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Research Article

Trace and Axial Anomalies on Equal Footing

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We discussed that for some particular nonsupersymmetric theories, a generalized symmetry that includes both the scale and axial transformations and leads to a single current may contain also a pseudoscalar term. The method, inspired by the superconformal anomalies, has important application for low-energy effective models where it allows the introduction of a single complex glueball field with a scalar and a pseudoscalar component on the same footing with the complex meson nonet fields made of quarks. Both axial and trace anomalies are satisfied in accordance to the meson structure and the QCD requirements.

1. Introduction

One of the most important tools in dealing with physical systems and especially quantum field theoretical models is the incorporation of symmetries, global or gauged or exact or approximate. We know that a standard Lagrangian must be real and thus Hermitian and Lorentz invariant. In some instances, the Lagrangian might be also scale invariant, at least at tree level, thus leading to new class of theories whose properties were explored in detail in the literature. The scale invariance is usually broken at the quantum level [1–4] by the renormalization group equations and at tree level by explicit noninvariant terms in the Lagrangian. One usually introduces the symmetric energy momentum tensor θ^{μ}_{ν} which may be derived with some complications or not from the canonical energy momentum tensor T^{μ}_{ν}. Then, the amount of breaking of the scale invariance is measured by

$$\partial_{\mu} D^{\mu} = \theta^{\mu}_{\mu}, \tag{1}$$

where $D^{\mu} = \theta^{\mu}_{\nu} x^{\nu}$ is the dilatation current and $\partial_{\mu}\theta^{\mu}_{\nu} = 0$.

The quantum anomalies of a theoretical particle physics model are crucial for describing its properties, and more so for the supersymmetric QCD and the QCD Lagrangians, where they are an important tool for constructing low-energy effective models of hadrons.

The effective approach based on the exact realization of axial and scale anomalies for supersymmetric QCD was analyzed by Seiberg [5] and Seiberg and Witten [6] leading to remarkable results regarding the phase structure of these theories. Applications of these methods for QCD were then implemented in [7, 8].

It is known that in supersymmetric gauge theories, at least in the holomorphic picture, the axial and trace anomalies belong to the same supermultiplet; and thus, the pseudoscalar and scalar glueballs can be regarded on the same footing. A unified description of both these anomalies for supersymmetric gauge theories was presented in [9]. For example, for the supersymmetric Yang-Mills ($F^a_{\mu\nu}$ is the gauge field tensor),

$$\begin{aligned} \theta^{\mu}_{\mu} &= 3N_c(F + F*) = -\frac{3N_c g^2}{32\pi^2} F^{a\mu\nu} F^a_{\mu\nu}, \\ \partial^{\mu} J^5_{\mu} &= 2iN_c(F - F^*) = \frac{N_c g^2}{32\pi^2} F^{a\mu\nu} \tilde{F}^a_{\mu\nu}. \end{aligned} \tag{2}$$

Here, $J^5_{\mu} = \bar{\lambda}\bar{\sigma}_{\mu}\lambda_a$ is the axial anomaly current and λ is the gluino field. Moreover the main ingredient of the effective theory is S, a composite chiral superfield with the structure

$$S(y) = \Phi(y) + \sqrt{2}\theta\Psi(y) + \theta^2 F(y), \tag{3}$$

where

$$\begin{aligned} \Phi &\approx \lambda^2, \\ \Psi &\approx \sigma^{\mu\nu}\lambda_a F^a_{\mu\nu,} \\ F &\approx -\frac{1}{2}F^{a\mu\nu}F^a_{\mu\nu} - \frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}F^{a\mu\nu}F^{a\rho\sigma}. \end{aligned} \tag{4}$$

Inspired by supersymmetric QCD, we will present here an approach for QCD where the pseudoscalar and scalar glueballs are treated as components of a single complex glueball field leaving for future work any possible direct connection with supersymmetry and the subtleties that might arise from it.

