ReviewArticle

Recent Developments in the Holographic Description of Quantum Chaos

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We review recent developments encompassing the description of quantum chaos in holography. We discuss the characterization of quantum chaos based on the late time vanishing of out-of-time-order correlators and explain how this is realized in the dual gravitational description. We also review the connections of chaos with the spreading of quantum entanglement and diffusion phenomena.

1. Introduction

The characterization of quantum chaos is fairly complicated. Possible approaches range from semiclassical methods to random matrix theory: in the first case one studies the semiclassical limit of a system whose classical dynamics is chaotic; in the later approach the characterization of quantum chaos is made by comparing the spectrum of energies of the system in question to the spectrum of random matrices [1]. Despite the insights provided by the above-mentioned approaches, a complete and more satisfactory understanding of quantum chaos remains elusive.

Surprisingly, new insights into quantum chaos have come from black holes physics! In the context of so-called gauge-gravity duality [2–4], black holes in asymptotically AdS spaces are dual to strongly coupled many-body quantum systems. It was recently shown that the chaotic nature of many-body quantum systems can be diagnosed with certain out-of-time-order correlation (OTOC) functions which, in the gravitational description, are related to the collision of shock waves close to the black hole horizon [5–9]. In addition to being useful for diagnosing chaos in holographic systems and providing a deeper understanding for the inner-working mechanisms of gauge-gravity duality, OTOCs have also proved useful in characterizing chaos in more general nonholographic systems, including some simple models like the kicked-rotor [10], the stadium billiard [11], and the Dicke model [12].

In this paper we review the recent developments in the holographic description of quantum chaos. We discuss the characterization of quantum chaos based on the late time vanishing of OTOCs and explain how this is realized in the dual gravitational description. We also review the connections of chaos with spreading of quantum entanglement and diffusion phenomena. We focus on the case of \( d \)-dimensional gravitational systems with \( d \geq 3 \), which excludes the case of gravity in \( AdS_2 \) and SYK-like models [13–16]. (Another interesting perspective on the characterization of chaos in the context of (regularized) \( AdS_2/CFT_1 \) is provided by [17–19].) Also, due the lack of the author’s expertise, we did not cover the recent developments in the direct field theory calculations of OTOCs. This includes calculations for CFTs [20], weakly coupled systems [21, 22], random unitary models [23–25], and spin chains [26–30].

2. A Bird Eye’s View on Classical Chaos

In this section we briefly review some basic aspects of classical chaos. For definiteness we consider the case of a
classical thermal system with phase space denoted as $X = (q, p)$, where $q$ and $p$ are multidimensional vectors denoting the coordinates and momenta of the phase space. We can quantify whether the system is chaotic or not by measuring the stability of a trajectory in phase space under small changes of the initial condition. Let us consider a reference trajectory in phase space, $X(t)$, with some initial condition $X(0) = X_0$. A small change in the initial condition $X_0 \rightarrow X_0 + \delta X_0$ leads to a new trajectory $X(t) \rightarrow X(t) + \delta X(t)$. This is illustrated in Figure 1. For a chaotic system, the distance between the new trajectory and the reference one increases exponentially with time

$$|\delta X(t)| \sim |\delta X_0| e^{\lambda t}$$

or

$$\frac{\partial X(t)}{\partial X_0} \sim e^{\lambda t},$$

(1)

where $\lambda$ is the so-called Lyapunov exponent. This should be contrasted with the behavior of nonchaotic systems, in which $\delta X(t)$ remains bounded or increases algebraically [31].

The exponential increase depends on the orientation of $\delta X_0$ and this leads to a spectrum of Lyapunov exponents, $\{\lambda_1, \lambda_2, \ldots, \lambda_K\}$, where $K$ is the dimensionality of the phase space. A useful parameter characterizing the trajectory instability is

$$\lambda_{\text{max}} = \lim_{t \to \infty} \lim_{\epsilon \to 0} \frac{1}{t} \log \left( \frac{\delta X(t)}{\delta X_0} \right),$$

(2)

which is called the maximum Lyapunov exponent. When the above limits exist and $\lambda_{\text{max}} > 0$, the trajectory shows sensitivity to initial conditions and the system is said to be chaotic [31].

The chaotic behavior can be a consequence of either a complicated Hamiltonian or simply the contact with a thermal heat bath. This is because chaos is a common property of thermal systems. For the latter to make contact with black holes physics, we consider the case of a classical thermal system with inverse temperature $\beta$. If $F(X)$ is some function of the phase space coordinates, we define its classical expectation value as

$$\langle F \rangle_{\beta} = \int \frac{dX e^{-\beta H(X)}}{\int dX e^{-\beta H(X)}} F(X)$$

(3)

where $H(X)$ is the system's Hamiltonian.

Classical thermal systems have two exponential behaviors that have analogues in terms of black holes physics: the Lyapunov behavior, characterizing the sensitive dependence on initial conditions, and the Ruelle behavior, characterizing the approach to thermal equilibrium [32, 33].

To quantify the sensitivity to initial conditions in a thermal system we need to consider thermal expectation values. Note that (1) can have either signs. To avoid cancellations in a thermal expectation values, we consider the square of this derivative

$$F(t) = \left\langle \left( \frac{\partial X(t)}{\partial X(0)} \right)^2 \right\rangle_{\beta}.$$  

(4)

The expected behavior of this quantity is the following [34]

$$F(t) \sim \sum_{k} c_k e^{2\lambda_k t},$$  

(5)

where $c_k$ are constants and $\lambda_k$ are the Lyapunov exponents. At later times the behavior is controlled by the maximum Lyapunov exponent $F \sim e^{\lambda_{\text{max}} t}$.

The approach to thermal equilibrium or, in other words, how fast the system forgets its initial condition can be quantified by two-point functions of the form

$$G(t) = \langle X(t) X(0) \rangle_{\beta} - \langle X \rangle_{\beta}^2,$$

(6)

whose expected behavior is [34]

$$G(t) \sim \sum_{f} b_j e^{-\mu_j t},$$

(7)

where $b_j$ are constants and $\mu_j$ are complex parameters called Ruelle resonances. The late time behavior is controlled by the smallest Ruelle resonance $G \sim e^{-\mu_{\text{min}} t}$.

### 3. Some Aspects of Quantum Chaos

In this section we review some aspects of quantum chaos. For a long time, the characterization of quantum chaos was made by comparing the spectrum of energies of the system in question to the spectrum of random matrices or using semiclassical methods [1]. Here we follow a different approach, which was first proposed by Larkin and Ovchinnikov [35] in the context of semiclassical systems, and it was recently developed by Shenker and Stanford [6–8] and by Kitaev [9].

For simplicity, let us consider the case of a one-dimensional system, with phase space variables $(q, p)$. Classically, we know that $\partial q(t)/\partial q(0)$ grows exponentially with time for a chaotic system. The quantum version of this quantity can be obtained by noting that

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\}_P.$$

(8)
where \( \{q(t), p(0)\}_{PB} \) denotes the Poisson bracket between the coordinate \( q(t) \) and the momentum \( p(0) \). The quantum version of \( \partial q(t)/\partial q(0) \) can then be obtained by promoting the Poisson bracket to a commutator

\[
[q(t), p(0)]_{PB} \rightarrow \frac{1}{i\hbar} [\hat{q}(t), \hat{p}(0)]
\]

(9)

where now \( \hat{q}(t) \) and \( \hat{p}(0) \) are Heisenberg operators.

