

Research Article

On Lorentz Invariant Complex Scalar Fields

Gustavo Rigolin 

Departamento de Física, Universidade Federal de São Carlos, São Carlos, SP 13565-905, Brazil

Correspondence should be addressed to Gustavo Rigolin; rigolin@ufscar.br

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We obtain a Lorentz covariant wave equation whose complex wave function transforms under a Lorentz boost according to the following rule, $\Psi(x) \rightarrow e^{(i\hbar)f(x)}\Psi(x)$. We show that the spacetime-dependent phase $f(x)$ is the most natural relativistic extension of the phase associated with the transformation rule for the nonrelativistic Schrödinger wave function when it is subjected to a Galilean transformation. We then generalize the previous analysis by postulating that $\Psi(x)$ transforms according to the above rule under proper Lorentz transformations (boosts or spatial rotations). This is the most general transformation rule compatible with a Lorentz invariant physical theory whose observables are bilinear functions of the field $\Psi(x)$. We use the previous wave equations to describe several physical systems. In particular, we solve the bound state and scattering problems of two particles which interact both electromagnetically and gravitationally (static electromagnetic and gravitational fields). The former interaction is modeled via the minimal coupling prescription while the latter enters via an external potential. We also formulate logically consistent classical and quantum field theories associated with these Lorentz covariant wave equations. We show that it is possible to make those theories equivalent to the Klein-Gordon theory whenever we have self-interacting terms that do not break their Lorentz invariance or if we introduce electromagnetic interactions via the minimal coupling prescription. For interactions that break Lorentz invariance, we show that the present theories imply that particles and antiparticles behave differently at decaying processes, with the latter being more unstable. This suggests a possible connection between Lorentz invariance-breaking interactions and the matter-antimatter asymmetry problem.

1. Introduction

A complex scalar field $\Phi(x)$ is usually defined as a function of the spacetime coordinates $x = (ct, \mathbf{r})$ such that it remains invariant under a symmetry operation, i.e., $\Phi(x) = \Phi'(x')$, where $\Phi'(x')$ is the field after we apply the symmetry operation. In its more general standard definition, one may also multiply the transformed field by a *constant* complex number of modulus one [1–5]. A paradigmatic example of a physical system described by a complex scalar field is a charged Klein-Gordon particle, which transforms according to the above rule under proper Lorentz transformations (boosts or space rotations).

And what about the nonrelativistic wave function, which is a solution to the Schrödinger equation [6]? Is it a complex scalar field in the above sense? Strictly speaking, it is not. This comes about because under a Galilean transformation,

it changes according to the following prescription [1], $\Psi(x) = e^{(i\hbar)\theta(x')}\Psi'(x')$, where the phase $\theta(x')$ is not a constant, being a function of the spacetime coordinates. We require $\Psi(x)$ to transform in this way in order to have the Schrödinger equation invariant under a Galilean transformation [1].

The Schrödinger field $\Psi(x)$ illustrates that it is perfectly possible to build a logically consistent field theory assuming a more general transformation law for a complex scalar field under a symmetry operation, where the phase θ depends on the spacetime coordinates. The extension of the latter observation to the relativistic domain is the leitmotif of the present work. We want to determine the wave equation whose wave function transforms similarly to Schrödinger's wave function and which is invariant (Lorentz covariant) under proper Lorentz transformations.

We can also understand the main idea of this work, as described above, as the search for the answer to the

following simple question: What is the Lorentz covariant wave equation associated with a complex scalar field $\Psi(x)$ whose bilinear $\Psi(x)\Psi^*(x) = |\Psi(x)|^2$, rather than $\Psi(x)$ itself, is assumed to be a Lorentz invariant (scalar)? Note that here we are demanding not $\Psi(x)$ but rather $\Psi(x)\Psi^*(x)$ to be a Lorentz scalar. The above question can be rephrased as follows: What is the Lorentz covariant wave equation associated with a complex scalar field $\Psi(x)$ that changes after a Lorentz transformation according to the following rule: $\Psi(x) \longrightarrow e^{(i\hbar)f(x)}\Psi(x)$? Here, $f(x)$ is an arbitrary real spacetime-dependent phase that is chosen to guarantee the Lorentz covariance of the wave equation.

The motivation underlying the above question is related to the fact that almost all observables in quantum field theory are bilinear functions of the fields. Thus, a scalar quantum field theory built from the start by assuming that the bilinear functions rather than the fields themselves are invariant should, in principle, be consistent with all known experimental facts.

The first step towards answering the above questions is a critical examination of the physical meaning of the nonrelativistic phase $\theta(x)$ above, which allows us to naturally infer its relativistic extension. With the relativistic transformation rule for $\Psi(x)$ at hand, we can search for the wave equation whose wave function transforms according to it and which is covariant under a Lorentz boost. By adding a few extra assumptions, such that we should recover the Schrödinger wave equation at the nonrelativistic limit, we show that the relativistic wave equation we obtain is unique. We also show its connection to the Klein-Gordon equation, and we apply it to the description of a variety of physical problems at the first quantization level.

We then move to the construction of the classical and quantum field theories related to this relativistic wave equation. As we show in the following pages, it is possible to build consistent relativistic classical and quantum field theories if we employ the more general definition of a complex scalar field as outlined above. We also determine the scenarios under which the complex scalar field theories here presented are equivalent to the Klein-Gordon theory. For Lorentz invariant self-interactions and for electromagnetic interactions, the predictions of the present theory and those derived from the Klein-Gordon theory are the same.

A few surprises appeared during our journey to the classical and, in particular, to the quantum field theories here presented. First, as a logical consequence of those theories, we noticed that we can ascribe “negative” masses to antiparticles, while still keeping their energies positive. We then proved that this intriguing result, naturally emerging from the present theory, does not lead to any conflict with present-day experimental facts. We also observed that particles and antiparticles with the same wave number no longer had the same energies and that we could make them have different momenta as well by properly tuning the free parameters of the field theories here developed. Despite all these peculiarities, and as we already noted above, at the level of electromagnetic interactions, the present theories were shown to be equivalent to the Klein-Gordon one. Moreover, we showed that the “negative” masses for antipar-

ticles lead to interesting experimental predictions if we add Lorentz invariance-breaking interactions to the theory. In this scenario, particles and antiparticles do not behave symmetrically and the process of baryogenesis emerges naturally (antiparticles are more unstable). This surprising and interesting result could not have been anticipated without fully developing the quantum field theory associated with the Lorentz covariant Schrödinger equation given in the first part of this work. We also noted that for gravitational interactions, the present theory can be tuned such that particles and antiparticles either attract or repel each other.

A few other subjects are addressed in this work, all connected in one way or another to the wave equations here derived. These results, together with the major ones highlighted above, are for ease of access systematically put together below before we start the more technical part of this work.

2. Outline of This Work

This work can be divided into three main parts. The first one, comprising Sections 3 to 5, develops the main idea of this manuscript in the framework of first quantization. Being more specific, the following results are achieved in the first part:

- (1) In Section 3, we derive a wave equation, which we dub Lorentz covariant Schrödinger equation, that is covariant under proper Lorentz transformations and whose wave function transforms under a Lorentz boost according to equation (18),

$$\Psi(\mathbf{r}, t) \longrightarrow e^{(i\hbar)\theta(\mathbf{r}, t)}\Psi(\mathbf{r}, t). \quad (1)$$

This is the relativistic extension of the transformation law (10) associated with the nonrelativistic Schrödinger equation

- (2) In Section 4, we obtain by elementary methods the four-current density associated with the Lorentz covariant Schrödinger equation. We also show how to write it in a manifestly covariant way, and we show its relation to the Klein-Gordon equation
- (3) In Section 5, we apply the Lorentz covariant Schrödinger equation to describe a multitude of physical systems. Possible applications to condensed matter systems are also briefly discussed. The main points of this section are as follows: (a) we investigate the free particle solutions (external potential $V = 0$) of the Lorentz covariant Schrödinger equation, which already hint towards the possibility of the existence of antiparticles possessing negative masses and positive energies; (b) we then study the case of a nonnull constant potential ($V \neq 0$), where the dependence of the particle’s mass on its value is highlighted; (c) for a constant potential, we also show that the Lorentz covariant Schrödinger equation can be seen as

formally equivalent to the complex variable extension of the telegraph equation. This implies that for certain values of V , we can have a distortionless and dispersionless wave packet evolution; (d) the bound state problem associated with a Coulomb-like potential V is presented and solved; (e) we apply the minimal coupling prescription to the Lorentz covariant Schrödinger equation in order to include electromagnetic interactions in a gauge invariant way. Of particular interest is the exact solution to the bound state problem where we have the simultaneous action of static electromagnetic and gravitational fields between two particles. The electromagnetic interaction is modeled via the minimal coupling prescription while the gravitational interaction enters via the external potential V ; and (f) we also perturbatively study the scattering between two scalar particles of different masses and charges when both the electromagnetic and gravitational interactions are simultaneously taken into account

The second part of this work, contained in Sections 6 and 7 and in Appendixes A and B, develops the classical and quantum field theories associated with the Lorentz covariant Schrödinger equation. Specifically,

- (1) in Section 6, we present the Lagrangian formulation of the Lorentz covariant Schrödinger equation and we also investigate the continuous and discrete symmetries related to it. Of particular notice is the need to modify the charge conjugation operation such that the mass changes its sign when we implement this symmetry operation. Only in this way can we properly exchange the roles of particles with antiparticles and, at the same time, guarantee the validity of the CPT theorem
- (2) in Section 7, we formulate the quantum field theory associated with the Lorentz covariant Schrödinger Lagrangian. We show that it is possible to build a canonically second quantized theory out of the Lorentz covariant Schrödinger fields that is logically consistent. In the framework of second quantization, the necessity to identify particles and antiparticles with masses having different signs is further clarified. It is shown that particles and antiparticles have different but positive-definite expressions describing their relativistic energies. We then prove the equivalence between the Lorentz covariant Schrödinger theory and the Klein-Gordon theory when we add relativistic invariant self-interacting terms to the Lagrangian density. We also show the equivalence between both theories when electromagnetic interactions are present. The details of those proofs are given in Appendix B. However, if we include interactions that do not respect Lorentz invariance, we prove by giving explicit examples that both theories lead to different predictions. We discuss a possible connection between those Lorentz invariance-breaking interactions and the asymmetry of matter and antimatter seen in the present-day universe

- (3) in Appendix A, we explore the main features associated with a noncanonical quantization of the Lorentz covariant Schrödinger fields, whose origin is traced back to requiring the relativistic energies associated with particles and antiparticles to be the same. This leads to the violation of the microcausality condition and, interestingly, to the emergence of an instantaneous gravitational-like interaction between the scalar particles of the theory. We also speculate about the possibility of arriving at a consistent quantum field theory of gravitation if we work with quantum field theories whose main assumption is the existence of particles and antiparticles having both positive energies but positive and negative masses

In Section 8, the third and last part of this work, we generalize the Lorentz covariant Schrödinger equation, presenting a new Lorentz covariant wave equation where the four spacetime coordinates and its derivatives appear on an equal footing. For the free particle case, we have

$$\partial_\mu \partial^\mu \Psi - 2i\kappa^\mu \partial_\mu \Psi = 0. \quad (2)$$

Note that κ^μ is not a four-vector (the four constants $\kappa^0, \kappa^1, \kappa^2, \kappa^3$ are invariants under proper Lorentz transformations, being the equivalent in the present theory of the mass “ m ” in the Klein-Gordon theory. In other words, it is a postulate of the present theory that $\kappa^0, \kappa^1, \kappa^2, \kappa^3$ are invariants under proper Lorentz transformations); it is a shorthand notation for the four constants $\kappa^0, \dots, \kappa^3$. The above equation is Lorentz covariant if we use the more general transformation law for Ψ as given below. This should be contrasted with the original Lorentz covariant Schrödinger equation developed in the first part of this work,

$$\partial_\mu \partial^\mu \Psi - i \frac{2mc}{\hbar} \partial_0 \Psi = 0, \quad (3)$$

where we have a first-order time derivative of the wave function and no first-order space derivatives. To arrive at the generalized Lorentz covariant Schrödinger equation, we assume that under a proper Lorentz transformation (boosts or spatial rotations), the wave function changes according to the following prescription:

$$\Psi(\mathbf{r}, t) \longrightarrow e^{(i\hbar)f(\mathbf{r}, t)} \Psi(\mathbf{r}, t), \quad (4)$$

with $f(\mathbf{r}, t)$ being uniquely determined by the requirement of the covariance of the wave equation under spatial rotations and boosts. In other words, in addition to assuming that $f(\mathbf{r}, t) \neq 0$ for Lorentz boosts, as we did in the first part of this manuscript, we now assume that $f(\mathbf{r}, t) \neq 0$ for spatial rotations as well.

We also show in Section 8 the conditions under which the generalized Lorentz covariant Schrödinger Lagrangian leads to the same predictions of the Klein-Gordon theory. It turns out that for Lorentz invariant self-interactions and electromagnetic interactions, we can make the two theories

agree by properly adjusting the values of the constant coefficients (κ'') appearing in the generalized Lagrangian density. Interestingly and contrary to the original Lorentz covariant Schrödinger Lagrangian, there still remains a free parameter at our disposal after enforcing the equivalence of the generalized theory and the Klein-Gordon one. Finally, we show that the inclusion of interaction terms in the Lagrangian that are not Lorentz invariant destroys the equivalence between both theories.

3. Galilean Invariance and Lorentz Covariance

Our main goal in this section is to derive a wave equation that is Lorentz covariant if its solution, the wave function, transforms similarly to the way the solution to the nonrelativistic Schrödinger equation transforms under a Galilean boost. To achieve that, we first review how and why the solution to the Schrödinger equation transforms according to equation (10). We then propose a natural relativistic generalization of this transformation rule and derive the most general linear wave equation that is covariant under proper Lorentz transformations if its wave function transforms according to the relativistic extension of equation (10).

3.1. Galilean Invariance of the Schrödinger Equation. First off, we need to clearly explain what one usually means by stating that the Schrödinger equation is invariant under a Galilean transformation [1].

Let S and S' be two inertial reference frames whose Cartesian coordinate systems are connected by the following Galilean boost:

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t', \quad (5)$$

$$t = t'. \quad (6)$$

Here, $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$ are the Cartesian coordinates of systems S and S' , respectively, while t and t' are the respective time coordinates measured in those inertial frames. The constant vector \mathbf{v} gives the relative velocity of S' with respect to S .

In reference frame S , the Schrödinger wave equation for a particle of mass m subjected to the potential $V(\mathbf{r}, t)$ is

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{r}, t) + V(\mathbf{r}, t) \Psi(\mathbf{r}, t), \quad (7)$$

where $\Psi(\mathbf{r}, t)$ is the particle wave function and $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$ is the Laplacian written in Cartesian coordinates.