In Section 2, we discuss from a new point of view the scale symmetry and the construction of the symmetric energy momentum tensor for real and complex scalar field theories and QED. In Section 3, we will slightly modify the scale transformation such that to incorporate also the axial transformation for QED [10–12] and the $U(1)$ transformation for the complex scalar field theory. Thus, we will add to the symmetric energy momentum tensor presumably a "pseudoscalar" tensor τ^μ_ν such that the associated full Noether current becomes

$$J^\mu = \theta^\mu_\nu x^\nu + \tau^\mu_\nu x^\nu, \tag{5}$$

with

$$\begin{aligned} \partial_\mu \theta^\mu_\nu &= 0, \\ \partial_\mu \tau^\mu_\nu &= 0. \end{aligned} \tag{6}$$

Finally, the conservation of currents for the extended transformation will be

$$\partial_\mu J^\mu = \theta^\mu_\mu + \tau^\mu_\mu. \tag{7}$$

In Section 4, we will show how this approach makes total sense for a low-energy QCD linear sigma model with two chiral nonets, the model is discussed in detail in [13–20]. Section 5 is dedicated to conclusions.

2. Scale Transformation Revisited

We will analyze in some detail the scale transformation for some simple theories with amendments that our findings can be generalized easily to more intricate models.

2.1. Scalar Field Theory. We start with a Lagrangian for the scalar field theory

$$L = \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi - V(\Phi). \tag{8}$$

Under the scale transformation, the scalar field transforms as

$$\Phi(x) \longrightarrow \exp[-\sigma]\Phi(x\exp[-\sigma]), \tag{9}$$

whereas the action changes to

$$\begin{aligned} \delta S = &\int \delta\left(d^4x\right)L(\Phi(x),x) + \int d^4x \frac{\partial L}{\partial x^\mu}\delta x^\mu \\ &+ \int d^4x \partial_\mu\left[\frac{\partial L}{\partial\partial^\mu\Phi(x)}\delta(\Phi(x))\right]. \end{aligned} \tag{10}$$

The conserved current can be determined to be

$$J_\mu = \frac{\partial L}{\partial\partial_\mu\Phi(x)}\Phi(x) + \left[\frac{\partial L}{\partial\partial_\mu\Phi(x)}\partial_\rho\Phi(x) - L\eta^\mu_\rho\right]x^\rho. \tag{11}$$

The quantity

$$\frac{\partial L}{\partial\left(\partial_\mu\Phi(x)\right)}\partial_\rho\Phi(x) - L\eta^\mu_\rho = T^\mu_\rho \tag{12}$$

is the canonical energy momentum tensor. There are different procedures for deriving the symmetric energy momentum tensor θ^μ_ρ out from the canonical energy momentum tensor in Eq. (12). Here, we will present a very simple way applicable to any theory in general. First, we observe that we need to add an extra term to it that can stem only from the first term in expression (11). We write

$$\frac{\partial L}{\partial\left(\partial_\mu\Phi\right)}\Phi = a\partial^\mu\Phi\Phi\partial_\nu x^\nu - a\partial_\rho\Phi\Phi\eta^\mu_\nu\partial_\rho x^\nu = 3a\partial^\mu\Phi\Phi. \tag{13}$$

Since Eq. (13) is an identity, we immediately determine $a = 1/3$. Then, one can write

$$\begin{aligned} \frac{\partial L}{\partial\left(\partial_\mu\Phi\right)}\Phi &= \frac{1}{3}\partial^\mu\Phi\Phi\partial_\nu x^\nu - \frac{1}{3}(\partial^\rho\Phi)\Phi\eta^\mu_\nu\partial_\rho x^\nu \\ &= \frac{1}{3}\partial_\nu[\partial^\mu\Phi\Phi x^\nu] - \frac{1}{3}\partial_\rho[(\partial^\rho\Phi)\Phi\eta^\mu_\nu x^\nu] \\ &\quad - \frac{1}{3}\partial_\nu[(\partial^\mu\Phi)\Phi]x^\nu + \frac{1}{3}\partial_\rho[(\partial^\rho\Phi)\Phi\eta^\mu_\nu]x^\nu. \end{aligned} \tag{14}$$

We drop the total derivatives from the current to get

$$\frac{\partial L}{\partial\left(\partial_\mu\Phi\right)}\Phi = \left[-\frac{1}{6}\partial_\nu\partial^\mu\left(\Phi^2\right) + \frac{1}{6}\partial^2\Phi^2\eta^\mu_\nu\right]x^\nu. \tag{15}$$

Then, the symmetric energy momentum tensor is just

$$\theta^\mu_\nu = T^\mu_\nu + \left[-\frac{1}{6}\partial_\nu\partial^\mu\left(\Phi^2\right) + \frac{1}{6}\partial^2\Phi^2\eta^\mu_\nu\right], \tag{16}$$

and this result is consistent with the standard formula for the symmetric energy momentum tensor in the literature.

