Research Article

Approximate Solutions, Thermal Properties, and Superstatistics Solutions to Schrödinger Equation

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In this work, we apply the parametric Nikiforov-Uvarov method to obtain eigensolutions and total normalized wave function of Schrödinger equation expressed in terms of Jacobi polynomial using Coulomb plus Screened Exponential Hyperbolic Potential (CPSEHP), where we obtained the probability density plots for the proposed potential for various orbital angular quantum number, as well as some special cases (Hellmann and Yukawa potential). The proposed potential is best suitable for smaller values of the screening parameter α . The resulting energy eigenvalue is presented in a close form and extended to study thermal properties and superstatistics expressed in terms of partition function (Z) and other thermodynamic properties such as vibrational mean energy (U), vibrational specific heat capacity (C), vibrational entropy (S), and vibrational free energy (F). Using the resulting energy equation and with the help of Matlab software, the numerical bound state solutions were obtained for various values of the screening parameter (α) as well as different expectation values via Hellmann-Feynman Theorem (HFT). The trend of the partition function and other thermodynamic properties obtained for both thermal properties and superstatistics were in excellent agreement with the existing literatures. Due to the analytical mathematical complexities, the superstatistics and thermal properties were evaluated using Mathematica 10.0 version software. The proposed potential model reduces to Hellmann potential, Yukawa potential, Screened Hyperbolic potential, and Coulomb potential as special cases.

1. Introduction

The approximate analytical solutions of one-dimensional radial Schrödinger equation with a multiple potential function have been studied using a suitable approximation scheme to the centrifugal term within the frame work of the parametric Nikiforov-Uvarov method [1]. The solutions to the wave equations in quantum mechanics and applied physics play a crucial role in understanding the importance of physical systems [2]. The two most important parts in studying Schrödinger equations are the total wave function

and energy eigenvalues [3]. The analytic solutions of wave equations for some physical potentials are possible for $l = 0$. For $l \neq 0$, special approximation schemes like the Greene-Aldrich and Pekeris approximations are employed to deal with the centrifugal barrier in order to obtain approximate bound state solutions [4–6]. The Greene-Aldrich approximation scheme is mostly applicable for short range potentials [7]. Eigensolutions for both relativistic and nonrelativistic wave equations have been studied with different methods which include the following: Exact quantisation, WKB, Nikiforov-Uvarov method (NU), Laplace transform technique,

asymptotic iteration method, proper quantisation, supersymmetric quantum mechanics approach, vibrational approach, formula method, factorisation method, and Shifted $1/N$ -expansion method [8–13]. Bound state solutions obtained from the Schrödinger equation has practical applications in investigating tunnelling rate of quantum mechanical systems [14] and mass spectra of quarkonia systems [15–19]. Among other goals achieved in this research article is to apply the Hellmann-Feynman Theorem (HFT) to eigenequation of the Schrödinger wave equations to obtain expectation values of $\langle r^{-1} \rangle_{nl}$, $\langle r^{-2} \rangle_{nl}$, $\langle T \rangle_{nl}$, and $\langle p^2 \rangle_{nl}$ analytically. The Hellmann-Feynman Theorem gives an insight about chemical bonding and other forces existing among atoms of molecules [20–25]. To engage HFT in calculating the expectation values, one needs to promote the fixed parameter which appears in the Hamiltonian to be a continuous variable in order to ease the mathematical purpose of taking the derivative [26]. Similarly, the application of Hellmann-Feynman Theorem provides a less mathematical approach of obtaining expectation values of a quantum mechanical systems [27, 28]. Some of the potential models considered within the framework of relativistic and nonrelativistic wave equations are Hulthen-Yukawa Inversely quadratic potential [29], noncentral Inversely quadratic potential [30], Modified Hylleraas potential [31], Yukawa, Hulthen, Eckart, Deng-Fan, Pseudoharmonic, Kratzer, Woods-Saxon, double ring shape, Coulomb, Tietz-Wei, Tietz-Hua, Deng-Fan, Manning-Rosen, trigonometric Rosen-Morse, hyperbolic scalar, and vector potential and exponential type potentials among others [32–50]. Coulomb, hyperbolic, and screened exponential type potentials have been of interest to researchers in recent times because of their enormous applications in both chemical and physical sciences. In view of this, Parmar [51] studied ultrageneralized exponential hyperbolic potential where he obtained energy eigenvalues, unnormalized wave function and the partition function. This potential reduces to Yukawa potential, Screened cosine Kratzer potential, Manning-Rosen potential, Hulthen plus Inversely quadratic exponential Mie-type potential, and many others. Diaf et al. [52], in their studies, obtained eigensolutions to the Schrödinger equation with trigonometric Inversely quadratic plus Coulombic hyperbolic potential where they obtained energy eigenvalue and normalized wave function using the Nikiforov-Uvarov method. Onate [53] examined bound state solutions of the Schrödinger equation with second Pöschl-Teller-like potential where he obtained vibrational partition function, mean energy, vibrational specific heat capacity, and mean free energy. In that work, the Pöschl-Teller-like potential was expressed in a hyperbolic form. The practical application of energy eigenvalue of Schrödinger equation in investigating the partition function, thermodynamic properties, and superstatistics arouses the interest of many researchers. Recently, Okon et al. [54] obtained the thermodynamic properties and bound state solutions of the Schrödinger equation using Mobius square plus screened Kratzer potential for two diatomic systems (carbon(II) oxide and scandium fluoride) within the framework of the Nikiforov-Uvarov method. Their results were in agreement to semiclassical WKB among others. They presented energy eigenvalue in a close form in order to obtain partition function and other thermodynamic properties. Omugbe et al. [55] recently studied the unified treatment of the nonrela-

tivistic bound state solutions, thermodynamic properties, and expectation values of exponential-type potentials where they obtained the thermodynamic properties within the framework of semiclassical WKB approach. The authors studied the special cases of the potential as Eckart, Manning-Rosen, and Hulthen potentials. Besides, Oyewumi et al. [56] studied the thermodynamic properties and the approximate solutions of the Schrödinger equation with shifted Deng-Fan potential model within the framework of asymptotic Iteration method where they apply Pekeris-type approximation to centrifugal term to obtain rotational-vibrational energy eigenvalues for selected diatomic systems. A lot of researches have been carried out by Ikot et al. These can be seen in Refs. [57–60]. Also, Boumali and Hassanabadi [61] studied thermal properties of a two-dimensional Dirac oscillator under an external magnetic field where they obtained relativistic spin-1/2 fermions subject to Dirac oscillator coupling and a constant magnetic field in both commutative and noncommutative spaces.

In this work, we propose a novel potential called Coulomb plus Screened Exponential Hyperbolic Potential to study bound state solutions, expectation values, superstatistics, and thermal properties within the framework of the parametric Nikiforov-Uvarov method [62]. This article is divided into 9 sections. The introduction is given in Section 1. The parametric Nikiforov-Uvarov method is presented in Section 2. The solutions of the radial Schrödinger equation are presented in Section 3. The application of Hellmann-Feynman Theorem to obtain expectation values is presented in Section 4. The thermodynamic properties and superstatistics formulations are presented in Sections 5 and 6, respectively. Numerical results and discussion are presented Sections 7 and 8, respectively, and the article is concluded in Section 9.

