

Research Article

Superstring Backgrounds in String Geometry

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String geometry theory is a candidate of the nonperturbative formulation of string theory. In order to determine the string vacuum, we need to clarify how superstring backgrounds are described in string geometry theory. In this paper, we show that all the type IIA, IIB, SO(32) type I, and SO(32) and $E_8 \times E_8$ heterotic superstring backgrounds are embedded in configurations of the fields of a single string geometry model. In particular, we show that the configurations satisfy the equations of motion of the string geometry model in $\alpha' \rightarrow 0$ if and only if the embedded string backgrounds satisfy the equations of motion of the supergravities, respectively. This means that classical dynamics of the string backgrounds are described as a part of classical dynamics in string geometry theory. Furthermore, we define an energy of the configurations in the string geometry model because they do not depend on the string geometry time. A string background can be determined by minimizing the energy.

1. Introduction

Superstring theory is a promising candidate of a unified theory including gravity. However, superstring theory is established at only the perturbative level as of this moment. The perturbative superstring theory lacks predictability because it has many perturbatively stable vacua.

String geometry theory is a candidate of nonperturbative formulation of superstring theory [1], which can determine a nonperturbatively stable vacuum. In string geometry theory, the path integral of the perturbative superstring theory on the flat string background is derived by taking a Newtonian limit of fluctuations around a fixed flat background in an Einstein-Hilbert action coupled with any field on string manifolds [1, 2]. A perturbative topological string theory is also derived from the topological sector of string geometry theory [3]. That is, the spectrum and all order scattering amplitudes in superstring theory on a flat background are derived from string geometry theory. However, perturbative string theory describes only propagation and interactions of strings in a

fixed classical string background and cannot describe dynamics of the classical string background itself. Only the consistency with the Weyl invariance requires that the string background satisfies the equations of motion of supergravity. That is, string backgrounds are treated as external fields in the perturbative string theory. In order to determine a string background, a nonperturbative string theory needs to be able to describe dynamics of the string backgrounds not in consequence of consistency.

In paper [4], as a first step to determine the string vacuum, the authors studied how arbitrary configurations of the bosonic string backgrounds are embedded in configurations of the fields of a bosonic string geometry model. In particular, the authors showed that the action of the string backgrounds is obtained by a consistent truncation of the action of the string geometry model; the configurations of the fields of string geometry model satisfy their equations of motion if and only if the embedded configurations of the string backgrounds satisfy their equations of motion. This means that classical dynamics of the string backgrounds are

described as a part of classical dynamics in string geometry theory. This fact supports the conjecture that string geometry theory is a nonperturbative formulation of string theory.

This truncation is valid without taking the $\alpha' \rightarrow 0$ limit, which corresponds to $X(\bar{\sigma}) \rightarrow x$. This fact will be important to derive the path integral of the nonlinear sigma model from fluctuations around the string background configurations in the string geometry theory, since the string backgrounds in the nonlinear sigma model depend not only on the string zero modes x but also on the other modes of $X(\sigma)$ [5–7]:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[\left(g^{ab} G_{\mu\nu}(X(\sigma)) + i\epsilon^{ab} B_{\mu\nu}(X(\sigma)) \right) \partial_a X^\mu \partial_b X^\nu + \alpha' R\phi(X(\sigma)) \right]. \quad (1)$$

In this paper, we generalize the results in [4] to the supersymmetric case, which possesses interesting problems as follows. In general, it is too difficult to define nonlinear sigma models in R-R backgrounds in the NS-R formalism of string theory. On the other hand, if one can perform a supersymmetric generalization of the above results as they are, there is an apparent contradiction that we can derive nonlinear sigma models in R-R backgrounds in the NS-R formalism from string geometry theory. We will see how this contradiction is resolved. There is another interesting problem: Chern-Simons terms cannot be defined in string geometry models because they are infinite dimensional, although supergravities, which should be reproduced from the models, possess Chern-Simons terms. Nevertheless, we will see that the type IIA and IIB supergravities are reproduced from a string geometry model.

The organization of this paper is as follows. In Section 2, we introduce a string geometry model. In Section 3, we identify type IIA and IIB string background configurations and obtain the equations of motions of the type IIA and IIB supergravities from the equations of motions of the string geometry model by a consistent truncation in the $\alpha' \rightarrow 0$ limit. In Section 4, we define an energy of the string background configurations because they do not depend on the string geometry time. A string background can be determined by minimizing the energy. In Section 5, all the five supergravities in the ten dimensions are derived from a single string geometry model by consistent truncations. In Section 6, we conclude and discuss our results.

2. String Geometry Model

We study a string geometry model whose action is given by,

$$S = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}\bar{\tau} \mathcal{D}\mathbf{X}_{\bar{D}} \sqrt{G} \left(e^{-2\Phi} \left(\mathbf{R} + 4\nabla_{\mathbf{I}} \Phi \nabla^{\mathbf{I}} \Phi - \frac{1}{2} |\mathbf{H}|^2 \right) - \frac{1}{2} \sum_{p=1}^9 |\tilde{\mathbf{F}}_p|^2 \right). \quad (2)$$

