Hindawi
Advances in High Energy Physics
Volume 2019, Article ID 5769564, 8 pages
https://doi.org/10.1155/2019/5769564

Research Article

Heun-Type Solutions of the Klein-Gordon and Dirac Equations in the Garfinkle-Horowitz-Strominger Dilaton Black Hole Background

Marina-Aura Dariescu,[1] **Ciprian Dariescu**,[1] **and Cristian Stelea**[2]

[1]*Faculty of Physics, "Alexandru Ioan Cuza" University, 11 Blvd. Carol I, Iasi 700506, Romania*
[2]*Research Department, Faculty of Physics, "Alexandru Ioan Cuza" University, 11 Blvd. Carol I, Iasi 700506, Romania*

Correspondence should be addressed to Cristian Stelea; cristian.stelea@uaic.ro

Received 20 December 2018; Revised 28 January 2019; Accepted 11 February 2019; Published 26 February 2019

Guest Editor: Subhajit Saha

We study the Klein-Gordon and the Dirac equations in the background of the Garfinkle-Horowitz-Strominger black hole in the Einstein frame. Using a $SO(3,1) \times U(1)$–gauge covariant approach, as an alternative to the Newman-Penrose formalism for the Dirac equation, it turns out that these solutions can be expressed in terms of Heun confluent functions and we discuss some of their properties.

1. Introduction

In recent years, black holes with electric or magnetic charge, in presence of a scalar field called *dilaton*, have been studied mainly in string theories. These charged black holes are solutions of the low-energy four-dimensional effective theories obtained by dimensional compactification of the heterotic string theories. Generically, the effective action of these theories describes a massless dilaton coupled to an abelian vector field [1]. Due to the dimensional compactification process, the dilaton is also nonminimally coupled to the Ricci scalar, with the effective solution being described in the so-called string frame. However, to facilitate comparison with the standard black holes in general relativity, it is convenient to go to the so-called Einstein frame by performing conformal rescaling of the metric (for a review, see [2]).

A remarkable black hole solution of the effective four-dimensional compactified theory was found by Gibbons and Maeda [3, 4] and independently rediscovered in a simpler form, few years later, by Garfinkle, Horowitz, and Strominger (GHS) [5] (for a review of its properties, see [6]). Even though, in terms of the string metric, the electric and magnetic black holes have very different properties, in the Einstein frame the metric does not change when we go from an electrically charged to a magnetically charged black hole (this is basically due to the electromagnetic duality present in the Einstein frame; in the string frame, the electromagnetic field strength is also modified by the dilaton field [7]).

Using the GHS metric in the Einstein frame, the present work is devoted to a study of the Klein-Gordon and Dirac equations, which describe charged particles evolving in the Garfinkle-Horowitz-Strominger (GHS) dilaton black hole spacetime. Within a $SO(3,1) \times U(1)$–gauge covariant approach, it turns out that the solutions can be expressed in terms of Heun confluent functions [8, 9]. Special attention is given to the resonant frequencies, which arise here by imposing a polynomial form of the Heun functions. In general, the so-called quasi-normal modes have discrete spectra of complex characteristic frequencies, with the real part representing the actual frequency of the oscillation and the imaginary part representing the damping. By comparing these modes with the gravitational waves observed in the universe, one should be able to identify the presence of a GHS black hole [10, 11] (see also in [12] the effect of the dilaton field imprint on the gravitational waves emitted in the collision of two GHS black holes).

When the parameter related to the dilaton field goes to zero, one obtains the Klein-Gordon and Dirac equations for the usual Schwarzschild metric, which have been intensively worked out both in their original form and in different types of extensions. For instance, recently, for the Schwarzschild metric in the presence of an electromagnetic field, the Klein-Gordon and Dirac equations for massless particles have been put into a Heun-type form [13, 14]. One should note that Heun functions are often encountered when studying the propagation of various test fields in the background of various black holes or relativistic stars [15–20] and also in cosmology in the context of extended effective field theories of inflation [21].

The method used in the present paper, while based on Cartan's formalism, is an alternative to the Newman-Penrose (NP) formalism [22], which is usually employed for solving Dirac equation describing fermions in the vicinity of different types of black holes [23–27].