The propose Coulomb plus Screened Hyperbolic Exponential Potential (CPSHEP) is given as

$$V(r) = -\frac{v_1}{r} + \left(\frac{B}{r} - \frac{v_2 \cosh \alpha}{r^2} \right) e^{-\alpha r}, \quad (1)$$

where v_1 and v_2 are the potential depths, B is a real constant parameter, and α is the adjustable screening parameter. The Pekeris-like approximation to the centrifugal term is given as

$$\frac{1}{r^2} = \frac{\alpha^2}{(1 - e^{-\alpha r})^2} \Rightarrow \frac{1}{r} = \frac{\alpha}{(1 - e^{-\alpha r})}. \quad (2)$$

The graph of Pekeris approximation to centrifugal term is given in Figure 1.

2. Parametric Nikiforov-Uvarov (NU) Method

The NU method is based on reducing second order linear differential equation to a generalized equation of hypergeometric type and provides exact solutions in terms of special orthogonal functions like Jacobi and Laguerre as well as corresponding energy eigenvalues [63–70]. The reference equation for parametric NU method according to Tezcan and Sever [71] is given as

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1 - c_3 s)} \Psi'(s) + \frac{1}{s^2(1 - c_3 s)^2} [-\Omega_1 s^2 + \Omega_2 s - \Omega_3] \Psi(s) = 0. \quad (3)$$

The condition for energy equation is given as [70].

$$c_2 n - (2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3 \sqrt{c_8}) + n(n - 1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0. \quad (4)$$

The total wave function is given as

$$\psi_{nl}(s) = N_{nl} s^{c_{12}} (1 - c_3 s)^{-c_{12} - (c_{13}/c_3)} P_n^{(c_{10}-1, (c_{11}/c_3) - c_{10}-1)}(1 - 2c_3 s). \quad (5)$$

The parametric constants are obtained as follows:

$$\begin{aligned} c_1 = c_2 = c_3 = 1, c_4 = \frac{1}{2}(1 - c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = c_5^2 + \Omega_1, \\ c_7 = 2c_4 c_5 - \Omega_2, c_8 = c_4^2 + \Omega_3, c_9 = c_3 c_7 + c_3^2 c_8 + c_6, c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \\ c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3 \sqrt{c_8}), c_{12} = c_4 + \sqrt{c_8}, c_{13} = c_5 - (\sqrt{c_9} + c_3 \sqrt{c_8}). \end{aligned} \quad (6)$$

3. The Radial Solution of Schrödinger Wave Equation

The radial Schrödinger wave equation with the centrifugal term is given as

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E - V(r) - \frac{\hbar^2 l(l+1)}{2\mu r^2} \right] R(r) = 0. \quad (7)$$

Equation (7) can only be solved analytically to obtain exact solution if the angular orbital quantum number $l = 0$. However, for $l > 0$, equation (7) can only be solve by using the approximations in (2) to the centrifugal term. Substituting equation (1) into (7) gives

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} + \frac{\nu_1}{r} - \frac{B e^{-\alpha r}}{r} + \frac{\nu_2 e^{-\alpha r} \cosh \alpha}{r^2} - \frac{\hbar^2 l(l+1)}{2\mu r^2} \right] R(r) = 0. \quad (8)$$

By substituting equation (2) into (8) gives the following equation:

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} + \frac{\nu_1 \alpha}{(1 - e^{-\alpha r})} - \frac{B \alpha e^{-\alpha r}}{(1 - e^{-\alpha r})} + \frac{\nu_2 \alpha^2 e^{-\alpha r} \cosh \alpha}{(1 - e^{-\alpha r})^2} - \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu (1 - e^{-\alpha r})^2} \right] R(r) = 0. \quad (9)$$

By defining $s = e^{-\alpha r}$ and with some simple algebraic simplification, equation (9) can be presented in the form

FIGURE 1: The graph of Pekeris approximation for various values of α .

$$\frac{d^2 R(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dR}{ds} + \frac{1}{s^2(1-s)^2} \left\{ \begin{aligned} &-(\varepsilon^2 - \chi_1)s^2 \\ &+(2\varepsilon^2 - \delta^2 - \chi_1 + \chi_2)s - (\varepsilon^2 - \delta^2 + l(l+1)) \end{aligned} \right\} R(s) = 0, \quad (10)$$

where

$$\varepsilon^2 = -\frac{2\mu E_{nl}}{\hbar^2 \alpha^2}, \delta^2 = \frac{2\mu \nu_1}{\hbar^2 \alpha}, \chi_1 = \frac{2\mu B}{\hbar^2 \alpha}, \chi_2 = \frac{2\mu \nu_2 \cosh \alpha}{\hbar^2}. \quad (11)$$

Comparing equation (10) to (3), the following polynomials were obtain:

$$\Omega_1 = (\varepsilon^2 - \chi_1), \Omega_2 = (2\varepsilon^2 - \delta^2 - \chi_1 + \chi_2), \Omega_3 = (\varepsilon^2 - \delta^2 + l(l+1)). \quad (12)$$

Using equation (6), other parametric constants are obtained as follows:

$$\begin{aligned} c_1 = c_2 = c_3 = 1; c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \varepsilon^2 - \chi_1, c_7 = -2\varepsilon^2 + \delta^2 + \chi_1 - \chi_2, \\ c_8 = \varepsilon^2 - \delta^2 + l(l+1), c_9 = \frac{1}{4} - \chi_2 + l(l+1), c_{10} = 1 + 2\sqrt{\varepsilon^2 - \delta^2 + l(l+1)}, \\ c_{11} = 2 + \sqrt{1 - 4\chi_2 + 4l(l+1)} + 2\sqrt{\varepsilon^2 - \delta^2 + l(l+1)}, c_{12} = \sqrt{\varepsilon^2 - \delta^2 + l(l+1)}, \\ c_{13} = -\frac{1}{2} - \left[\frac{1}{2} \sqrt{1 - 4\chi_2 + 4l(l+1)} + \sqrt{\varepsilon^2 - \delta^2 + l(l+1)} \right]. \end{aligned} \quad (13)$$

Using equations (4), (12), and (13) with much algebraic simplification, the energy eigenvalue for the proposed potential is given as

$$E_{nl} = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} - v_1 \alpha + \frac{\hbar^2 \alpha^2}{2\mu} \cdot \left\{ \frac{(n^2 + n + (1/2)) + (n + (1/2)) \sqrt{1 + 4l(l+1) - (8v_2\mu \cosh \alpha/\hbar^2) + (2\mu B/\hbar^2 \alpha) - (2\mu v_1/\hbar^2 \alpha) - (2v_2\mu \cosh \alpha/\hbar^2) + 2l(l+1)}}{(2n+1) + \sqrt{1 + 4l(l+1) - (2v_2\mu \cosh \alpha/\hbar^2)}} \right\}^2 \quad (14)$$