Now, using equations (5) and (6) together with the chain rule, we get

$$\nabla^2 = \nabla'^2, \quad (8)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \mathbf{v} \cdot \nabla', \quad (9)$$

where the dot means the scalar product and ∇' is the gradient operator in system S' , namely, $\nabla' = (\partial/\partial x', \partial/\partial y', \partial/\partial z')$. Substituting equations (8) and (9) into (7) and assuming that the wave function in S is connected to the wave function in S' according to the following prescription,

$$\Psi(\mathbf{r}, t) = e^{(i\hbar)\theta(\mathbf{r}', t')} \Psi'(\mathbf{r}', t'), \quad (10)$$

we obtain the invariance of the Schrödinger equation,

$$i\hbar \frac{\partial \Psi'(\mathbf{r}', t')}{\partial t'} = -\frac{\hbar^2}{2m} \nabla'^2 \Psi'(\mathbf{r}', t') + V'(\mathbf{r}', t') \Psi'(\mathbf{r}', t'), \quad (11)$$

if, and only if,

$$\theta(\mathbf{r}', t') = \frac{mv^2}{2} t' + m\mathbf{v} \cdot \mathbf{r}' + \text{cte}, \quad (12)$$

where cte is a real constant that is usually assumed to be zero and $v^2 = \mathbf{v} \cdot \mathbf{v}$. We also assume that the potential transforms as $V(\mathbf{r}, t) = V'(\mathbf{r}', t')$. The Coulomb potential, for instance, is the paradigmatic example of a potential that transforms according to the above prescription under a Galilean boost.

Note that strictly speaking, we can only make the Schrödinger equation invariant under a Galilean boost by imposing that the wave function “fails” to be invariant thereunder. In other words, the prescription $\Psi(\mathbf{r}, t) = \Psi'(\mathbf{r}', t')$ connecting the wave functions in coordinate systems S and S' does not suffice to make the Schrödinger equation invariant under a Galilean transformation. Nevertheless, the probability density $|\Psi(\mathbf{r}, t)|^2$ is invariant under the transformation given in equation (10) and thus invariant under a Galilean boost.

We can also understand equation (10) as a way to obtain the solution to the Schrödinger equation in reference frame S' if the solution in S is already known,

$$\Psi'(\mathbf{r}', t') = e^{-(i\hbar)\theta(\mathbf{r}', t')} \Psi(\mathbf{r}, t) = e^{-(i\hbar)\theta(\mathbf{r}', t')} \Psi(\mathbf{r}' + \mathbf{v}t', t'). \quad (13)$$

This prescription relating the solution in S' to the one in S is crucial for the consistency of the description of the same physical system in different Galilean inertial frames. The following simple example illustrates this point and further clarifies how to apply equation (13) in order to get $\Psi'(\mathbf{r}', t')$ from $\Psi(\mathbf{r}, t)$.

3.1.1. Example: The Plane Wave. Let us assume that in S , we have a free particle with momentum p . For simplicity, we set $V(\mathbf{r}, t) = 0$ and assume, without loss of generality, that we are dealing with a one-dimensional problem. Up to an overall global phase, the solution to the Schrödinger equation (7) describing such a particle is the plane wave

$$\Psi(x, t) = e^{i(kx - \omega t)} = e^{(i\hbar)(px - (p^2/2m)t)}, \quad (14)$$

where $p^2/2m$ is the particle's energy.

Let us now solve the same problem in the reference frame S' , which we assume is moving away from S with constant velocity $\mathbf{v} = (v, 0, 0)$, with $v > 0$. This means that S' is moving along the positive direction of the x -axis. In this frame, the momentum of the particle is no longer p but $p - mv$ and the solution to the Schrödinger equation (11) is

$$\Psi'(x', t') = e^{i(k'x' - \omega't')} = e^{(i\hbar)((p-mv)x' - ((p-mv)^2/2m)t')}, \quad (15)$$

where now $(p - mv)^2/2m$ is the particle's energy.

It is not difficult to see that the naive prescription $\Psi(x, t) \rightarrow \Psi'(x', t')$ connecting the wave functions from these two reference frames will not transform the right-hand side of equation (14) into the right-hand side of (15). However, noting that equation (12) for this specific problem is

$$\theta(x', t') = \frac{mv^2}{2}t' + mvx', \quad (16)$$

where we have set $\text{cte} = 0$, we get after equation (13)

$$\begin{aligned} \Psi'(x', t') &= e^{-(i\hbar)\theta(x', t')} \Psi(x, t) = e^{-(i\hbar)\theta(x', t')} \Psi(x' + vt', t') \\ &= e^{-(i\hbar)((mv^2/2)t' + mvx')} e^{(i\hbar)(p(x' + vt') - (p^2/2m)t')} \\ &= e^{(i\hbar)((p-mv)x' - ((p-mv)^2/2m)t')}, \end{aligned} \quad (17)$$

which is exactly equation (15). This simple example clearly illustrates why prescription (10) is mandatory to properly connect the solutions to the Schrödinger equation obtained in two different inertial reference frames. (We employed the term invariance to denote that the Schrödinger equation does not change its form under a Galilean boost because this is the usual practice in nonrelativistic treatises on quantum mechanics (see ref. [1], for example.) From now on, we reserve the term invariant to quantities such as the rest mass of a particle, the speed of light, the electric charge, or any scalar quantity that has the same numerical value in any inertial reference frame. Equations that look the same in all inertial frames after we apply a given transformation will now be called covariant.)

3.2. Lorentz Covariant Schrödinger Equation. Looking carefully at equation (12), we can understand the terms multiplying t' and \mathbf{r}' , namely, $mv^2/2$ and $m\mathbf{v}$, as the contribution to the energy and momentum of the particle of mass m that is related to the fact that reference frame S' is moving from S with constant velocity \mathbf{v} . For example, assume the particle is at rest in S' . From the point of view of S , it is moving away with constant velocity \mathbf{v} and thus with kinetic energy $mv^2/2$ and momentum $m\mathbf{v}$.

The above interpretation is the key that opens the door to the relativistic version of equations (10) and (12), which will ultimately allow us to obtain a natural modification to the Schrödinger equation that makes it Lorentz covariant.

Calling $\gamma = 1/\sqrt{1 - v^2/c^2}$ the Lorentz factor, m the particle's rest mass, and c the speed of light in vacuum, the relativistic kinetic energy and momentum for a particle of mass m are, respectively, $(\gamma - 1)mc^2$ and $\gamma m\mathbf{v}$, where \mathbf{v} is the particle's velocity. With that in mind, we postulate that under a Lorentz transformation connecting two different inertial frames S and S' with relative velocity \mathbf{v} , the wave functions describing a particle of mass m in S and S' are connected by the following relation:

$$\Psi(\mathbf{r}, t) = e^{(i\hbar)\theta(\mathbf{r}', t')} \Psi'(\mathbf{r}', t'), \quad (18)$$

with

$$\theta(\mathbf{r}', t') = (\gamma - 1)mc^2 t' + \gamma m\mathbf{v} \cdot \mathbf{r}' + \text{cte}. \quad (19)$$

Note that we can set the real constant cte to zero without losing in generality. Equation (18) is formally identical to equation (10) and when $c \rightarrow \infty$ equation (19) tends to (12) since $\lim_{c \rightarrow \infty} \gamma = 1$ and $\lim_{c \rightarrow \infty} [(\gamma - 1)mc^2] = mv^2/2$.

With the wave function's transformation law given by equations (18) and (19), we are naturally led to ask the following question:

What is the form of the wave equation that is covariant under a Lorentz boost and whose wave function transforms under such a boost according to equations (18) and (19)?

We can answer that question unambiguously; i.e., we can get one and only one wave equation, if we add the following extra very natural assumptions in our search for the Lorentz covariant Schrödinger equation:

- (i) It tends to the nonrelativistic Schrödinger equation when $c \rightarrow \infty$
- (ii) It is isotropic, namely, covariant under three-dimensional spatial rotations in the same sense as the nonrelativistic Schrödinger equation is
- (iii) It is a homogeneous linear partial differential equation of order not greater than two and with constant coefficients multiplying the derivatives

The first assumption above guarantees that we recover the nonrelativistic quantum mechanics when the physical system studied moves with small velocities. The second assumption assumes that there is no privileged orientation in space, a symmetry that is also respected by the nonrelativistic Schrödinger equation. The third and last extra assumption keeps the superposition principle valid in the relativistic domain and restricts our search to the simplest wave equations, namely, those that are homogeneous and that have at most second-order derivatives and constant coefficients multiplying those derivatives. Note that we allow the coefficient multiplying the wave function $\Psi(\mathbf{r}, t)$ to be

nonconstant since we want to recover the term $V(\mathbf{r}, t)\Psi(\mathbf{r}, t)$ of the nonrelativistic Schrödinger equation. Using this terminology, the nonrelativistic Schrödinger equation is a homogeneous linear partial differential equation of order two (in the spatial variables) with constant coefficients multiplying the derivatives.

Before we start our search for the Lorentz covariant Schrödinger equation, we need first to set up some conventions and notation. The contravariant four-vector x^μ is defined such that $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Working with the metric $g_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$, where $g_{\mu\nu}$ is a diagonal 4×4 matrix, the covariant four-vector is $x_\mu = g_{\mu\nu}x^\nu = (ct, -x, -y, -z)$. We assume the Einstein summation convention with Greek indexes running from 0 to 3 and Latin ones running from 1 to 3. The scalar product between two four-vectors is defined as $x^\mu y_\mu$, which makes it invariant under Lorentz boosts and spatial rotations. We also define the covariant four-gradient as $\partial_\mu = (\partial/\partial x^0, \partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3)$ and the contravariant four-gradient by $\partial^\mu = g^{\mu\nu}\partial_\nu$, where $g^{\mu\nu} = g_{\mu\nu}$ since we are dealing with Minkowski spacetime.

In the four-vector notation just defined, the most general homogeneous linear partial differential equation in the variables x^μ , of order less or equal to two, and with constant coefficients multiplying the derivatives is

$$a^{\mu\nu}\partial_\mu\partial_\nu\Psi(x) + b^\mu\partial_\mu\Psi(x) + f(x)\Psi(x) = 0. \quad (20)$$

Here, $\Psi(x) = \Psi(\mathbf{r}, t)$, $a^{\mu\nu}$, and b^μ are constants (relativistic invariants), and $f(x) = f(\mathbf{r}, t)$; i.e., it may depend on time and on the spatial coordinates. The function $f(x)$ is proportional to the relativistic version of the potential that affects a particle described by the nonrelativistic Schrödinger equation. Later, we will fix its value using the first extra assumption above.

By invoking the isotropic condition for the free particle case, i.e., the wave equation should be covariant under any spatial orthogonal transformation belonging to the group $S O(3)$ when $f(x) = 0$, we can show that several of the $a^{\mu\nu}$ and b^μ constants are zero. To see this note that the free particle Schrödinger equation is covariant under spatial rotations if

$$\Psi(\mathbf{r}, t) = e^{i\alpha}\Psi'(\mathbf{r}', t), \quad (21)$$

where α is a constant (usually set to zero) and \mathbf{r}' is connected to \mathbf{r} by an orthogonal transformation belonging to the $SO(3)$ group, namely, $\mathbf{r}' = M\mathbf{r}$, with $M \in SO(3)$.

The covariance of the free particle Schrödinger equation under transformations belonging to the group $SO(3)$ can be easily proved using equation (21) and by noting that the Laplacian does not change under such transformations. We are thus led to assume that the wave functions that are solutions to the Lorentz covariant Schrödinger equation also obey equation (21) when we spatially rotate the system of coordinates. This requirement is our second extra assumption pointed out above.

Let us start analyzing how the term $b^\mu\partial_\mu\Psi(x)$ of equation (20) changes under a spatial rotation. Writing this term explicitly, we have

$$b^0\partial_0\Psi(\mathbf{r}, t) + b^1\partial_1\Psi(\mathbf{r}, t) + b^2\partial_2\Psi(\mathbf{r}, t) + b^3\partial_3\Psi(\mathbf{r}, t). \quad (22)$$

If we rotate the system of coordinates counterclockwise by π radians about the x^3 -axis, it is not difficult to see that new components of the vector x^μ becomes $x^{\mu'} = (x^0, x^1, x^{2'}, x^{3'}) = (x^0, -x^1, -x^2, x^3)$. Applying the chain rule, this implies that

$$\partial_{\mu'} = (\partial_{0'}, \partial_{1'}, \partial_{2'}, \partial_{3'}) = (\partial_0, -\partial_1, -\partial_2, \partial_3). \quad (23)$$

Substituting equations (21) and (23) into (22) and neglecting the irrelevant constant phase, we obtain

$$b^0\partial_{0'}\Psi'(\mathbf{r}', t) - b^1\partial_{1'}\Psi'(\mathbf{r}', t) - b^2\partial_{2'}\Psi'(\mathbf{r}', t) + b^3\partial_{3'}\Psi'(\mathbf{r}', t). \quad (24)$$

We can only make equations (24) and (22) look the same, i.e., guarantee covariance under this particular rotation, if $b^1 = -b^1$ and $b^2 = -b^2$. This implies that $b^1 = b^2 = 0$. A similar analysis, where we fix, for instance, the x^1 -axis and rotate the other two spatial axis, leads to $b^3 = 0$. This reduces the term given by equation (22) to

$$b^0\partial_0\Psi(\mathbf{r}, t). \quad (25)$$

Turning our attention to the term $a^{\mu\nu}\partial_\mu\partial_\nu\Psi(x)$ of equation (20), we note that the same reasoning that led to $b^j = 0$ leads to $a^{0j} + a^{j0} = 0$, for $j = 1, 2, 3$. This must be the case since the derivatives accompanying the constants a^{0j} or a^{j0} are of the form $\partial_0\partial_j$ and $\partial_j\partial_0$, with only one spatial coordinate. Assuming that we can change the order of the derivatives and this can always be done for well-behaved wave functions $\Psi(\mathbf{r}, t)$, we get $(a^{0j} + a^{j0})\partial_0\partial_j\Psi(\mathbf{r}, t)$ for that piece of the wave equation where one and only one of the superscript indexes is zero. Using the same arguments that led to equation (23), we can always rotate the axis to get $\partial_{j'} = -\partial_j$ for a given j . For example, if we rotate the system of coordinates counterclockwise by π radians about the x^3 -axis, we get that $(a^{01} + a^{10})\partial_0\partial_1\Psi(\mathbf{r}, t)$ changes to $-(a^{01} + a^{10})\partial_{0'}\partial_{1'}\Psi'(\mathbf{r}', t)$ (and similarly for $j = 2$). The covariance can only be guaranteed if $a^{01} + a^{10} = 0$ and $a^{02} + a^{20} = 0$. By rotating the system about the x^1 -axis by π radians, we see that $a^{03} + a^{30} = 0$ in order to preserve the covariance of the wave equation under this particular rotation.