The structure of this paper is as follows: in the next section, we present the solutions of the Klein-Gordon and Dirac equations in the background of the GHS dilatonic black hole. In Section 3, we discuss the solutions of the massless Dirac equations in this background and show how to recover the expression of the Hawking temperature. The final section is dedicated to conclusions.

2. Klein-Gordon and Dirac Equations on the GHS Dilaton Black Hole Metric

In Einstein frame, the static and spherically symmetric GHS dilaton black hole metric is given by [5]

$$ds^2 = -R\,dt^2 + \frac{dr^2}{R} + r\,(r-a)\left[d\theta^2 + \sin^2\theta\,d\varphi^2\right], \tag{1}$$

where

$$R = 1 - \frac{2M}{r}, \quad \text{and } a = \frac{Q^2}{M}, \tag{2}$$

with M and Q being the mass and the charge of this black hole, which has an event horizon at $r = 2M$ and two singularities located at $r = 0$ and $r = a$. Obviously, if the electric charge of the GHS black hole is zero, the metric in (1) reduces to the Schwarzschild one.

The parameter a is related to the dilaton field ϕ as (Note that we set the asymptotic value of the dilaton field $\phi_0 = 0$.)

$$e^{-2\phi} = 1 \mp \frac{a}{r}, \tag{3}$$

where the *minus* and *plus* signs are for, respectively, the magnetically electrically charged black holes.

Within the $SO(3,1)$–gauge covariant formulation, we introduce the pseudoorthonormal frame $\{E_a\}_{(a=\overline{1,4})}$; that is,

$$\begin{aligned} E_1 &= \sqrt{R}\partial_r, \\ E_2 &= \frac{1}{\sqrt{r\,(r-a)}}\partial_\theta, \\ E_3 &= \frac{1}{\sqrt{r\,(r-a)}\sin\theta}\partial_\varphi, \\ E_4 &= \frac{1}{\sqrt{R}}\partial_t, \end{aligned} \tag{4}$$

whose corresponding dual base is

$$\begin{aligned} \omega^1 &= \frac{1}{\sqrt{R}}dr, \\ \omega^2 &= \sqrt{r\,(r-a)}d\theta, \\ \omega^3 &= \sqrt{r\,(r-a)}\sin\theta\,d\varphi, \\ \omega^4 &= \sqrt{R}\,dt, \end{aligned} \tag{5}$$

so that the metric in (1) becomes the usual Minkowsky metric $ds^2 = \eta_{ab}\,\omega^a\omega^b$, with $\eta_{ab} = \mathrm{diag}[1,1,1,-1]$.

Using the first Cartan's equation,

$$d\omega^a = \Gamma^a_{.[bc]}\omega^b \wedge \omega^c, \tag{6}$$

with $1 \le b < c \le 4$ and $\Gamma^a_{.[bc]} = \Gamma^a_{.bc} - \Gamma^a_{.cb}$, we obtain the following connection one-forms $\Gamma_{ab} = \Gamma_{abc}\omega^c$, where $\Gamma_{abc} = -\Gamma_{bac}$; namely,

$$\begin{aligned} \Gamma_{212} = \Gamma_{313} &= \frac{r - a/2}{r\,(r-a)}\sqrt{R}, \\ \Gamma_{323} &= \frac{\cot\theta}{\sqrt{r\,(r-a)}}, \\ \Gamma_{414} &= -\frac{M}{r^2\sqrt{R}} \end{aligned} \tag{7}$$

In the pseudoorthonormal bases (with $\eta_{44} = -1$), the fourth component of the one-form potential is

$$A_4 = -\frac{1}{\sqrt{R}}\frac{Q}{r}, \tag{8}$$

and it corresponds to an electric field:

$$F_{14} = E_1A_4 - E_4A_1 + A_c\Gamma^c{}_{ab} - A_c\Gamma^c{}_{ba} = \frac{Q}{r^2}. \tag{9}$$

2.1. The Klein-Gordon Equation. For the complex scalar field of mass m_0, minimally coupled to gravity, the Klein-Gordon equation has the general $SO(3,1)\times U(1)$-gauge covariant form

$$\eta^{ab}\Phi_{;ab} - m_0^2\Phi = 0; \tag{10}$$

that is,

$$\eta^{ab}\Phi_{|ab} - \eta^{ab}\Phi_{|c}\Gamma^c{}_{ab} = m_0^2\Phi + 2iqA^a\Phi_{|a} + q^2A_aA^a\Phi, \tag{11}$$

where

$$\Phi_{;a} = \Phi_{|a} - iqA_a\Phi, \tag{12}$$

with $\Phi_{|a} = E_a\Phi$.