Using equation (5), the total unnormalized wave function is given as

$$\Psi_{nl}(s) = N_{nl} s^{\sqrt{(e^2 - \delta^2 + l(l+1))}} (1-s)^{(1/2) + (1/2)\sqrt{1 + 4l(l+1) - 4\chi_2}} P_n^{\left[\left(2\sqrt{(e^2 - \delta^2 + l(l+1))} \right), \left(\sqrt{1 + 4l(l+1) - 4\chi_2} \right) \right]} (1-2s). \quad (15)$$

To obtain the normalization constant of equation (15), we employ the normalization condition

$$\int_0^\infty |R_{nl}(r)|^2 dr = 1 \Rightarrow \int_0^\infty [N_{nl} s^\beta (1-s)^\eta P_n^{(2\beta, 2\eta-1)}(1-2s)]^2 ds = 1, \quad (16)$$

where

$$\beta = \sqrt{(e^2 - \delta^2 + l(l+1))}, \quad (17)$$

$$\eta = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4l(l+1) - 4\chi_2}. \quad (18)$$

The wave function is assumed to be in bound at $r \in (0, \infty)$ and $s = e^{-\alpha r} \in (1, 0)$.

Equation (15) reduces to

$$-\frac{N_{nl}^2}{\alpha} \int_1^0 s^{2\beta} (1-s)^{2\eta} [P_n^{(2\beta, 2\eta-1)}(1-2s)]^2 \frac{ds}{s} = 1. \quad (19)$$

Let $z = (1-2s)$ such that the boundary of integration of equation (19) changes from $s \in (1, 0)$ to $z \in (-1, 1)$. Then, equation (19) reduces to

$$\frac{N_{nl}^2}{2\alpha} \int_{-1}^1 \left(\frac{1-z}{2} \right)^{2\beta-1} \left(\frac{1+z}{2} \right)^{2\eta} [P_n^{(2\beta, 2\eta-1)}(z)]^2 dz = 1. \quad (20)$$

Using the standard integral,

$$\int_{-1}^1 \left(\frac{1-w}{2} \right)^x \left(\frac{1+w}{2} \right)^y [P_n^{(x, y-1)}(w)]^2 dw = \frac{2^{x+y+1} \Gamma(x+n+1) \Gamma(y+n+1)}{n! \Gamma(x+y+n+1) \Gamma(x+y+2n+1)}. \quad (21)$$

Let $z = w, x = 2\beta - 1, y = 2\eta$. Then, using equation (20),

the normalization constant can be obtained as

$$N_{nl} = \sqrt{\frac{2\alpha(n!) \Gamma(2\beta + 2\eta + n) \Gamma(2\beta + 2\eta + 2n)}{2^{(2\beta+2\eta)} \Gamma(2\beta + n) \Gamma(2\eta + n + 1)}}. \quad (22)$$

Hence, the total normalized wave function is given as

$$R_{nl}(s) = \sqrt{\frac{2\alpha(n!) \Gamma(2\beta + 2\eta + n) \Gamma(2\beta + 2\eta + 2n)}{2^{(2\beta+2\eta)} \Gamma(2\beta + n) \Gamma(2\eta + n + 1)}} s^\beta (1-s)^\eta P_n^{(2\beta, 2\eta-1)}(1-2s). \quad (23)$$

4. Expectation Values Using Hellmann-Feynman Theorem

In this section, some expectation values are obtain using Hellmann-Feynman Theorem (HFT). According to the Hellmann-Feynman Theorem, the Hamiltonian H for a particular quantum mechanical system is expressed as a function of some parameters q . Let $E(q)$ and $\Psi(q)$ be the eigenvalues and eigenfunction of the Hamiltonian. Then,

$$\frac{\partial E_{nl}}{\partial q} = \left\langle \Psi(q) \left| \frac{\partial H(q)}{\partial q} \right| \Psi(q) \right\rangle. \quad (24)$$

For the purpose of clarity, the Hamiltonian for the propose potential using HFT is

$$H = -\frac{\hbar^2}{2\mu} \frac{d}{dr^2} + \frac{\hbar^2 l(l+1)}{2\mu r^2} - \frac{v_1}{r} + \left(\frac{B}{r} - \frac{v_2 \cosh \alpha}{r^2} \right) e^{-\alpha r}. \quad (25)$$

4.1. Expectation Value for $\langle r^{-2} \rangle_{nl}$. To obtain the expectation value for $\langle r^{-2} \rangle_{nl}$, we set $q = l$, to have

$$\langle r^{-2} \rangle_{nl} = \alpha^2 - \frac{\alpha^2(2l+1)}{2} \left(\frac{1}{2} (2l+1) \left(\sqrt{\left(l + \frac{1}{2} \right)^2 - \frac{2v_2\mu \cosh \alpha}{\hbar^2}} \right)^{-1/2} + Q_6 \right) Q_7, \quad (26)$$

where

$$Q_6 = \frac{(2l+1)\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)}\right) - ((2\mu B/\hbar^2\alpha) - (2\mu\nu_1/\hbar^2\alpha) + l(l+1))(1/2)(2l+1)\left(\sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)}\right)^{-1/2}}{\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)}\right)^2},$$

$$Q_7 = \left\{ \left(n + (1/2) + \sqrt{(l + (1/2))^2 - \frac{2v_2\mu \cosh \alpha}{\hbar^2}} \right) + \frac{((2\mu B/\hbar^2\alpha) - (2\mu\nu_1/\hbar^2\alpha) + l(l+1))}{\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)}\right)} \right\}. \quad (27)$$

4.2. *Expectation Value for $\langle r^{-1} \rangle_{nl}$.* Setting $q = B$, then, the expectation value for $\langle r^{-1} \rangle_{nl}$ becomes

$$\langle r^{-1} \rangle = - \frac{\hbar^2 \alpha^2 Q_7 \times e^{\alpha r}}{4\mu} \cdot \left[\frac{(2\mu/\hbar^2 \alpha^2) \left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)} \right)}{\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)} \right)^2} \right]. \quad (28)$$

4.3. *Expectation Value for $\langle T \rangle_{nl}$.* Setting $q = \mu$, then,

$$\langle T \rangle_{nl} = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} - \frac{\hbar^2 \alpha^2 Q_2}{8\mu} + \frac{\hbar^2 \alpha^2 Q_7}{4} \cdot \left[- \frac{v_2 \cosh \alpha}{\hbar^2 \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)}} + \frac{((2\mu B/\hbar^2\alpha) - (2\mu\nu_1/\hbar^2\alpha) + l(l+1))}{\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)} \right)^2} \right]. \quad (29)$$