To deal with the remaining purely spatial terms $a^{ij}\partial_i\partial_j\Psi(x)$, we rotate the system of coordinates by $\pi/2$ instead of π radians. For instance, if we rotate the system of coordinates counterclockwise by $\pi/2$ radians about the x^3 -axis, the components of the vector x^μ in the rotated frame is $x^{\mu'} = (x^0, x^1, x^{2'}, x^{3'}) = (x^0, x^2, -x^1, x^3)$. And if we now

apply the chain rule, we get

$$\partial_{\mu'} = (\partial_{0'}, \partial_{1'}, \partial_{2'}, \partial_{3'}) = (\partial_0, \partial_2, -\partial_1, \partial_3). \quad (26)$$

Substituting equation (26) into

$$(a^{12} + a^{21})\partial_1\partial_2\Psi(\mathbf{r}, t) \quad (27)$$

and using equation (21), we obtain, up to an irrelevant global phase, equation (27) in the rotated frame,

$$-(a^{12} + a^{21})\partial_2'\partial_1'\Psi'(\mathbf{r}', t). \quad (28)$$

Since the order of the derivatives can be exchanged freely for physical wave functions, we can only have covariance under this rotation if $a^{12} + a^{21} = 0$. Repeating the previous analysis where we rotate the system either about x^2 or x^1 by $\pi/2$ radians, we get that we should have $a^{13} + a^{31} = 0$ and $a^{23} + a^{32} = 0$ to keep the rotational covariance of the wave equation (20).

By a similar argument, we can show that $a^{11} = a^{22} = a^{33}$. For example, the counterclockwise $\pi/2$ rotation about the x^3 -axis leads to equation (26). This implies, together with equation (21), that the quantity

$$a^{11}\partial_1\partial_1\Psi(\mathbf{r}, t) + a^{22}\partial_2\partial_2\Psi(\mathbf{r}, t) \quad (29)$$

changes to

$$a^{11}\partial_2'\partial_2'\Psi'(\mathbf{r}', t) + a^{22}\partial_1'\partial_1'\Psi'(\mathbf{r}', t) \quad (30)$$

after this rotation. By demanding the covariance under this rotation, we get that $a^{11} = a^{22}$. Repeating the previous argument for a rotation of $\pi/2$ about the x^1 -axis gives $a^{22} = a^{33}$.

Putting all those results together, equation (20) becomes

$$a^{00}\partial_0\partial_0\Psi(x) - a^{11}\partial_j\partial^j\Psi(x) + b^0\partial_0\Psi(x) + f(x)\Psi(x) = 0, \quad (31)$$

where we have used that $\partial_j = -\partial^j$. Note that this equation is covariant under all rotations since the only remaining spatial derivatives, namely, $-\partial_j\partial^j = \nabla^2$, is the Laplacian, which does not change its form under rotations.

So far, we have not investigated what restrictions a Lorentz boost impose on the remaining constants shown in equation (31). In order to do that, it is convenient to rewrite equation (31) as follows:

$$A\partial_0^2\Psi(x) - B\partial_j\partial^j\Psi(x) + C\partial_0\Psi(x) + f(x)\Psi(x) = 0. \quad (32)$$

In this notation, our task is to determine what are the values and relations among A , B , and C that arise after we impose the covariance of equation (32) and assume that $\Psi(x)$ transforms according to the prescription (18) under a Lorentz boost. Note that we are assuming that $f(x)$ is a rel-

ativistic scalar; namely, it transforms to $f'(x')$ under a Lorentz boost.

Since we already proved the covariance of equation (32) under three-dimensional spatial rotations, we do not lose in generality by assuming that the velocity v of the inertial reference frame S' with respect to S is directed along the x^1 -axis. With such a choice for \mathbf{v} , the variables $x^{\mu'}$ are connected to x^{μ} by the following Lorentz transformation:

$$x^0 = \gamma(x^{0'} + \beta x^{1'}), \quad (33)$$

$$x^1 = \gamma(x^{1'} + \beta x^{0'}), \quad (34)$$

$$x^2 = x^{2'}, \quad (35)$$

$$x^3 = x^{3'}, \quad (36)$$

where

$$\beta = \frac{v}{c}, \quad (37)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

By applying the chain rule, we get

$$\partial_0 = \gamma(\partial_{0'} - \beta\partial_{1'}), \quad (38)$$

$$\partial_1 = \gamma(\partial_{1'} - \beta\partial_{0'}), \quad (39)$$

$$\partial_2 = \partial_{2'}, \quad (40)$$

$$\partial_3 = \partial_{3'}. \quad (41)$$

Inserting equations (38)–(41) into (32), using equations (18) and (19), and carrying out the derivatives, the wave equation (32) can be written up to an overall phase as follows:

$$\begin{aligned} A'\partial_0'^2\Psi'(x') + B'\partial_1'^2\Psi'(x') + B\partial_2'^2\Psi'(x') + B\partial_3'^2\Psi'(x') \\ + C'\partial_0'\Psi'(x') + D'\partial_0'\partial_1'\Psi'(x') + E'\partial_1'\Psi'(x') \\ + F'\Psi'(x') + f'(x')\Psi'(x') = 0, \end{aligned} \quad (42)$$

where

$$A' = \gamma^2(A + \beta^2 B), \quad (43)$$

$$B' = \gamma^2(\beta^2 A + B), \quad (44)$$

$$C' = \gamma \left\{ \frac{i2mc}{\hbar} [(1 - \gamma)A - \gamma\beta^2 B] + C \right\}, \quad (45)$$

$$D' = -2\gamma^2\beta(A + B), \quad (46)$$

$$E' = -\gamma\beta \left\{ \frac{i2mc}{\hbar} [(1 - \gamma)A - \gamma B] + C \right\}, \quad (47)$$

$$F' = \frac{imc}{\hbar} \left\{ \frac{imc}{\hbar} [(1-\gamma)^2 A + \gamma^2 \beta^2 B] + (1-\gamma)C \right\}. \quad (48)$$

Equation (42) is what wave equation (32) becomes after a Lorentz boost for arbitrary values of the constants A , B , and C . In order to make equations (42) and (32) look the same, i.e., in order to have Lorentz covariance, we must have $D' = 0$, $E' = 0$, and $F' = 0$. The first condition, $D' = 0$, implies $B = -A$, as can be seen by looking at equation (46). Inserting this last relation into equation (47), we get that $E' = 0$ if $C = -i2mcA/\hbar$. With this value for C and using that $B = -A$, we automatically get $F' = 0$. Also, using that $A = -B$, we get $C' = C = -i2mcA/\hbar$, $A' = A$, and $B' = -A = B$. By gathering all these results together, we can write equation (42) as follows:

$$A\partial_0^2\Psi'(x') - B\partial_{j'}\partial^{j'}\Psi'(x') + C\partial_0\Psi'(x') + f'(x')\Psi'(x') = 0, \quad (49)$$

which explicitly shows the Lorentz covariance of the wave equation (32). If we now use that

$$B = -A \text{ and } C = -i\frac{2mc}{\hbar}A, \quad (50)$$

and that we must have $A \neq 0$, we can cast equation (32) as follows:

$$\partial_\mu\partial^\mu\Psi(x) - i\frac{2mc}{\hbar}\partial_0\Psi(x) + \frac{f(x)}{A}\Psi(x) = 0. \quad (51)$$

To fix the value of A , we need the first extra assumption, namely, we need to impose that equation (32), or equivalently (51), tends to the nonrelativistic Schrödinger equation when $c \rightarrow \infty$. This can be accomplished more easily by first rewriting equation (51) in the nonrelativistic notation. Noting that the d'Alembertian $\partial_\mu\partial^\mu = (1/c^2)(\partial^2/\partial t^2) - \nabla^2$, $\partial_0 = (1/c)(\partial/\partial t)$, and $\Psi(x) = \Psi(\mathbf{r}, t)$, we can rewrite equation (51) as

$$\frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{r}, t)}{\partial t^2} - \nabla^2\Psi(\mathbf{r}, t) - i\frac{2m}{\hbar}\frac{\partial\Psi(\mathbf{r}, t)}{\partial t} + \frac{f(\mathbf{r}, t)}{A}\Psi(\mathbf{r}, t) = 0. \quad (52)$$

If we take the nonrelativistic limit of equation (52), assuming that $\Psi(\mathbf{r}, t)$ and its first- and second-order derivatives does not diverge, the limit $c \rightarrow \infty$ gives

$$-\nabla^2\Psi(\mathbf{r}, t) - i\frac{2m}{\hbar}\frac{\partial\Psi(\mathbf{r}, t)}{\partial t} + \lim_{c \rightarrow \infty} \left[\frac{f(\mathbf{r}, t)}{A} \right] \Psi(\mathbf{r}, t) = 0. \quad (53)$$

On the other hand, the nonrelativistic Schrödinger equation (7) can be put in the following form:

$$-\nabla^2\Psi(\mathbf{r}, t) - i\frac{2m}{\hbar}\frac{\partial\Psi(\mathbf{r}, t)}{\partial t} + \frac{2m}{\hbar^2}V(\mathbf{r}, t)\Psi(\mathbf{r}, t) = 0. \quad (54)$$

Comparing equations (53) and (54), we see that they are equal if

$$\lim_{c \rightarrow \infty} \left[\frac{f(\mathbf{r}, t)}{A} \right] = \frac{2m}{\hbar^2}V(\mathbf{r}, t). \quad (55)$$

Since A does not depend on \mathbf{r} and t , we can without loss of generality set

$$\lim_{c \rightarrow \infty} f(\mathbf{r}, t) = \frac{2m}{\hbar^2}V(\mathbf{r}, t), \quad (56)$$

$$\lim_{c \rightarrow \infty} A = 1, \quad (57)$$

as the conditions upon $f(\mathbf{r}, t)$ and A that allow us to get the nonrelativistic Schrödinger equation as the exact nonrelativistic limit of the Lorentz covariant Schrödinger equation.

We can go even further and prove that $A = 1$, for whatever value of c , if we note that equation (57) implies that

$$A = 1 + \sum_{j=1}^N \frac{A_j}{c^j}, \quad (58)$$

where $N \geq 1$, A_j is a constant, and c^j is a positive power of the speed of light c . In addition to that, equation (56) implies that A must be dimensionless, since otherwise $(f(\mathbf{r}, t)/A)\Psi(\mathbf{r}, t)$ will have a different dimension when compared to the other three terms of equation (52). Now, the only dimensional constants appearing so far, and in particular in the free particle case, are the mass of the particle m , the speed of light c , and Planck's constant \hbar . This means that $A_j = \alpha_j m^{x_j} \hbar^{y_j}$, where α_j is a pure number and x_j, y_j are exponents to be determined such that A_j/c^j becomes dimensionless.

The dimensions of m , c , and \hbar are $[m] = M$, $[c] = LT^{-1}$, and $[\hbar] = ML^2T^{-1}$, where M means mass, L length, and T time. Thus, $[A_j/c^j] = \alpha_j [m^{x_j}][\hbar^{y_j}][c^{-j}] = M^{x_j+y_j}L^{2y_j-j}T^{-y_j+j}$. A_j/c^j is dimensionless if $x_j + y_j = 0$, $2y_j - j = 0$, and $-y_j + j = 0$. The last two equations give $j = 2y_j$ and $j = y_j$, which is only possible if $j = y_j = 0$. This result when inserted into the first relation gives $x_j = 0$. We see then that $A_j/c^j = \alpha_j$, a pure number, and that equation (58) reads

$$A = 1 + \sum_{j=1}^N \alpha_j. \quad (59)$$

But since $\lim_{c \rightarrow \infty} A = 1$, we must have

$$\lim_{c \rightarrow \infty} \sum_{j=1}^N \alpha_j = 0. \quad (60)$$

This implies that

$$\sum_{j=1}^N \alpha_j = 0, \quad (61)$$

for any value of c because α_j , being a pure number, does not depend on c . Equation (61) together with (59) leads to the desired result, namely, $A = 1$ for any value of c .

Using that $A = 1$, we can write the Lorentz covariant Schrödinger equation (52) as

$$\frac{1}{c^2} \frac{\partial^2 \Psi(\mathbf{r}, t)}{\partial t^2} - \nabla^2 \Psi(\mathbf{r}, t) - i \frac{2m}{\hbar} \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} + f(\mathbf{r}, t) \Psi(\mathbf{r}, t) = 0, \quad (62)$$

where the nonrelativistic limit of $f(\mathbf{r}, t)$ is given by equation (56). Finally, if we abuse notation and make the convention that $V(\mathbf{r}, t)$ represents both the nonrelativistic potential as well as its relativistic extension, we can write the Lorentz covariant Schrödinger equation (52) as follows:

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi - i \frac{2m}{\hbar} \frac{\partial \Psi}{\partial t} + \frac{2mV}{\hbar^2} \Psi = 0, \quad (63)$$

where we have written Ψ and V instead of $\Psi(\mathbf{r}, t)$ and $V(\mathbf{r}, t)$ to simplify notation. Equation (63) is the Lorentz covariant wave equation that we were searching for and that satisfies all the assumptions laid out in the beginning of this section. (We should remark that the results here reported can be readily adapted to two independent real fields. This is true because we can map a single complex field to two real and independent scalar fields. See, for instance, Sections 2.2 and 3.2 of ref. [4] and Section 4.2 of ref. [5].)

Note that all time derivatives above are associated with the “geometrical time” of the reference frame S , where all measurements and observables are made and defined for a given experiment in S . Putting it simply, t is the time an observer at the inertial frame S records looking at his or her watch. This should be compared with the following free particle wave equation, derived in refs. [7–10] using a whole set of different assumptions than those employed here:

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi - i \frac{2m}{\hbar} \frac{\partial \Psi}{\partial \tau} = 0. \quad (64)$$

In equation (64), τ is an invariant free parameter usually associated with the proper time of the system being studied. As such, it does not change under a Lorentz boost and equation (64) is Lorentz covariant if its wave function transforms according to the standard definition of a complex scalar function, i.e., $\Psi'(\mathbf{r}', t') = \Psi(\mathbf{r}, t)$. In the context of the Lorentz covariant Schrödinger equation, however, τ is no longer a free parameter and we must set $\tau = t$ in order to have a consistent Lorentz covariant wave equation whose wave function transforms according to equation (18) under a Lorentz boost.

It is important to stress at this point that we have made no attempt to uniquely determine the relativistic invariant $f(\mathbf{r}, t)$, or equivalently, $V(\mathbf{r}, t)$. In this sense, wave equation (63) is not strictly unique. However, for the free particle case, this point is not relevant and we do have a unique wave equation stemming from the three assumptions given at

the beginning of this section. On the other hand, when we have an external field, we just need equation (56) to be satisfied in order to recover the nonrelativistic Schrödinger equation from the wave equation (62). Equivalently, we must have $V(\mathbf{r}, t)$ tending to the potential energy associated to this external field when $c \rightarrow \infty$ to recover the nonrelativistic Schrödinger equation from (63).

Most of the time, in this work, we will be dealing with the free particle case, in particular when we second quantize the Lorentz covariant Schrödinger equation, and with the minimal coupling prescription, when modeling how a charged particle interacts with electromagnetic fields. Therefore, in those instances, the issue of the nonuniqueness of the relativistic invariant $f(\mathbf{r}, t)$ is not a problem since we will be assuming $f(\mathbf{r}, t) = 0$.