The action of string geometry theory is not determined as of this moment. On this stage, we should consider various

possible actions. Then, we call each action a string geometry model and call the whole formulation string geometry theory. In [1], the perturbative superstring theory on the flat spacetime is derived from a gravitational model coupled with a $u(1)$ field on a Riemannian string manifold, whereas in [2], it is derived from gravitational models coupled with arbitrary fields on a Riemannian string manifold. Thus, the perturbative superstring theory on the flat spacetime is derived from this model. In this action, G_N is a constant, $\mathbf{I} = \{d, (\mu\bar{\sigma}\bar{\theta})\}$, $|\mathbf{H}|^2 := (1/3!) \mathbf{G}^{\mathbf{I}_1 \mathbf{I}_2} \mathbf{G}^{\mathbf{I}_3 \mathbf{I}_4} \mathbf{H}_{\mathbf{I}_1 \mathbf{I}_2 \mathbf{I}_3} \mathbf{H}_{\mathbf{I}_1 \mathbf{I}_2 \mathbf{I}_4}$, and we use the Einstein notation for the index \mathbf{I} . Action (2) consists of a scalar curvature \mathbf{R} of a metric $\mathbf{G}_{\mathbf{I}_1 \mathbf{I}_2}$, a scalar field Φ , a field strength $\mathbf{H}_{\mathbf{I}_1 \mathbf{I}_2 \mathbf{I}_3}$ of a two-form field $\mathbf{B}_{\mathbf{I}_1 \mathbf{I}_2}$ and $\tilde{\mathbf{F}}_p$. $\tilde{\mathbf{F}}_p$ are defined by $\sum_{p=1}^9 \tilde{\mathbf{F}}_p = e^{-B_2} \wedge \sum_{k=1}^9 \mathbf{F}_k$, where \mathbf{F}_k are field strengths of $(k-1)$ -form fields \mathbf{A}_{k-1} . For example, $\tilde{\mathbf{F}}_5 = \mathbf{F}_5 - \mathbf{B}_2 \wedge \mathbf{F}_3 + 1/2 \mathbf{B}_2 \wedge \mathbf{B}_2 \wedge \mathbf{F}_1$. They are defined on a *Riemannian string manifold*, whose definition is given in [1]. String manifold is constructed by patching open sets in string model space E , whose definition is summarized as follows. First, a global time $\bar{\tau}$ is defined canonically and uniquely on a super Riemann surface $\bar{\Sigma}$ by the real part of the integral of an Abelian differential uniquely defined on $\bar{\Sigma}$ [8, 9]. We restrict $\bar{\Sigma}$ to a $\bar{\tau}$ constant line and obtain $\bar{\Sigma}|_{\bar{\tau}}$. An embedding of $\bar{\Sigma}|_{\bar{\tau}}$ to \mathbb{R}^d represents a many-body state of superstrings in \mathbb{R}^d and is parametrized by coordinates $(\bar{\mathbf{E}}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}), \bar{\tau})$ where $\bar{\mathbf{E}}$ is a super vierbein on $\bar{\Sigma}$ and $\mathbf{X}_{\bar{D}_T}(\bar{\tau})$ is a map from $\bar{\Sigma}|_{\bar{\tau}}$ to \mathbb{R}^d . $\bar{\Sigma}$ represents a representative of the super diffeomorphism and super Weyl transformation on the worldsheet. Giving a super Riemann surface $\bar{\Sigma}$ is equivalent to giving a supervierbein $\bar{\mathbf{E}}$ up to super diffeomorphism and super Weyl transformations. \bar{D}_T represents all the backgrounds except for the target metric that consists of the B-field, the dilaton, and the R-R fields. String model space E is defined by the collection of the string states by considering all the $\bar{\Sigma}$, all the values of $\bar{\tau}$, and all the $\mathbf{X}_{\bar{D}_T}(\bar{\tau})$. How near the two string states is defined by how near the values of $\bar{\tau}$ and how near $\mathbf{X}_{\bar{D}_T}(\bar{\tau})$. An ε -open neighborhood of $[\bar{\Sigma}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}_s), \bar{\tau}_s]$ is defined by $U([\bar{\Sigma}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}_s), \bar{\tau}_s], \varepsilon) := \{[\bar{\Sigma}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}), \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|\mathbf{X}_{\bar{D}_T}(\bar{\tau}) - \mathbf{X}_{\bar{D}_T}(\bar{\tau}_s)\|^2} < \varepsilon\}$, where $\bar{\mathbf{E}}$ is a discrete variable in the topology of string geometry. As a result, $d\bar{\mathbf{E}}$ cannot be a part of basis that span the cotangent space in (3), whereas fields are functionals of $\bar{\mathbf{E}}$ as in (4). The precise definition of the string topology is given in the Section 2 in [1]. By this definition, arbitrary two string states on a connected super Riemann surface in E are connected continuously. Thus, there is a one-to-one correspondence between a super Riemann surface in \mathbb{R}^d and a curve parametrized by $\bar{\tau}$ from $\bar{\tau} = -\infty$ to $\bar{\tau} = \infty$ on E . That is, curves that represent asymptotic processes on E reproduce the right moduli space of the super Riemann surfaces in \mathbb{R}^d . Therefore, a string geometry model possesses all-order information of superstring theory. Actually, all order perturbative scattering amplitudes of the superstrings in the flat space-time are derived from the string geometry theory as in [1, 2]. The consistency of the perturbation theory determines $d = 10$ (the critical dimension). The cotangent space is spanned by

$$\begin{aligned} d\mathbf{X}_{\bar{D}_T}^d &:= d\bar{\tau}, \\ d\mathbf{X}_{\bar{D}_T}^{(\mu\bar{\sigma}\bar{\theta})} &:= d\mathbf{X}_{\bar{D}_T}^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \end{aligned} \quad (3)$$