We will be interested in thermal systems, so we would like to calculate the expectation value of \( [\hat{q}(t), \hat{p}(0)] \) in a thermal state. However, this commutator might have either signs in a thermal expectation value and this might lead to cancellations. To overcome this problem, we consider the expectation value of the square of this commutator

\[
C(t) = \langle [\hat{q}(t), \hat{p}(0)]^2 \rangle_{\beta}
\]

(10)

where \( \beta \) is the system’s inverse temperature and the overall sign is introduced to make \( C(t) \) positive. More generally, one might replace \( \hat{q}(t) \) and \( \hat{p}(0) \) by two generic Hermitian operators \( V \) and \( W \) and quantify chaos with the double commutator

\[
C(t) = \langle -[W(t), V(0)]^2 \rangle_{\beta}.
\]

(11)

This quantity measures how much an early perturbation \( V \) affects the later measurement of \( W \). As chaos means sensitive dependence on initial conditions, we expect \( C(t) \) to be ‘small’ in nonchaotic system and ‘large’ if the dynamics is chaotic. In the following we give a precise meaning for the adjectives ‘small’ and ‘large’.

For some class of systems the quantum behavior of \( C(t) \) has a lot of similarities with the classical behavior of \( \langle (\partial q(t)/\partial q(0))^2 \rangle \). However, the analogy between the classical and quantum quantities is not perfect because there is not always a good notion of a small perturbation in the quantum case (remember that classical chaos is characterized by the fact that a small perturbation in the past has important consequences in the future). If we start with some reference state and then perturb it, we easily produce a state that is orthogonal to the original state, even when we change just a few quantum numbers. Because of that it seems unnatural to quantify the perturbation as small. Fortunately, there are some quantum systems in which the notion of a small perturbation makes perfect sense. An example is provided by systems with a large number of degrees of freedom. In this case a perturbation involving just a few degrees of freedom is naturally a small perturbation.

For some class of chaotic systems, which include holographic systems, \( C(t) \) is expected to behave as (see [30,36] for a discussion of different possible OTOC growth forms)

\[
C(t) \sim \begin{cases} N_{\text{dof}}^{-1} & \text{for } t < t_d, \\ N_{\text{dof}}^{-1} \exp(\lambda_L t) & \text{for } t_d < t < t_s, \\ \Theta(1) & \text{for } t > t_s, \end{cases}
\]

(12)

where \( N_{\text{dof}} \) is the number of degrees of freedom of the system. Here, we have assumed \( V \) and \( W \) to be unitary and Hermitian operators, so that \( VV = WW = 1 \). The exponential growth of \( C(t) \) is characterized by the Lyapunov exponent (this is actually the quantum analogue of the classical Lyapunov exponent; the two quantities are not necessarily the same in the classical limit [21]; here we stick to the physicists long standing tradition of using misnomers and just refer to \( \lambda_L \) as the Lyapunov exponent) \( \lambda_L \) and takes place at intermediate time scales bounded by the dissipation time \( t_d \) and the scrambling time \( t_s \). The dissipation time is related to the classical Ruelle resonances (\( t_d \sim \mu^{-1} \)) and it characterizes the exponential decay of two-point correlators, e.g., \( \langle V(0)V(t) \rangle \sim e^{-t/\lambda_L} \). The dissipation time also controls the late time behavior of \( C(t) \). The scrambling time \( t_s \sim \lambda_L^{-1} \log N_{\text{dof}} \) is defined as the time at which \( C(t) \) becomes of order \( \Theta(1) \). See Figure 2. The scrambling time controls how fast the chaotic system scrambles information. If we perturb the system with an operator that involves only a few degrees of freedom, the information about this operator will spread among the other degrees of freedom of the system. After a scrambling time, the information will be scrambled among all the degrees of freedom and the operator will have a large commutator with almost any other operator.

To understand how the above behavior relates to chaos, we write the double commutator as

\[
C(t) = \langle -[W(t), V(0)]^2 \rangle_{\beta}
\]

(13)

\[
= 2 - 2 \langle W(t) V(0) W(t) V(0) \rangle_{\beta},
\]

(14)

where we made the assumption that \( W \) and \( V \) are Hermitian and unitary operators. Note that all the relevant information about \( C(t) \) is contained in the OTOC:

\[
\text{OTO}(t) = \langle W(t) V(0) W(t) V(0) \rangle.
\]

(15)

The fact that \( C(t) \) approaches 2 at later times implies that the OTO(t) should vanish in that limit. To understand why this is related to chaos we think of \( \text{OTO}(t) \) as an inner-product of two states

\[
\text{OTO}(t) = \langle \psi_2 | \psi_1 \rangle,
\]

(16)

where

\[
| \psi_1 \rangle = W(-t) V(0) | \beta \rangle,
\]

(17)

\[
| \psi_2 \rangle = V(0) W(-t) | \beta \rangle.
\]
where $|\beta\rangle$ is some thermal state and we replace $t \rightarrow -t$ to make easier the comparison with black holes physics.

If $[V(0), W(t)] \approx 0$ for any value of $t$, the two states are approximately the same, and $\langle \psi_1 | \psi_2 \rangle \approx 1$, implying $C(t) \approx 0$. That means the system displays no chaos—the early measurement of $V$ has no effect on the later measurement of $W$. If, on the other hand, $[V(0), W(t)] \neq 0$, the states $|\psi_1\rangle$ and $|\psi_2\rangle$ will have a small superposition $\langle \psi_1 | \psi_2 \rangle \approx 0$, implying $C(t) \approx 2$. That means that $V$ has a large effect on the later measurement of $W$.

In Figure 3 we construct the states $|\psi_1\rangle$ and $|\psi_2\rangle$ and explain why $\langle \psi_1 | \psi_2 \rangle = 0$ for large $t$ means chaos. Let us start by constructing the state $|\psi_1\rangle = W(-t) V(0) |\beta\rangle = e^{-iHt} W(0) e^{iHt} V(0) |\beta\rangle$. The unperturbed thermal state is represented by a horizontal line. We initially consider the state $V(0) |\beta\rangle$, which is the thermal state perturbed by $V$. If we evolve the system backwards in time (applying the operator $e^{-iHt}$) for some time which is larger than the dissipation time, the system will thermalize and it will no longer display the perturbation $V$. After that, we apply the operator $W$, which should be thought of as a small perturbation, and then we evolve the system forwards in time (applying the operator $e^{iHt}$). The final results of this set of operations depend on the nature of the system. If the system is chaotic, the perturbation $W$ will have a large effect after a scrambling time, and the perturbation $V$ will (at least partially) rematerialize at $t = 0$. If, on the other hand, $[V(0), W(t)] \neq 0$, the states $|\psi_1\rangle$ and $|\psi_2\rangle$ have a large superposition, i.e., $\langle \psi_1 | \psi_2 \rangle \approx 1$.

In this construction we assumed the operators $V(0)$ and $W(-t)$ to be separated by a scrambling time, i.e., $|t| > t_\ast$. This is important because, at earlier times, the two operators, which in general involve different degrees of freedom of the system, generically commute. The operators manage to have a nonzero commutator at later times because of the phenomenon of operation growth that we will describe in the next section.