On the other hand, we will also solve several bound state or scattering problems where we will assume that $V(\mathbf{r}, t)$ is given by its nonrelativistic version. For particles with small velocities, this is a very good approximation. Also, this is the approach we will follow when dealing with an external gravitational field. (Although we do not need in this work the exact relativistic expression for $V(\mathbf{r}, t)$, we make the following conjecture aiming at an effective invariant modeling of it. What we show below is just a sketch and should not be taken as the final and definitive prescription to the relativistic version of $V(\mathbf{r}, t)$. The argument goes as follows. When we are presented with the nonrelativistic Schrödinger equation (7), it is implicitly assumed that the external source associated with the potential energy V is at rest. In this rest frame, we can define a unitary and dimensionless four-vector $n^\mu = (1, 0, 0, 0)$ that is proportional to the four-momentum of the source (the source momentum is zero since it is at rest). Since V has dimension of energy, we can also postulate the existence of a four-vector v^μ such that in this frame, it is given by $v^\mu = (V, 0, 0, 0)$. Now, it is clear that $n^\mu v_\mu = V$ and it is tempting to replace V in equation (63) by $n^\mu v_\mu$ as the relativistic scalar whose nonrelativistic limit tends to the potential V appearing in the Schrödinger equation.)

It is worth mentioning that we can also write equation (63) in two other useful ways, one that shows its resemblance to the nonrelativistic Schrödinger equation,

$$-\frac{\hbar^2}{2mc^2} \frac{\partial^2 \Psi}{\partial t^2} + i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi, \quad (65)$$

and the other using the four-vector notation,

$$\partial_\mu \partial^\mu \Psi - i \frac{2mc}{\hbar} \partial_0 \Psi + \frac{2mV}{\hbar^2} \Psi = 0. \quad (66)$$

Remark 1. If we were dealing strictly with the free particle case, we would not be able to fix the value of A . In this scenario, $f(\mathbf{r}, t) = 0$ and any value of $A \neq 0$ would give equation (63), with $V(\mathbf{r}, t) = 0$, from equation (52).

Remark 2. Again, if we restrict ourselves to the free particle case, the nonrelativistic Schrödinger equation follows

directly from equation (63) when $c \rightarrow \infty$. There is no need to go through the discussion contained between equations (52) and (63).

Remark 3. We can write equation (63) more compactly if we define $\tilde{\partial}_\mu = \partial_\mu - (mc/\hbar)\delta_\mu$ and $\tilde{\partial}^\mu = \partial^\mu - (mc/\hbar)\varepsilon^\mu$, where $\delta_\mu = \varepsilon^\mu = (i, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Note that $\tilde{\partial}^\mu$ is not $g^{\mu\nu}\tilde{\partial}_\nu$. With this convention, equation (66) is $\tilde{\partial}_\mu\tilde{\partial}^\mu\Psi + (2mV/\hbar^2)\Psi = 0$.

Remark 4. It is possible, though, to redefine ∂_μ such that $\partial^\mu = g^{\mu\nu}\partial_\nu$ is still valid. This is accomplished if $\tilde{\partial}_\mu = \partial_\mu + (i mc/\hbar)\zeta_\mu$, with $\zeta_\mu = (-1, 0, 0, 0)$, and if we measure the potential energy from a different origin, starting from $mc^2/2$ instead of zero. In other words, using $\tilde{\partial}_\mu$ and $\bar{V} = V + mc^2/2$, equation (63) becomes $\tilde{\partial}_\mu\tilde{\partial}^\mu\Psi + (2m\bar{V}/\hbar^2)\Psi = 0$, where $\tilde{\partial}^\mu = g^{\mu\nu}\tilde{\partial}_\nu$.

3.2.1. Example: The Plane Wave. It is instructive at this point to study the free particle case in the light of equation (63), the Lorentz covariant Schrödinger equation. For simplicity, we deal with the one-dimensional case, assume that in the reference frame S the particle of mass m has momentum $\mathbf{p} = (p, 0, 0)$, and that S' moves away from S with velocity $\mathbf{v} = (v, 0, 0)$, where $v > 0$.

Setting $V(x, t) = 0$ and writing

$$\Psi(x, t) = e^{(i/\hbar)(px - Kt)}, \quad (67)$$

it is not difficult to see that equation (67) is a solution to equation (63) if

$$K = -mc^2 + \sqrt{m^2c^4 + p^2c^2}. \quad (68)$$

Note that we also have another possible relation between K and p giving a solution to equation (63), i.e., $K = -mc^2 - \sqrt{m^2c^4 + p^2c^2}$. We will deal with this other possible solution later, where we will attempt to give a physical interpretation to it that has intriguing consequences.

Looking at equation (68) and remembering that a relativistic particle satisfies

$$E^2 = m^2c^4 + p^2c^2, \quad (69)$$

E being the total energy of the particle, we immediately see that K is the particle's kinetic energy, where we have subtracted from E the particle's rest energy mc^2 . We can thus rewrite equation (67) as

$$\Psi(x, t) = e^{(i/\hbar)(m^2c^2t + px - Et)}. \quad (70)$$

Using equations (18) and (19), the wave function in the

reference frame S' is, up to a constant phase,

$$\begin{aligned} \Psi'(x', t') &= e^{-(i/\hbar)[(y-1)mc^2t' + \gamma mvx']} \Psi(x, t) \\ &= e^{-(i/\hbar)[(y-1)mc^2t' + \gamma mvx']} \Psi\left(\gamma(x' + vt'), \gamma\left(t' + \frac{vx'}{c^2}\right)\right), \end{aligned} \quad (71)$$

where we have used equations (33) and (34) to obtain the last line. Employing equation (70) to evaluate equation (71), we arrive at

$$\Psi'(x', t') = e^{(i/\hbar)[mc^2t' + \gamma(p^1 - \beta p^0)x' - c\gamma(p^0 - \beta p^1)t']}, \quad (72)$$

where $p^0 = E/c$ and $p^1 = p$ are the first two components of the four-momentum $p^\mu = (p^0, p^1, p^2, p^3) = (E/c, p_x, p_y, p_z)$.

Under a Lorentz boost in the x -direction, the four-momentum p^μ transforms as

$$p^{0'} = \gamma(p^0 - \beta p^1), \quad (73)$$

$$p^{1'} = \gamma(p^1 - \beta p^0), \quad (74)$$

$$p^{2'} = p^2, \quad (75)$$

$$p^{3'} = p^3. \quad (76)$$

Using equations (73) and (74), equation (72) becomes

$$\Psi'(x', t') = e^{(i/\hbar)[m^2c^2t' + p^{1'}x' - cp^{0'}t']} = e^{(i/\hbar)[m^2c^2t' + p^{1'}x' - E't']}, \quad (77)$$

where $p^{1'} = p^{1'}$ and $E' = p^{0'}c$ are, respectively, the particle's momentum and total energy in reference frame S' .

As expected, equation (77), which is a solution to the Lorentz covariant Schrödinger equation in the reference frame S' , can be obtained from equation (70) by simply changing the unprimed quantities x, t, p , and E to the respective primed ones.

4. Basic Properties of the Lorentz Covariant Schrödinger Equation

4.1. Probability Four-Current Density. If we write the wave function $\Psi(\mathbf{r}, t)$ in its polar form,

$$\Psi(\mathbf{r}, t) = R(\mathbf{r}, t)e^{iS(\mathbf{r}, t)/\hbar} = \sqrt{\rho(\mathbf{r}, t)}e^{iS(\mathbf{r}, t)/\hbar}, \quad (78)$$

the transformation law (18) for the wave function under a Lorentz boost becomes

$$R(\mathbf{r}, t) = R'(\mathbf{r}', t') \text{ or } \rho(\mathbf{r}, t) = \rho'(\mathbf{r}', t'), \quad (79)$$

$$S(\mathbf{r}, t) = S'(\mathbf{r}', t') + (\gamma - 1)mc^2 t' + \gamma m \mathbf{v} \cdot \mathbf{r}', \quad (80)$$

where we have set cte = 0 in equation (19).

Using equations (79) and (80), we want to prove that the following object,

$$\begin{aligned} J^\mu &= \left(\rho c - \frac{\rho}{m} \frac{\partial S}{c \partial t}, \frac{\rho}{m} \frac{\partial S}{\partial x}, \frac{\rho}{m} \frac{\partial S}{\partial y}, \frac{\rho}{m} \frac{\partial S}{\partial z} \right) \\ &= \left(\rho c - \frac{\rho}{m} \partial_0 S, \frac{\rho}{m} \partial_1 S, \frac{\rho}{m} \partial_2 S, \frac{\rho}{m} \partial_3 S \right), \end{aligned} \quad (81)$$

is a contravariant four-vector and that it satisfies the continuity equation, $\partial_\mu J^\mu = 0$, whenever the potential $V(\mathbf{r}, t)$ is real. These two properties allow us to identify J^μ as the probability four-current density of the Lorentz covariant Schrödinger equation.

Note that the three spatial terms of J^μ , namely, $\mathbf{J} = (J^1, J^2, J^3)$, are formally identical to the three vector components of the probability current density of the nonrelativistic Schrödinger equation [1],

$$\mathbf{J} = -\frac{i\hbar}{2m} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*). \quad (82)$$

Equivalently, \mathbf{J} can also be seen as formally equal to the three spatial terms of the Klein-Gordon equation four-current density [2].

The time component of J^μ , on the other hand, can be written as

$$J^0 = \rho c + \frac{i\hbar}{2m} (\Psi^* \partial_0 \Psi - \Psi \partial_0 \Psi^*). \quad (83)$$

The time component of the Klein-Gordon four-current density [2] is formally equal to the second term of the right-hand side of equation (83). The first term, ρc , is the extra ingredient we need to guarantee that J^μ is a four-vector if $\Psi(\mathbf{r}, t)$ transforms according to equations (79) and (80).

Let us start the proof that J^μ is a four-vector. For simplicity, and without loss of generality, we assume a Lorentz boost along the x^1 -direction. This implies that $\mathbf{v} = (v, 0, 0)$. Thus, using equations (38), (79), and (80) and carrying out the derivatives, the time component of J^μ can be written as follows,

$$J^0 = \rho' c - \frac{\rho'}{m} \gamma \left[\partial_0' S' + (\gamma - 1)mc - \beta \partial_1' S' - \beta \gamma m v \right]. \quad (84)$$

Noting that $v = \beta c$ and that $\gamma^2(1 - \beta^2) = 1$, we get after a couple of simplifications

$$\begin{aligned} J^0 &= \rho' c - \frac{\rho'}{m} \gamma \left[\frac{mc}{\gamma} (1 - \gamma) + \partial_0' S' - \beta \partial_1' S' \right] \\ &= \gamma \left[\left(\rho' c - \frac{\rho'}{m} \partial_0' S' \right) + \beta \left(\frac{\rho'}{m} \partial_1' S' \right) \right] = \gamma [J^{0'} + \beta J^{1'}]. \end{aligned} \quad (85)$$

In a similar way, we get

$$J^1 = \gamma (J^{1'} + \beta J^{0'}), \quad (86)$$

$$J^2 = J^{2'}, \quad (87)$$

$$J^3 = J^{3'}. \quad (88)$$

Equations (85)-(88) are the laws a contravariant vector must obey after a Lorentz boost in the x^1 -direction (cf. equations (33)-(36)), proving that J^μ is indeed a four-vector.

Let us move to the proof that J^μ is a conserved current. Computing the four-divergence of equation (81), we get

$$\partial_\mu J^\mu = c \partial_0 \rho - \frac{1}{m} \partial_\mu \rho \partial^\mu S - \frac{\rho}{m} \partial_\mu \partial^\mu S. \quad (89)$$

To make progress, we need to insert the polar form of the wave function, equation (78), into the Lorentz covariant Schrödinger equation, equation (63). Carrying out the derivatives and noting that $R = \sqrt{\rho}$, we can show that equation (63) is equivalent to the following two coupled equations:

$$\rho \partial_\mu \partial^\mu \rho - \frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{2\rho^2}{\hbar^2} (\partial_\mu S \partial^\mu S - 2mc \partial_0 S - 2mV_R) = 0, \quad (90)$$

$$\rho \partial_\mu \partial^\mu S + \partial_\mu \rho \partial^\mu S - mc \partial_0 \rho + \frac{2m\rho V_I}{\hbar} = 0. \quad (91)$$

Equations (90) and (91) are, respectively, what we get by equating the real and imaginary parts of equation (63) to zero. We have also written

$$V = V_R + iV_I, \quad (92)$$

since a complex potential can be used to phenomenologically model, for example, dissipative processes.

If we divide equation (91) by m , we can write it as

$$c \partial_0 \rho - \frac{1}{m} \partial_\mu \rho \partial^\mu S - \frac{\rho}{m} \partial_\mu \partial^\mu S = \frac{2\rho V_I}{\hbar}. \quad (93)$$

Comparing equations (93) and (89), we see that

$$\partial_\mu J^\mu = \frac{2\rho}{\hbar} V_I. \quad (94)$$

Equation (94) is the continuity equation with a source or sink modeled by the complex part of the potential V . If V is real, as one would expect for closed systems described by Hermitian operators, we arrive at

$$\partial_\mu J^\mu = 0, \quad (95)$$

proving that J^μ is the conserved current of the Lorentz covariant Schrödinger equation.

It is worth mentioning that $\partial_\mu J^\mu$ tends exactly to the continuity equation of the nonrelativistic Schrödinger

equation. This can be seen writing equation (95) explicitly in terms of its derivatives and using the definition of J^0 as given in equation (81),

$$\partial_\mu J^\mu = \frac{1}{c} \frac{\partial J^0}{\partial t} + \nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t} - \frac{1}{mc^2} \frac{\partial}{\partial t} \left(\rho \frac{\partial S}{\partial t} \right) + \nabla \cdot \mathbf{J} = 0. \quad (96)$$

Taking the limit where $c \rightarrow \infty$, we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (97)$$

which is the continuity equation associated with the nonrelativistic Schrödinger equation.

4.2. Manifest Covariance. Our goal here is to recast the Lorentz covariant Schrödinger equation in a manifestly covariant way. This can be accomplished by noting that the Lorentz covariant Schrödinger equation, equation (63), is equivalent to equations (90) and (91). Moreover, equation (91) is equivalent to the continuity equation (94), which is already written in a manifestly covariant form,

$$\partial_\mu J^\mu = \frac{2\rho}{\hbar} V_I. \quad (98)$$

What remains to be done is to rewrite equation (90) in a manifestly covariant way. Using equation (81), it is not difficult to see that

$$J_\mu J^\mu = \rho^2 c^2 - \frac{2\rho^2 c}{m} \partial_0 S + \frac{\rho^2}{m^2} \partial_\mu S \partial^\mu S. \quad (99)$$

After multiplying equation (99) by m^2/ρ^2 , we arrive at

$$\frac{m^2}{\rho^2} (J_\mu J^\mu - \rho^2 c^2) = \partial_\mu S \partial^\mu S - 2mc \partial_0 S. \quad (100)$$

Using equation (100), we can rewrite equation (90) as

$$\rho \partial_\mu \partial^\mu \rho - \frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{2m^2}{\hbar^2} (J_\mu J^\mu - \rho^2 c^2) + \frac{4m\rho^2}{\hbar^2} V_R = 0. \quad (101)$$

Equations (98) and (101) are the manifestly covariant equations we were looking for. They are, respectively, the imaginary and real parts of the Lorentz covariant Schrödinger equation, written in a manifestly covariant way, that one obtains after inserting the wave function in its polar form (78) into equation (63).