where $\mu = 0, \dots, d-1$. The summation over $(\bar{\sigma}, \bar{\theta})$ is defined by $\int d\bar{\sigma} d^2\bar{\theta} \hat{\mathbf{E}}(\bar{\sigma}, \bar{\tau}, \bar{\theta})$. $\hat{\mathbf{E}}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) := (1/\bar{n})\bar{\mathbf{E}}(\bar{\sigma}, \bar{\tau}, \bar{\theta})$, where \bar{n} is the lapse function of the two-dimensional metric. This summation is transformed as a scalar under $\bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}))$ and invariant under a supersymmetry transformation $(\bar{\sigma}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta}))$. As a result, action (2) is invariant under this $\mathcal{N} = (1, 1)$ supersymmetry transformation. An explicit form of the line element is given by

$$\begin{aligned} ds^2(\bar{\mathbf{E}}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}), \bar{\tau}) &= G(\bar{\mathbf{E}}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}), \bar{\tau})_{dd} (d\bar{\tau})^2 \\ &+ 2d\bar{\tau} \int d\bar{\sigma} d^2\bar{\theta} \bar{\mathbf{E}} \sum_{\mu} G(\bar{\mathbf{E}}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}), \bar{\tau})_{d(\mu\bar{\sigma}\bar{\theta})} d\mathbf{X}_{\bar{D}_T}^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \\ &+ \int d\bar{\sigma} d^2\bar{\theta} \bar{\mathbf{E}} \int d\bar{\sigma}' d^2\bar{\theta}' \bar{\mathbf{E}}' \sum_{\mu, \mu'} G(\bar{\mathbf{E}}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}), \bar{\tau})_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} d\mathbf{X}_{\bar{D}_T}^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}) d\mathbf{X}_{\bar{D}_T}^{\mu'}(\bar{\sigma}', \bar{\tau}, \bar{\theta}'). \end{aligned} \quad (4)$$

The inverse metric $\mathbf{G}^{IJ}(\bar{\mathbf{E}}, \mathbf{X}_{\bar{D}_T}(\bar{\tau}), \bar{\tau})$ is defined by $\mathbf{G}_{IJ} \mathbf{G}^{JK} = \mathbf{G}^{KI} \mathbf{G}^{IJ} = \delta_I^K$, where $\delta_d^d = 1$ and $\delta_{\mu\bar{\sigma}\bar{\theta}}^{\mu'\bar{\sigma}'\bar{\theta}'}$ where $\delta_{\bar{\sigma}\bar{\theta}}^{\bar{\sigma}'\bar{\theta}'} = \delta_{(\bar{\sigma}\bar{\theta})(\bar{\sigma}'\bar{\theta}')} = (1/\bar{\mathbf{E}})\delta(\bar{\sigma} - \bar{\sigma}')\delta^2(\bar{\theta} - \bar{\theta}')$.

3. Consistent Truncations to Type IIA and IIB Supergravities

In this section, we will show that we can consistently truncate the string geometry model (2) to the type IIA and IIB supergravities if we apply appropriate configurations to the model and take $\alpha' \rightarrow 0$.

From equation (2), the equations of motion of \mathbf{G}_{IJ} , Φ , $\mathbf{B}_{L_1 L_2}$, and $\mathbf{A}_{L_1 \dots L_{p-1}}$ are derived as

$$\begin{aligned} \mathbf{R}_{IJ} + 2\nabla_I \nabla_J \Phi - \frac{1}{4} \mathbf{H}_{IL_1 L_2} \mathbf{H}_J^{L_1 L_2} - \frac{1}{2} \mathbf{G}_{IJ} \\ \cdot \left(\mathbf{R} + 4\nabla_I \nabla^I \Phi - 4\partial_I \Phi \partial^I \Phi - \frac{1}{2} |\mathbf{H}_3|^2 \right) \\ - \frac{1}{2} e^{2\Phi} \sum_{p=1}^9 \left[\frac{1}{(p-1)!} \tilde{\mathbf{F}}_{IL_1 \dots L_{p-1}} \tilde{\mathbf{F}}_J^{L_1 \dots L_{p-1}} - \frac{1}{2} \mathbf{G}_{IJ} |\tilde{\mathbf{F}}_p|^2 \right] = 0, \end{aligned} \quad (5)$$

$$\mathbf{R} + 4\nabla_I \nabla^I \Phi - 4\partial_I \Phi \partial^I \Phi - \frac{1}{2} |\mathbf{H}_3|^2 = 0, \quad (6)$$

$$\begin{aligned} \sum_{p=3}^9 \sum_{n=0}^{[(p-3)/2]} \frac{1}{2^{n+1} \cdot (p-2)!} \mathbf{F}_{I_1 \dots I_{p-2-2n}} \mathbf{B}_{J_1 K_1} \dots \mathbf{B}_{J_n K_n} \tilde{\mathbf{F}}^{I_1 \dots I_{p-2-2n} J_1 K_1 \dots J_n K_n L_1 L_2} \\ + \nabla_I (e^{-2\Phi} \mathbf{H}^{IL_1 L_2}) = 0, \end{aligned} \quad (7)$$

$$\nabla_I \tilde{\mathbf{F}}^{IL_1 \dots L_{p-1}} + \left(\frac{1}{2}\right)^n \nabla_I \left[\mathbf{B}_{J_1 K_1} \dots \mathbf{B}_{J_n K_n} \tilde{\mathbf{F}}^{J_1 K_1 \dots J_n K_n L_1 \dots L_{p-1}} \right] = 0, \quad (8)$$

respectively.