The two terms in the left-hand side of relation (11) are, respectively, given by

$$\begin{aligned}\eta^{ab}\Phi_{|ab} = {} & R\frac{\partial^2\Phi}{\partial r^2} + \frac{M}{r^2}\frac{\partial\Phi}{\partial r} \\ & + \frac{1}{r(r-a)}\left[\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\right] - \frac{1}{R}\frac{\partial^2\Phi}{\partial t^2}\end{aligned} \tag{13}$$

and

$$\begin{aligned}\eta^{ab}\Phi_{|c}\Gamma^c{}_{ab} = {} & -\left[\frac{M}{r^2} + \frac{R(2r-a)}{r(r-a)}\right]\frac{\partial\Phi}{\partial r} \\ & - \frac{\cot\theta}{r(r-a)}\frac{\partial\Phi}{\partial\theta},\end{aligned} \tag{14}$$

and the Klein-Gordon equation in (11) can be cast into the following explicit form:

$$\begin{aligned}& r(r-a)R\frac{\partial^2\Phi}{\partial r^2} + (2r-2M-a)\frac{\partial\Phi}{\partial r} \\ & + \left[\frac{\partial^2\Phi}{\partial\theta^2} + \cot\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\right] \\ & - \left[r(r-a)m_0^2 + \frac{r(r-a)}{R}\left(\frac{\partial}{\partial t} + i\frac{qQ}{r}\right)^2\right]\Phi \\ & = 0.\end{aligned} \tag{15}$$

Using the separation of the variables with the ansatz

$$\Phi(r,\theta,\varphi,t) = G(r)\,Y_\ell^m(\theta,\varphi)\,e^{-i\omega t}, \tag{16}$$

where Y_ℓ^m are the spherical harmonics, it turns out that the unknown function $G(r)$ is the solution of the differential equation

$$\begin{aligned}& r(r-a)R\frac{d^2G}{dr^2} + (2r-2M-a)\frac{dG}{dr} \\ & - \left[\ell(\ell+1) + r(r-a)m_0^2 - \frac{r-a}{r-2M}(\omega r - qQ)^2\right]G \\ & = 0.\end{aligned} \tag{17}$$

This equation can be solved exactly, with its solutions being expressed in terms of the confluent Heun functions [8, 9] as

$$\begin{aligned}G = {} & e^{\alpha x/2}\left\{C_1 x^{\beta/2} HeunC[\alpha,\beta,\gamma,\delta,\eta,x]\right. \\ & \left. + C_2 x^{-\beta/2} HeunC[\alpha,-\beta,\gamma,\delta,\eta,x]\right\}\end{aligned} \tag{18}$$

with the variable

$$x = \frac{r-2M}{a-2M} \tag{19}$$

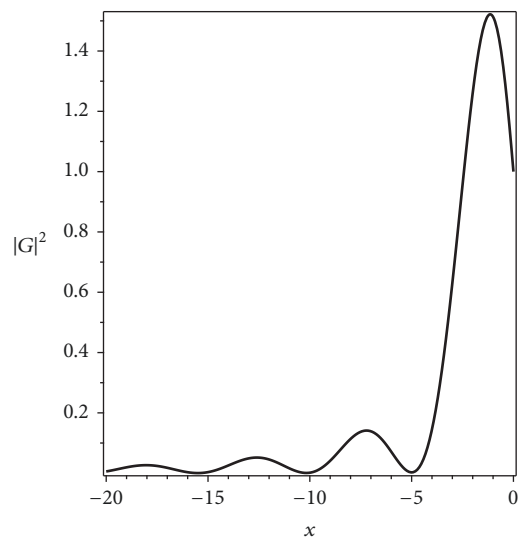

Figure 1: The square modulus of the function $G(r)$ given in (18).