4.3. Connection to the Klein-Gordon Equation. Looking at equation (63), it is not difficult to realize that the nonrelativistic limit of the Lorentz covariant Schrödinger equation gives exactly the usual Schrödinger equation. Indeed, taking the limit where the speed of light $c \rightarrow \infty$, the first term of equation (63) tends to zero and what remains is the nonrelativistic Schrödinger equation. This is most clearly seen using equation (65), where the remaining terms after the limit in which c

$\rightarrow \infty$ is the nonrelativistic Schrödinger equation written in its usual form. Note that to obtain the nonrelativistic Schrödinger equation from the Klein-Gordon equation, a much more involved limiting process is needed [2].

On the other hand, if the mass $m \rightarrow 0$, equation (63) is the standard relativistic invariant wave equation for a massless scalar particle or, equivalent, the massless Klein-Gordon equation. It is worth mentioning that when $m = 0$, Ψ is a relativistic scalar in the conventional way, i.e., $\Psi \rightarrow \Psi$ after a Lorentz transformation (cf. equations (18) and (19) for $m = 0$).

Our goal now is to search for the phase transformation that we must implement on Ψ to go from the Lorentz covariant Schrödinger equation, equation (63), to the Klein-Gordon equation with $m \neq 0$ [2]. In order to determine the correct phase transformation, we make the following ansatz:

$$\Psi(\mathbf{r}, t) = e^{-i(\hbar)f(t)} \Phi(\mathbf{r}, t), \quad (102)$$

where $f(t)$ depends only on the time. The reason to work with a function f that does not depend on the position vector \mathbf{r} is related to the fact that the spatial derivatives of equation (63) are already the ones appearing in the Klein-Gordon equation. To arrive at the Klein-Gordon equation starting with equation (63), we just need to be able to find an f that allows us to get rid of the first-order time derivative after the transformation above.

Inserting equation (102) into equation (63) and carrying out the derivatives, we get, up to a global phase,

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi - \frac{2i}{\hbar} \left(m + \frac{\dot{f}}{c^2} \right) \frac{\partial \Phi}{\partial t} - \left(\frac{i\ddot{f}}{c^2 \hbar} + \frac{\dot{f}^2}{c^2 \hbar^2} + \frac{2mf}{\hbar^2} - \frac{2mV}{\hbar^2} \right) \Phi = 0, \quad (103)$$

where the dot and double dots over f denote the first- and second-order derivatives with respect to time t . To obtain the Klein-Gordon equation, we must first impose that

$$m + \frac{\dot{f}}{c^2} = 0, \quad (104)$$

which leads to

$$f(t) = -mc^2 t + \text{cte}. \quad (105)$$

This condition guarantees that equation (103) has no first-order derivatives of Φ with respect to time.

Inserting equation (105) into equation (103), we get

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi + \left(\frac{m^2 c^2}{\hbar^2} + \frac{2mV}{\hbar^2} \right) \Phi = 0. \quad (106)$$

Equation (106) can be seen as a generalization of the Klein-Gordon equation, effectively describing a scalar field Φ in the presence of an external potential V . If we set $V = 0$, we recover the Klein-Gordon equation for massive scalar fields [2],

$$\begin{array}{c}
\text{Massless Klein-Gordon equation} \\
\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi = 0 \\
\uparrow m=0 \\
\boxed{\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi - i \frac{2m}{\hbar} \frac{\partial \Psi}{\partial t} + \frac{2mV}{\hbar^2} \Psi = 0} \xrightarrow{c \rightarrow \infty} \text{Non-relativistic Schrödinger equation} \\
\downarrow V=0 \\
\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi + \frac{m^2 c^2}{\hbar^2} \Phi = 0 \\
\text{Klein-Gordon equation} \\
\Psi(\mathbf{r}, t) = e^{\frac{i}{\hbar} m c^2 t} \Phi(\mathbf{r}, t)
\end{array}$$

FIGURE 1: Relationship among the Lorentz covariant Schrödinger equation and the other standard scalar wave equations of quantum mechanics.

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi + \frac{m^2 c^2}{\hbar^2} \Phi = 0. \quad (107)$$

We have thus proved that if Ψ satisfies the Lorentz covariant Schrödinger equation (63), the wave function Φ , connected to Ψ by the phase transformation

$$\Psi(\mathbf{r}, t) = e^{(i/\hbar) m c^2 t} \Phi(\mathbf{r}, t), \quad (108)$$

satisfies the generalized Klein-Gordon equation (99). In Figure 1, we show pictorially how the Lorentz covariant Schrödinger equation is connected to the nonrelativistic Schrödinger equation and to the Klein-Gordon equation.

5. Predictions of the Lorentz Covariant Schrödinger Equation

5.1. The Free Particle. Here and in Sections 5.2, 5.3, and 5.4, we assume, without losing generality, that we deal with a one-dimensional problem.

The free particle solution (plane wave solution) to the Lorentz covariant Schrödinger equation is obtained assuming $V(x, t) = 0$. Inserting the ansatz

$$\Psi(x, t) = e^{(i/\hbar)(px - Kt)} \quad (109)$$

into equation (63), we see that equation (109) is a solution to equation (63) if

$$K = K_{\pm} = -m c^2 \pm \sqrt{m^2 c^4 + p^2 c^2}. \quad (110)$$

The physical meaning of K_+ is straightforward. It is the relativistic kinetic energy of a free particle of mass $m > 0$, where $E = K_+ + m c^2 = \sqrt{m^2 c^4 + p^2 c^2}$ is its total relativistic energy. The physical interpretation of K_- is subtler than that of K_+ , at least at the stage of first quantization. A rigorous justification of what will be said about K_- here and in Section 5.2 will be given in Section 7, where we develop the quantum field theory of the Lorentz covariant Schrödinger equation. Our goal here is to motivate and to qualitatively explore the main features of equation (63) that lead to the results outlined below and rigorously established when we

“second quantize” the Lorentz covariant Schrödinger equation.

If we could redefine the zero of the potential V in such a way that K_{\pm} changed to $\tilde{K}_{\pm} = \pm \sqrt{m^2 c^4 + p^2 c^2}$, the same interpretation associated with the negative and positive energy solutions of the Klein-Gordon equation would be valid here. We would associate $\tilde{K}_+ > 0$, the positive energy solution of the wave equation, to a particle of mass m , energy \tilde{K}_+ , and traveling forward in time ($t > 0$). The negative solution would be reinterpreted as an antiparticle with the same mass m and positive energy $-\tilde{K}_- = \tilde{K}_+$ traveling backwards in time:

$$e^{(i/\hbar)(px - \tilde{K}_- t)} = e^{(i/\hbar)[px + \tilde{K}_-(-t)]} = e^{(i/\hbar)[px - \tilde{K}_+(-t)]}. \quad (111)$$

However, as we show in Section 5.2, we cannot redefine the zero of the potential independently of \mathbf{p} to get $\tilde{K}_{\pm} = \pm \sqrt{m^2 c^4 + p^2 c^2}$. This means that for each value of the magnitude of \mathbf{p} , we would need to work with a different zero of the potential for this interpretation to make sense. We believe this solution is unsatisfactory, and in order to have a \mathbf{p} -independent interpretation of K_{\pm} , we are obliged to take the following route.

If we stick with the zero of the potential at $V = 0$, we readily see that $K_- \neq -K_+$ and the previous interpretation attributed to the negative energy K_- does not follow. This can be remedied by the following procedure. Looking at equation (111) and comparing the far left term with the far right one, we note that $\tilde{K}_- t = \tilde{K}_+(-t)$. This last equality is clearly not satisfied by K_- and K_+ . We can only satisfy it if the antiparticle that travels backwards in time also has a negative mass, namely, K_- changes to $\bar{K}_- = m c^2 - \sqrt{m^2 c^4 + p^2 c^2}$. In this scenario, we have $\bar{K}_- t = K_+(-t)$ and the particle of mass m , positive energy K_+ , and traveling forward in time must have an antiparticle with a negative mass traveling backwards in time with energy $-\bar{K}_- = K_+$.

This interpretation is not too wide of the mark if we study the solutions to the Lorentz covariant Schrödinger equation when we set $V = 0$ and change $m \rightarrow -m$. In this case, it is not difficult to see that equation (63) has the

following solutions:

$$\Psi(x, t) = e^{(i/\hbar)(px - \bar{K}t)}, \quad (112)$$

with

$$\bar{K} = \bar{K}_{\pm} = mc^2 \pm \sqrt{m^2c^4 + p^2c^2}. \quad (113)$$

We now have that $\bar{K}_- = -K_+$, which allows us to write

$$e^{(i/\hbar)(px - \bar{K}_-t)} = e^{(i/\hbar)[px + \bar{K}_-(-t)]} = e^{(i/\hbar)[px - K_+(-t)]}. \quad (114)$$

Therefore, to the particle with mass m , energy K_+ , traveling forward in time, we must have an antiparticle with mass $-m$, energy $-\bar{K}_- = K_+$, traveling backwards in time. The same interpretation can be attached to the pair of solutions with energies given by \bar{K}_+ and K_- since here we also have $-K_- = \bar{K}_+$. The solution with $\bar{K}_+ = mc^2 + \sqrt{m^2c^4 + p^2c^2}$ describes a particle of mass $-m$, traveling forward in time, and with positive energy \bar{K}_+ , while the solution with K_- is its antiparticle with mass m traveling backwards in time.

We can intuitively understand why this interpretation will rigorously appear in the second quantization of the Lorentz covariant Schrödinger equation by the following argument. Looking at equation (63) when $V = 0$, we note that if we change m to $-m$, we get the following equation:

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi + i \frac{2m}{\hbar} \frac{\partial \Psi}{\partial t} = 0. \quad (115)$$

On the other hand, by taking the complex conjugate of equation (63), we arrive at

$$\frac{1}{c^2} \frac{\partial^2 \Psi^*}{\partial t^2} - \nabla^2 \Psi^* + i \frac{2m}{\hbar} \frac{\partial \Psi^*}{\partial t} = 0. \quad (116)$$

Comparing equations (115) and (116), we see that they are formally the same, having the same mathematical structure. In other words, the solution Ψ^* of the complex conjugate equation is equivalent to the solution Ψ of the negative mass equation. In this way, when second quantizing the complex Lagrangian leading to equation (63) and its complex conjugate version, the conserved charge obtained via Noether's theorem will be formed by particles with positive and negative masses.

It is instructive to remark that the Klein-Gordon equation does not possess this feature, even when we deal with its complex version. Looking at equation (107), we see that the Klein-Gordon equation is unchanged if $m \rightarrow -m$ and that its complex conjugate version is formally equivalent to equation (107). The key ingredient that makes the Lorentz covariant Schrödinger equation change its structure when $m \rightarrow -m$, or when we take its complex conjugate, is the presence of the term

$$i \frac{2m}{\hbar} \frac{\partial \Psi}{\partial t}, \quad (117)$$

which has the imaginary number i and which is linear in the mass m . The Klein-Gordon equation has no term linear in m or explicitly depending on i . (In the context of Foldy-Wouthuysen transformations, the interpretation of the positive and negative energy solutions of the Dirac equation as describing particles with positive and “negative” masses is also possible for weak external fields [11]. Furthermore, for optical and quantum hydrodynamic systems, effective negative masses associated with particle-like excitations in those systems were experimentally reported [12–16].)

5.2. Particle in a Constant Potential. We now assume that the potential $V(x, t)$ is a nonnull constant V . Inserting the ansatz given by equation (109) into (63), we have that it is a solution to equation (63) if

$$K = K_{\pm} = -mc^2 \pm \sqrt{m^2c^4 + p^2c^2 + 2mc^2V}. \quad (118)$$

Adding mc^2 to both sides of equation (118) and squaring it lead to

$$E^2 - p^2c^2 = m^2c^4 + 2mc^2V, \quad (119)$$

where we have defined

$$E = K + mc^2. \quad (120)$$

The right-hand side of equation (119) is a Lorentz invariant and can be compactly written as

$$p^\mu p_\mu = m^2c^2 + 2mV = m^2c^2 \left(1 + \frac{2V}{mc^2}\right), \quad (121)$$

where $p^\mu = (E/c, \mathbf{p})$ is the four-momentum.

In the particle's rest frame $\mathbf{p} = 0$ and evaluating equation (121) in this frame, we get for the particle's rest energy,

$$E_0 = mc^2 \sqrt{1 + \frac{2V}{mc^2}} = m'c^2. \quad (122)$$

We see that the presence of the potential V changes the rest mass of the particle to

$$m' = m \sqrt{1 + \frac{2V}{mc^2}}. \quad (123)$$

For small values of V , such that

$$\frac{V}{mc^2} \ll 1, \quad (124)$$

we have

$$m' \approx m + \frac{V}{c^2}. \quad (125)$$

Equation (125) means that the mass of any particle in the presence of a small potential changes equally by the same quantity, namely, V/c^2 . For high values of V , the changes in

the mass are different for each type of particle (it depends on the value of m) and are given by equation (123). These results tell us that the mass of a particle described by the Lorentz covariant Schrödinger equation depends on the background external field to which it is subjected and which is phenomenologically modeled by the potential V . This feature might be useful in the effective description of condensed matter systems, where the “spacetime” structure is given by an effective Minkowski spacetime with limiting velocity $c_{\text{eff}} < c$ and effective masses differing from the “bare” masses defined when $V = 0$. Note that this feature is also seen in the generalized Klein-Gordon equation (106) since it also depends on the potential V . (The possibility of an external potential that is not related to electromagnetic or gravitational fields and that may change the rest mass of a particle is introduced by Stueckelberg in ref. [17].)