### 3.1. Operator Growth and Scrambling

The operators $V$ and $W$ act generically at different parts of the physical system, yet they can have a nonzero commutator at later times. This
is possible because in chaotic systems the time evolution of an operator makes it more and more complicated, involving and increasing number of degrees of freedom. As a result, an operator that initially involves just a few degrees of freedom becomes delocalized over a region that grows with time. The growth of the operator $W(t)$ is maybe more evident from the point of view of the Baker-Campbell-Hausdorff (BCH) formula, in terms of which we can write

$$W(t) = e^{iHt} W(0) e^{-iHt} = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} [H, [H, \ldots [H, W(0)] \ldots]].$$

From the above formula it is clear that, at each order in $t$, there is a more complicated contribution to $W(t)$. In chaotic systems the operator becomes more and more delocalized as the time evolves, and it eventually becomes delocalized over the entire system. The time scale at which this occurs is the so-called scrambling time $t_\star$. After the scrambling time the operator $W(t)$ manages to have a nonzero and large commutator with almost any other operator, even operators involving only a few degrees of freedom.

This can be clearly illustrated in the case of a spin chain. Let us follow [7] and consider an Ising-like model with Hamiltonian

$$H = -\sum_i (Z_i Z_{i+1} + gX_i + hZ_i),$$

where $X_i, Y_i$, and $Z_i$ denote Pauli matrices acting on the $i$th site of the spin chain. The above system is integrable if we take $g = 1$ and $h = 0$, but it is strongly chaotic if we choose $g = -1.05$ and $h = 0.5$.

To illustrate the concept of scrambling, we consider the time evolution of the operator $Z_1$. Using the BCH formula we can write the following.

$$Z_1(t) = Z_1 - it [H, Z_1] - \frac{t^2}{2!} [H, [H, Z_1]] + \ldots$$

Ignoring multiplicative constants and signs we can write the above terms (schematically) as follows.

$$[H, Z_1] \sim Y_1$$

$$[H, [H, Z_1]] \sim Y_1 + X_1 Z_2$$

$$[H, [H, [H, Z_1]]] \sim Y_1 + Y_2 X_1 + Y_1 Z_2$$

$$[H, [H, [H, [H, Z_1]]]] \sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2$$

As the time evolves, higher order terms become important in series (20), and the operator $Z_1(t)$ becomes more and more complicated, involving terms in an increasing number of sites. For large enough $t$ the operator will involve all the sites of the spin chain and it will manage to have a nonzero commutator with a Pauli operator in any other site of the system. In this situation the information about $Z_1$ is essentially scramble among all the degrees of freedom of the system. As discussed before, this occurs after a scrambling time. Above this time the double commutator $C(t)$ saturates to a constant value. This should be contrasted to what happens for an integrable system. In this case the operator grows, but it also decreases at later times. In the chaotic case, the operator remains large at later times [7].

### 3.2. Probing Chaos with Local Operators.

In quantum field theories we can upgrade (11) to the case where the operators are separated in space

$$C(t, x) = \langle -[V(0,0), W(t, x)]^2 \rangle_\beta.$$ (22)

Strictly speaking, the above expression is generically divergent, but it can be regularized by adding imaginary times to the time arguments of the operators $V$ and $W$. For a large class of spin chains, higher-dimensional SYK-models, and CFTs, the above commutator is roughly given by

$$C(t, x) \sim \exp \left[ A_L \left( t - t_\star - \frac{|x|}{v_\beta} \right) \right].$$ (23)

where $v_\beta$ is the so-called butterfly velocity. (Actually, $v_\beta$ represents the “velocity of the butterfly effect”. Here we continue to follow the tradition of using misnomers.) This velocity describes the growth of the operator $W$ in physical space and it acts as a low-energy Lieb-Robinson velocity [37], which sets a bound for the rate of transfer of quantum information. From the above formula, we can see that there is an additional delay in scrambling due to the physical separation between the operators. The butterfly velocity defines an effective light-cone for commutator (22). Inside the cone, for $t - t_\star \gg |x|/v_\beta$, we have $C(t, x) \sim \mathcal{O}(1)$, whereas for outside the cone, for $t - t_\star < |x|/v_\beta$, the commutator is small, $C(t, x) \sim 1/N_{\text{dof}} \ll 1$. Outside the light-cone the Lorentz invariance implies a zero commutator. The light-cone and the butterfly effect cone are illustrated in Figure 5.

### 4. Chaos and Holography

In this section we review how the chaotic properties of holographic theories can be described in terms of black holes physics. Black holes behave as thermal systems and thermal systems generically display chaos. This implies that black holes are somehow chaotic. This statement has a precise realization in the context of the gauge/gravity duality. According to this duality, some strongly coupled nongravitational systems are dual to higher-dimensional gravitational systems. In the most known and studied example of this duality the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory living in $\mathbb{R}^{1,3}$ is dual to type IIB supergravity in $AdS_5 \times S^5$. More generally, a $d$–dimensional nongravitational theory living in $\mathbb{R}^{d-1,1}$ is dual to a gravity theory living in a higher-dimensional space.
of the form $AdS_{d+1} \times \mathcal{M}$, where $\mathcal{M}$ is generically a compact manifold. The nongravitational theory can be thought of as living in the boundary of $AdS_{d+1}$ and because of that is usually called the boundary theory. The gravitational theory is also called the bulk theory.

There is a dictionary relating physical quantities in the boundary and bulk description [3, 4]. An example is provided by the operators of the boundary theory, which are related to bulk fields. The boundary theory at finite temperature can be described by introducing a black hole in the bulk. The thermalization properties of the boundary theory have a nice visualization in terms of black hole physics. By applying a local operator in the boundary theory we produce some perturbation that describes a small deviation from the thermal equilibrium. The information about some perturbation that describes a small deviation from the thermal equilibrium is provided by the blue-shift suffered by the in-falling quanta or, equivalently, the red shift suffered by the quanta escaping from the black hole. The blue-shift suffered by the in-falling quanta is determined by the black hole’s temperature. If the quanta asymptotic energy is $E_0$, this energy increases exponentially with time

$$E = E_0 e^{(2\pi/\hbar)\beta t},$$

where $\beta$ is the Hawking’s inverse temperature. Later we will see that this exponential increase in the energy of the in-falling quanta gives rise to the Lyapunov behavior of $C(t,x)$ of holographic theories.

4.1. Holographic Setup

The TFD State & Two-Sided Black Holes. In the study of chaos it is convenient to consider a thermofield double state made out of two identical copies of the boundary theory

$$|\text{TFD}⟩ = \frac{1}{Z^{1/2}} \sum_n e^{-\beta E_n/2} |n⟩_1 |n⟩_R,$$

where $L$ and $R$ label the states of the two copies, which we call QFT$_L$ and QFT$_R$, respectively. The two boundary theories do not interact and only know about each other through their entanglement. This state is dual to an eternal (two-sided) black hole, with two asymptotic boundaries, where the boundary theories live [38]. This is a wormhole geometry, with an Einstein-Rosen bridge connecting the two sides of the geometry. The wormhole is not traversable, which is consistent with the fact that the two boundary theories do not interact.