and parameters

$$\begin{aligned}\alpha &= 2i(a-2M)\sqrt{\omega^2 - m_0^2}, \\ \beta &= 2i(2M\omega - qQ), \\ \gamma &= 0, \\ \delta &= 2\left[Mm_0^2 - 2M\omega^2 + qQ\omega\right](2M-a), \\ \eta &= -\delta - \ell(\ell+1).\end{aligned} \tag{20}$$

The parameters α and β are purely imaginary, and the radial part of the density probability is given by the square modulus of the Heun functions in (18). The two independent solutions have the generic behavior represented in Figure 1, for $r > 2M > a$. The main features of the probability curve are quite nice; that is, it satisfies all the text-book requirements imposed to physically meaningful wave functions. In this respect, $|G(r)|^2 = 1$ on the horizon and it gets a series of local decreasing maxima, finally vanishing rapidly, at the spatial infinity. If the number of these maxima was finite, the state would be bounded. Otherwise, it could asymptotically radiate. More details on the physical phenomena related to these properties are thoroughly discussed in [28], where the authors are computing the complex values of the energy spectrum coming from the polynomial condition imposed on the Heun functions. In [19, 28], the authors were working in coordinate bases, with

$$A_4^{(c)} = -\frac{Q}{r}, \tag{21}$$

so that $A_4^{(c)}dt = A_4\omega^4$, where A_4 is given in (8).

In the particular case $a = 0$, corresponding to the familiar Schwarzschild black hole, the function G has the same expression as in (18) but with the variable and parameters computed for $a = 0$.

2.2. The Dirac Equation. The spinor of mass μ minimally coupled to gravity is described by the Dirac equation

$$\gamma^a \Psi_{;a} + \mu\Psi = 0 \tag{22}$$

with

$$\Psi_{;a} = \Psi_{|a} + \frac{1}{4}\Gamma_{bca}\gamma^b\gamma^c\Psi - iqA_a\Psi. \tag{23}$$

In contrast to the Klein-Gordon case, the situation is more complicated in the case of the Dirac equation (22) and this complication is basically due to the square root $\sqrt{r(r-a)}$, which appears in the expressions of E_2 and E_3. Thus, with the term expressing the Ricci spin-connection given by

$$\begin{aligned}\frac{1}{4}\Gamma_{bca}\gamma^a\gamma^b\gamma^c = \frac{1}{2}\left[\frac{2r-a}{r(r-a)}\sqrt{R} + \frac{M}{r^2\sqrt{R}}\right]\gamma^1 \\ + \frac{\cot\theta}{2\sqrt{r(r-a)}}\gamma^2,\end{aligned} \tag{24}$$

the Dirac equation becomes

$$\begin{aligned}&\gamma^1\left[\sqrt{R}\frac{\partial\Psi}{\partial r} + \frac{2r-a-3M+aM/r}{2\sqrt{R}r(r-a)}\Psi\right] \\ &+ \frac{\gamma^2}{\sqrt{r(r-a)}}\left[\frac{\partial\Psi}{\partial\theta} + \frac{\cot\theta}{2}\Psi\right] \\ &+ \frac{\gamma^3}{\sqrt{r(r-a)}\sin\theta}\frac{\partial\Psi}{\partial\varphi} + \frac{\gamma^4}{\sqrt{R}}\left[\frac{\partial\Psi}{\partial t} + iq\frac{Q}{r}\Psi\right] \\ &+ \mu\Psi = 0.\end{aligned} \tag{25}$$

As in the previous Klein-Gordon case, one can use the separation of the variables

$$\Psi = \psi(r,\theta)\, e^{i(m\varphi-\omega t)}, \tag{26}$$

with the function $\psi(r,\theta)$ defined as

$$\psi(r,\theta) = \left[r(r-a)\sqrt{R}\right]^{-1/2}\chi(r,\theta) \tag{27}$$

and one obtains the explicit expression of the differential equation satisfied by $\chi(r,\theta)$:

$$\begin{aligned}&\sqrt{Rr(r-a)}\gamma^1\frac{\partial\chi}{\partial r} + \gamma^2 D_\theta\chi + \frac{im}{\sin\theta}\gamma^3\chi \\ &+ i\sqrt{\frac{r(r-a)}{R}}\left(\frac{qQ}{r} - \omega\right)\gamma^4\chi + \mu\sqrt{r(r-a)}\chi \\ &= 0,\end{aligned} \tag{28}$$