4.4. *Expectation Value for $\langle p^2 \rangle_{nl}$.* The relationship between T and p^2 is given as $T = p^2/2\mu$, therefore

$$\langle p^2 \rangle_{nl} = \hbar^2 \alpha^2 l(l+1) - \frac{\hbar^2 \alpha^2 Q_2}{4} + \frac{\hbar^2 \alpha^2 \mu Q_7}{2} \cdot \left[- \frac{v_2 \cosh \alpha}{\hbar^2 \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)}} + \frac{((2\mu B/\hbar^2\alpha) - (2\mu\nu_1/\hbar^2\alpha) + l(l+1))}{\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)} \right)^2} \right]. \quad (30)$$

5. Thermodynamic Properties

In this section, we present the thermodynamic properties for the potential model. The thermodynamic properties of quantum systems can be obtained from the exact partition function given by

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta E_n}, \quad (31)$$

where λ is an upper bound of the vibrational quantum number obtained from the numerical solution of $dE_n/dn = 0$, given as $\lambda = -\delta + \sqrt{Q_3}$, $\beta = 1/kT$, where k and T are Boltz-

mann constant and absolute temperature, respectively. In the classical limit, the summation in equation (31) can be replaced with an integral:

$$Z(\beta) = \int_0^{\lambda} e^{-\beta E_n} dn. \quad (32)$$

In order to obtain the partition function, the energy equation (14) can be presented in a close and compact form as

$$E_{nl} = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} - v_1 \alpha - \frac{\hbar^2 \alpha^2}{8\mu} \cdot \left\{ \left(n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2} \right)^2 - \frac{2v_2\mu \cosh \alpha}{\hbar^2}} \right) + \frac{((2\mu B/\hbar^2\alpha) - (2\mu\nu_1/\hbar^2\alpha) + l(l+1))}{\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2\mu \cosh \alpha/\hbar^2)} \right)} \right\}^2. \quad (33)$$

Equation (33) can further be simplified to

$$E_{nl} = Q_1 - Q_2 \left\{ (n + \delta) + \frac{Q_3}{(n + \delta)} \right\}^2, \quad (34)$$

where

$$Q_1 = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} - v_1 \alpha, \quad Q_2 = \frac{\hbar^2 \alpha^2}{8\mu},$$

$$Q_3 = \left(\frac{2\mu B}{\hbar^2 \alpha} - \frac{2\mu\nu_1}{\hbar^2 \alpha} + l(l+1) \right), \quad (35)$$

$$\delta = \frac{1}{2} + \sqrt{\left(l + \frac{1}{2} \right)^2 - \frac{2v_2\mu \cosh \alpha}{\hbar^2}},$$

and equation (34) can be represented in the form

$$E_{nl} = Q_1 - Q_2 \left[\rho + \frac{Q_3}{\rho} \right]^2 \Rightarrow - \left[Q_2 \rho^2 + \frac{Q_2 Q_3^2}{\rho^2} \right] - (2Q_2 Q_3 - Q_1), \quad (36)$$

where

$$\rho = n + \delta. \quad (37)$$

Using Mathematica 10.0 version, the partition function of equation (38) is given as

(i) Partition function is obtained as follows:

$$Z(\beta) = \frac{e^{-\beta Q_1 + 2\beta Q_2 Q_3 - 2N\sqrt{-\beta Q_2}} \sqrt{\pi} \{2 \operatorname{erf}(N_0 - (N/\delta))\}}{4\sqrt{-\beta Q_2}}. \quad (39)$$

Substituting equation (36) into equation (32) taking note of changes in the integration boundaries using equation (37) gives the partition function as

$$Z(\beta) = e^{\beta(2Q_2 Q_3 - Q_1)} \int_{\delta}^{\lambda+\delta} e^{\beta(Q_2 \rho^2 + (Q_2 Q_3^2 / \rho^2))} d\rho. \quad (38)$$

(ii) Vibrational mean energy is given as

$$U(\beta) = -\frac{\partial \ln Z(\beta)}{\partial \beta} = \left\{ \frac{2Q_2 e^{2N\sqrt{-\beta Q_2}} \left[-e^{(\beta Q_2 (\delta^4 + Q_3^2) / \delta^2)} \delta + e^{(\beta Q_2 ((\delta+\lambda)^4 + Q_3^2) / (\delta+\lambda)^2)} \lambda + 2Q_2 e^{2N\sqrt{-\beta Q_2}} \sqrt{\pi} \operatorname{erf}(N(N_2 - N_0 - (N/\delta))) \right]}{\sqrt{\pi} \left\{ \operatorname{erf}(N_0 - (N/\delta)) - e^{4N\sqrt{-\beta Q_2}} \operatorname{erf}(N_0 + (N/\delta)) + \operatorname{erf}(N_1 - (N/(\lambda + \delta))) + e^{4N\sqrt{-\beta Q_2}} \operatorname{erf}(N_1 + (N/(\lambda + \delta))) \right\}} \right\}. \quad (40)$$

(iii) Vibrational entropy is given as

$$S(\beta) = k \ln Z(\beta) - k\beta \frac{\partial \ln Z(\beta)}{\partial \beta} = k \ln \frac{1}{4\sqrt{-\beta Q_2}} e^{-\beta Q_1 + 2\beta Q_2 Q_3 - 2N\sqrt{-\beta Q_2}} \sqrt{\pi} \left\{ \operatorname{erf}\left(N_0 - \frac{N}{\delta}\right) + e^{4N\sqrt{-\beta Q_2}} \operatorname{erf}\left(N_1 + \frac{N}{(\lambda + \delta)} - N_0 - \frac{N}{\delta}\right) + \operatorname{erf}\left(N_1 - \frac{N}{(\lambda + \delta)}\right) \right\} - k\beta \left\{ \frac{2Q_2 e^{2\sqrt{-\beta Q_2} \sqrt{-\beta Q_2 Q_3^2}} \left[-e^{(\beta Q_2 (\delta^4 + Q_3^2) / \delta^2)} \delta + e^{(\beta Q_2 ((\delta+\lambda)^4 + Q_3^2) / (\delta+\lambda)^2)} \lambda + 2Q_2 e^{2N\sqrt{-\beta Q_2}} \sqrt{\pi} \operatorname{erf}(N(N_2 - N_0 - (N/\delta))) \right]}{\sqrt{\pi} \{2 \operatorname{erf}(N_0 - (N/\delta))\}} \right\}. \quad (41)$$

(iv) Vibrational free energy is given as

$$F(\beta) = -\frac{1}{\beta} \ln Z(\beta) = -\frac{1}{\beta} \ln \left(\frac{e^{-\beta Q_1 + 2\beta Q_2 Q_3 - 2N\sqrt{-\beta Q_2}} \sqrt{\pi}}{4\sqrt{-\beta Q_2}} \left\{ \operatorname{erf}\left(N_1 - \frac{N}{(\lambda + \delta)} + N_0 - \frac{N}{\delta}\right) + e^{4N\sqrt{-\beta Q_2}} \operatorname{erf}\left(N_1 + \frac{N}{(\lambda + \delta)} - N_0 - \frac{N}{\delta}\right) \right\} \right). \quad (42)$$