For definitiveness we assume a metric of the form

$$ds^2 = -G_{tt}(r) dt^2 + G_{rr}(r) dr^2 + G_{ij}(r, x^k) dx^i dx^j,$$

where the boundary is located at $r = \infty$, where the above metric is assumed to asymptote $AdS_{d+1}$. We take the horizon as located at $r = r_H$, where $G_{tt}$ vanishes and $G_{rr}$ has a first order pole. For future purposes, let $\beta$ be the Hawking’s inverse temperature, and $S_{BH}$ be the Bekenstein-Hawking entropy.

In the study of shock waves it is more convenient to work with Kruskal-Szekeres coordinates, since these coordinates cover smoothly the globally extended spacetime. We first define the tortoise coordinate

$$dr_\ast = \sqrt{\frac{G_{rr}}{G_{tt}}} dr,$$

and then we introduce the Kruskal-Szekeres coordinates $U, V$ as follows.

$$U = e^{(2\pi/\hbar)(r - t)},$$

$$V = e^{(2\pi/\hbar)(r + t)}$$

(Left exterior region)
Figure 6: Bulk picture of thermalization. The figure represents an asymptotically \textit{AdS} black hole geometry. The boundary is at the top edge, while the black hole horizon is at the bottom edge. The black hole’s interior is shown in gray. The boundary operator $V$ is dual to the bulk field $\phi$. From the point of view of the boundary theory the perturbation produced by $V$ is initially localized around the point $x$, but it gets delocalized over a region that increases with time. In the bulk description this is described by a particle (field excitation) that is initially close to the boundary and then falls into the black hole.

$$U = -e^{(2\pi/\beta)(r,-t)},$$
$$V = +e^{(2\pi/\beta)(r,+t)}$$
(right exterior region)

$$U = +e^{(2\pi/\beta)(r,-t)},$$
$$V = +e^{(2\pi/\beta)(r,+t)}$$
(future interior region)

$$U = -e^{(2\pi/\beta)(r,-t)},$$
$$V = -e^{(2\pi/\beta)(r,+t)}$$
(past interior region)

In terms of these coordinates the metric reads

$$ds^2 = 2A(UV)dUdV + G_{ij}(UV)dx^idx^j,$$  \hspace{1cm} (30)

where

$$A(UV) = \frac{\beta^2}{8\pi^2} \frac{G_{tt}(UV)}{UV}.$$  \hspace{1cm} (31)

In these coordinates the horizon is located at $U = 0$ or at $V = 0$. The left and right boundaries are located at $UV = -1$ and the past and future singularities at $UV = 1$. The Penrose diagram for this metric is shown in Figure 7.

The global extended spacetime can also be described in terms of complexified coordinates [39]. In this case one defines the complexified Schwarzschild time

$$t = t_L + it_E,$$  \hspace{1cm} (32)

where $t_L$ and $t_E$ are the Lorentzian and Euclidean times, and then one describes the time in each of the four patches (left and right exterior regions, and the future and past interior regions) as having a constant imaginary part.

$$t_E = 0 \quad \text{(right exterior region)}$$
$$t_E = -\frac{\beta}{4} \quad \text{(future interior region)}$$
$$t_E = -\frac{\beta}{2} \quad \text{(left exterior region)}$$
$$t_E = +\frac{\beta}{4} \quad \text{(past interior region)}$$  \hspace{1cm} (33)

The Euclidean time has a period of $\beta$. The Lorentzian time increases upward (downward) in the right (left) exterior region, and to the right (left) in the future (past) interior.

Note that, with the complexified time, one can obtain an operator acting on the left boundary theory by adding (or subtracting) $i\beta/2$ to the time of an operator acting on the right boundary theory.

\textbf{Perturbations of the TFD State & Shock Wave Geometries.} We now turn to the description of states of the form

$$W(t)|\text{TFD}\rangle$$  \hspace{1cm} (34)

where $W$ is a thermal scale operator that acts on the right boundary theory. This state can be describe by a ‘particle’
(field excitation) in the bulk that comes out of the past horizon, reaches the right boundary at time \( t \), and then falls into the future horizon, as illustrated in Figure 8.

If \(|t|\) is not too large, the state \( W(t)|\text{TFD}\rangle \) will represent just a small perturbation of the TFD state and the corresponding description in the bulk will be just an eternal two-sided black hole geometry slightly perturbed by the presence of a probe particle. This is no longer the case if \(|t|\) is large. In this case there is a nontrivial modification of the geometry. A very early perturbation, for example, is described in the bulk in terms of a particle that falls towards the future horizon for a very long time and gets highly blue-shifted in the process. If the particle's energy is \( E_0 \) in the asymptotic past, this energy will be exponentially larger from the point of view of the \( t = 0 \) slice of the geometry, i.e., \( E = E_0 e^{(2\pi/\beta) t} \). Therefore, for large enough \(|t|\), the particle's energy will be very large and one needs to include the corresponding back-reaction.

The back-reaction of a very early (or very late) perturbation is actually very simple—it corresponds to a shock wave geometry [40, 41]. To understand that, we first need to notice that, under boundary time evolution, the stress energy of a generic perturbation \( W \) gets compressed in the \( V \)-direction and stretched in the \( U \)-direction. For large enough \(|t|\) we can approximate the stress tensor of the \( W \)-particle as

\[
T_{VV} \sim p^U \delta(V) a(\vec{x}),
\]

where \( p^U \sim \beta^{-1} e^{(2\pi/\beta)t} \) is the momentum of the \( W \)-particle in the \( U \)-direction and \( a(\vec{x}) \) is some generic function that specifies the location of the perturbation in the spatial directions of the right boundary. Note that \( T_{VV} \) is completely localized at \( V = 0 \) and homogeneous along the \( U \)-direction. Besides, even if the \( W \)-particle is massive, the exponential blue-shift will make it follow an almost null trajectory, as shown in Figure 9.

The shock wave geometry produced by the \( W \)-particle is described by the metric

\[
ds^2 = 2A(UV) dUdV + G_{ij}(UV) dx^i dx^j
\]

\[
-2A(UV) h(t, \vec{x}) \delta(V) dV^2,
\]

which is completely specified by the shock wave transverse profile \( h(t, \vec{x}) \). This geometry can be seen as two pieces of an eternal black hole glued together along \( V = 0 \) with a shift of magnitude \( h(t, \vec{x}) \) in the \( U \)-direction. We find it useful to represent this geometry with the same Penrose diagram of the unperturbed geometry, but with the prescription that any trajectory crossing the shock wave gets shifted in the \( U \)-direction as \( U \rightarrow U + h(t, \vec{x}) \). See Figure 9.

The precise form of \( h(t, \vec{x}) \) can be determined by solving the \( VV \)-component of Einstein’s equation. For a local perturbation, i.e., \( a(\vec{x}) = \delta^{d-1}(\vec{x}) \), the solution reads

\[
h(t, \vec{x}) \sim G_{N} e^{(2\pi/\beta) t - |\vec{x}|/\nu_0},
\]

with \( \mu = \frac{2\mu}{\beta} \sqrt{\frac{(d - 1)}{G_{N}}} \frac{G_{I} (r_H)}{G_{I} (r_H)} \).