where

$$D_\theta = \frac{\partial}{\partial\theta} + \frac{\cot\theta}{2}. \tag{29}$$

Using the Weyl representation for the γ^i matrices,

$$\begin{aligned}\gamma^1 &= -i\beta\alpha^3, \\ \gamma^2 &= -i\beta\alpha^1, \\ \gamma^3 &= -i\beta\alpha^2, \\ \gamma^4 &= -i\beta,\end{aligned} \tag{30}$$

with

$$\begin{aligned}\alpha^\mu &= \begin{pmatrix}\sigma^\mu & 0 \\ 0 & -\sigma^\mu\end{pmatrix}, \\ \beta &= \begin{pmatrix}0 & -\mathrm{I} \\ -\mathrm{I} & 0\end{pmatrix}, \\ \text{so that } \gamma^5 &= \begin{pmatrix}\mathrm{I} & 0 \\ 0 & -\mathrm{I}\end{pmatrix},\end{aligned} \tag{31}$$

where σ^μ denote the usual Pauli matrices and (28) becomes

$$\begin{aligned}&\sqrt{Rr(r-a)}\alpha^3\frac{\partial\chi}{\partial r} + \alpha^1 D_\theta\chi + \frac{im}{\sin\theta}\alpha^2\chi \\ &+ i\sqrt{\frac{r(r-a)}{R}}\left(\frac{qQ}{r} - \omega\right)\chi + i\mu\sqrt{r(r-a)}\beta\chi \\ &= 0,\end{aligned} \tag{32}$$

and one may use again the standard procedure based on the separation of the variables. Thus, with the bi-spinor χ written in terms of two components spinors as

$$\chi(r,\theta) = \begin{bmatrix}\zeta(r,\theta) \\ \eta(r,\theta)\end{bmatrix}, \tag{33}$$

where

$$\begin{aligned}\zeta_1 &= S_1(r)\,T_1(\theta), \\ \zeta_2 &= S_2(r)\,T_2(\theta), \\ \eta_1 &= S_2(r)\,T_1(\theta), \\ \eta_2 &= S_1(r)\,T_2(\theta),\end{aligned} \tag{34}$$

one obtains the following system of coupled radial equations for the components S_1 and S_2, that is,

$$\begin{aligned}S_1' + \frac{i(qQ-\omega r)}{rR}S_1 + \frac{1}{\sqrt{R}}\left[\frac{\lambda}{\sqrt{r(r-a)}} - i\mu\right]S_2 = 0, \\ S_2' - \frac{i(qQ-\omega r)}{rR}S_2 + \frac{1}{\sqrt{R}}\left[\frac{\lambda}{\sqrt{r(r-a)}} + i\mu\right]S_1 = 0,\end{aligned} \tag{35}$$

and take into account the following essential relations:

$$\begin{aligned}\left[\frac{d}{d\theta} + \frac{\cot\theta}{2} + \frac{m}{\sin\theta}\right]T_2 = \lambda T_1, \\ \left[\frac{d}{d\theta} + \frac{\cot\theta}{2} - \frac{m}{\sin\theta}\right]T_1 = -\lambda T_2.\end{aligned} \tag{36}$$

Thus, the angular parts T_A, with $A = 1, 2$, are satisfying the decoupled equations

$$\frac{d^2 T_A}{d\theta^2} + \cot\theta \frac{dT_A}{d\theta} - \left[\frac{(\cos\theta \mp 2m)^2}{4\sin^2\theta} - \lambda^2 + \frac{1}{2}\right] T_A = 0, \tag{37}$$

with the solutions given by the spin-weighted spherical harmonics [29], for $\lambda = \ell + 1/2$.

