\rho + 3p < 0$ (it is notable that in [16] the author discussed the contributions of the matter with $|E| = (\rho + 3p)V$ in the bulk degrees of freedom and the derivation of Friedmann equation was unaffected). Subsequently, one can reduce (2) into

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \quad (5)$$

which is the standard dynamical Friedmann equation of flat FRW universe in general relativity. Furthermore, recalling the continuity equation,

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (6)$$

and integrating (5) gives us the standard Friedmann equation

$$H^2 = \frac{8\pi G\rho}{3}. \quad (7)$$

Note that in [13] the integration result is $H^2 + k/a^2 = 8\pi G\rho/3$ with general geometry, where the authors interpreted the integration constant k as the spatial curvature of the FRW universe.

3. The Quantum Effect on Friedmann Equation in Cosmic Space of the FRW Universe

In this section, we will apply the proposal described in last section to study the quantum effect on the Friedmann equation. We only consider the quantum effect at the phenomenological level and borrow the modified dispersion relation (MDR) [14]:

$$\frac{\sin(\eta l_p E)}{\eta l_p} = \sqrt{p^2 + m^2}. \quad (8)$$

Here p and E are the momentum and energy of a particle with mass m , respectively. The Planck length is $l_p = \sqrt{8\pi G} = 1/M_p$, where M_p is the Planck mass. η is a dimensionless parameter and $\eta \rightarrow 0$ goes to the standard dispersion relation $E^2 = p^2 + m^2$.

With the use of thermodynamical description on the apparent horizon, the authors of [14] derived the modified Friedmann equation of a spatially flat universe from MDR (8). Later, the extended study in general FRW universe with $k = 0, \pm 1$ was presented in [15].

Here, we will derive the modified Friedmann equation by following the steps of emergent cosmic space shown in last section. According to the study in [15], MDR (8) modified the entropy for the first energy branch as

$$S_M = \frac{A}{4G} \sqrt{1 - \frac{4\pi\eta^2 l_p^2}{A}} + \frac{\pi\eta^2 l_p^2}{G} \ln \left[\sqrt{\frac{A}{4\pi\eta^2 l_p^2}} + \sqrt{\frac{A}{4\pi\eta^2 l_p^2} - 1} \right], \quad (9)$$

where $A = 4\pi r_A^2 = 4\pi/(H^2 + k/a^2)$ is the area of the apparent horizon at the classical level. To proceed, we define an effective apparent horizon area with the quantum effect

$$\begin{aligned} \widetilde{A} &= 4GS_M \\ &= A \sqrt{1 - \frac{4\pi\eta^2 l_p^2}{A}} \\ &\quad + 4\pi\eta^2 l_p^2 \ln \left[\sqrt{\frac{A}{4\pi\eta^2 l_p^2}} + \sqrt{\frac{A}{4\pi\eta^2 l_p^2} - 1} \right] \quad (10) \\ &= 4\pi r_A^2 \sqrt{1 - \frac{\eta^2 l_p^2}{r_A^2}} \\ &\quad + 4\pi\eta^2 l_p^2 \ln \left[\frac{r_A}{\eta l_p} + \sqrt{\frac{r_A^2}{\eta^2 l_p^2} - 1} \right]. \end{aligned}$$

Note that when $\eta \rightarrow 0$, \widetilde{A} is equal to A and recovers the usual result.

Moreover, the volume (V) and the area (A) of the apparent horizon of an n -sphere with radius r_A satisfy [34]

$$\frac{dV}{dA} = \frac{r_A}{n-1}. \quad (11)$$

Then one can think that the change of the effective volume mainly stems from the change of the effective area, so that we have the time evolution of the effective volume of the FRW universe [34]

$$\frac{d\widetilde{V}}{dt} = \frac{r_A}{2} \frac{d\widetilde{A}}{dt} = \frac{4\pi r_A^3 \dot{r}_A}{\eta l_p \sqrt{r_A^2/\eta^2 l_p^2 - 1}}, \quad (12)$$

from which we can obtain the effective volume

$$\widetilde{V} = \frac{4\pi\eta^3 l_p^3}{3} \left(2 + \frac{r_A^2}{\eta^2 l_p^2} \right) \sqrt{\frac{r_A^2}{\eta^2 l_p^2} - 1}. \quad (13)$$

Also, when $\eta \rightarrow 0$, $\widetilde{V} = 4\pi r_A^3/3$ is the usual Hubble volume.

We move on to calculate N_{bulk} in the bulk and N_{sur} in the boundary. Considering the Hawking temperature (similar to (12), we ignore the direct correction to the radius in the Hawking temperature, and the changes of numbers of degrees of freedom directly stem from the corrections of the area of the apparent horizon) $T = 1/2\pi r_A$ and $E = -(\rho + 3p)\widetilde{V}$ with dark energy in the bulk, we obtain

$$\begin{aligned} N_{\text{bulk}} &= \frac{2E}{T} \\ &= -\frac{16\pi^2 (\rho + 3p) \eta^3 l_p^3 r_A}{3} \left(2 + \frac{r_A^2}{\eta^2 l_p^2} \right) \sqrt{\frac{r_A^2}{\eta^2 l_p^2} - 1}. \quad (14) \end{aligned}$$