Interestingly, the shock wave profile contains information about the parameters characterizing the chaotic behavior of the boundary theory. Indeed, the double commutator has a region of exponential growth at which \( C(t, \vec{x}) \sim h(t, \vec{x}) \). From this identification, we can write

\[
h(t, \vec{x}) \sim e^{(2\pi/\beta) (t - t_* - |\vec{x}|/\nu_0)}
\]

where (the leading order contribution to) the scrambling time scales logarithmically with the Bekenstein-Hawking entropy

\[
t_* \sim \frac{\beta}{2\pi} \log \frac{1}{G_{N}} \sim \frac{\beta}{2\pi} \log S_{\text{BH}} \tag{39}
\]

while the Lyapunov exponent is proportional to the Hawking's temperature.

\[
\lambda_L = \frac{2\pi}{\beta}. \tag{40}
\]
The butterfly velocity is determined from the near-horizon geometry. (Here we are assuming isotropy. In the case of anisotropic metrics the formula for $v_B$ is a little bit more complicated. See, for instance, Appendix A of [42] or Appendix B of [43].)

\[ v_B^2 = \frac{C_{11}^d (r_H)}{(d - 1) G_0^d (r_H)}. \]  

(41)

4.2. Bulk Picture for the Behavior of OTOCs. In this section we present the bulk perspective for the vanishing of OTOCs at later times. In order to do that, we write the OTOC as a superposition of two states

\[ \text{OTO} (t) = \langle \text{TFD} | W (−t) V (0) W (−t) V (0) | \text{TFD} \rangle \]

\[ = \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle, \]

(42)

where the ‘in’ and ‘out’ states are given by the following.

\[ \psi_{\text{in}} = W (−t) V (0) | \text{TFD} \rangle, \]

\[ \psi_{\text{out}} = V^\dagger (0) W^\dagger (−t) | \text{TFD} \rangle \]

(43)

The interpretation of a vanishing OTOC in terms of the bulk theory is actually very simple. Let us go step by step and construct first the state $V(0)|\beta\rangle$. This state is described by a particle that comes out of the past horizon, reaches the boundary at $t = 0$, and then falls back into the future horizon. See the left panel of Figure 10.

Now the ‘in’ state can be obtained as

\[ \psi_{\text{in}} = W (−t) V (0) | \text{TFD} \rangle \]

\[ = e^{-iHt} W (0) e^{iHt} V (0) | \text{TFD} \rangle. \]

(44)

This amounts to the following: evolving the state $V(0)|\text{TFD}\rangle$ backwards in time, applying the operator $W$, and then evolving the system forwards in time. The corresponding description in the bulk is shown in the right panel of Figure 10. From this picture we can see that the perturbation $W$ produces a shock wave that causes a shift in the trajectory of the $V$-particle, which no longer reaches the boundary at $t = 0$, but rather with some time delay. The physical interpretation is that a small perturbation in the asymptotic past (represented by $W$) is amplified over time and destroys the initial configuration (represented by the state $V(0)|\text{TFD}\rangle$).

The bulk description of the ‘out’ state $|\psi_{\text{out}}\rangle = V(0)W(−t)|\text{TFD}\rangle$. The $W$-particle produces the shock wave geometry. The trajectory of the $V$-particle is such that, after suffering the shift $U \rightarrow U + h(t, \vec{x})$, it reaches the boundary at time $t = 0$, producing the perturbation $V$.

Comparing the bulk description of the state $|\psi_{\text{in}}\rangle$ (shown in the right panel of Figure 10) with the description of the state $|\psi_{\text{out}}\rangle$ (shown in Figure 11) we can see that these states are indistinguishable when $h(t, \vec{x})$ is zero, but they become more and more different for large values of $h(t, \vec{x})$. As a consequence, the overlap $C(t) = \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ is equal to one when $h = 0$, but it decreases to zero as we increase the value of $h$.
The exponential behavior of $h(t, \vec{x})$ implies that an early enough perturbation can produce a very large shift in the $V$-particle’s trajectory, causing it to be captured by the black hole and preventing the materialization of the $V$ perturbation at the boundary. See Figure 12. This should be compared with the physical picture given in Figure 3.

The physical picture of the process described in Figure 12 is quite simple. The state $V(0)|\text{TFD}$ can be represented by a black hole geometry in which a particle (the $V$-particle) escapes from the black hole and reaches the boundary at time $t = 0$. The state $W(-t)V(0)|\text{TFD}$ is obtained by perturbing the state $V(0)|\text{TFD}$ in the asymptotic past. This corresponds to the addition of a $W$-particle to the system in the asymptotic past. This particle gets highly blue-shifted as it falls towards the black hole. The black hole captures the $W$-particle and becomes bigger. The $V$-particle fails to escape from the bigger black hole and never reaches the boundary to produce the $V$ perturbation. This physical picture is illustrated in Figure 13.

The precise form of the above OTOC can be obtained by calculating the overlap $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ using the Eikonal approximation [8], in which the Eikonal phase $\delta$ is proportional to the shock wave profile $\delta \sim h(t, \vec{x})$. The OTOC can be written as an integral of the phase $e^{i\delta}$ weighted by kinematical factors which are basically Fourier transforms of bulk-to-boundary propagators for the $V$ and $W$ operators.

The result for Rindler AdS$_3$ reads (the below result assumes $\Delta_W \gg \Delta_Y$)

$$\frac{\langle V(\text{in}_1) W(t + i\epsilon_2) V(\text{in}_3) W(t + i\epsilon_4) \rangle}{\langle V(\text{in}_1) V(\text{in}_3) \rangle \langle W(\text{in}_2) W(\text{in}_4) \rangle} \sim \Delta_Y$$

$$\sim \frac{1}{1 - (8\pi i G_N \Delta_W/\epsilon_{13} e_{24}) e^{(2\pi/\beta)(t - |\vec{x}|/\gamma_0)}}$$

(45)

where $\Delta_Y$ and $\Delta_W$ are the scaling dimensions of the operators $V$ and $W$, respectively, and $\epsilon_{ij} = i(e^{i\phi} - e^{i\phi'})$. For this system $\beta = 2\pi$ and $\gamma_0 = 1$. This formula matches the direct CFT calculation (the CFT perspective for the onset of chaos has been widely discussed in [44]; other references in this direction include, for instance, [45–48]) obtained in [20]. It can also be derived using the geodesic approximation for two-sided correlators in a shock wave background [5, 20].

Expanding the above result for small values of $G_N e^{(2\pi/\beta)(t - |\vec{x}|/\gamma_0)}$, we obtain

$$\text{OTO}(t) = 1 - 8\pi i G_N \Delta_Y \Delta_W e^{(2\pi/\beta)(t - |\vec{x}|/\gamma_0)}$$

(46)

and, since $h(t, \vec{x}) \sim G_N e^{(2\pi/\beta)(t - |\vec{x}|/\gamma_0)}$, the above result implies

$$C(t, \vec{x}) \sim h(t, \vec{x})$$

(47)

The above result is valid for small (in AdS/CFT the Newton constant is related to the rank of the gauge group of dual CFT as $G_N \sim 1/N^3$, where $a$ is a positive number that depends on the dimensionality of the bulk space time (cf. section 7.2 of [49]); our classical gravity calculations are only valid in the large-$N$ limit (that suppresses quantum corrections) so it is natural to consider $G_N$ as a small parameter) values of $G_N$, or for any value of $G_N$, but for times in the range $t_d < t < t_*$, where $t_* = (\beta/2\pi) \log(1/G_N)$.