As for the radial equations, we employ the auxiliary function method and consider S_1 and S_2 as

$$S_1 = e^{i\omega r}\left(\frac{r}{2M} - 1\right)^{2i\omega M - iqQ} \Sigma_1(r),$$
$$S_2 = e^{-i\omega r}\left(\frac{r}{2M} - 1\right)^{-2i\omega M + iqQ} \Sigma_2(r), \tag{38}$$

so that system (35) leads to the following simpler equations for the unknown functions Σ_A:

$$\Sigma_1' + \left(\frac{r}{2M} - 1\right)^{-4i\omega M + 2iqQ} \frac{e^{-2i\omega r}}{\sqrt{R}} \left[\frac{\lambda}{\sqrt{r(r-a)}} - i\mu\right] \cdot \Sigma_2 = 0,$$
$$\Sigma_2' + \left(\frac{r}{2M} - 1\right)^{4i\omega M - 2iqQ} \frac{e^{2i\omega r}}{\sqrt{R}} \left[\frac{\lambda}{\sqrt{r(r-a)}} + i\mu\right] \Sigma_1 = 0. \tag{39}$$

The differential equation for Σ_1, that is,

$$\Sigma_1'' + \left[\frac{4i\omega r - 4iqQ + 1}{2(r-2M)} + \frac{1}{2(r-a)} + \frac{i\mu(2r-a)}{2\sqrt{r(r-a)}\left[\lambda - i\mu\sqrt{r(r-a)}\right]}\right]\Sigma_1' - \frac{r}{r-2M}\left[\frac{\lambda^2}{r(r-a)} + \mu^2\right]\Sigma_1 = 0, \tag{40}$$

can not be analytically solved. Numerically, using Mathematica [30], with the initial conditions

$$\Sigma_1(2M_+) = 0,$$
$$\Sigma_1'(2M_+) = 1, \tag{41}$$

the absolute value of the radial part of $|\psi|^2$ given in (26), namely,

$$F(r) = \frac{1}{r(r-a)\sqrt{R}} |S_1|^2, \tag{42}$$

is represented in Figure 2, for $r > 2M > a$.

This is describing the fermionic ground state in the outer region, with just one maximum (as it should) and exponentially vanishing at infinity. We have not analyzed the corresponding modes located within the black hole, since, in the limit $a \longrightarrow 0$, the area of the sphere $r = a$ is zero so that this surface is singular. Once Q increases, the singular surface moves towards the event horizon $r = 2M$, and one has to solve the problems related to the physically meaningful boundary conditions.

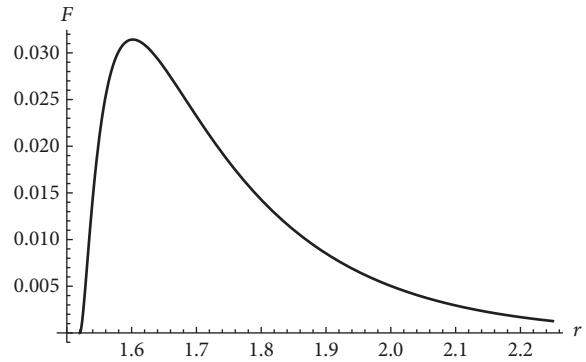

Figure 2: Function (42), for $a = 1$ and $2M = 1.5$.

If one imposes $\lambda \ll \mu r$ and performs a series expansion of the last term multiplying Σ_1' in (40), to first order in a/r, this last term can be approximated to $1/(r-a)$ and the solution is given by the Heun confluent functions [8, 9] as

$$\Sigma_1 = C_1 e^{-\mu r} HeunC\left[\alpha, \beta, \gamma, \delta, \eta, \frac{r-2M}{a-2M}\right] + C_2 e^{-\mu r}\left(\frac{r}{2M} - 1\right)^{-2i\omega M + 2iqQ + 1/2} \cdot HeunC\left[\alpha, -\beta, \gamma, \delta, \eta, \frac{r-2M}{a-2M}\right], \tag{43}$$

with the parameters

$$\alpha = 2\mu(2M - a),$$
$$\beta = 2i(\omega M - qQ) - \frac{1}{2},$$
$$\gamma = -\frac{3}{2},$$
$$\delta = 2\mu^2 M(2M - a)$$
$$\eta = -\delta + \frac{i(\omega M - qQ)}{2} + \frac{5}{8} - \lambda^2. \tag{44}$$

Near the exterior event horizon, $r \longrightarrow 2M$; that is, $x \longrightarrow 0$; the Heun confluent functions in (43) have a polynomial form if their parameters are satisfying the condition [8, 9]

$$\frac{\delta}{\alpha} = -\left[n + 1 + \frac{\beta + \gamma}{2}\right]. \tag{45}$$

By replacing the expressions of (44) in (45), one gets the relation

$$i(\omega M - qQ) + \mu M = -n, \tag{46}$$

for the Heun function multiplied by C_1, and the relation

$$i(\omega M - qQ) - \mu M = n + \frac{1}{2}, \tag{47}$$

for the one multiplied by C_2. These relations are pointing out a quantized part of the imaginary part of ω, which corresponds to resonant frequencies [28].