We can also obtain a stronger condition than (124) to the validity of equation (125) and in a way that connects the magnitude of the potential V with the magnitude of the particle’s momentum $|\mathbf{p}| = |p|$. This is achieved by rewriting equation (118) as

$$\left(\frac{E}{pc}\right)^2 = \left(\frac{mc^2}{pc} + \frac{V}{pc}\right)^2 + 1 - \left(\frac{V}{pc}\right)^2 \quad (126)$$

and noting that if

$$\left(\frac{V}{pc}\right)^2 \ll 1, \quad (127)$$

we get that $1 - (V/pc)^2 \approx 1$ and thus

$$\left(\frac{E}{pc}\right)^2 \approx \left(\frac{mc^2}{pc} + \frac{V}{pc}\right)^2 + 1. \quad (128)$$

Rewriting equation (128) as

$$E^2 - p^2 c^2 \approx \left(m + \frac{V}{c^2}\right)^2 c^4 = m'^2 c^4, \quad (129)$$

we immediately see that it is the energy-momentum relation for a particle of mass m' , where m' is given by equation (125). Therefore, if

$$|V| \ll |\mathbf{p}|c, \quad (130)$$

a particle with mass m when subjected to the constant field V will behave like a free particle of mass $m + V/c^2$.

In order to better appreciate the meaning of K_+ , as given by equation (118), we expand it in powers of $1/c^2$ and compare the resulting expansion with what one would expect in the nonrelativistic limit. A simple calculation leads to

$$K_+ = \frac{p^2}{2m} + V + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (131)$$

where $\mathcal{O}(1/c^2)$ denotes terms of order higher or equal to $1/c^2$.

Looking at equation (131), we see that in the limit of $c \rightarrow \infty$, we obtain that K_+ is the nonrelativistic kinetic energy plus the potential energy of the particle. This is consistent with assigning to K_+ the interpretation that it is the total relativistic energy of the particle minus its “bare” rest energy mc^2 when we get back to the relativistic domain. A similar interpretation is attached to K_- if we remember that we should consider it the solution to the Lorentz covariant Schrödinger equation with a negative mass. In other words, K_- can only be interpreted as the particle’s relativistic kinetic energy when $V = 0$.

We end this section proving, as claimed in Section 5.1, that it is not possible to redefine the zero of the potential independently of \mathbf{p} to get $K_{\pm} = \pm\sqrt{m^2 c^4 + p^2 c^2}$. If we try to obtain a scenario where $K_{\pm} = \pm\sqrt{m^2 c^4 + p^2 c^2}$, we have to solve for V the following equation (cf. equation (118)):

$$-mc^2 \pm \sqrt{m^2 c^4 + p^2 c^2 + 2mc^2 V} = \pm\sqrt{m^2 c^4 + p^2 c^2}. \quad (132)$$

Its solutions are

$$V = \frac{mc^2}{2} \pm \sqrt{m^2 c^4 + p^2 c^2}. \quad (133)$$

Looking at equation (133), we see that it is not possible to have V independent of the magnitude of $|\mathbf{p}|$. Moreover, even allowing for a p -dependence, we cannot get the job done since we need different values of V to set either K_+ or K_- to be $\sqrt{m^2 c^4 + p^2 c^2}$ or $-\sqrt{m^2 c^4 + p^2 c^2}$, respectively (note the \pm in the expression for V above). This completes the proof of the claim given in Section 5.1.

5.3. Complex Variable Telegraph Equation. If we look at equation (121), we note that as we decrease the value of the external potential V , we decrease the value of the Lorentz invariant $p^\mu p_\mu$. Eventually, for a sufficiently low potential, we will get $p^\mu p_\mu = 0$. In this scenario, when $V = -mc^2/2$, it is expected that a particle of “bare” mass m behaves like a massless particle. Moreover, a wave packet in this case is expected to propagate without distortion and without spreading. This is indeed the case as we prove below.

The one-dimensional Lorentz covariant Schrödinger equation, equation (63), can be written as

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} - i \frac{2m}{\hbar} \frac{\partial \Psi}{\partial t} + \frac{2mV}{\hbar^2} \Psi = 0. \quad (134)$$

This equation has the same structure as the Heaviside telegraph equation [18, 19], a real partial differential equation which under certain conditions has a solution in which a distortionless propagation of a wave packet is possible. Equation (134) is a complex version of the telegraph equation, and it inherits the main features of the real Heaviside telegraph equation. As we show below, under a particular condition for the external potential V , equation (134) has a solution in which a wave packet propagates distortionless and without dispersion.

We should also mention that when $V = 0$, equation (134) is similar to the complex telegraph equation postulated by Sancho in his search for a nonrelativistic wave equation leading to a noninstantaneous spread of a localized wave packet [20],

$$\frac{2m\tau}{\hbar} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} - i \frac{2m}{\hbar} \frac{\partial \Psi}{\partial t} = 0, \quad (135)$$

with τ being a free parameter with dimension of time. Equation (126) was obtained by extending to the complex plane the real telegraph equation [18, 19] and adjusting the coefficients multiplying the derivatives such that the Schrödinger equation is obtained for an “instantaneous” relaxation time ($\tau = 0$). In ref. [20], it was shown that we indeed have a noninstantaneous spread of the wave function for a localized initial condition whenever $\tau \neq 0$. Comparing equations (134) and (135), we see that they become equal if $\tau = \hbar/(2mc^2)$.

If we insert the ansatz (cf. equation (108))

$$\Psi(x, t) = e^{(i\hbar)mc^2 t} \Phi(x, t) \quad (136)$$

into equation (134), we get

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \left(\frac{m^2 c^2}{\hbar^2} + \frac{2mV}{\hbar^2} \right) \Phi = 0. \quad (137)$$

Choosing

$$V = -\frac{mc^2}{2}, \quad (138)$$

equation (137) becomes

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} = 0. \quad (139)$$

This is the standard wave equation for a massless scalar particle in one dimension. Its general solution is

$$\Phi(x, t) = F(x - ct) + G(x + ct), \quad (140)$$

where $F(x - ct)$ and $G(x + ct)$ are nondispersive traveling waves to the right and to the left, respectively.

The general solution to equation (134) is thus

$$\Psi(x, t) = e^{(i\hbar)mc^2 t} [F(x - ct) + G(x + ct)]. \quad (141)$$

For a wave packet traveling to the right, we have $G(x + ct) = 0$ and thus

$$\Psi(x, t) = e^{(i\hbar)mc^2 t} F(x - ct). \quad (142)$$

This leads to a probability density that propagates without dispersion and distortion to the right and with the speed of light c ,

$$\rho(x, t) = \Psi^*(x, t)\Psi(x, t) = |F(x - ct)|^2. \quad (143)$$

We can also test the previous result by studying the propagation of a wave packet for decreasing values of the potential V . As we decrease V , less dispersive is the evolution of the wave packet, until we reach the critical value $V = -mc^2/2$ where a dispersionless evolution of the wave packet is achieved.

A possible way to achieve such a huge negative potential would be to work with a scalar charged particle subjected to a constant electric potential φ . If q is its charge, $V = q\varphi$ and equating it to $-mc^2/2$ gives $\varphi = -mc^2/(2q)$. We can estimate the order of magnitude of the electric potential using the rest mass of the electron and its charge. This gives $\varphi \approx -2.54 \times 10^5 V$. In other words, one needs an electric potential of magnitude of the order of 100 thousands volts to start to see a dispersionless propagation.

However, if we work with greater charges or lower masses, the lower the voltage needed. For some condensed matter systems such as a semiconductor, the effective mass of the electron can be as low as 1% of its rest mass. In these materials, an electric potential of the order of $-10^3 V$ would be enough to see a dispersionless evolution.

5.4. Free Particle Wave Packets. We now investigate how wave packets evolve according to the Lorentz covariant Schrödinger equation when we set $V = 0$, comparing its evolution to the ones predicted by the Klein-Gordon equation and by the nonrelativistic Schrödinger equation when we set the Hamiltonian equal to $\sqrt{m^2 c^4 + p^2 c^2}$, with $\hat{p} = -i\hbar \partial / \partial x$. In all cases, we will be dealing with the positive energy solutions.

If we write the wave function as

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int f_{\Psi}(p) e^{(i\hbar)[px - K(p)t]} dp, \quad (144)$$

$f_{\Psi}(p)$ is the momentum distribution of the wave packet and $K(p)$, a function of the momentum p , is determined by inserting equation (144) into the corresponding wave equation and imposing that $\Psi(x, t)$ should be a solution to the wave equation. The integration above includes the entire real line; i.e., p runs from $-\infty$ to ∞ .

If we insert equation (144) into the one-dimensional Lorentz covariant Schrödinger equation for a free particle, namely, equation (134) with $V = 0$, we guarantee that equation (144) is a solution to it if

$$K(p) = -mc^2 + \sqrt{m^2 c^4 + p^2 c^2}, \quad (145)$$

where $K(p)$ is the positive energy solution. Similarly, inserting equation (144) into equation (137) with $V = 0$, where the latter equation is the one-dimensional Klein-Gordon equation for a particle with mass m , we get

$$K(p) = \sqrt{m^2 c^4 + p^2 c^2}. \quad (146)$$

Comparing the two dispersion relations above, equations (145) and (146), we note that they differ by $-mc^2$, which does not depend on p . This means that the probability

density $|\Psi(x, t)|^2$ computed from equation (144) is the same whether we work with the Lorentz covariant Schrödinger equation or with the Klein-Gordon equation.

It is also instructive if we compute the dispersion relation for the following wave equation:

$$i\hbar \frac{d}{dt} |\Psi\rangle = \sqrt{m^2 c^4 + \hat{p}^2 c^2} |\Psi\rangle, \quad (147)$$

which is the standard Schrödinger equation written in the bra-ket notation with $\hat{H} = \sqrt{m^2 c^4 + \hat{p}^2 c^2}$. Here, \hat{H} is the relativistic Hamiltonian for a free particle where the momentum p is replaced by the operator $\hat{p} = -i\hbar \partial / \partial x$. If we project it onto the momentum space, we can easily operate with $\sqrt{m^2 c^4 + \hat{p}^2 c^2}$ on the momentum eigenstates and solve the corresponding differential equation. Fourier transforming back to the position representation, we realize that $\Psi(x, t)$ evolves according to equation (144) and with a dispersion relation given by equation (146). In other words, the Lorentz covariant Schrödinger equation, the Klein-Gordon equation, and the wave equation (147) predict the same wave-packet dynamics in the free particle regime.

5.5. Time-Independent Potentials. We proceed with the investigation of the main predictions of the Lorentz covariant Schrödinger equation, studying now its general features when the particle of mass m is subjected to a time independent, and not necessarily constant, potential. Specifically, we want to address the bound state solutions to equation (63) when $V(\mathbf{r}, t) = V(\mathbf{r})$.

Inserting the ansatz

$$\Psi(\mathbf{r}, t) = e^{-iKt/\hbar} \psi(\mathbf{r}) \quad (148)$$

into equation (63), where K is a constant, we get the following time-independent equation:

$$\frac{-\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + V(\mathbf{r}) \psi(\mathbf{r}) = \lambda \psi(\mathbf{r}), \quad (149)$$

where

$$\lambda = \left(\frac{K^2}{2mc^2} + K \right). \quad (150)$$

Equation (149) is formally identical to the time-independent nonrelativistic Schrödinger equation, with λ being its eigenvalue. We thus see that any solution to the latter is also a solution to equation (149), the time-independent Lorentz covariant Schrödinger equation. Due to the linearity of equation (63), its general solution for time-independent potentials is the superposition of the stationary solutions (148),

$$\Psi(\mathbf{r}, t) = \sum_n e^{-iK_n t/\hbar} \psi_n(\mathbf{r}), \quad (151)$$

where $\psi_n(\mathbf{r})$ is an eigenvector of equation (149) with eigenvalue λ_n and, for bound state problems, K_n is the following solution

to equation (150):

$$K_n = -mc^2 + \sqrt{m^2 c^4 + 2mc^2 \lambda_n}. \quad (152)$$

To better appreciate the physical meaning of K , we expand it in powers of $1/c^2$,

$$K = \lambda - \frac{1}{2} \frac{\lambda^2}{mc^2} + \mathcal{O}(1/c^4). \quad (153)$$

In the nonrelativistic limit, when $c \rightarrow \infty$, $K = \lambda$ and we recognize it as the energy of the nonrelativistic system, with the rest energy already absent. The standard notation in the nonrelativistic regime is to employ the letter E instead of λ to represent the eigenvalues of equation (149) since in this regime, they are the system's total energy (no rest energy included). Similarly, we can understand K as the relativistic bound energy of the system, with the rest energy already excluded.

5.5.1. Example: The Hydrogen Atom. Let us study the solution to equation (149) when a particle of mass m and charge $-e$ (an electron, for instance) is subjected to the electrostatic Coulomb potential generated by a particle of mass M and charge $e > 0$ (a proton, for example). In the SI units, the potential is

$$V(\mathbf{r}) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}, \quad (154)$$

where r is the distance between the two particles and ϵ_0 is the vacuum permittivity. If $M \gg m$, we can consider the mass M at rest and m in equation (149) is, for all practical purposes, equal to the mass of the lightest particle. Otherwise, we should understand m in equation (149) as the reduced mass $mM/(m+M)$. We are also leaving out of our analysis any spinorial properties of the electron and proton. We are effectively dealing with charged scalar particles.

Inserting equation (154) into (149), we see that we formally have the nonrelativistic Schrödinger equation for the Hydrogen atom, whose eigenvalues are [2]

$$\lambda_n = -\frac{mc^2 \alpha^2}{2n^2}. \quad (155)$$

Here, $n \geq 1$ is a positive integer and α , the fine structure constant, is

$$\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c}. \quad (156)$$

Inserting equation (154) into (149), we get

$$K_n = -mc^2 \left(1 - \sqrt{1 - \frac{\alpha^2}{n^2}} \right) \quad (157)$$

$$= -\frac{mc^2 \alpha^2}{2n^2} \left(1 + \frac{\alpha^2}{4n^2} \right) + \mathcal{O}(\alpha^6), \quad (158)$$

where the last line is the expansion of the exact value of K_n up to fourth order in the fine structure constant.

It is interesting to compare the previous result with the solution to the same problem using the Dirac equation minimally coupled to the electromagnetic field. In this case, we get after subtracting the particle's rest energy [2]

$$K_{n,j+1/2}^D = -mc^2 \left\{ 1 - \left[1 + \left(\frac{\alpha}{n - (j + 1/2) + \sqrt{(j + 1/2)^2 - \alpha^2}} \right)^2 \right]^{-1/2} \right\} \\ = -\frac{mc^2 \alpha^2}{2 n^2} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + 1/2} - \frac{3}{4} \right) \right] + \mathcal{O}(\alpha^6), \quad (159)$$

where $n \geq 1$ is a positive integer and $j + 1/2 = 1, 2, \dots, n$. The last expression above is $K_{n,j+1/2}^D$ expanded up to fourth order in α .

Computing the energy of the ground state predicted by the Lorentz covariant Schrödinger equation, equation (157) with $n = 1$, and by the Dirac equation, equation (159) with $n = 1$ and $j + 1/2 = 1$, we get the same result to all orders of α ,

$$K_1 = K_{1,1}^D = -mc^2 \left(1 - \sqrt{1 - \alpha^2} \right). \quad (160)$$

Moreover, if we compute the energies predicted by the Dirac equation for $n \geq 1$ using the highest possible value for the total angular momentum; i.e., if we set $j + 1/2 = n$ in equation (159), we get

$$K_{n,n}^D = -mc^2 \left(1 - \sqrt{1 - \frac{\alpha^2}{n^2}} \right) = K_n. \quad (161)$$

These are exactly the energies predicted by the Lorentz covariant Schrödinger equation (cf. equation (157)), a remarkable match with the most energetic bound energies predicted by the Dirac equation for a given n .