Despite being true in the Rindler AdS$_3$ case, the proportionality between the double commutator and the shock wave profile has not been demonstrated in more general cases. However, the authors of [8] argued that, in regions of moderate scattering between the $V$- and $W$-particle, the identification $C(t, \vec{x}) \sim h(t, \vec{x})$ is approximately valid.

At very late times, the behavior of the OTO($t$) is expected to be controlled by the black hole quasinormal modes. Indeed, in the case of a compact space it is possible to show that

$$C(t) \sim e^{-2\omega(t - t_*, -R/\gamma_0)}$$

with $\text{Im}(\omega) < 0$,

(48)

where $R$ is the diameter of the compact space and $\omega$ is the system lowest quasinormal frequency [8].

4.2.1. Stringy Corrections. In this section we briefly discuss the effects of stringy corrections to the Einstein gravity results for OTOCs. We start by reviewing the Einstein gravity results from the perspective of scattering amplitudes. In the framework of the Eikonal approximation, the phase shift suffered by the $V$-particle is given by

$$\delta = -P^V h(t, \vec{x}) \sim G_N s$$

(49)

where we used the fact that $h(t, \vec{x}) \sim G_N P^V$ and introduced a Mandelstam-like variable $s = 2A(0) P^V$. In a small-$G_N$ expansion the double commutator $C(t)$ and the phase shift $\delta$ scale with $s$ in the same way, namely,

$$C(t) \sim G_N s$$

(50)

where $s \sim \beta^{-2} e^{(2\pi/\beta)t}$.

The string corrections can be incorporated using the standard Veneziano formula for the relativistic scattering
amplitude $\mathcal{A} \sim s \delta$. The phase shift can then be schematically written as an infinite sum

$$\delta \sim \sum_f G_N s_f J_f^{-1}, \quad (51)$$

where each term corresponds to the contribution due to the exchange of a spin-$J$ field. In Einstein gravity the dominant contribution comes from the exchange of a spin-2 field, the graviton. In string theory, we have to include an infinite tower of higher spin fields. Naively, it looks like these higher spin contributions will increase the development of chaos. However, the resummation of the above sum actually leads to a decrease in the development of chaos. The string-corrected phase shift has a milder dependence with $s$, namely,

$$\delta \sim G_N s_{\text{eff}}^{-1}, \quad (52)$$

with the effective spin given by [8]

$$J_{\text{eff}} = 2 - d (d-1) \ell_s^2 / 4 \ell_{\text{AdS}}^2 \quad (53)$$

where $\ell_s$ is the string length, $\ell_{\text{AdS}}$ is the AdS length scale, and $d$ is the number of dimensions of the boundary theory. As a result, the string-corrected double commutator grows in time with an effective smaller Lyapunov exponent

$$C_{\text{string}}(t) \sim e^{(2n/\beta)(1 - d(d-1)\ell_s^2/4 \ell_{\text{AdS}}^2) t}, \quad (54)$$

and this leads to a larger scrambling time. (At small scales, the string-corrected shock wave has a Gaussian profile, and the concept of butterfly velocity is not meaningful. It was recently shown, however, that at larger scales is possible to define a string-corrected butterfly velocity. The result for $N = 4$ SYM theory reads [50] $v_B = \sqrt{2/3(1 + (2\zeta(3)/16)/(1/\lambda^{3/2}))}$, where $\lambda$ is the 't Hooft coupling, which can be written in terms of string length scale as $\lambda = (\ell_{\text{AdS}} / \ell_s)^{3/2}$.)

The above discussion implies that for a theory with a finite number of high-spin fields ($J > 2$) chaos would develop faster than in Einstein gravity. These theories, however, are known to violate causality [51]. It is then natural to speculate that the Lyapunov exponent obtained in Einstein gravity has the maximal possible value allowed by causality. This is indeed true and this is the topic of the next section.

4.2.2. Bounds on Chaos. One of the remarkable insights that came from the holographic description of quantum chaos is the fact that there is a bound on chaos—the quantum Lyapunov exponent is bounded from above, while the scrambling time is bounded from below. A distinct feature of holographic systems is that they saturate these two bounds.

Let us follow the historical order and start by discussing the lower bound on the scrambling time. In black holes physics the scrambling time defines how fast the information that has fallen into a black hole can be recovered from the emitted Hawking radiation. (This assumes that half of the black hole’s initial entropy has been radiated [52]..) In the context of the Hayden-Preskill thought experiment, the scrambling time is barely compatible with black hole complementarity [52], since a smaller scrambling time would lead to a violation of the no-cloning principle. This led Susskind and Sekino to conjecture that black holes are the fastest scramblers in nature; i.e., they have the smallest possible scrambling time [53]. The lower bound on the scrambling time of a generic many-body quantum system can be written as

$$t_* \geq C(\beta) \log N_{\text{dof}} \quad (56)$$

where $C(\beta)$ is some function of the inverse temperature. In the case of black holes this function is simply given by $C(\beta) = \beta / 2\pi$.

The scrambling time defines a stronger notion of thermalization and should not be confused with the dissipation time. In fact, for black holes, one expects the dissipation time to be given by the black hole quasinormal modes (this is true in the case of low dimension operators) $t_d \sim \beta$, while the scrambling
time is parametrically larger $t_\star \sim \beta \log N_{\text{dof}}$. This brings us to the second bound on chaos: for systems with such a large hierarchy between the scrambling and the dissipation, time is possible to derive an upper bound for the Lyapunov exponent [54]:

$$\lambda_L \leq \frac{2\pi}{\beta}.$$  \hspace{1cm} (57)

One should emphasize that this bound does not depend on the existence of a holographic dual. It can be derived for generic many-body quantum systems under some very reasonable assumptions.

The fact that black hole always has a maximum Lyapunov exponent led to the speculation that the saturation of the chaos bound might be a sufficient condition for a system to have an Einstein gravity dual [9, 54]. In fact, there have been many attempts to use the saturation of the chaos bound as a criterion to discriminate holographic CFTs from the nonholographic ones [20, 44–48, 55, 56]. It was recently shown, however, that this criterion, though necessary, is insufficient to guarantee a dual description purely in terms of Einstein gravity [57, 58].

Since $v_B$ defines the speed at which information propagates, it is natural to question whether this quantity is also bounded. From the perspective of the boundary theory, causality implies

$$v_B \leq 1,$$  \hspace{1cm} (58)

meaning that information should not propagate faster than the speed of light. Indeed, the above bound can be derived in the context of Einstein gravity by using Null Energy Condition (NEC) and assuming an asymptotically AdS geometry (this derivation uses an alternative definition for $v_B$, which is based on entanglement wedge subregion duality [59]) [60]. This is consistent with the expectation that gravity theories in asymptotically AdS geometries are dual to relativistic theories. In contrast, for geometries which are not asymptotically AdS, the butterfly velocity can surpass the speed of light [42, 60], which is consistent with the non-Lorentz invariance of the corresponding boundary theories.