3. The Massless Case

The Dirac equation has been worked out for several physically important metrics, mainly using the NP formalism [25] and some of the solutions, especially in the massless case, have been expressed in terms of Heun confluent functions [13].

In view of the analysis developed in the previous section, the massless and chargeless fermions are described by the radial equations coming from system (35); namely,

$$\begin{aligned} S_1' - \frac{i\omega}{R} S_1 + \frac{\lambda}{\sqrt{Rr(r-a)}} S_2 = 0, \\ S_2' + \frac{i\omega}{R} S_2 + \frac{\lambda}{\sqrt{Rr(r-a)}} S_1 = 0, \end{aligned} \tag{48}$$

with

$$\begin{aligned} S_1 &= e^{i\omega r}\left(\frac{r}{2M} - 1\right)^{2i\omega M} \Sigma_1(r), \\ S_2 &= e^{-i\omega r}\left(\frac{r}{2M} - 1\right)^{-2i\omega M} \Sigma_2(r). \end{aligned} \tag{49}$$

Thus, system (48) turns into the simpler form

$$\begin{aligned} \Sigma_1' + e^{-2i\omega r}\left(\frac{r}{2M} - 1\right)^{-4i\omega M} \frac{\lambda}{\sqrt{Rr(r-a)}} \Sigma_2 = 0, \\ \Sigma_2' + e^{2i\omega r}\left(\frac{r}{2M} - 1\right)^{4i\omega M} \frac{\lambda}{\sqrt{Rr(r-a)}} \Sigma_1 = 0, \end{aligned} \tag{50}$$

which leads to the following differential equation for Σ_1:

$$\begin{aligned} \Sigma_1'' + \left[\frac{4i\omega r + 1}{2(r-2M)} + \frac{1}{2(r-a)}\right] \Sigma_1' \\ - \frac{\lambda^2}{(r-2M)(r-a)} \Sigma_1 = 0 \end{aligned} \tag{51}$$

and similarly for Σ_2. The solution of this equation is expressed in terms of Heun confluent functions as

$$\begin{aligned} \Sigma_1 = e^{-2i\omega r} \{ C_1 HeunC[\alpha, \beta, \gamma, \delta, \eta, x] \\ + C_2 x^{-\beta} HeunC[\alpha, -\beta, \gamma, \delta, \eta, x] \} \end{aligned} \tag{52}$$

where the variable is

$$x = \frac{r - 2M}{a - 2M} \tag{53}$$

and the corresponding parameters are

$$\begin{aligned} \alpha &= 2i\omega(2M - a), \\ \beta &= 4i\omega M - \frac{1}{2}, \\ \gamma &= -\frac{1}{2}, \\ \delta &= i\omega(4i\omega M + 1)(2M - a) \\ \eta &= -\delta - \frac{i\omega a}{2} + \frac{3}{8} - \lambda^2. \end{aligned} \tag{54}$$

The solutions to Heun confluent equations are computed as power series expansions around the regular singular point $x = 0$; that is, $r = 2M$. The series converges for $r < a$ (the second regular singularity) and the analytic continuation is obtained by expanding the solution around the regular singularity $r = a$ and overlapping the series.