We consider particular configurations, which we call IIA and IIB string background configurations,

Metric:

$$\begin{aligned} \mathbf{G}_{dd}(\bar{\tau}, \mathbf{X}) &= -1, \\ \mathbf{G}_{(\mu_1 \bar{\sigma}_1 \bar{\theta}_1)(\mu_2 \bar{\sigma}_2 \bar{\theta}_2)}(\bar{\tau}, \mathbf{X}) &= G_{\mu_1 \mu_2}(\mathbf{X}(\bar{\sigma}_1 \bar{\theta}_1)) \delta_{(\bar{\sigma}_1 \bar{\theta}_1)(\bar{\sigma}_2 \bar{\theta}_2)} \delta_{(\bar{\sigma}_1 \bar{\theta}_1)(\bar{\sigma}_1 \bar{\theta}_1)}, \\ \text{the others} &= 0. \end{aligned} \quad (9)$$

Scalar field:

$$\Phi(\bar{\tau}, \mathbf{X}) = \int d\bar{\sigma} d^2\bar{\theta} \bar{\mathbf{E}} \delta_{(\bar{\sigma}, \bar{\theta})} \phi(\mathbf{X}(\bar{\sigma}, \bar{\theta})). \quad (10)$$

B field:

$$\begin{aligned} \mathbf{B}_{(\mu_1 \bar{\sigma}_1 \bar{\theta}_1)(\mu_2 \bar{\sigma}_2 \bar{\theta}_2)}(\bar{\tau}, \mathbf{X}) &= B_{\mu_1 \mu_2}(\mathbf{X}(\bar{\sigma}_1 \bar{\theta}_1)) \delta_{(\bar{\sigma}_1 \bar{\theta}_1)(\bar{\sigma}_2 \bar{\theta}_2)} \delta_{(\bar{\sigma}_1 \bar{\theta}_1)(\bar{\sigma}_1 \bar{\theta}_1)}, \\ \text{the others} &= 0. \end{aligned} \quad (11)$$

p-form field:

$$\begin{aligned} \mathbf{A}_{(\mu_1 \bar{\sigma}_1 \bar{\theta}_1) \dots (\mu_p \bar{\sigma}_p \bar{\theta}_p)}(\bar{\tau}, \mathbf{X}) &= A_{\mu_1 \dots \mu_p} \\ &\cdot (\mathbf{X}(\bar{\sigma}_1 \bar{\theta}_1)) \delta_{(\bar{\sigma}_1 \bar{\theta}_1)(\bar{\sigma}_2 \bar{\theta}_2)} \dots \delta_{(\bar{\sigma}_{p-1} \bar{\theta}_{p-1})(\bar{\sigma}_p \bar{\theta}_p)} \delta_{(\bar{\sigma}_1 \bar{\theta}_1)(\bar{\sigma}_1 \bar{\theta}_1)}, \\ \text{the others} &= 0, \end{aligned} \quad (12)$$

where

$$\begin{aligned} A_{\mu_1 \dots \mu_p} &= 0 (p : \text{even}), \\ \tilde{F}_8 &= - * \tilde{F}_2, \\ \tilde{F}_6 &= * \tilde{F}_4, \\ A_1 &= C_1, \\ A_3 &= C_3 + B_2 \wedge C_1, \end{aligned} \quad (13)$$

for IIA string background configuration, or

$$\begin{aligned} A_{\mu_1 \dots \mu_p} &= 0 (p : \text{odd}), \\ \tilde{F}_9 &= * \tilde{F}_1, \\ \tilde{F}_7 &= - * \tilde{F}_3, \\ \tilde{F}_5 &= * \tilde{F}_5, \\ A_0 &= C_0, \\ A_2 &= C_2 + B_2 C_0, \\ A_4 &= C_4 + \frac{1}{2} B_2 \wedge C_2 + \frac{1}{2} B_2 \wedge B_2 C_0, \end{aligned} \quad (14)$$

for IIB string background configuration. $G_{\mu_1\mu_2}(x)$ is a symmetric tensor field, $\phi(x)$ is a scalar field, $B_{\mu_1\mu_2}(x)$ is a B field, and $C_{\mu_1\cdots\mu_p}(x)$ are p-form fields on a 10-dimensional space-time. We will show that the IIA and IIB configurations satisfy the equations of motion of the string geometry model in $\alpha' \rightarrow 0$ if and only if the 10-dimensional fields satisfy the equations of motion of the IIA and IIB supergravities.

We remark that the string background configuration has a nontrivial dependence on the worldsheet. The consistent truncation will be ensured due to the relation between the worldsheet dependence of the fields and of the indices of the string geometry fields. For example, see $(\bar{\sigma}_1\bar{\theta}_1)$ dependence on the string background configuration for the metric. In addition, the factor $\delta_{(\bar{\sigma}_1\bar{\theta}_1)(\bar{\sigma}_1\bar{\theta}_1)}$ reflects that the point particle limit is a field theory.

The $\alpha' \rightarrow 0$ limits of the equations of motions (5), (6), (7), and (8) with the IIA string background configuration are equivalent to the equations of motion of the type IIA supergravity

$$S_{\text{IIA}} = \frac{1}{2\kappa_{10}^2} \left(\int d^{10}x \sqrt{-G} \left(e^{-2\phi} \left(R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} |H|^2 \right) - \frac{1}{2} |\tilde{F}_2|^2 - \frac{1}{2} |\tilde{F}_4|^2 \right) - \frac{1}{2} \int B \wedge dC_3 \wedge dC_3 \right). \quad (15)$$