The statistical physics has shown that N_{sur} can be calculated from the entropy [18]

$$\begin{aligned} N_{\text{sur}} &= 4S_M \\ &= \frac{4\pi r_A^2}{G} \sqrt{1 - \frac{\eta^2 l_p^2}{r_A^2}} \\ &\quad + \frac{4\pi\eta^2 l_p^2}{G} \ln \left[\frac{r_A}{\eta l_p} + \sqrt{\frac{r_A^2}{\eta^2 l_p^2} - 1} \right]. \end{aligned} \quad (15)$$

Substituting (12), (14), and (15) into (2), we get

$$\begin{aligned} &\frac{4\pi r_A^3 \dot{r}_A}{\eta l_p \sqrt{r_A^2/\eta^2 l_p^2 - 1}} \\ &= 4\pi r_A^2 \sqrt{1 - \frac{\eta^2 l_p^2}{r_A^2}} + 4\pi\eta^2 l_p^2 \ln \left[\frac{r_A}{\eta l_p} + \sqrt{\frac{r_A^2}{\eta^2 l_p^2} - 1} \right] \quad (16) \\ &\quad + \frac{16\pi^2 G (\rho + 3p) \eta^3 l_p^3 r_A}{3} \left(2 + \frac{r_A^2}{\eta^2 l_p^2} \right) \sqrt{\frac{r_A^2}{\eta^2 l_p^2} - 1}. \end{aligned}$$

The expression above looks very complicated; however, with $k = 0$, we have $\dot{r}_A = 1 - (\ddot{a}/a)r_A^2$, so that (16) can be reduced into (with $k = \pm 1$, we have $\dot{r}_A = 1 - ((H\dot{H} - k\dot{a})/a^3 + \sqrt{(H^2 + k/a^2)^3})/\sqrt{(H^2 + k/a^2)}r_A^2$ which makes it difficult to simplify (16); we hope to solve this problem in near future)

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G (\rho + 3p)}{3} \left(1 + \frac{\eta^2 l_p^2}{r_A^2} - \frac{2\eta^4 l_p^4}{r_A^4} \right) \\ &\quad + \frac{\eta^2 l_p^2 [2 + \ln(\eta^2 l_p^2/4r_A^2)]}{2r_A^4} \\ &\simeq -\frac{4\pi G (\rho + 3p)}{3} \left(1 + \frac{\eta^2 l_p^2}{r_A^2} \right) + \frac{\eta^2 l_p^2}{r_A^4}, \end{aligned} \quad (17)$$

where, in the third line, we have approximately expanded the expression to the order $\eta^2 l_p^2/r_A^2$ because it is a small quantity. Form (17) is the modified dynamical Friedmann equation for the flat FRW universe, which reduces to the standard dynamical Friedmann equation when $\eta \rightarrow 0$. Further combining continuity equation (6) with (17), we obtain the other modified Friedmann equation

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G\rho}{3} + \frac{8\pi G\eta^2 l_p^2}{3a^2} \int \left(\dot{a}^2 \dot{\rho} + \frac{2\dot{a}^3 \rho}{a} \right) dt \\ &\quad + \frac{2\eta^2 l_p^2}{a^2} \int \frac{\dot{a}^5}{a^3} dt. \end{aligned} \quad (18)$$

Here we also set the integration constant to be vanished. Again, $\eta \rightarrow 0$ in (18) reproduces the standard result of flat FRW universe.

In order to analyze whether (18) admits a bounce solution, we define

$$\rho_c = -\frac{\rho^2}{(\eta^2 l_p^2/a^2) \int (\dot{a}^2 \dot{\rho} + 2\dot{a}^3 \rho/a) dt + (3\eta^2 l_p^2/4\pi G \rho a^2) \int (\dot{a}^5/a^3) dt}, \quad (19)$$

so that (18) can be rewritten as (1) for bounce solution. The unsolved integral in ρ_c makes it difficult to give a reliable conclusion; however, we can at least give some discussions. First, without the quantum correction, that is, $\eta \rightarrow 0$, ρ_c goes to infinity, so bounce case (1) recovers standard case (7) without bounce. Then, when ρ_c in (19) is positive, (17) and (18) fulfill the bouncing conditions, that is, $a > 0$, $\dot{a} = 0$, and $\ddot{a} > 0$; then (18) admits a bounce solution. Finally, when ρ_c is nonpositive, we can not have any bounce solution.

We note that, for the second energy branch whose entropy is $-S_M$ [15], the procedures above are straightforward and the modified Friedmann equation is the same as (17) and (18). However, for this energy branch, the effective volume (\bar{V}) and the area (\bar{A}) of the apparent horizon are negative, which are not physical.

4. Summary and Discussions

In this paper, we studied the quantum effect on the Friedmann equation for the flat FRW universe with the use of Padmanabhan's conjecture in the emergent universe. We obtained modified Friedmann equations (17) and (18) with the quantum correction on dispersion relation (8). For the closed ($k = 1$) and open ($k = -1$) universes, we found it is difficult to simplify the dynamical equation in our process, but we still see the quantum effect on (16) which is supposed to be the modified Friedmann equation. We also argued the condition under which modified Friedmann equation (18) admits a bounce solution in the flat universe.

It is worth pointing out that, in our paper, the modified Friedmann equation was only obtained in the flat FRW universe with $k = 0$, which may imply that key equation (2) is not the basic equation of the emergent universe and it may have to be corrected at the quantum level. This is an interesting point we will study in the near future. On the other hand, the experimental testing of quantum effect in bouncing cosmology is another interesting aspect. There are many literatures discussed this topic, for example, [35, 36] comparing the quantum theory with experimental data and [37, 38] probing the quantum gravity with modified dispersion relation by cold atoms.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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