For large x values, one may use the formula [8, 28]

$$\begin{aligned} &HeunC[\alpha, \beta, \gamma, \delta, \eta, x] \\ &\approx D_1 x^{-[(\beta+\gamma+2)/2+\delta/\alpha]} + D_2 e^{-\alpha x} x^{-[(\beta+\gamma+2)/2-\delta/\alpha]} \\ &= e^{-\alpha x/2} x^{-(\beta+\gamma+2)/2} \left\{ D_1 e^{\alpha x/2} x^{-\delta/\alpha} + D_2 e^{-\alpha x/2} x^{\delta/\alpha} \right\} \\ &= D e^{-\alpha x/2} x^{-(\beta+\gamma+2)/2} \sin\left[-\frac{i\alpha x}{2} + \frac{i\delta}{\alpha} \ln x + \sigma \right], \end{aligned} \tag{55}$$

where D is an arbitrary constant and σ is the phase shift. With the parameters given in (54), the component S_1 from (49) gets the asymptotic form

$$S_1 \approx \frac{D}{\sqrt{r}} \sin\left[\omega r + 2\omega M \ln r - \frac{i}{2} \ln r + \sigma\right] \tag{56}$$

which, for large r values, behaves like

$$S_1 \sim \exp[i(\omega r + 2\omega M \ln r + \sigma)]. \tag{57}$$

The necessary condition for a polynomial form of the Heun confluent functions in (45) leads to the following quantized imaginary quasi-spectrum:

$$\omega = i\frac{(n+1)}{4M}. \tag{58}$$

In order to study the radiation emitted by the GHS black hole, one has to take the radial solution near the exterior event horizon, $r \longrightarrow r_h = 2M$. For $x \longrightarrow 0$, the Heun functions can be approximated to 1 and the (radial) components of Ψ defined in (26) and (33), with S_A given in (49), can be written as

$$\begin{aligned} \Psi_1 \approx e^{-i\omega t} e^{-i\omega r} \frac{1}{\sqrt{r(r-a)}} \{ C_1 (r-2M)^{2i\omega M - 1/4} \\ + C_2 (r-2M)^{-2i\omega M + 1/4} \} \end{aligned} \tag{59}$$

and

$$\Psi_2 \approx e^{-i\omega t} e^{i\omega r} \frac{1}{\sqrt{r(r-a)}} \left\{ C_1 (r-2M)^{-2i\omega M - 1/4} + C_2 (r-2M)^{2i\omega M + 1/4} \right\}, \tag{60}$$

pointing out the *in* and *out* modes

$$\Psi_{in} \sim e^{-i\omega t} (r-2M)^{-2i\omega M + 1/4}, \quad \Psi_{out} \sim e^{-i\omega t} (r-2M)^{2i\omega M - 1/4}. \tag{61}$$

By definition, the component ψ_{out} should asymptotically have the form

$$\Psi_{out} \sim (r - r_h)^{(i/2\kappa_h)(\omega - \omega_h)} \tag{62}$$

so that the relative scattering probability at the exterior event horizon surface is given by

$$\Gamma = \left| \frac{\Psi_{out}(r > 2M)}{\Psi_{out}(r < 2M)} \right|^2 = \exp\left[-\frac{2\pi}{\kappa_h} (\omega - \omega_h) \right]. \tag{63}$$

Inspecting the above relations, it yields the well-known results: $\kappa_h = 1/(4M)$,

$$\Gamma = e^{-8\pi M \omega} \tag{64}$$

and the mean number of emitted particles

$$N = \frac{\Gamma}{1-\Gamma} = \frac{1}{e^{\omega/T_h} - 1}, \tag{65}$$

where $T_h = 1/(8\pi M)$ is the Hawking temperature.

4. Conclusions

In the present paper, we have used the free-of-coordinates formalism to write down both the Klein-Gordon and the Dirac equations, in their $SO(3,1) \times U(1)$ expression, for the GHS metric in (1).

Unlike the case for bosons, it turns out that, for the charged massive fermions interacting with the GHS dilaton black hole, the radial equation (40) does not have an analytic solution. However, to first order in a/r, the corresponding equations are satisfied by the Heun confluent functions (43).

In the massless case, the Dirac equation can be analytically solved and the derived solution, given by (52), is valid for the whole space, which includes not only the near-horizon region but also the far-away-from-the-black-hole region. Once the relation in (45) among the Heun function's parameters is imposed, the confluent Heun functions can be cast into a polynomial form and the energy spectrum is given by the imaginary quantized expression (58).

Finally, by identifying the *out* modes near the event horizon, we identified the Hawking black body radiation and the expected Hawking temperature is correctly recovered.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by a grant of Ministry of Research and Innovation, CNCS-UEFISCDI, Project no. PN-III-P4-ID-PCE-2016-0131, within PNCDI III.

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