It is worth noting that we have neglected the spin properties of the electron and of the proton and we have not minimally coupled the Lorentz covariant Schrödinger equation with the electromagnetic field to arrive at those results. We just have inserted the Coulomb potential as given above directly into the Lorentz covariant Schrödinger equation. It is really impressive that such a simple approach leads exactly to a subset of the bound energies predicted by the Dirac equation. And of course, to break the degeneracy and get bound energies depending on the total angular momentum of the electron, we either have to add the spin-orbit coupling by hand in the Lorentz covariant Schrödinger equation or deal directly with a spin-1/2 relativistic wave equation as Dirac did.

Before we move on to the next section, we want to compare the bound energies given by the Lorentz covariant Schrödinger equation with the ones given by solving the Klein-Gordon equation under the same conditions and minimally coupled with the electromagnetic field. Subtracting the rest energy, we have [2]

$$K_{n,l}^{KG} = -mc^2 \left\{ 1 - \left[1 + \left(\frac{\alpha}{n - (l + 1/2) + \sqrt{(l + 1/2)^2 - \alpha^2}} \right)^2 \right]^{-1/2} \right\} \\ = -\frac{mc^2 \alpha^2}{2 n^2} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{l + 1/2} - \frac{3}{4} \right) \right] + \mathcal{O}(\alpha^6), \quad (162)$$

where $n \geq 1$ is a positive integer and $l = 0, 1, 2, \dots, n - 1$. The last line is $K_{n,l}^{KG}$ expanded up to fourth order in α . Equation (162) is similar to (159), with $j + 1/2$ changed to $l + 1/2$, where l labels the orbital angular momentum of the mass m .

Calculating the ground state energy predicted by the Klein-Gordon equation, namely, equation (162) with $n = 1$ and $l = 0$, we obtain

$$K_{1,0}^{KG} = -mc^2 \left[1 - \sqrt{\frac{1}{2} \left(1 + \sqrt{1 - 4\alpha^2} \right)} \right] \quad (163)$$

$$= -\frac{mc^2}{2} \alpha^2 \left(1 + \frac{5}{4} \alpha^2 \right) + \mathcal{O}(\alpha^6). \quad (164)$$

Looking at equation (164), we see that the ground state energy predicted by the Klein-Gordon equation minimally coupled to the electromagnetic field is not equal to equation (160), the ground state energy given by the Dirac equation and by the Lorentz covariant Schrödinger equation. Even to order of α^4 , the values of the ground state energies are already different. In other words, at the level of the ground state energy, the Lorentz covariant Schrödinger equation gives a better description of the hydrogen atom than the Klein-Gordon equation minimally coupled to the electromagnetic field. It is worth mentioning, nevertheless, that if we insert V given by equation (154) into the generalized Klein-Gordon equation, equation (106), we will get the same bound energies reported here for the Lorentz covariant Schrödinger equation (equation (157)).

5.6. Minimal Coupling with the Electromagnetic Field. Our goal now is to study the Lorentz covariant Schrödinger equation, as given by equation (63), minimally coupled to an electromagnetic field. We also want to study the simultaneous action of the electromagnetic and gravitational fields on a particle of mass m and charge q , where the static Newtonian gravitational potential enters in equation (63) via the potential energy $V(\mathbf{r})$. (See refs. [21–24] for general-relativistic and post-Newtonian corrections in hydrogen-like systems when the spin of the interacting particles is also included in the analysis.)

The electromagnetic minimal coupling, in SI units and in the metric signature we have been using, is obtained by replacing in the wave equation all derivatives ∂_μ according to the following prescription [2]:

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + \frac{iq}{\hbar} A_\mu, \quad (165)$$

where the covariant four-vector potential is

$$A_\mu = \left(\frac{\varphi}{c}, -\mathbf{A} \right). \quad (166)$$

Here, φ and $\mathbf{A} = (A^1, A^2, A^3)$ are, respectively, the electric and vector potentials characterizing an electromagnetic field.

Applying prescription (166) to equation (66), we have

$$D_\mu D^\mu \Psi - i \frac{2mc}{\hbar} D_0 \Psi + \frac{2mV}{\hbar^2} \Psi = 0, \quad (167)$$

where V should be thought as the particle's potential energy associated with an external field whose origin is not electromagnetic. All electromagnetic interactions are embedded in the minimal coupling assumption.

Noting that after a Lorentz boost, D_μ transforms as a covariant four-vector and that $D_\mu D^\mu$ is a Lorentz invariant, we can show that equation (167) is Lorentz covariant if Ψ transforms according to equation (18). In addition to that, it is not difficult to see that if we implement the gauge transformation

$$\tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu \chi(x), \quad (168)$$

we get

$$\tilde{D}_\mu \tilde{D}^\mu \tilde{\Psi} - i \frac{2mc}{\hbar} \tilde{D}_0 \tilde{\Psi} + \frac{2mV}{\hbar^2} \tilde{\Psi} = 0, \quad (169)$$

where

$$\tilde{D}_\mu = \partial_\mu + \frac{iq}{\hbar} \tilde{A}_\mu, \quad (170)$$

$$\tilde{\Psi}(x) = e^{-(iq/\hbar)\chi(x)} \Psi(x). \quad (171)$$

In other words, the Lorentz covariant Schrödinger equation minimally coupled to the electromagnetic field is invariant with respect to a local gauge transformation given by equation (156) if the wave function transforms according to equation (171).

5.6.1. Bound State Solutions in a "Pure" Static Electric Field.

We now set $V = 0$; i.e., we have a "pure" electromagnetic problem, and assume that the particle of mass m and charge $q = -e < 0$ is subjected to an attractive Coulomb potential generated by a "fixed" charge e . This is essentially the hydrogen atom where we disregard the spinorial aspects of the electron and the proton. In this scenario,

$$A^\mu = (\varphi/c, 0, 0, 0), \quad (172)$$

with

$$\varphi = \frac{1}{4\pi\epsilon_0} \frac{e}{r}. \quad (173)$$

Inserting equations (172) and (173) into equation (167) and separating the time variable from the spatial ones using the ansatz given by equation (148), we get

$$-\hbar^2 c^2 \nabla^2 \psi(\mathbf{r}) = \left[(K - \mathcal{V}(\mathbf{r}) + mc^2)^2 - m^2 c^4 \right] \psi(\mathbf{r}), \quad (174)$$

where

$$\mathcal{V}(\mathbf{r}) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} = -\hbar c \frac{\alpha}{r} \quad (175)$$

is the particle's electrostatic potential energy.

If we make the substitution $K + mc^2 \rightarrow E$, equation (174) becomes the stationary Klein-Gordon equation minimally coupled to the electromagnetic field via the Coulomb potential [2]. Therefore, the bound energies for the present problem are simply $K_{n,l} = E_{n,l}^{\text{KG}} - mc^2$, where $E_{n,l}^{\text{KG}}$ are the bound energies for the Klein-Gordon equation. The explicit form of $K_{n,l}$ is given by equation (162).

5.6.2. Bound State Solutions in a "Pure" Gravitational Field.

We now set $A^\mu = 0$ and

$$V(\mathbf{r}) = -\frac{GmM}{r} = -\hbar c \frac{\tilde{\alpha}}{r}, \quad (176)$$

where G is the gravitational constant, M is the mass of the "fixed" particle generating the gravitational field that acts on m , and $\tilde{\alpha} = GmM/\hbar c$ is the "gravitational fine structure constant."

Inserting equation (176) into equation (167) and using that $A^\mu = 0$, the stationary equation we get is given by equation (149), whose bound state energies are given by equation (157) with α changed to $\tilde{\alpha}$. Note that for the hydrogen atom, the leading term in the energy is of the order $\tilde{\alpha}^2 mc^2 \sim 10^{-83} mc^2$, irrelevant if compared to $\alpha^2 mc^2 \sim 10^{-5} mc^2$, the leading contribution coming from the Coulomb potential.

5.6.3. Bound State Solutions in Static Electric and Gravitational Fields.

It is very instructive to study the case in which we have a charged scalar particle described by the Lorentz covariant Schrödinger equation minimally coupled to the electromagnetic field and also subjected to a gravitational field. The leading contribution coming from a static gravitational field is obtained setting $V(\mathbf{r})$ as given by equation (176). The Coulomb potential is modeled via equations (172) and (173).

As we will show below, the bound energies to this problem are not simply independent contributions coming from the electromagnetic and gravitational fields. We will see that the way the system "feels" the gravitational field *depends* on the electromagnetic field acting on it. In other words, the bound energies depend not only on isolated functions of α and $\tilde{\alpha}$ but also on functions of the product $\alpha\tilde{\alpha}$. The simultaneous presence of both fields leads to bound energies that couple the electromagnetic and gravitational fields in a non-trivial way.

Inserting equations (172), (173), and (176) into (167) and employing the ansatz (148), the stationary Lorentz covariant Schrödinger equation can be written as

$$\nabla^2 \psi(\mathbf{r}) = -\frac{1}{\hbar^2 c^2} \{ [K - \mathcal{V}(\mathbf{r})]^2 + 2mc^2 [K - \mathcal{V}(\mathbf{r}) - V(\mathbf{r})] \} \psi(\mathbf{r}). \quad (177)$$

If we express the Laplacian operator in spherical coordinates and write $\psi(\mathbf{r}) = \psi_{n,l,m}(\mathbf{r}) = u_{n,l}(r) Y_l^m(\theta, \phi)$, where $r = |\mathbf{r}|$ and θ and ϕ are, respectively, the polar and azimuthal angles of the spherical-polar coordinates, equation (177) decouples into the following two equations:

$$\frac{1}{u_{n,l}} \frac{d}{dr} \left(r^2 \frac{du_{n,l}}{dr} \right) + \frac{r^2 f(r)}{\hbar^2 c^2} = l(l+1), \quad (178)$$

$$\frac{1}{Y_l^m} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y_l^m}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_l^m}{\partial \phi^2} \right] = -l(l+1). \quad (179)$$

The solutions to equation (179) are the spherical harmonics [3], where $l = 0, 1, 2, \dots$ and $m = 0, \pm 1, \pm 2, \dots, \pm l$.

The radial equation (178), where

$$f(r) = [K - \mathcal{V}(\mathbf{r})]^2 + 2mc^2 [K - \mathcal{V}(\mathbf{r}) - V(\mathbf{r})], \quad (180)$$

becomes

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{f(r)}{\hbar^2 c^2} \right] R_{n,l} = 0 \quad (181)$$

after using that

$$u_{n,l}(r) = \frac{R_{n,l}(r)}{r}. \quad (182)$$

Computing explicitly $f(r)$, we can rewrite equation (181) as

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1) - \alpha^2}{r^2} + \frac{2(E\alpha + mc^2 \tilde{\alpha})}{\hbar c r} - \frac{m^2 c^4 - E^2}{\hbar^2 c^2} \right] R_{n,l} = 0, \quad (183)$$

where

$$E = K + mc^2. \quad (184)$$

We will restrict ourselves to bound energies such that $-2mc^2 < K < 0$. This is equivalent to working with $|E| < m c^2$. With such values for E , the constant

$$B = \sqrt{\frac{4(m^2 c^4 - E^2)}{\hbar^2 c^2}} \quad (185)$$

is a real number. Now, defining

$$\rho = Br, \quad (186)$$

$$\lambda = \frac{2(E\alpha + mc^2 \tilde{\alpha})}{B\hbar c}, \quad (187)$$

$$\mu = \sqrt{\left(l + \frac{1}{2} \right)^2 - \alpha^2}, \quad (188)$$

equation (183) becomes

$$\left[\frac{d^2}{d\rho^2} - \frac{\mu^2 - 1/4}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4} \right] R_{n,l}(\rho) = 0. \quad (189)$$

Equation (189) is formally the same one obtains when solving the Klein-Gordon equation minimally coupled to the electromagnetic field [2]. The only difference between the present problem and the one given in ref. [2] is the absence of the term proportional to $\tilde{\alpha}$ in equation (187). Therefore, we can follow the same steps given in ref. [2] to arrive at the following solution to equation (189):

$$R_{n,l}(\rho) = N e^{-\rho/2} \rho^{\mu+1/2} {}_1F_1(a, b; \rho), \quad (190)$$

where ${}_1F_1(a, b; \rho)$ is the confluent hypergeometric function, N a normalization constant, and

$$b = 2\mu + 1, \quad (191)$$

$$a = \mu + \frac{1}{2} - \lambda. \quad (192)$$

The function ${}_1F_1(a, b; \rho) \rightarrow \infty$ when $\rho \rightarrow \infty$, and thus, in order to get a finite and normalizable solution, we have to truncate the series that defines the confluent hypergeometric function. This is achieved if [2]

$$a = -n' = 0, 1, 2, 3, \dots \quad (193)$$

Equation (193) leads to the quantization of the bound energies, whose values are given by solving for E equation (192),

$$\mu + \frac{1}{2} - \lambda = -n', \quad (194)$$

where λ is given by equation (187). Picking the solution in which $|E| < mc^2$ and using equation (184), we finally get

$$K_{n,l} = -mc^2 \left\{ 1 + \frac{\alpha \tilde{\alpha}}{w^2 + \alpha^2} - \left[\left(1 + \frac{\alpha^2}{w^2} \right) \left(1 + \frac{\tilde{\alpha}^2}{w^2 + \alpha^2 - \tilde{\alpha}^2} \right) \right]^{-1/2} \right\}, \quad (195)$$

where

$$w = n - \left(l + \frac{1}{2}\right) + \sqrt{\left(l + \frac{1}{2}\right)^2 - \alpha^2}, \quad (196)$$

$$n = n' + l + 1. \quad (197)$$

Equation (197) together with the fact that $l = 0, 1, 2, \dots$ leads to $n = 1, 2, 3, \dots$ and $l = 0, 1, 2, \dots, n - 1$.

It is not difficult to see that if $\tilde{\alpha} = 0$, we recover the pure electromagnetic solution (162) and that if $\alpha = 0$, a simple calculation shows that we get back to the pure gravitational solution, namely, equation (157), with α changed to $\tilde{\alpha}$. The more interesting scenario occurs when both the electric and gravitational fields are turned on. In this case, the effect of the gravitational field acting on the particle m is affected by the presence of the electric field. This is clear looking at equation (195), where we see that the bound energies depend also on products of $\alpha\tilde{\alpha}$.