(v) Vibrational specific heat capacity is given as

$$C(\beta) = k\beta^2 \left(\frac{\partial^2 \ln Z(\beta)}{\partial \beta^2} \right) = k\sqrt{-\beta Q_2} e^{2N\sqrt{-\beta Q_2}} \times \left\{ \frac{\left[-\operatorname{erf}(\Lambda_1) - e^{2N\sqrt{-\beta Q_2}} \operatorname{erf}(\Lambda_2) + \operatorname{erf}(\Lambda_3) + e^{2N\sqrt{-\beta Q_2}} \operatorname{erf}(\Lambda_4) \right] N_1 + \left(N_2^2 / (4\sqrt{\pi} \sqrt{-\beta Q_2}) \right) - 4N \left[\operatorname{erf}(\Lambda_1 - \Lambda_2) + e^{2N\sqrt{-\beta Q_2}} \operatorname{erf}(\Lambda_2 - \Lambda_4) \right] \sqrt{-\beta Q_2} N_2 - \left(N_4 N_5 / (\delta(\lambda + \delta) \sqrt{-\beta Q_2})^2 \right) + \Lambda_5 \left(-\delta^2 \sqrt{-\beta Q_2} + 2N \right) - 2\beta Q_2 N \Lambda_6 + \delta(\lambda + \delta) N_7}{\sqrt{\pi} \left[\operatorname{erf}(\Lambda_1) + e^{2N\sqrt{-\beta Q_2}} \operatorname{erf}(\Lambda_2) - \operatorname{erf}(\Lambda_3) - e^{2N\sqrt{-\beta Q_2}} \operatorname{erf}(\Lambda_4) \right]^2} \right\}. \quad (43)$$

where

$$\left. \begin{aligned} \Lambda_1 &= N_0 - \frac{N}{\delta}, \Lambda_2 = N_0 + \frac{N}{\delta}, \Lambda_3 = N_1 - \frac{N}{(\lambda + \delta)}, \Lambda_4 = N_1 + \frac{N}{(\lambda + \delta)}, \Lambda_5 = e^{(\beta Q_2 (\delta^4 + Q_3^2) / \delta^2)} \delta, \\ \Lambda_6 &= e^{(\beta Q_2 (\delta + \lambda)^4 + Q_3^2) / (\delta + \lambda)^2} \delta, N = \sqrt{-\beta Q_2 Q_3^2}, N_0 = \sqrt{-\beta Q_2} \delta, N_1 = \sqrt{-\beta Q_2} (\lambda + \delta), \\ N_2 &= N_0 \delta + 2\delta \lambda \sqrt{-\beta Q_2} + \frac{\lambda^2 \sqrt{-\beta Q_2} + N}{\lambda + \delta}, N_3 = 2N e^{2N\sqrt{-\beta Q_2}} \sqrt{\pi}, N_4 = -\Lambda_5 + 2\Lambda_6 + N_3 \operatorname{erf}(\Lambda_8 - \Lambda_1), \\ N_5 &= \Lambda_5 - 2\Lambda_6 + N_3 \operatorname{erf}(\Lambda_2 - \Lambda_8), N_6 = Q_2^2 Q_3^2 - e^{2N\sqrt{-\beta Q_2}} \sqrt{\pi} \delta (\lambda + \delta) \operatorname{erf}(\Lambda_8 - \Lambda_2) N \sqrt{-\beta Q_2}, \\ N_7 &= \operatorname{erf}(\Lambda_6) \left(\delta^2 \sqrt{-\beta Q_2} + 2\delta \lambda \sqrt{-\beta Q_2} + \lambda^2 \sqrt{-\beta Q_2} - 2N \right), \\ N_8 &= -\operatorname{erf}(\Lambda_3) + e^{4N\sqrt{-\beta Q_2}} \operatorname{erf}(\Lambda_2 - \Lambda_3) - 4\sqrt{\pi} \beta^2 e^{4N\sqrt{-\beta Q_2}} \delta (\lambda + \delta) \operatorname{erf}(\Lambda_8). \end{aligned} \right\} \quad (44)$$

6. Superstatistics Formulation

Superstatistics is the superposition of two different statistics which is applicable to driven nonequilibrium systems to statistical intensive parameter (β) fluctuation [72]. This intensive parameter which undergoes spatiotemporal fluctuations includes chemical potential and energy fluctuation which is basically describe in terms of effective Boltzmann factor [73]. According to Edet et.al [74], the effective Boltzmann factor is given as

$$B(E) = \int_0^\infty e^{-\beta' E} f(\beta', \beta) d\beta', \quad (45)$$

where $f(\beta', \beta) = \delta(\beta - \beta')$ is the Dirac delta function. However, the generalized Boltzmann factor expressed in terms of deformation parameter q is given as

$$B(E) = e^{-\beta E} \left(1 + \frac{q}{2} \beta^2 E^2 \right). \quad (46)$$

The partition function for superstatistics formalism is

then given as

$$Z_s = \int_0^\infty B(E) dn. \quad (47)$$

Substituting equation (34) into equation (46) gives the generalized Boltzmann factor equation as

$$B(E) = \left[1 + \frac{q}{2} \beta^2 \left(-\left(Q_2 \rho^2 + \frac{Q_2 Q_3^2}{\rho^2} \right) - (2Q_2 Q_3 - Q_1) \right) \right]^2 e^{-\beta \left[(Q_2 \rho^2 + (Q_2 Q_3^2 / \rho^2)) - (2Q_2 Q_3 - Q_1) \right]}. \quad (48)$$

Using equation (47), the superstatistics partition function equation is given as

$$Z_s = e^{\beta(2Q_2 Q_3 - Q_1)} \int_0^\infty \left[1 + \frac{q}{2} \beta^2 \left(-\left(Q_2 \rho^2 + \frac{Q_2 Q_3^2}{\rho^2} \right) - (2Q_2 Q_3 - Q_1) \right) \right]^2 e^{\beta(Q_2 \rho^2 + (Q_2 Q_3^2 / \rho^2))} d\rho. \quad (49)$$

Using Mathematica 10.0 version, the partition obtain from equation (47) is

$$Z_s = \frac{1}{16\sqrt{-\beta Q_2 Q_3^2 \beta Q_2}} \left\{ e^{-\beta Q_1 + 2\beta Q_2 Q_3 - 2\sqrt{-\beta Q_2} \sqrt{-\beta Q_2 Q_3^2}} \sqrt{\pi} \left[-32q\beta^3 Q_2^3 Q_3^2 + (8 + 3q + 4q\beta Q_1 + 4q\beta^2 Q_1^2) \sqrt{-\beta Q_2} \sqrt{-\beta Q_2 Q_3^2} + 8q(1 + 2\beta Q_1)(-\beta Q_2)^{3/2} Q_3 \sqrt{-\beta Q_2 Q_3^2} + 8q^2 \beta^2 Q_2^2 Q_3^2 \times \left(1 + 2\beta Q_1 + 4\sqrt{-\beta Q_2} \sqrt{-\beta Q_2 Q_3^2} \right) \right] \right\}. \quad (50)$$