Next, we consider a complex scalar field theory with the Lagrangian

$$\mathscr{L} = \partial^\mu\Phi^*\partial_\mu\Phi - V(\Phi,\Phi^*). \tag{17}$$

We consider an extension of the scale transformation in which $x' = \exp[\sigma]x$, and the fields transform as

$$\Phi(x) \longrightarrow \Phi'\left(x'\right) = \exp[-\sigma(1+i)]\Phi(\exp[-\sigma]x),$$
$$\Phi^*(x) \longrightarrow \Phi'^*\left(x'\right) = \exp[-\sigma(1-i)]\Phi^*(\exp[-\sigma]x). \quad (18)$$

Note that the above transformation is just the scale transformation associated with a $U(1)$ transformation with the same infinitesimal parameter σ. Then, it can be easily determined that

$$J^\mu = \frac{\partial L}{\partial(\partial_\mu\Phi)}\Phi + \frac{\partial L}{\partial(\partial_\mu\Phi^*)}\Phi^* + \left[\frac{\partial L}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi + \frac{\partial L}{\partial(\partial_\mu\Phi^*)}\partial_\nu\Phi^* - L\eta^\mu_\nu\right]x^\nu + i\frac{\partial L}{\partial(\partial_\mu\Phi)}\Phi - i\frac{\partial L}{\partial(\partial_\mu\Phi^*)}\Phi^*. \quad (19)$$

Finally, one can apply the procedure in Eq. (16) to obtain

$$J^\mu = \theta^\mu_\nu x^\nu - K^\mu, \quad (20)$$

where

$$K^\mu = -i(\partial^\mu\Phi^*)\Phi + i(\partial^\mu\Phi)\Phi^*. \quad (21)$$

Our aim is to generalize the trace anomaly such that to contain also a contribution from the term in Eq. (21). Specifically, we want to find τ^μ_ν such that $\partial_\mu\tau^\mu_\nu = 0$ and

$$K^\mu = -\tau^\mu_\nu x^\nu. \quad (22)$$

We find using the procedure in Eq. (14)

$$\tau^\mu_\nu = \left[-\frac{i}{3}\partial_\nu(\partial^\mu\Phi^*\Phi) + \frac{i}{3}\partial^\rho\left(\partial_\rho\Phi^*\Phi\right)\eta^\mu_\nu\right] + h.c. \quad (23)$$

It can be easily checked that

$$K^\mu = -\tau^\mu_\nu x^\nu,$$
$$\partial_\mu\tau^\mu_\nu = 0. \quad (24)$$

Finally, one can compute the trace of the tensor in Eq. (24) as

$$-\partial_\mu K^\mu = \tau^\mu_\mu = -i\left[\frac{\partial V}{\partial\Phi}\Phi - \frac{\partial V}{\partial\Phi^*}\Phi^*\right], \quad (25)$$

where we applied the equation of motion.

2.2. QED. Now, we shall state the results for τ^μ_ν briefly for QED [10–12] (for a detailed review, see [21]). Consider the transformation of the fields

$$\Psi_L(x) \longrightarrow \exp\left[-\sigma\left(\frac{3}{2}+i\right)\right]\Psi_L\left(\exp[-\sigma]x'\right),$$
$$\Psi_R(x) \longrightarrow \exp[-\sigma(3/2-i)]\Psi_R\left(\exp[-\sigma]x'\right), \quad (26)$$

together with the corresponding transformation for the conjugate fields. Since the scale transformation contribution in QED is well known, we will consider only the extra contribution coming from the axial transformation included in Eq. (26). Thus, the conserved current has the form

$$J^{\mu\prime}_L = J^\mu_L - i\frac{\partial L}{\partial(\partial_\mu\Psi_L)}\Psi_L,$$
$$J^{\mu\prime}_R = J^\mu_R + i\frac{\partial L}{\partial(\partial_\mu\Psi_R)}\Psi_R. \quad (27)$$