If we further assume isotropy, it is possible to derive a stronger bound for $v_B$ [61]

$$v_B \leq v_B^{Sch} = \sqrt{\frac{d}{2(d-1)}},$$  \hspace{1cm} (59)

where $v_B^{Sch}$ is the value of the butterfly velocity for an AdS-Schwarzschild black brane in $d + 1$ dimensions. This is also the butterfly velocity for a $d$-dimensional thermal CFT.

The above formula shows that, for thermal CFTs, $v_B$ does not depend on the temperature. However, if we deform the CFT, $v_B$ acquires a temperature dependence as we move along the corresponding renormalization group (RG) flow. In fact, by considering deformations that break the rotational symmetry, it was noticed that the butterfly velocity violates the above bound, but remains bounded from above by its value at the infrared (IR) fixed point, never surpassing the speed of light [62–64]. The above bound can also be violated by higher curvature corrections, but $v_B$ remains bounded by the speed of light as long as causality is respected. (For instance, in 4-dimensional Gauss-Bonnet (GB) gravity, the butterfly velocity surpasses the speed of light for $\lambda_{\text{GB}} < -3/4$, but causality requires $\lambda_{\text{GB}} > -0.19$ [65, 66]. Moreover, it was recently shown that, unless one adds an infinite tower of extra higher spin fields, GB gravity might be inconsistent with causality for any value of the GB coupling [51].) The violation of the bound given in (59) by anisotropy or higher curvature corrections is reminiscent of the well-known violation of the shear viscosity to entropy density ratio bound [67–72].

### 4.3. Chaos and Entanglement Spreading.

The thermofield double state displays a very atypical left-right pattern of entanglement that results from nonzero correlations between subsystems of QFT\textsubscript{L} and QFT\textsubscript{R} at $t = 0$. The chaotic nature of the boundary theories is manifested by the fact that small perturbations added to the system in the asymptotic past destroy this delicate correlations [5].

The special pattern of entanglement can be efficiently diagnosed by considering the mutual information $I(A,B)$ between spatial subsystems $A \subset$ QFT\textsubscript{L} and $B \subset$ QFT\textsubscript{R}, defined as

$$I(A, B) = S_A + S_B - S_{A\cup B},$$  \hspace{1cm} (60)

where $S_A$ is the entanglement entropy of the subsystem $A$ and so on. The mutual information is always positive and provides an upper bound for correlations between operators $O_L$ and $O_R$ defined on $A$ and $B$, respectively [73],

$$I(A, B) \geq \frac{\langle\langle O_L O_R \rangle \rangle - \langle\langle O_L \rangle \rangle^2 \langle\langle O_R \rangle \rangle^2}{2 \langle\langle O_L^2 \rangle \rangle \langle\langle O_R^2 \rangle \rangle}.$$  \hspace{1cm} (61)

The thermofield double state has nonzero mutual information between large (for small subsystems, the mutual information is zero) subsystems of the left and right boundary, signaling the existence of left-right correlations. These correlations can be destroyed by small perturbations in the asymptotic past, meaning that initially positive mutual information drops to zero when we add a very early perturbation to the system.

Interestingly, the vanishing of the mutual information can be connected to the vanishing of the OTOCs discussed earlier. If, for simplicity, we assume that $O_L$ and $O_R$ have zero thermal one point function, then the disruption of the mutual information implies the vanishing of the following four-point function

$$\langle O_L O_R \rangle_W = \langle \text{TFD} | W_L^I O_L^I O_R^I W_R^I | \text{TFD} \rangle = 0,$$  \hspace{1cm} (62)

which is related by analytic continuation to the one-sided out-of-time-order correlator introduced earlier. (To obtain an OTOC with operators acting only on the right boundary theory, one just needs to add $i\beta/2$ to time argument of the operator $O_L$ in the above formula.)

The disruption of the mutual information has very simple geometrical realization in the bulk. The entanglement entropies that appear in the definition of $I(A, B)$ can be
holographically calculated using the HRRT prescription [74, 75]

\[ S_A = \frac{\text{Area}(\gamma_A)}{4G_N}, \]  

where \( \gamma_A \) is an extremal surface whose boundary coincides with the boundary of the region \( A \). There is an analogous formula for \( S_B \). Both \( \gamma_A \) and \( \gamma_B \) are U-shaped surfaces lying outside of the event horizon, in the left and right side of the geometry, respectively. There are two candidates for the extremal surface that computes \( S_{A:B} \): the surface \( \gamma_A \cup \gamma_B \) or the surface \( \gamma_{\text{wormhole}} \), that connects the two asymptotic boundaries of the geometry. See Figure 14. According to the RT prescription, we should pick the surface with less area. If \( \gamma_A \cup \gamma_B \) has less area than \( \gamma_{\text{wormhole}} \), then \( I(A, B) = 0 \), because \( \text{Area}(\gamma_A \cup \gamma_B) = \text{Area}(\gamma_A) + \text{Area}(\gamma_B) \). On the other hand, if \( \gamma_{\text{wormhole}} \) has less area than \( \gamma_A \cup \gamma_B \), i.e., \( \text{Area}(\gamma_{\text{wormhole}}) < \text{Area}(\gamma_A) + \text{Area}(\gamma_B) \), then we have a positive mutual information

\[ I(A, B) = \frac{1}{4G_N} \left[ \text{Area}(\gamma_A) + \text{Area}(\gamma_B) - \text{Area}(\gamma_{\text{wormhole}}) \right], \]  

\[ > 0. \]

Now, an early perturbation of the thermofield double state gives rise to a shock wave geometry in which the wormhole becomes longer. As a consequence, the area of the surface \( \gamma_{\text{wormhole}} \) increases, resulting in a smaller mutual information. It is then clear that the mutual information will drop to zero if the wormhole is longer enough. The length of the wormhole depends on the strength of the shock wave, which, by its turn, depends on how early the perturbation is producing it. Therefore, an early enough perturbation will produce a very long wormhole in which the mutual information will be zero. The fact that the shock wave geometry produces a longer wormhole (along the \( t = 0 \) slice of the geometry) is clearly seen if we represent the shock wave geometry with a tilted Penrose diagram. See, for instance, Figure 3 of [76].

The mutual information \( I(A, B) \) decreases as a function of the time \( t_0 \) at which we perturbed the system. For \( t_0 \geq t_* \), the mutual information decreases linearly with behavior controlled by the so-called entanglement velocity \( v_E \) [63]

\[ \frac{dI(A, B)}{dt_0} = -\frac{dS_{A:B}}{dt_0} = -v_E s_{th} \text{Area}(A \cup B), \]  

where \( s_{th} \) is the thermal entropy density and \( \text{Area}(A \cup B) \) is the area of \( A \cup B \) (or the volume of the boundary of this region). The two-sided black hole geometry with a shock wave can be thought of as an additional example of a holographic quench protocol [63], and the time-dependence of entanglement entropy can be understood in terms of the so-called ‘entanglement tsunami’ picture. See [77] for field theory calculations and [78–82] for holographic calculations. However, it was recently shown that the entanglement tsunami picture is not very sharp. See [59] for further details. In [81, 82], the entanglement velocity was conjectured to be bounded as

\[ v_E \leq v_E^{\text{Sch}} = \frac{\sqrt{d(d-1)^{1/2-1/d}}}{(d-1)^{1/2-1/d}}, \]

where \( v_E^{\text{Sch}} \) is the entanglement velocity for a \( (d+1) \)-dimensional Schwarzschild black brane or, equivalently, the value of \( v_E \) for a \( d \)-dimensional thermal CFT. This bound can be derived in the context of Einstein gravity assuming an asymptotically AdS geometry, isotropy, and NEC [61]. Just like in the case of \( v_{\text{gs}} \), the entanglement velocity in thermal CFTs does not depend on the temperature. But \( v_E \) acquires a temperature dependence if we deform the CFT and move along the corresponding RG flow [63, 64]. In these cases, \( v_E \) violates the above bound, but it remains bounded by its corresponding value at the IR fixed point, never surpassing the speed of light.