TABLE 1: Numerical bound state solutions for CPSHEP: $B = 0.2$, $v_1 = 0.1V$, $v_2 = 0.2V$, $\hbar = \mu = 1$.

n	l	$E_{nl}, \alpha = 0.01$ (eV)	$E_{nl}, \alpha = 0.02$ (eV)	$E_{nl}, \alpha = 0.03$ (eV)	$E_{nl}, \alpha = 0.04$ (eV)
0	0	-0.03061781	-0.033096257	-0.035560349	-0.038010115
1	0	-0.01893207	-0.021509527	-0.024138612	-0.026819316
	1	-0.00842605	-0.011467973	-0.014626685	-0.017902168
2	0	-0.00927784	-0.012006400	-0.014887315	-0.017920574
	1	-0.00586094	-0.008981149	-0.012380582	-0.016059205
	2	-0.00490303	-0.008288097	-0.012015183	-0.016084269
3	0	-0.00619246	-0.009146159	-0.012402307	-0.015960892
	1	-0.00477947	-0.008148623	-0.0120120250	-0.016371067
	2	-0.00434223	-0.008001341	-0.012261886	-0.017123836
	3	-0.00414695	-0.008039833	-0.012612300	-0.017864352
	0	-0.04924686	-0.008178413	-0.011934614	-0.016193276
4	1	-0.00427845	-0.008007979	-0.012498900	-0.017751159
	2	-0.00408585	-0.008147119	-0.013123860	-0.019016034
	3	-0.00401576	-0.008352259	-0.013722838	-0.020127466
	4	-0.00400000	-0.008568537	-0.014266689	-0.021094435
	0	-0.00434190	-0.007970639	-0.012351858	-0.017485540
5	1	-0.00405927	-0.008241005	-0.013501346	-0.019840219
	2	-0.00400216	-0.008568666	-0.014417121	-0.021547480
	3	-0.00400991	-0.008901268	-0.015236763	-0.023016365
	4	-0.00404605	-0.009216668	-0.015965876	-0.024293643
	5	-0.00409483	-0.009507448	-0.016612192	-0.025409045

We use the same procedure of thermodynamics (Section 5) to obtain superstatistics vibrational mean energy (U_s), vibrational specific heat capacity (C_s), vibrational entropy (S_s), and vibrational free energy (F_s) from the partition equation (41). However, this solution is not included in the article because of the lengthy and bulky analytical equations.

7. Numerical Results

Using Matlab 10.0 version, the numerical bound state solutions for the proposed potential were calculated using (14) for different quantum state. Also, using equations (18), (20), (21), and (22), the expectation values for $\langle r^{-2} \rangle_{nl}$, $\langle r^{-1} \rangle_{nl}$, $\langle T \rangle_{nl}$, and $\langle p^2 \rangle_{nl}$, respectively, were calculated as shown in Tables 1–5.

7.1. Special Cases

- (a) Hellmann potential: substituting $v_2 = 0$ into equation (1), then, the potential reduces to Hellmann potential

$$v(r) = -\frac{v_1}{r} + \frac{Be^{-\alpha r}}{r}. \quad (51)$$

The required energy equation is

$$E_{nl} = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} - v_1 \alpha - \frac{\hbar^2 \alpha^2}{8\mu} \left\{ (n+l+1) + \frac{((2\mu/\hbar^2 \alpha)(B-v_1) + l(l+1))}{(n+l+1)} \right\}^2. \quad (52)$$

The result of equation (52) agrees excellently with Ref. [75].

- (b) Yukawa potential: if $v_1 = v_2 = 0$, then, equation (1) reduces to Yukawa potential

$$v(r) = \frac{Be^{-\alpha r}}{r}. \quad (53)$$

The corresponding energy eigenequation is

$$E_{nl} = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} - \frac{\hbar^2 \alpha^2}{8\mu} \left\{ (n+l+1) + \frac{((2\mu/\hbar^2 \alpha)(B-v_1) + l(l+1))}{(n+l+1)} \right\}^2. \quad (54)$$

Equation (54) is consistent with result obtain in Ref. [76] in order to prove the mathematical accuracy of our analytical calculation.

- (c) Screened-hyperbolic inversely quadratic potential: substituting $B = v_1 = 0$ into equation (1), the potential reduces to screened-hyperbolic inversely quadratic potential

$$v(r) = \frac{v_2 e^{-\alpha r} \cosh \alpha}{r^2}. \quad (55)$$

TABLE 2: Expectation values for $\langle r^{-2} \rangle_{nl}$: $B = 0.2$, $v_1 = 0.1V$, $v_2 = 0.2V$, $\hbar = \mu = 1$.

n	l	$\langle r^{-2} \rangle_{nl}$ $\alpha = 0.01 (\text{\AA}^{-2})$	$\langle r^{-2} \rangle_{nl}$ $\alpha = 0.03 (\text{\AA}^{-2})$	$\langle r^{-2} \rangle_{nl}$ $\alpha = 0.03 (\text{\AA}^{-2})$	$\langle r^{-2} \rangle_{nl}$ $\alpha = 0.04 (\text{\AA}^{-2})$
0	0	-0.180845600	-0.19312890	-0.20559450	-0.21823660
1	0	-0.007804320	-0.01146716	-0.01567526	-0.02042717
	1	0.031534220	0.03193252	0.03112350	0.02910763
2	0	-0.003148090	-0.00654506	-0.01084855	-0.01605580
	1	0.010561720	0.00855289	0.00383228	-0.00359942
	2	0.057017470	0.04310601	0.01225219	-0.03554203
	0	-0.002633730	-0.00636279	-0.01136044	-0.01762263
3	1	0.004432820	-0.00033896	-0.00987264	-0.02416721
	2	0.021856240	-0.01200783	-0.07495907	-0.16699480
	3	0.051656280	-0.03959544	-0.20947110	-0.45796650
	0	-0.002613291	-0.00679560	-0.01260871	-0.02004732
	1	0.001361432	-0.00701901	-0.02276403	-0.04587216
4	2	0.003256901	-0.05268207	-0.01526563	-0.02966622
	3	0.002156285	-0.15152770	-0.42284930	-0.81180300
	4	0.000077152	-0.30591300	-0.84750470	-1.62484500
	0	-0.002708980	-0.00738565	-0.01405538	-0.02271165
	1	-0.000934536	-0.01380659	-0.03719615	-0.07110123
5	2	-0.010539190	-0.09245381	-0.23627330	-0.44199310
	3	-0.034490940	-0.25675980	-0.64214770	-1.19064800
	4	-0.074756000	-0.52092780	-1.29216600	-2.38846200
	5	-0.129402700	-0.88828570	-2.20300600	-4.07355300

The resulting energy eigenvalue is

$$E_{nl} = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} - \frac{\hbar^2 \alpha^2}{8\mu} \left\{ \left(n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2} \right)^2 - \frac{2v_2 \mu \cosh \alpha}{\hbar^2}} \right) + \frac{l(l+1)}{\left(n + (1/2) + \sqrt{(l + (1/2))^2 - (2v_2 \mu \cosh \alpha / \hbar^2)} \right)} \right\}^2. \quad (56)$$

(d) Coulomb potential: substituting $\alpha = 0$ into equation (53), then, the potential reduces to Coulomb potential

$$v(r) = \frac{B}{r}. \quad (57)$$

By substituting $\alpha = 0$ into equation (54), it gives the corresponding energy eigenvalue for Coulomb's potential as

$$E_{nl} = -\frac{\hbar^2 \mu B^2}{2(n+l+1)^2}. \quad (58)$$

Equation (58) agrees we with result obtain in Ref [76].