Here, J^μ_L and J^μ_R are the standard scale transformation currents. We denote

$$K^\mu = i\frac{\partial L}{\partial(\partial_\mu\Psi_L)}\Psi_L - i\frac{\partial L}{\partial(\partial_\mu\Psi_R)}\Psi_R = \bar{\Psi}\gamma^\mu\gamma^5\Psi, \quad (28)$$

where we applied the equation of motion. Thus, we obtained the standard axial current. Next, we want to determine τ^μ_ν such that

$$K^\mu = -\tau^\mu_\nu x^\nu,$$
$$\partial_\mu\tau^\mu_\nu = 0. \quad (29)$$

For that, there is only one possible term

$$\tau^\mu_\nu = \frac{1}{3}\partial_\nu\left(\bar{\Psi}\gamma^\mu\gamma^5\Psi\right) - \frac{1}{3}\partial_\rho\left(\bar{\Psi}\gamma^\rho\gamma^5\Psi\right)\eta^\mu_\nu, \quad (30)$$

where it is clear that the second equation in Eq. (29) is satisfied if one applies the equation of motion in the absence of quark masses. Thus, the full current will be

$$J^{\mu\prime} = [\theta^\mu_\nu + \tau^\mu_\nu]x^\nu, \quad (31)$$

and we were able to write a total current that encapsulates both scale and axial transformations. Finally, the conservation law is

$$\partial_\mu J'^\mu = \theta^\mu_\mu + \tau^\mu_\mu \quad (32)$$

and includes both a scalar and a pseudoscalar part.

3. Application to a Generalized Linear Sigma Model

In this section, we will show in detail how one can introduce in a consistent way from the point of view presented here the scalar and pseudoscalar glueballs in a linear sigma model which is a low-energy description of QCD. The presence and properties of the scalar and pseudoscalar glueballs in the QCD spectrum were the subjects of relevant works in the literature. In [22, 23], the trace anomaly in low-energy QCD and some features of the scalar glueball were discussed in different frameworks. Furthermore, in [24–26], the axial anomaly and spectrum and decays of the pseudoscalar glueball were analyzed in low-energy QCD.

As an application of our method, we consider a generalized linear sigma model [13–19] with two chiral meson nonets, one with a quark antiquark structure and the other one with a four quark composition. We denote this nonets by M and M', where $M = S + i\Phi$, $M' = S' + i\Phi'$, and S and S' represent the scalar states and Φ and Φ' the pseudoscalar ones. The fields M and M' transform in the same way under chiral SU(3) transformations

$$\begin{aligned} M &\longrightarrow U_L\, M\, U_R^\dagger, \\ M' &\longrightarrow U_L\, M'\, U_R^\dagger \end{aligned} \tag{33}$$

but transform differently under $U(1)_A$ transformation

$$\begin{aligned} M &\longrightarrow e^{2i\nu} M, \\ M' &\longrightarrow e^{-4i\nu} M'. \end{aligned} \tag{34}$$

In order for the model to be consistent, it is useful to introduce terms that mock up the axial and scale anomalies which read

$$\begin{aligned} \partial^\mu J_\mu^5 &= \frac{g^2}{16\pi^2} N_F \tilde{F} F = G, \\ \theta_\mu^\mu &= \partial^\mu D_\mu = -\frac{\beta(g)}{2g} F F = H. \end{aligned} \tag{35}$$

Note that in this section we use a different convention for the metric tensor with respect to Section 2. Here, F is the $SU(3)_C$ field tensor, $\tilde{F}$ is its dual, N_F is the number of flavors, $\beta(g)$ is the beta function for the coupling constant, J_μ^5 is the axial current, and D_μ is the dilatation current. The field H may be associated in low-energy QCD with the scalar glueball whereas G with the pseudoscalar one. The Lagrangian has the form

$$\begin{aligned} L = &-\frac{1}{2}\mathrm{Tr}\left(\partial_\mu M \partial_\mu M^\dagger\right) - \frac{1}{2}\mathrm{Tr}\left(\partial_\mu M' \partial_\mu M'^\dagger\right) - V_0\left(M, M'\right) \\ &- V_{SB}, \end{aligned} \tag{36}$$