The $\alpha' \rightarrow 0$ limits of the equations of motions (5), (6), (7), and (8) with the IIB string background configuration are also equivalent to the equations of motion of the type IIB supergravity,

$$S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \left(\int d^{10}x \sqrt{-G} \left(e^{-2\phi} \left(R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} |H|^2 \right) - \frac{1}{2} |\tilde{F}_1|^2 - \frac{1}{2} |\tilde{F}_3|^2 - \frac{1}{4} |\tilde{F}_5|^2 \right) - \frac{1}{2} \int C_4 \wedge H \wedge dC_2 \right), \quad (16)$$

and the self-dual condition,

$$\tilde{F}_5 = * \tilde{F}_5. \quad (17)$$

Here, we display a mechanism how the $\alpha' \rightarrow 0$ limit of the equation of motion of $\mathbf{G}_{(\mu\bar{\sigma}_1\bar{\theta}_1)(\nu\bar{\sigma}_2\bar{\theta}_2)}$ with the string background configuration is equivalent to the equation of motion of $G_{\mu\nu}$. By substituting the string background configuration, the left-hand side of the Einstein equation becomes

$$\begin{aligned} \mathbf{R}_{(\mu\bar{\sigma}_1\bar{\theta}_1)(\nu\bar{\sigma}_2\bar{\theta}_2)} - \frac{1}{2} \mathbf{G}_{(\mu\bar{\sigma}_1\bar{\theta}_1)(\nu\bar{\sigma}_2\bar{\theta}_2)} \mathbf{R} &= \delta_{(\bar{\sigma}_1\bar{\theta}_1)(\bar{\sigma}_2\bar{\theta}_2)} \delta_{(\bar{\sigma}_1\bar{\theta}_1)(\bar{\sigma}_1\bar{\theta}_1)} \\ &\cdot \left(R_{\mu\nu} \left(\mathbf{X}_{\bar{D}_T} \left(\bar{\sigma}_1\bar{\theta}_1 \right) \right) - \frac{1}{2} G_{\mu\nu} \left(\mathbf{X}_{\bar{D}_T} \left(\bar{\sigma}_1\bar{\theta}_1 \right) \right) \right) \int d\bar{\sigma} d^2\bar{\theta} \tilde{\mathbf{E}} \\ &\cdot \left(\bar{\sigma}\bar{\theta} \right) \delta_{(\bar{\sigma}\bar{\theta})(\bar{\sigma}\bar{\theta})} R \left(\mathbf{X}_{\bar{D}_T} \left(\bar{\sigma}\bar{\theta} \right) \right). \end{aligned} \quad (18)$$

As one can see in this formula, if an equation of motion includes a trace (in \mathbf{R} in this case), the reduced equation of motion includes an extra summation $\int d\bar{\sigma} d^2\bar{\theta} \tilde{\mathbf{E}}_{(\bar{\sigma}\bar{\theta})} \delta_{(\bar{\sigma}\bar{\theta})(\bar{\sigma}\bar{\theta})}$ against the equation of motion of the string backgrounds. Fortunately, the terms including the extra summation vanish by using the string background configuration and the equation of motion of the scalar, as one can see below. Actually, by substituting the string background configuration into the equation of motion of $\mathbf{G}_{(\mu\bar{\sigma}_1\bar{\theta}_1)(\nu\bar{\sigma}_2\bar{\theta}_2)}$, we obtain

$$\begin{aligned} 0 &= R_{(\mu\bar{\sigma}_1\bar{\theta}_1)(\nu\bar{\sigma}_2\bar{\theta}_2)} - \frac{1}{4} \mathbf{H}_{(\mu\bar{\sigma}_1\bar{\theta}_1)I_1I_2} \mathbf{H}_{(\nu\bar{\sigma}_2\bar{\theta}_2)}^{I_1I_2} + 2\nabla_{(\mu\bar{\sigma}_1\bar{\theta}_1)} \nabla_{(\nu\bar{\sigma}_2\bar{\theta}_2)} \Phi \\ &- \frac{1}{2} \mathbf{G}_{(\mu\bar{\sigma}_1\bar{\theta}_1)(\nu\bar{\sigma}_2\bar{\theta}_2)} \left(\mathbf{R} + 4\nabla_1 \nabla^1 \Phi - 4\partial_1 \Phi \partial^1 \Phi - \frac{1}{2} |\mathbf{H}_3|^2 \right) \\ &- \frac{1}{2} e^{2\phi} \sum_{p=1}^9 \left(\frac{1}{(p-1)!} \tilde{\mathbf{F}}_{(\mu\bar{\sigma}_1\bar{\theta}_1)I_1\cdots I_{p-1}} \tilde{\mathbf{F}}_{(\nu\bar{\sigma}_2\bar{\theta}_2)}^{I_1\cdots I_{p-1}} - \frac{1}{2} \mathbf{G}_{(\mu\bar{\sigma}_1\bar{\theta}_1)(\nu\bar{\sigma}_2\bar{\theta}_2)} |\tilde{\mathbf{F}}_p|^2 \right) \\ &= \delta_{(\bar{\sigma}_1\bar{\theta}_1)(\bar{\sigma}_2\bar{\theta}_2)} \delta_{(\bar{\sigma}_1\bar{\theta}_1)(\bar{\sigma}_1\bar{\theta}_1)} \left(R_{\mu\nu} \left(\mathbf{X}(\bar{\sigma}, \bar{\theta}) \right) - \frac{1}{4} H_{\mu\mu_1\mu_2} H_{\nu\nu_1\nu_2} + 2\nabla_\mu \nabla_\nu \phi \right. \\ &- \frac{1}{2} e^2 \int d\bar{\sigma} d^2\bar{\theta} \tilde{\mathbf{E}}_{(\bar{\sigma}\bar{\theta})} \delta_{(\bar{\sigma}\bar{\theta})(\bar{\sigma}\bar{\theta})} \phi \left(\mathbf{X}(\bar{\sigma}, \bar{\theta}) \right) \sum_{p=1}^9 \frac{1}{(p-1)!} \tilde{\mathbf{F}}_{\mu\mu_1\cdots\mu_{p-1}} \tilde{\mathbf{F}}_{\nu\nu_1\cdots\nu_{p-1}} \\ &\left. - \frac{1}{2} G_{\mu\nu} \int d\bar{\sigma} d^2\bar{\theta} \tilde{\mathbf{E}}_{(\bar{\sigma}\bar{\theta})} \delta_{(\bar{\sigma}\bar{\theta})(\bar{\sigma}\bar{\theta})} \sum_{p=1}^9 |\tilde{\mathbf{F}}_p \left(\mathbf{X}(\bar{\sigma}, \bar{\theta}) \right)|^2 \right). \end{aligned} \quad (19)$$