One can also prove that the entanglement velocity is also bounded by the speed of light. (See [83, 84] for a discussion about small subsystems.) This can be done by using positivity of the mutual information [85] or using inequalities involving the relative entropy [86]. More generally, the authors of [59] conjecture that \( v_E \leq v_{\text{gs}} \), which implies the bound \( v_E \leq 1 \) in the cases where \( v_{\text{gs}} \) is bounded. However, both [85, 86] assumed that the theory is Lorentz invariant. In the case of

**Figure 14**: Illustration of the entangling surfaces in the \( t = 0 \) slice of a two-sided black brane geometry. The U-shaped surfaces \( \gamma_A \) and \( \gamma_B \) are represented by blue curves. The surface stretching through the wormhole is given by the union of the two red surfaces \( \gamma_{\text{wormhole}} = \gamma_1 \cup \gamma_2 \). In the left panel we represent the unperturbed geometry, in which the two horizons coincide. In the right panel we represent the geometry in the presence of a shock wave added at some time \( t_0 \) in the past. In this case the size of the wormhole is effectively larger, and the two horizons no longer coincide.
non-Lorentz invariant theories (e.g., noncommutative gauge theories) the entanglement velocity can surpass the speed of light. This has been verified both in holography calculations [42] and in field theory calculations [87].

Finally, we mention that other concepts from information theory can also be used to diagnose chaos in holography. It has been shown, for instance, that the relative entropy is also a useful tool to diagnose chaotic behavior [88]. For a connection between chaos and computational complexity, see, for instance [89, 90].

4.4. Chaos and Hydrodynamics. Recently, there has been a growing interest in the connection between chaos and hydrodynamics [91–99]. Here we briefly review some interesting connection between chaos and diffusion phenomena.

A longstanding goal of quantum condensed matter physics is to have a deeper understanding of the so-called ‘strange metals’. These are strongly correlated materials that do not have a description in terms of quasiparticles excitations and whose transport properties display a remarkable degree of universality. In [100, 101] Sachdev and Damle proposed that such a universal behavior could be explained by a fundamental dissipative timescale

$$\tau_p \sim \frac{\hbar}{2\pi k_B T},$$  \hspace{1cm} (67)

which would govern the transport in such systems.

Interestingly, the Lyapunov exponent defines a time scale $\tau_L = 1/\lambda_L$, and the upper bound on $\lambda_L$ translates into a lower bound for $\tau_L$ that precisely coincides with $\tau_p$

$$\tau_L \geq \frac{\hbar \beta}{2\pi k_B},$$  \hspace{1cm} (68)

where we reintroduced $\hbar$ and the Boltzmann constant in the expression for the bound on the Lyapunov exponent. (In systems of units where $\hbar$ and $k_B$ are not equal to one, the bound on the Lyapunov exponent reads $\lambda_L \leq 2\pi k_B/\hbar \beta$.) Holographic systems saturate the above bound, and this explains the universality observed in the transport properties of these systems.

A prototypical example of universality is the linear resistivity of strange metals. In [102], Hartnoll proposed that the linear resistivity could be explained by the existence of a universal lower bound on the diffusion constants related to the collective diffusion of charge and energy

$$D \geq \frac{\hbar v^2}{(k_B T)},$$  \hspace{1cm} (69)

where $v$ is some characteristic velocity of the system. As $D$ is inversely proportional to the resistivity, systems saturating the above bound would display linear resistivity behavior. (See [103] for a recent successful holographic description of linear resistivity at high temperature.)

One should think of (69) as a reformulation of the Kovtun-Son-Starinets (KSS) bound [104]

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \frac{\hbar}{k_B},$$  \hspace{1cm} (70)

which also relies on the idea of a fundamental dissipative timescale $\tau_L \sim \hbar/(k_B T)$ controlling transport in strongly interacting systems. Naively, the observed violations of the KSS bound would seem to indicate the existence of systems in which the bound (68) is violated. The bound (69) saves the idea of a fundamental dissipative timescale by introducing an additional parameter in the game, namely, the characteristic velocity $v$. The fact that $\eta/s$ can be made arbitrarily small in some systems corresponds to the fact that the characteristic velocity is highly suppressed in those cases. See [91] for further details.

In [91, 92] Blake proposed that, at least for holographic systems with particle-hole symmetry, the characteristic velocity $v$ should be replaced by the butterfly velocity. More precisely

$$D_e \geq C_e \nu_B^2 \tau_L,$$  \hspace{1cm} (71)

where $D_e$ is the electric diffusivity and $C_e$ is a constant that depends on the universality class of theory. This proposal was motivated by the fact that both $D_e$ and $\nu_B$ are determined by the dynamics close to the black hole horizon in the aforementioned systems. Despite working well for systems where energy and charge diffuse independently, this proposal was shown to fail in more general cases [93, 105–108]. This is related to the fact that, in more general cases, the diffusion of energy and charge is coupled, and the corresponding transport coefficients are not given only in terms of the geometry close to the black hole horizon. Hence, there is no reason for these coefficients to be related to the butterfly velocity, which is always determined solely by the near-horizon geometry.

There is, however, a universal piece of the diffusivity matrix that can be related to the chaos parameters at infrared fixed points. This is the thermal diffusion constant [94]

$$D_T \geq C_T \nu_B^2 \tau_L,$$  \hspace{1cm} (72)

where $C_T$ is a universality constant (different from $C_e$). This proposal was shown to be valid even for systems with spatial anisotropy [109]. The above relation is not well defined when the system’s dynamical critical exponent $z$ is equal to one, but it can be extended in this case (we thank Hyun-Sik Jeong for calling our attention to this) [110].

Finally, we mention that there is an interesting relation between chaos and hydrodynamics that manifests itself in the so-called ‘pole-skipping’ phenomenon. See [95–97] for further details.

5. Closing Remarks

The holographic description of quantum chaos not only has provided new insights into the inner-workings of gauge-gravity duality, but also has given insights outside the scope of holography: some examples include the characterization of chaos with OTOCs, the definition of a quantum Lyapunov exponent, and the existence of a bound for chaos.

The success of this new approach to quantum chaos explains the growing experimental interest that OTOCs
have been received. Indeed, several protocols for measuring OTOCs have been proposed, and there are already a few experimental results. See [111] and references therein.

Finally, one of the remarkable features of quantum chaos is level statistics described by random matrices. The fact that this is present in the infrared limit of the SYK model [112–114] suggests that it should also be present in quantum black holes (we thank A. M. García-García for calling our attention to this), although this has not yet been verified [115].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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