where $V_0(M, M')$ is a function made from $SU(3)_L \times SU(3)_R$ (but not necessarily $U(1)_A$) invariants formed out of M and M'. The leading choice of terms corresponding to eight or fewer underlying quarks plus antiquark lines at each effective vertex reads [17]

$$\begin{aligned} V_0 = &-c_2\,\mathrm{Tr}\left(M M^\dagger\right) + c_4^a\,\mathrm{Tr}\left(M M^\dagger M M^\dagger\right) + d_2\,\mathrm{Tr}\left(M' M'^\dagger\right) \\ &+ e_3^a\left(\varepsilon_{abc}\varepsilon^{def} M_d^a M_e^b M'^c_f + \mathrm{H.c.}\right) \\ &+ c_3\left[\gamma_1 \ln\left(\frac{\det M}{\det M^\dagger}\right) + (1-\gamma_1)\ln\frac{\mathrm{Tr}\left(M M'^\dagger\right)}{\mathrm{Tr}\left(M' M^\dagger\right)}\right]^2. \end{aligned} \tag{37}$$

All the terms except the last two (which mock up the axial anomaly) have been chosen to also possess the $U(1)_A$ invariance.

For a scalar and pseudoscalar Lagrangian of the type in Eq. (36) but also contains scalar and pseudoscalar glueballs, the trace anomaly reads [20]

$$\theta_\mu^\mu = M\frac{\partial V}{\partial M} + \frac{\partial V}{\partial M^\dagger} M^\dagger + 4H\frac{\partial V}{\partial H} + 4G\frac{\partial V}{\partial G} - 4V, \tag{38}$$

where we applied the equation of motion in the results of Section 2.

From Eq. (23), we obtained in the previous section in terms of Φ

$$\begin{aligned} \tau_\mu^\mu &= \left[-\frac{i}{3}\partial_\mu\left(\partial^\mu \Phi^* \Phi\right) + \frac{i}{3}\partial^\rho\left(\partial_\rho \Phi^* \Phi\right)\eta_\mu^\mu\right] + h.c. \\ &= -i\left[\frac{\partial V}{\partial \Phi}\Phi - \frac{\partial V}{\partial \Phi^*}\Phi^*\right]. \end{aligned} \tag{39}$$

Adjusted for the generalized linear sigma model, the axial anomaly contribution in the regular case for a single chiral nonet M is (note that for this case, the kinetic term has negative sign)

$$\tau_\mu^\mu = i\left[M\frac{\partial V}{\partial M} - M^\dagger\frac{\partial V}{\partial M^\dagger}\right]. \tag{40}$$

Now, we shall extend the scale transformation to include the axial one as discussed in Section 2. For that, we denote $Q = H + iG$ and consider the generalized transformation

$$\begin{aligned} M &\longrightarrow \exp\left[-(1+i)\sigma\right] M(x\exp\left[-\sigma\right]), \\ M^\dagger &\longrightarrow \exp\left[-(1-i)\sigma\right] M^\dagger(x\exp[-\sigma]), \\ M' &\longrightarrow \exp\left[-(1-2i)\sigma\right] M'^{(x\exp\left[-\sigma\right])}, \\ M'^\dagger &\longrightarrow \exp\left[-(1+2i)\sigma\right] M'^\dagger(x\exp\left[-\sigma\right]), \\ Q &\longrightarrow \exp\left[-4\sigma\right] Q(x\exp\left[-\sigma\right]), \\ Q^* &\longrightarrow \exp\left[-4\sigma\right] Q^*(x\exp\left[-\sigma\right]). \end{aligned} \tag{41}$$

The transformation in Eq. (41) makes sense as an expansion in the infinitesimal parameter σ.