In the second equality, we have used the equation of motion of scalar (6), and the terms proportional to the equation of motion, which includes a part of the extra summation, vanish. By using the common property in the IIA and IIB string background configurations,

$$\tilde{F}_{10-n} = \pm * \tilde{F}_n, \quad (20)$$

we obtain

$$|\tilde{F}_{10-n}|^2 = -|\tilde{F}_n|^2. \quad (21)$$

If we substitute this relation into (19), the last term, which includes the remaining part of the extra summation, vanishes. Furthermore, if we take

$$\alpha' \rightarrow 0, \quad (22)$$

we obtain

$$e^2 \int d\bar{\sigma} d^2\bar{\theta} \tilde{\mathbf{E}}_{(\bar{\sigma}\bar{\theta})} \delta_{(\bar{\sigma}\bar{\theta})(\bar{\sigma}\bar{\theta})} \phi \left(\mathbf{X}(\bar{\sigma}, \bar{\theta}) \right) \rightarrow e^{2\phi(x)} \quad (23)$$

and thus, (19) gets to be equivalent to

$$\begin{aligned} 0 &= R_{\mu\nu}(x) - \frac{1}{4} H_{\mu\mu_1\mu_2} H_{\nu\nu_1\nu_2} + 2\nabla_\mu \nabla_\nu \phi \\ &- \frac{1}{2} e^{2\phi} \sum_{p=1}^9 \frac{1}{(p-1)!} \tilde{F}_{\mu\mu_1\cdots\mu_{p-1}} \tilde{F}_{\nu\nu_1\cdots\nu_{p-1}}. \end{aligned} \quad (24)$$

This formula is equivalent to the equation of motion of the metric of

$$S' = \frac{1}{2\kappa_{10}^2} \left(\int d^{10}x \sqrt{-G} \left(e^{-2\phi} \left(R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} |H|^2 \right) - \frac{1}{2} \sum_{p=1}^9 |\tilde{F}_p|^2 \right) \right), \quad (25)$$

under the equation of motion of the scalar of this action and (21). The same applies to the other fields. Furthermore, the equations of motion of (25) where the zero mode part of the IIA or IIB string background configurations (13) and (14) are substituted are equivalent to the equations of motion of IIA or IIB supergravities (15) and (16), as one can see a proof in an appendix of [10]. Thus, the IIA and IIB supergravities, which possess the Chern-Simons terms, are derived from the string geometry model, which does not possess the Chern-Simons term. Therefore, we conclude that the string backgrounds can be embedded into the string geometry model in the sense of the consistent truncation in $\alpha' \rightarrow 0$.

In the NS-NS (bosonic) sector of the string geometry theory, the above discussion is valid without taking $\alpha' \rightarrow 0$ limit as in [4]. We can see this from the fact that the last two lines are absent in (19) in the NS-NS sector, for example. This fact will be important to derive the nonlinear sigma model on the NS-NS backgrounds as we mentioned in the introduction. We do not need the special mechanism that the extra summation vanishes as in (19) to derive the supergravities because the extra summation automatically vanishes in $\alpha' \rightarrow 0$ as in (23).

We could not reproduce the R-R sector of the supergravities from string geometry theory without the $\alpha' \rightarrow 0$ limit. This suggests that the string geometry theory cannot reproduce nonlinear sigma models on R-R backgrounds in the NS-R formalism. This is consistent with the fact that it is too difficult to formulate nonlinear sigma models on R-R backgrounds in the NS-R formalism.

4. Equations That Determine a String Background

Because the string background configurations (9), (10), (11), (12), (13), and (14) are stationary with respect to the string geometry time $\bar{\tau}$, the energy of it is defined as