8. Discussion

Figure 1 is the graph of Pekeris approximation against the screening parameter α . The nature of the graph shows that the approximation is suitable for the proposed potential. Variation of the probability density against the internuclear separation at various quantum state for $l=0$ and $l=1$, respectively, is shown in Figures 2(a) and 2(b), respectively. In Figure 2(a), the probability density curves produce thermal curves with regular peaks compacted close to the origin with uneven peaks at various internuclear distance. This curve shows that for orbital angular quantum number $l=0$, there is more concentration of the electron density at the origin for all the quantum state studied. The same situation is also observed for $l=1$ as shown in Figure 2(b). It can also be observed that at every value of the internuclear distance, the probability density for $l=0$ is higher than the probability density for $l=1$. Figure 3 shows variation of the probability density against the internuclear separation at various quantum state for $l=0$ and $l=1$ for Hellmann potential presented in Figures 3(a) and 3(b), respectively. More concentration of the electron density is observed at the origin in both cases. It is also seen that the probability density obtained for $l=0$ is lower than the probability density obtained for $l=1$. In Figure 4, we presented the variation of the probability density against the internuclear separation at various quantum state for $l=0$ and $l=1$ for Yukawa potential as shown in Figures 4(a) and 4(b), respectively. Here, there are more concentration and localization of

TABLE 3: Expectation values for $\langle r^{-1} \rangle_{nl}$: $B = 0.2$, $v_1 = 0.1V$, $v_2 = 0.2V$, $\hbar = \mu = 1$.

n	l	$\langle r^{-1} \rangle_{nl}$ $\alpha = 0.01 (\text{\AA}^{-1})$	$\langle r^{-1} \rangle_{nl}$ $\alpha = 0.02 (\text{\AA}^{-1})$	$\langle r^{-1} \rangle_{nl}$ $\alpha = 0.03 (\text{\AA}^{-1})$	$\langle r^{-1} \rangle_{nl}$ $\alpha = 0.04 (\text{\AA}^{-1})$
0	0	-0.19225140	-0.19688090	-0.20138910	-0.20577640
1	0	-0.01142578	-0.01191251	-0.01239767	-0.01288129
	1	-0.04285259	-0.04902030	-0.05517621	-0.06132020
2	0	-0.04962766	-0.05456679	-0.05949515	-0.06441276
	1	-0.02577863	-0.03141289	-0.03703604	-0.04264812
	2	-0.01862148	-0.02483049	-0.03102713	-0.03721144
3	0	-0.02857951	-0.03354061	-0.03849154	-0.04343230
	1	-0.01810582	-0.02350037	-0.02888420	-0.03425735
	2	-0.01440590	-0.02023623	-0.02605494	-0.03186203
	3	-0.01315972	-0.01949603	-0.02581969	-0.03213072
4	0	-0.01947010	-0.02444052	-0.02940090	-0.03435125
	1	-0.01401303	-0.01927972	-0.02453591	-0.02978164
	2	-0.01188206	-0.01748570	-0.02307814	-0.02865941
	3	-0.01069620	-0.01662480	-0.02254153	-0.02844645
	4	-0.00997767	-0.01620214	-0.02241417	-0.02861378
5	0	-0.01475625	-0.01973146	-0.02469666	-0.02965188
	1	-0.01157554	-0.01676609	-0.02194628	-0.02711614
	2	-0.01025245	-0.01570970	-0.02115605	-0.02659152
	3	-0.00949240	-0.01522176	-0.02093966	-0.02664612
	4	-0.00902694	-0.01501492	-0.02099092	-0.02695497
	5	-0.19225140	-0.19688090	-0.02117606	-0.02737791

TABLE 4: Expectation values for $\langle T \rangle_{nl}$: $B = 0.2$, $v_1 = 0.1V$, $v_2 = 0.2V$, $\hbar = \mu = 1$.

n	l	$\langle T \rangle_{nl}$ $\alpha = 0.01 (eV)$	$\langle T \rangle_{nl}$ $\alpha = 0.02 (eV)$	$\langle T \rangle_{nl}$ $\alpha = 0.03 (eV)$	$\langle T \rangle_{nl}$ $\alpha = 0.04 (eV)$
0	0	-0.026806820	-0.025402520	-0.02391260	-0.022337650
1	0	-0.015431540	-0.013981940	-0.01205726	-0.009657998
	1	-0.006571382	-0.007487332	-0.00824097	-0.008824540
2	0	-0.005631741	-0.003353308	0.00013309	0.004825102
	1	-0.004014817	-0.004921358	-0.00572940	-0.006428803
	2	-0.002395157	-0.002365251	-0.00173894	-0.000484917
3	0	-0.002066055	0.001684466	0.00770820	0.015999570
	1	-0.002916120	-0.003925660	-0.00491908	-0.005881812
	2	-0.001738771	-0.001662435	-0.00102536	0.002026249
	3	-0.001048483	0.000019672	0.00235305	0.006033901
4	0	-0.000071586	0.005705386	0.01515297	0.028261000
	1	-0.002372722	-0.003544271	-0.00480446	-0.006132816
	2	-0.001362593	-0.001292758	-0.00069870	0.000451413
	3	-0.000758604	0.000354150	0.02700911	0.006357665
	4	-0.002157670	0.002154345	0.00672056	0.013654220
5	0	0.001455576	0.009784670	0.02351393	0.042627300
	1	-0.002090302	-0.003469073	-0.00506461	-0.006849102
	2	-0.001134154	-0.001102620	-0.00058791	0.000445057
	3	-0.000568497	0.000558786	0.00289301	0.006508060
	4	-0.000048447	0.002360339	0.00693053	0.013820340
	5	0.000506994	0.004490706	0.01188615	0.023002670

TABLE 5: Expectation values for $\langle p^2 \rangle_{nl}$: $B = 0.2$, $v_1 = 0.1V$, $v_2 = 0.2V$, $\hbar = \mu = 1$.