Then, the conservation of current should be of the form

$$\theta^{\mu}_{\mu} + \tau^{\mu}_{\mu} = H - G, \tag{42}$$

where we used Eq. (35). Moreover, we fixed the transformation law for the field M' to correspond to the correct axial anomaly. A detailed discussion of the most general structure of the possible terms that mock up the scale and trace anomalies is given in [20]. Here, we will present a first-order term that leads to the correct conservation law

$$V = V_1 + V_1^{\dagger}, \tag{43}$$

where

$$\begin{aligned} V_1 &= Q\left[\lambda_1 \ln\left[\frac{Q}{\Lambda^4}\right] + \lambda_2 \ln\left[\frac{\det M}{\Lambda^3}\right] + \lambda_3 \ln\left[\frac{\mathrm{T}rMM'^{\dagger}}{\Lambda^3}\right]\right], \\ V_1^{\dagger} &= Q^*\left[\ln\left[\lambda_1 \frac{Q^*}{\Lambda^4}\right] + \lambda_2 \ln\left[\frac{\det M^{\dagger}}{\Lambda^3}\right] + \lambda_3 \ln\left[\frac{\mathrm{T}rM'M^{\dagger}}{\Lambda^3}\right]\right], \\ V &= Q\left[\lambda_1 \ln\left[\frac{Q}{\Lambda^4}\right] + \lambda_2 \ln\left[\frac{\det M}{\Lambda^3}\right] + \lambda_3 \ln\left[\frac{\mathrm{T}rMM'^{\dagger}}{\Lambda^3}\right]\right] \\ &\quad + Q^*\left[\ln\left[\lambda_1 \frac{Q^*}{\Lambda^4}\right] + \lambda_2 \ln\left[\frac{\det M^{\dagger}}{\Lambda^3}\right] + \lambda_3 \ln\left[\frac{\mathrm{T}rM'M^{\dagger}}{\Lambda^3}\right]\right]. \end{aligned} \tag{44}$$

For the transformation introduced in Eq. (41), the trace of the scale and axial currents read

$$\begin{aligned} \theta^{\mu}_{\mu} &= \left[M\frac{\partial V}{\partial M} + M^{\dagger}\frac{\partial V}{\partial M^{\dagger}} + M'\frac{\partial V}{\partial M'} + M'^{\dagger}\frac{\partial V}{\partial M'^{\dagger}} + 4Q\frac{\partial V}{\partial Q} \right. \\ &\quad \left. + 4Q^*\frac{\partial V}{\partial Q^*}\right] - 4V, \\ \tau^{\mu}_{\mu} &= i\left[M\frac{\partial V}{\partial M} - M^{\dagger}\frac{\partial V}{\partial M^{\dagger}} - 2M'\frac{\partial V}{\partial M'} + 2M'^{\dagger}\frac{\partial V}{\partial M'^{\dagger}}\right]. \end{aligned} \tag{45}$$

We list below the contributions of various terms

$$M\frac{\partial V_1}{\partial M} = Q(3\lambda_2 + \lambda_3),$$

$$\frac{\partial V_1^{\dagger}}{\partial M^{\dagger}}M^{\dagger} = Q^*(3\lambda_2 + \lambda_3),$$

$$M'^{\partial V_1^{\dagger}/\partial M'} = Q^*\lambda_3,$$

$$\frac{\partial V_1}{\partial M'^{\dagger}}M'^{\dagger} = Q\lambda_3,$$

$$\begin{aligned} 4Q\frac{\partial V_1}{\partial Q} &= 4Q\left[\lambda_1 \ln\left[\frac{Q}{\Lambda^4}\right] + \lambda_2 \ln\left[\frac{\det M}{\Lambda^2}\right] \right. \\ &\quad \left. + \lambda_3 \ln\left[\frac{\mathrm{T}rMM'^{\dagger}}{\Lambda^2}\right]\right] + 4Q\lambda_1, \\ 4Q^*\frac{\partial V_1^{\dagger}}{\partial Q^*} &= 4Q^*\left[\lambda_1 \ln\left[\frac{Q^*}{\Lambda^4}\right] + \lambda_2 \ln\left[\frac{\det M^{\dagger}}{\Lambda^2}\right] \right. \\ &\quad \left. + \lambda_3 \ln\left[\frac{\mathrm{T}rM'M^{\dagger}}{\Lambda^2}\right]\right] + 4Q^*\lambda_1. \end{aligned} \tag{46}$$