$$\begin{aligned} E &= \int \mathcal{D}E \mathcal{D}X T_{dd} \\ &= \int \mathcal{D}E \mathcal{D}X \left(-2\nabla_d \nabla^d \Phi + \frac{1}{4} \mathbf{H}_{dL_1 L_2} \mathbf{H}_d^{L_1 L_2} \right. \\ &\quad \left. + \frac{1}{2} e^{2\Phi} \sum_{p=1}^9 \frac{1}{(p-1)!} \tilde{\mathbf{F}}_{d_1 \dots d_{p-1}} \tilde{\mathbf{F}}_d^{d_1 \dots d_{p-1}} + G_{dd} \right. \\ &\quad \left. \cdot \left(-2\nabla_I \Phi \nabla^I \Phi + 2\nabla_I \nabla^I \Phi - \frac{1}{4} |H|^2 - \frac{1}{4} e^{2\Phi} \sum_{p=1}^9 |\tilde{\mathbf{F}}_p|^2 \right) \right) \\ &= \int \mathcal{D}X \int \mathcal{D}E \int d\bar{\sigma} d^2 \bar{\theta} \tilde{\mathbf{E}} \delta_{(\bar{\sigma}, \bar{\theta})}(\bar{\sigma}, \bar{\theta}) \\ &\quad \cdot \left(2\nabla_\mu \phi \left(\mathbf{X}(\bar{\sigma}, \bar{\theta}) \right) \nabla^\mu \phi - 2\nabla_\mu \nabla^\mu \phi + \frac{1}{4} |H|^2 \right) \\ &= \int \mathcal{D}X \int \mathcal{D}e \int d\bar{\sigma} \bar{e}(\bar{\sigma}) \delta_{\bar{\sigma}\bar{\sigma}} \\ &\quad \cdot \left(2\nabla_\mu \phi \left(X(\bar{\sigma}) \right) \nabla^\mu \phi - 2\nabla_\mu \nabla^\mu \phi + \frac{1}{4} |H|^2 \right) \\ &= \int d^{10}x \sqrt{-G(x)} \left(2\nabla_\mu \phi(x) \nabla^\mu \phi - 2\nabla_\mu \nabla^\mu \phi + \frac{1}{4} |H|^2 \right). \end{aligned} \quad (26)$$

On the first and second lines in the above formula, we have substituted (9), (10), (11), (12), (13), and (14) and obtained the third line. On the third line, because $\delta_{(\bar{\sigma}, \bar{\theta})}(\bar{\sigma}, \bar{\theta}) \propto \bar{\theta} \bar{\theta}$, if one integrates $\bar{\theta}$ and $\bar{\theta}$, only the bosonic leading terms remain and we obtained the fourth line. On the fourth line, we have regularized the integral over the embedding function as

$$\begin{aligned} \int \mathcal{D}X &= \prod_{j=1}^N \int d^{10}x_j \sqrt{-G(x_j)}, \\ \int d^{10}x_j \sqrt{-G(x_j)} &= 1, \\ \int \mathcal{D}e \int d\bar{\sigma} \bar{e}(\bar{\sigma}) \delta_{\bar{\sigma}\bar{\sigma}} &= \frac{1}{N} \sum_{i=1}^N, \end{aligned} \quad (27)$$

and obtained the fifth line.

Therefore, in string geometry theory, a string background is determined by minimizing the energy (26) of the solutions to the IIA or IIB equations of motions. In other words, in the IIA case, by using the method of Lagrange multiplier, the equations that determine string backgrounds are obtained by differentiating

$$\begin{aligned} \tilde{E} &= E + \int d^{10}x \sqrt{-G(x)} \left(\lambda_G^{\mu\nu}(x) f_{\mu\nu}^G(x) + \lambda_\phi(x) f^\phi(x) \right. \\ &\quad \left. + \lambda_B^{\mu\nu}(x) f_{\mu\nu}^B(x) + \lambda_{C_1}^\mu(x) f_\mu^{C_1}(x) + \lambda_{C_3}^{\mu_1 \mu_2 \mu_3}(x) f_{\mu_1 \mu_2 \mu_3}^{C_3}(x) \right), \end{aligned} \quad (28)$$

with respect to the IIA string backgrounds $G_{\mu\nu}(x)$, $\phi(x)$, $B_{\mu\nu}(x)$, $C_1(x)$, and $C_3(x)$ and the Lagrange multipliers $\lambda_G^{\mu\nu}(x)$, $\lambda_\phi(x)$, $\lambda_B^{\mu\nu}(x)$, $\lambda_{C_1}^\mu(x)$, and $\lambda_{C_3}^{\mu_1 \mu_2 \mu_3}(x)$, where $f_G^{\mu\nu}(x) = 0$, $f^\phi(x) = 0$, $f_{\mu\nu}^B(x) = 0$, $f_\mu^{C_1}(x) = 0$, and $f_{\mu_1 \mu_2 \mu_3}^{C_3}(x) = 0$ represent the IIA equations of motions, respectively. The same applies to the IIB case.

5. Consistent Truncations to Heterotic and Type I Supergravities

Let us generalize the model (2) and consider

$$\begin{aligned} S &= \int \mathcal{D}E \mathcal{D}\bar{\tau} \mathcal{D}X_{\bar{D}} \sqrt{G} \left(e^{-2\Phi} \left(\mathbf{R} + 4\nabla_I \Phi \nabla^I \Phi - \frac{1}{2} |\tilde{\mathbf{H}}|^2 \right. \right. \\ &\quad \left. \left. - \text{tr}(|\mathbf{F}|^2) \right) - \frac{1}{2} \sum_{p=1}^9 |\tilde{\mathbf{F}}_p|^2 \right), \end{aligned} \quad (29)$$

where $\tilde{\mathbf{H}} = d\mathbf{B} - \omega_3$, $\omega_3 = \text{tr}(\mathbf{A} \wedge d\mathbf{A} - (2i/3)\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A})$, and \mathbf{A} is a $N \times N$ Hermitian gauge field, whose field strength is given by \mathbf{F} . \mathbf{A} can be consistently truncated to 0 in this model, and model (2) is obtained. Thus, the IIA and IIB supergravities are derived by consistent truncations of model (29).