n	l	$\langle p^2 \rangle_{nl}$ $\alpha = 0.01(eV/c)^2$	$\langle p^2 \rangle_{nl}$ $\alpha = 0.02(eV/c)^2$	$\langle p^2 \rangle_{nl}$ $\alpha = 0.03(eV/c)^2$	$\langle p^2 \rangle_{nl}$ $\alpha = 0.04(eV/c)^2$
0	0	-0.053613630	-0.050805050	-0.04782520	-0.04467531
1	0	-0.030863080	-0.027963880	-0.02411452	-0.01931600
	1	-0.013142760	-0.014974660	-0.01648195	-0.01764908
2	0	-0.011263480	-0.006706617	0.00026618	0.00965020
	1	-0.008029633	-0.000984271	-0.01145882	-0.01285761
	2	-0.004790315	-0.004730502	-0.00347789	-0.00096983
3	0	-0.004132109	0.003368932	0.015416410	0.03199915
	1	-0.005832247	-0.007851326	-0.009838166	-0.01176362
	2	-0.003477542	-0.003324870	-0.002050733	0.00040524
	3	-0.002096967	0.000393442	0.004706108	0.01206780
4	0	-0.000143173	0.011410770	0.030305940	0.05652200
	1	-0.004745445	-0.007088541	-0.000960893	-0.01226563
	2	-0.002725186	-0.002585515	-0.001397411	0.00090282
	3	-0.001517209	0.000708300	0.005401821	0.01271533
	4	-0.000431534	0.004308690	0.013441140	0.02730844
5	0	0.002911151	0.019569340	0.047027870	0.08525461
	1	-0.004180604	-0.006938146	-0.010129220	-0.01369820
	2	-0.002268308	-0.002205244	-0.001175838	0.00089011
	3	-0.001136990	0.001117574	0.005786028	0.01301610
	4	-0.000096894	0.004720678	0.013861070	0.02764067
	5	0.001013988	0.008981412	0.023772290	0.04600535

FIGURE 2: Variation of the probability density plots against the internuclear separation of CPSHEP. (a) Probability density plot for fixed $l = 0$ and (b) probability density plot for fixed $l = 1$.

FIGURE 3: Variation of the probability density plots against the internuclear separation of Hellmann potential. (a) Probability density plot for fixed $l=0$ and (b) probability density plot for fixed $l=1$.

FIGURE 4: Variation of the probability density plots against the internuclear separation of Yukawa potential. (a) Probability density plot for fixed $l=0$ and (b) probability density plot for fixed $l=1$.

electron density at the origin for $l=0$ with uneven distribution of peak curves as shown in Figure 4(a), but this situation is not the same for $l=1$ as presented in Figure 4(b). The probability density at the second excited state for $l=1$ remains constant with uniform distribution of the density curves for all values of the internuclear separation.

In Figure 5, the variation of partition function for thermodynamic properties and superstatistics with the temperature parameter β were observed. Here, the partition function increases nonlinearly with β . However, in Figure 5(a), the partition function diverged as β increases, but later con-

verges for the superstatistics as shown in Figure 5(b). It can also be observed that the partition function converges as β becomes positive in the superstatistics in Figure 5(b). In Figure 6, we presented the variation of vibrational mean energy against β for thermodynamic properties and superstatistics, respectively, as shown in Figures 6(a) and 6(b). For the thermal property, the mean energy increases as β goes up for all values of λ as shown in Figure 6(a), and at higher values of β , the mean energy for various λ tends to converge. The superstatistics mean energy rises as the temperature of the system decreases as shown in Figure 6(b).

FIGURE 5: (a) Variation of partition function with β for thermodynamic property and (b) variation of partition function with β for superstatistics for CPSHEP, respectively.

FIGURE 6: (a) Variation of vibrational mean energy with β for thermodynamic property and (b) variation of vibrational mean energy with β for superstatistics for CPSHEP, respectively.

However, at a certain absolute temperature, the superstatistics mean energy increases vertically for various values of the deformation parameter (q). The variation is opposite in

Figure 7 which is the variation of the heat capacity against β for different values of λ and deformed parameter q is shown in Figures 7(a) and 7(b), respectively. In each case,

FIGURE 7: (a) Variation of vibrational heat capacity with β for thermodynamic property and (b) variation of vibrational heat capacity with β for superstatistics for CPSHEP, respectively.

FIGURE 8: (a) Variation of vibrational entropy with β for thermodynamic property and (b) variation of vibrational entropy with β for superstatistics for CPSHEP, respectively.

the heat capacity decreases monotonically with an increasing β for the nonsuperstatistics (thermodynamic property). At zero value of β , the heat capacity for various λ converged and diverges as β increases gradually. For the superstatistics, the heat capacity for various deformed parameter rises while the temperature cools down. The specific heat capacity has a turning point when β equals -150 as shown in Figure 7(b) while the specific heat capacity for the nonsuperstatistics

has a maximum turning point at about 0.15 as shown in Figure 7(a). In Figure 8, the vibrational entropy decreases and diverges while the temperature of the system decreases (β increases) in the nonsuperstatistics as shown in Figure 8(a). This decrease is sharper for negative values of the entropy when β is almost constant as shown in Figure 8(b). The superstatistics entropy varies inversely with β (directly with temperature). This means that when the

FIGURE 9: (a) Variation of vibrational free energy with β for thermodynamic property and (b) variation of vibrational free energy with β for superstatistics for CPSEHP, respectively.

temperature of the system is raised, the disorderliness of the system also increases for every value of the deformed parameter. The entropy for the superstatistics converged as the temperature parameter tends to zero. In Figure 9, the variation of free energy against the β is seen to be two different steps in the case of a nonsuperstatistics as shown in Figure 9(a). Here, between 0 and 60 values of β , the free energy increases steadily for the various values of λ but beyond this range; the free energy increases sharply at constant β . For the superstatistics, the free energy increases monotonically as the temperature of the system reduces gradually. The free energy is always higher when the deformed parameter is increased as shown in Figure 9(b).

Table 1 is the numerical bound state solutions for CPSEHP. The numerical bound state solutions increases with an increase in quantum state, but decreases with an increase in the screening parameter. Tables 2–5 are expectation values for $\langle r^{-2} \rangle_{nl}$, $\langle r^{-1} \rangle_{nl}$, $\langle T \rangle_{nl}$, and $\langle p^2 \rangle_{nl}$, respectively. Here, the numerical values in all cases decrease with an increase in the screening parameter.

9. Conclusion

In this work, we apply the parametric Nikiforov-Uvarov method to obtain the bound state solutions of Coulomb plus Screened Exponential Hyperbolic Potential. The resulting energy eigenvalues were presented in a close and compact form. The research work was extended to study thermal properties, superstatistics, and various expectation values. The proposed potential also reduced to Hellmann potential, Yukawa potential, Screened Hyperbolic potential, and Cou-

lomb potential as special cases. The normalized wave function for the mother potential and that of the Hellmann potential are similar, but the normalized wave function of the Yukawa potential seems different. The trend of the thermodynamic and superstatistics curves is in agreement to the results of an existing literature. The results of the thermodynamic properties and superstatistics revealed that the effect of the temperature on the thermodynamic properties and the superstatistics are similar. Finally, this research work has practical applications in physical and chemical sciences.

Data Availability

The numerical data used for this work are generated using Matlab programme.

Disclosure

An earlier version of this article has been deposited in arxiv in Cornell University which is accessible at <https://arxiv.org/abs/2110.09896>.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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