The final results for the combined trace and axial anomalies are

$$\begin{aligned} \theta^{\mu}_{\mu} &= (H + iG)[3\lambda_2 + 2\lambda_3 + 4\lambda_1] + (H - iG)[3\lambda_2 + 3\lambda_3 + 4\lambda_1] \\ &= H[6\lambda_2 + 4\lambda_3 + 8\lambda_1], \\ \tau^{\mu}_{\mu} &= i(H + iG)[3\lambda_2 + 3\lambda_3] - i(H - iG)[3\lambda_2 + 3\lambda_3] \\ &= -G[6\lambda_2 + 6\lambda_3]. \end{aligned} \tag{47}$$

Then, a sufficient and necessary condition for Eq. (42) to be fulfilled is

$$\begin{aligned} 6\lambda_2 + 4\lambda_3 + 8\lambda_1 &= 1, \\ 6\lambda_2 + 6\lambda_3 &= 1. \end{aligned} \tag{48}$$

As a consistency check, it is important to notice that the trace and axial anomalies would be equally satisfied by the potential in Eq. (44) also when the standard transformation for axial and trace anomalies would be applied.

4. Proposal for a Lagrangian Containing a Complex Glueball Field

In [20], an effective generalized linear sigma model that contained both scalar and pseudoscalar glueballs was introduced. There, H and G were introduced separately as individual states. Here, we will consider a new version of that Lagrangian where the scalar and pseudoscalar glueballs are parts of the same complex field $Q = H + iG$. This Lagrangian is consistent with the symmetries and axial and scale anomalies of low-energy QCD and has the form

$$\begin{aligned} L &= -\frac{1}{2}\mathrm{Tr}\left[\partial^{\mu}M\partial_{\mu}M^{\dagger}\right] - \frac{1}{2}\mathrm{Tr}\left[\partial^{\mu}M'\partial_{\mu}M'^{\dagger}\right] \\ &\quad - \frac{1}{2}[QQ^*]^{-3/4}\left[\partial^{\mu}Q\partial_{\mu}Q^*\right] + f + f_{A,S} + V_{SB}. \end{aligned} \tag{49}$$

In Eq. (49), the kinetic term for the glueball field Q is written such that it is scale invariant. The term f can be

$$f = -c_2 \operatorname{Tr}\left(MM^{\dagger}\right) + c_4^a \operatorname{Tr}\left(MM^{\dagger}MM^{\dagger}\right) + d_2 \operatorname{Tr}\left(M'M'^{\dagger}\right) + e_3^a\left(\varepsilon_{abc}\varepsilon^{def} M_d^a M_e^b M'^c_f + \text{H.}c.\right) + g_1 \frac{QQ^*}{\Lambda^4}, \quad (50)$$

where g_1 is a dimensionless coupling constant and Λ is the QCD scale.

Furthermore, the axial and scale anomaly terms are encapsulated in $f_{A,S} = V$, where a detailed expression of V is given in Eqs. (43) and (44).

The symmetry breaking terms can be

$$V_{SB} = \operatorname{Tr}\left[A\left(M + M^{\dagger}\right)\right], \quad (51)$$

where A is a 3×3 diagonal constant matrix proportional to the light quark masses.

Note that for each term in the full potential, we chose the minimal ones from a multitude of possible terms.

5. Conclusions

In this work, we explored some features of the scale transformation in the context of some simple theories in order to illustrate the possibility of incorporating both the scale and axial anomalies in a single larger transformation. Our method is inspired by the supersymmetric theories, where it is known that all anomalies including the trace and axial ones are part of the same supermultiplet structure [27]. In order to express this at the level of currents, we introduce a new tensor τ_{ν}^{μ} that has the correct properties such that the net anomalous conservation of current is given by the sum $\theta_{\mu}^{\mu} + \tau_{\nu}^{\mu}$. In some theories, τ_{μ}^{μ} is a pseudoscalar.

This approach allowed us to introduce a comprehensive term that mocks up both the trace and axial anomalies in an effective low-energy QCD model. Consequently, the scalar H and pseudoscalar G glueballs that may be present in a generalized linear sigma model with scalar and pseudoscalar meson nonets can be arranged in a singlet complex field $H + iG$ on the same footing with the quark composite mesons. In this direction, we also proposed a low-energy effective Lagrangian based on the derivations and results in this work. This method may have important application in low-energy QCD models and besides that, in any effective model that displays both trace and axial anomalies.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that she has no conflicts of interest.

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