On the other hand, the heterotic supergravity

$$S_{het} = \frac{1}{\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left(R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} |\tilde{H}|^2 - \frac{\kappa_{10}^2}{g_{10}^2} \text{tr}(|F|^2) \right), \quad (30)$$

where $\tilde{H} = dB - (\kappa_{10}^2/g_{10}^2)\omega_3$ is derived by consistently truncating model (29) as in Section 3. Here, we have introduced the constants κ_{10} and g_{10} by shifting the dilaton and rescaling in the 10 dimensions. The gauge field can be truncated to the adjoint representation of $\text{SO}(32)$ and $E_8 \times E_8$.

The heterotic supergravity can be shown to be equivalent to the type I supergravity

$$S_I = \frac{1}{\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(e^{-2\phi} (R + 4\nabla_\mu \phi \nabla^\mu \phi) - \frac{1}{2} |\tilde{F}_3|^2 \right) - \frac{1}{g_{10}^2} \int d^{10}x \sqrt{-G} e^{-\phi} \text{tr}(|F|^2), \quad (31)$$

where $\tilde{F}_3 = dC_2 - (\kappa_{10}^2/g_{10}^2)\omega_3$, by a transformation: $B \rightarrow C_2$, $G_{\mu\nu} \rightarrow e^{-\phi} G_{\mu\nu}$, and $\phi \rightarrow -\phi$. The gauge field can be truncated to the adjoint representation of $\text{SO}(32)$. As a result, the supergravities on all the ten-dimensional vacua in string theory are derived from the single model (29).

The gauge fields in the string backgrounds are not only able to be truncated to but also chosen automatically as $\text{SO}(32)$ or $E_8 \times E_8$ by consistency of the perturbative strings obtained from the Newtonian limit of the fluctuations around the string background configurations in string geometry theory. Actually, path integrals of nonlinear sigma models on the string backgrounds will be derived from the Newtonian limit of the fluctuations. Actually, the path-integrals of the topological sigma models on Kähler manifolds are derived from Newtonian limits of fluctuations around a class of backgrounds in the topological sector of string geometry theory [11]. The contributions of the background gauge fields to the path integrals are

$$\exp \left(\frac{1}{4\pi} \int d^2z \left(\lambda^A (\partial_z \lambda^A - i A_\mu^{AB}(X) \partial_z X^\mu \lambda^B) + \frac{i}{2} F_{\mu\nu}^{AB}(X) \lambda^A \lambda^B \psi^\mu \psi^\nu \right) \right), \quad (32)$$

in the heterotic theory, and

$$\text{tr} P \exp \left(i \int d\sigma \frac{dX^\mu}{d\tau} A_\mu(X) \right), \quad (33)$$

in the type I theory. If the gauge fields are small, these contributions are insertions of the vertex operators representing gauge fields,

$$a_\mu^{m\prime m} t_{AB}^m \int d^2z \lambda^A \lambda^B \left(i \partial_z X^\mu + \frac{1}{2} k_\nu \psi^\nu \psi^\mu \right) e^{ikX}, \quad (34)$$

in the heterotic theory, and

$$a_\mu^{m\prime m} t_{AB}^m \int d\sigma \frac{dX^\mu}{d\tau} e^{ikX}, \quad (35)$$

in the type I theory. On the other hand, the path integrals of the heterotic and type I perturbative strings on the flat background are derived from string geometry theory [1], and the vertex operators (34) and (35) can be constructed. The consistency of the perturbative strings chooses only $\text{SO}(32)$ and $E_8 \times E_8$ in the heterotic case and $\text{SO}(32)$ in the type I case. As a result, the background gauge fields must be chosen as the same, because (34) and (35) are obtained from (32) and (33), respectively.

6. Conclusion and Discussion

In this paper, we have shown that the supergravities on all the ten-dimensional vacua in superstring theory are derived from a single string geometry model by consistent truncations. That is, arbitrary configurations of all the type IIA, IIB, $\text{SO}(32)$ type I, and $\text{SO}(32)$ and $E_8 \times E_8$ heterotic superstring backgrounds are embedded in configurations of fields of a single string geometry model. In particular, the single action of the string geometry model is consistently truncated to the supergravity actions by applying the corresponding superstring background configurations to the model and taking $\alpha' \rightarrow 0$. That is, the $\alpha' \rightarrow 0$ limits of the equations of motions with those string background configurations are equivalent to the equations of motion of the corresponding supergravities. This means that classical dynamics of all the type IIA, IIB, $\text{SO}(32)$ type I, and $\text{SO}(32)$ and $E_8 \times E_8$ heterotic superstring backgrounds are described as a part of classical dynamics in string geometry theory. This fact supports the conjecture that string geometry theory is a nonperturbative formulation of string theory.

The above results are consistent with the fact that one can derive both the type IIA and IIB perturbative string theories on the flat background from a single string geometry model as shown in [1]: in this case, the configurations of the backgrounds are formally the same, whereas the charts that cover the backgrounds are different (IIA and IIB charts). These results strongly indicate that string geometry theory does not depend on string backgrounds. Here, we comment on supersymmetry. Although the single action of the ten-dimensional gravity (25) possesses both fields in type IIA and IIB supergravities, namely, all the R-R fields with odd and even degrees, the action cannot be generalized to be supersymmetric even if fermions are coupled. However, in the string geometry theory, an arbitrary action can be generalized to be supersymmetric, as one can see in [1]. Actually, the string geometry models (2) and (29) are supersymmetric, although they possess the tensor fields that include all the R-R fields with odd and even degrees.

Furthermore, we have defined an energy of the superstring background configurations, because they are stationary with respect to the string geometry time $\bar{\tau}$. Thus, a superstring background can be determined by minimizing the energy of the solutions to the equations of motions of

the superstring backgrounds. Therefore, we conclude that string geometry theory includes a nonperturbative effect that determines a superstring background.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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