We can better appreciate this feature if we expand equation (195) in terms of α and $\tilde{\alpha}$,

$$K_{n,l} = -\frac{mc^2\alpha^2}{2n^2} - \frac{mc^2\alpha^4}{2n^4} \left(\frac{n}{l+1/2} - \frac{3}{4}\right) + \mathcal{O}(\alpha^6) - \frac{mc^2\alpha\tilde{\alpha}}{n^2} + \mathcal{O}(\alpha^3\tilde{\alpha}) - \frac{mc^2\tilde{\alpha}^2}{2n^2} + \mathcal{O}(\tilde{\alpha}^4). \quad (198)$$

The first two terms in the right-hand side of equation (198) are, respectively, the nonrelativistic energy and the dominant relativistic correction due to the presence of a Coulomb potential alone, as discussed in Section 5.6.1. The sixth term in equation (198) is the nonrelativistic energy due to the presence of a gravitational field alone (cf. Section 5.6.2). The fourth term in equation (198),

$$-\frac{mc^2\alpha\tilde{\alpha}}{n^2}, \quad (199)$$

is the dominant contribution to the bound energy coming from the simultaneous presence of an electric field and a gravitational field. For the hydrogen atom, we have

$$\alpha\tilde{\alpha}mc^2 \sim 10^{-44}mc^2 \sim \alpha^{20}mc^2, \quad (200)$$

which is too small to be detected using today's technology. On the other hand, the simultaneous existence of electric and gravitational fields leads to a contribution to the bound energies about 40 orders of magnitude greater than the expected one due to the presence of a gravitational field alone, which is of order $\tilde{\alpha}^2mc^2 \sim 10^{-83}mc^2$.

In order to have a gravitational contribution to the bound energies of order α^4 , which is the order of magnitude of the dominant relativistic correction to the bound energies due to an electric field, we would need $\tilde{\alpha} \sim \alpha^3$. This is achieved with a "heavy proton" of mass of the order of 10^8 kg.

From a fundamental point of view, however, the fact that this simple model leads to a nontrivial influence of the gravitational field on how an electromagnetic field acts upon a particle of mass m deserves further investigation. It is not unlikely, as we show in Section 5.7, that the scattering between two charged particles, one of which is very massive, might lead to detectable predictions within the present model. (Note that by solving the standard Schrödinger equation for a particle subjected simultaneously to static electric and gravitational fields, we get for the bound energies $K_n^{Sch} = -mc^2\alpha^2/(2n^2) - mc^2\tilde{\alpha}^2/(2n^2) - mc^2\alpha\tilde{\alpha}/n^2$. In other words, the mixed term given by equation (199) is already present at the nonrelativistic level. Equation (195), on the other hand, is the relativistic bound energies $K_{n,l}$ we obtain by solving the Lorentz covariant Schrödinger equation for the same physical system. As can be seen by looking at equation (198), $K_{n,l}$ tends to K_n^{Sch} for small values of α and $\tilde{\alpha}$.)

5.6.4. Bound State Solutions in a Static Electric Field and a Constant Gravitational Field. If we have a Coulomb field minimally coupled to a charged particle of mass m , as given by equations (172) and (173), together with a constant gravitational field acting on it, which is achieved by setting V a constant in equation (176), we can repeat the steps detailed above and get the following expression for the bound energies:

$$K_{n,l} = -mc^2 \left[1 - \left(1 + \frac{\alpha^2}{w^2}\right)^{-1/2} \left(1 + \frac{2V}{mc^2}\right)^{1/2} \right], \quad (201)$$

where w was defined in equation (196).

Looking at equation (201), we see that it is similar to equation (162), the pure electrostatic solution to the Lorentz covariant Schrödinger equation. However, the term $(1 + \alpha^2/w^2)^{-1/2}$ does not appear alone now. It comes multiplied by $[1 + 2V/(mc^2)]^{1/2}$, whose origin stems from the presence of a constant gravitational field. Also, if $V \sim mc^2\alpha^2$, we would get corrections to the bound energies of order α^4 , which could in principle be detected.

5.6.5. Bound State Solutions in a Gravitational Field and a Constant Electric Field. For completeness, we present the bound energies when we have a constant electric field, which is achieved by setting equation (175) to a constant and using equation (176) to model the gravitational field. The bound energies in this case are

$$K_{n,l} = \mathcal{V} - mc^2 \left(1 - \sqrt{1 - \frac{\tilde{\alpha}^2}{n^2}} \right), \quad (202)$$

where $n = 1, 2, 3, \dots$. This case is less interesting since it is simply the pure gravitational case with bound energies displaced by \mathcal{V} , the electric potential energy due to a constant electric field.

5.6.6. Solutions in Constant Electric and Gravitational Fields. We are now interested in the plane wave solutions arising

from solving equation (167) when both the electric and gravitational fields acting on the particle are constant. The time-independent equation we need to solve is given by equation (177), with \mathcal{V} and V treated as two constants. Inserting the ansatz

$$\psi(\mathbf{r}) = e^{(i/\hbar)\mathbf{p}\cdot\mathbf{r}} \quad (203)$$

into equation (177), we get

$$K_{\pm} = \mathcal{V} - mc^2 \pm \sqrt{p^2 c^2 + m^2 c^4 + 2mc^2 V}, \quad (204)$$

where $p = |\mathbf{p}|$. Defining $m' = m(1 + 2V/(mc^2))^{1/2}$, $E = K_{\pm} + mc^2$, and squaring equation (204), we get

$$p^{\mu} p_{\mu} = m'^2 c^2, \quad (205)$$

where

$$p^{\mu} = \left(\frac{E - \mathcal{V}}{c}, \mathbf{p} \right). \quad (206)$$

Looking at equation (204), we see that the constant electric field affects K_{+} and K_{-} equally, displacing both energies by the same quantity \mathcal{V} . On the other hand, the constant gravitational field affects K_{+} and K_{-} differently, leading to the same results already discussed in Section 5.2.

It is worth mentioning the following two points. First, we can make $K_{+} = -K_{-}$ working with the Lorentz covariant Schrödinger equation minimally coupled to a constant electric field. Specifically, if we set $\mathcal{V} = mc^2$, we accomplish this task. This feature cannot be achieved without working in the minimal coupling scenario or by applying an external gravitational field, as we can see by looking at equation (204). See also Sections 5.1 and 5.2 for more details.

Second, equation (204) leads to a dispersion relation whose dependence on the value of the external constant gravitational field affects a particle's wave packet dynamics in a nontrivial way. This point becomes clearer inserting $K(p)$ as given by equation (204) into equation (144), which gives the time evolution of a one-dimensional wave packet in the presence of constant electric and gravitational fields. The wave packet dynamics will differ from that of a free particle ($\mathcal{V} = V = 0$) due to the presence of the term $2mc^2 V$ inside the square root of the dispersion relation (204). This factor introduces nontrivial changes when we integrate equation (144) to obtain the wave packet dynamics. Note that the effect of a constant electric field is trivial. It does not change the wave packet dynamics since it only adds a global phase to $\Psi(x, t)$.

5.7. Coulomb and Gravitational Scattering. Our goal here is to compute perturbatively the differential cross section for a beam of charged particles of mass m hitting charged particles of mass M at rest. The charges of the incident and target particles are respectively q and Q , the kinetic energy of the incident particles are $K < mc^2$, and we model the interaction of those particles via static electric and gravitational fields.

The stationary Lorentz covariant Schrödinger equation describing the simultaneous action of the Coulomb and gravitational fields is given by equation (177), which can be rewritten as

$$\nabla^2 \psi(\mathbf{r}) + k^2 \psi(\mathbf{r}) = J(\mathbf{r}) \psi(\mathbf{r}), \quad (207)$$

with

$$k^2 = (E^2 - m^2 c^4) / (\hbar^2 c^2), \quad (208)$$

$$J(\mathbf{r}) = \frac{-[\mathcal{V}'(\mathbf{r})]^2 + 2E\mathcal{V}'(\mathbf{r}) + 2mc^2 V(\mathbf{r})}{\hbar^2 c^2}. \quad (209)$$

For the present problem, we have

$$\mathcal{V}'(\mathbf{r}) = \frac{qQ}{4\pi\epsilon_0 r} = -\hbar c \frac{\alpha}{r} = \mathcal{V}(r), \quad (210)$$

$$V(\mathbf{r}) = -\frac{GmM}{r} = -\hbar c \frac{\tilde{\alpha}}{r} = V(r), \quad (211)$$

where $|\mathbf{r}| = r$. Note that now, we have that $\alpha = -qQ/(4\pi\epsilon_0 \hbar c)$. For $\alpha > 0$, the charges attract each other, and for $\alpha < 0$, we have a repulsive electrostatic interaction. The same interpretation for the sign of $\tilde{\alpha}$ applies. For ordinary matter, we always have $\tilde{\alpha} > 0$.

Using elementary techniques, we can transform equation (195) to its integral form [3],

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int \mathcal{G}(\mathbf{r} - \mathbf{r}') J(\mathbf{r}') \psi(\mathbf{r}') d^3 r', \quad (212)$$

where $\psi_0(\mathbf{r})$ is the solution to equation (207) with $J(\mathbf{r}) = 0$ and

$$\mathcal{G}(\mathbf{r}) = -\frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|} \quad (213)$$

is the Green function of the Helmholtz equation, i.e., solution to $\nabla^2 \mathcal{G}(\mathbf{r}) + k^2 \mathcal{G}(\mathbf{r}) = \delta^{(3)}(\mathbf{r})$, with $\delta^{(3)}(\mathbf{r})$ being the three-dimensional Dirac delta function.

Assuming we are dealing with localized potentials situated at the origin ($\mathbf{r}' = 0$) and noting that in a scattering experiment, we want the wave function far away from the target, i.e., $|\mathbf{r}| \gg |\mathbf{r}'|$, we can write the Green function as [3]

$$\mathcal{G}(\mathbf{r} - \mathbf{r}') = -\frac{e^{ikr}}{4\pi r} e^{-i\mathbf{k}\cdot\mathbf{r}'} + \mathcal{O}\left(\frac{1}{|\mathbf{r}|^2}\right), \quad (214)$$

where $\mathbf{k} = k\hat{\mathbf{r}}$. Moreover, considering that the incident particles move along the z -axis with a well-defined momentum and energy, the stationary wave function describing them can be written as

$$\psi_0(\mathbf{r}) = e^{i\mathbf{k}'\cdot\mathbf{r}}, \quad (215)$$

where $\mathbf{k}' = k\hat{\mathbf{z}}$.

Now, employing the first Born approximation (weak potential approximation), which implies that

$$\psi(\mathbf{r}) \approx \psi_0(\mathbf{r}) \quad (216)$$

inside the integral of equation (212), we get after equations (209) and (214)

$$\psi(\mathbf{r}) = e^{ikz} + f(\theta) \frac{e^{ikr}}{r}. \quad (217)$$

The scattering amplitude is given by

$$\begin{aligned} f(\theta) = & \frac{1}{\hbar^2 c^2 \kappa} \int_0^\infty r \sin(\kappa r) [\mathcal{V}(r)]^2 dr \\ & - \frac{2E}{\hbar^2 c^2 \kappa} \int_0^\infty r \sin(\kappa r) \mathcal{V}(r) dr \\ & - \frac{2m}{\hbar^2 \kappa} \int_0^\infty r \sin(\kappa r) V(r) dr, \end{aligned} \quad (218)$$

where

$$\kappa = 2k \sin(\theta/2) \quad (219)$$

and θ is the polar angle with respect to the incident z -direction, defining the polar angular position of the detector measuring the scattered particles. To arrive at equation (218), we relied on the fact that the potentials are spherically symmetric. This allowed us to straightforwardly integrate over the solid angle $d\Omega' = \sin(\theta') d\theta' d\phi'$, where $d^3 r' = r'^2 dr' d\Omega'$, $\boldsymbol{\kappa} \cdot \mathbf{r} = \kappa r \cos \theta$, and $\boldsymbol{\kappa} = \mathbf{k}' - \mathbf{k}$ in equation (212).

It is worth mentioning that the origin of the first and second terms of equation (218) can be traced back to applying the minimal coupling prescription to the Lorentz covariant Schrödinger equation, while the third one is related to the existence of an external potential whose origin is not electromagnetic. Moreover, for small energies ($E \ll mc^2$), the second and third terms of equation (218) tend, at the first Born approximation level, to the scattering amplitude obtained by solving the nonrelativistic Schrödinger equation for a particle subjected to the external potential $\mathcal{V}(r) + V(r)$. The first term, on the other hand, is absent from any nonrelativistic treatment of this problem, being a purely relativistic contribution to the scattering amplitude.

By inserting equations (210) and (211) into (218), we can explicitly compute the three remaining integrals. This leads to

$$\begin{aligned} f(\theta) = & \frac{\pi \alpha^2}{2\kappa} + \frac{2E\alpha}{\hbar c \kappa^2} + \frac{2mc^2 \tilde{\alpha}}{\hbar c \kappa^2} = \frac{\pi \hbar c \alpha^2}{4\sqrt{E^2 - m^2 c^4} \sin(\theta/2)} \\ & + \frac{\hbar c (E\alpha + mc^2 \tilde{\alpha})}{2(E^2 - m^2 c^4) \sin^2(\theta/2)}, \end{aligned} \quad (220)$$

where we used equations (208) and (219) to arrive at the last line. Note that we have a term that is second order in α in

equation (220). This means that we must also compute the second Born approximation to obtain all the other second-order terms contributing to the scattering amplitude.

With the aid of equation (220), the differential cross section can be written as [3]

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \quad (221)$$

In order to compare equation (220) with its nonrelativistic version, the one coming from solving the standard Schrödinger equation at the same level of approximation, and to properly identify its dominant relativistic corrections, it is convenient to write equation (220) in terms of the kinetic energy of the particle,

$$K = E - mc^2. \quad (222)$$

Using equation (222), expanding up to first order in $K/(mc^2)$, and retaining terms of lowest order in α and $\tilde{\alpha}$, equation (220) becomes

$$\begin{aligned} f(\theta) = & \frac{\hbar c}{4K \sin^2(\theta/2)} \left[\alpha + \tilde{\alpha} + \frac{K}{2mc^2} (\alpha - \tilde{\alpha}) \right] \\ & + \mathcal{O} \left[\alpha^o \tilde{\alpha}^s \left(\frac{K}{mc^2} \right)^2 \right] + \mathcal{O}(\alpha^n \tilde{\alpha}^q), \end{aligned} \quad (223)$$

where o, s, n, q are integers such that $o + s = 1$ and $n + q \geq 2$. If we solve the nonrelativistic Schrödinger equation for a particle subjected to the potential $\mathcal{V}(r) + V(r)$, we get for the first Born approximation

$$f^{Sch}(\theta) = \frac{\hbar c}{4K \sin^2(\theta/2)} (\alpha + \tilde{\alpha}) = \frac{1}{4K \sin^2(\theta/2)} \left(-\frac{qQ}{4\pi\epsilon_0} + GmM \right). \quad (224)$$

Comparing equations (223) and (224), we see that the dominant relativistic correction to the scattering amplitude is

$$f^{Rel}(\theta) = \frac{\hbar c}{8mc^2 \sin^2(\theta/2)} (\alpha - \tilde{\alpha}) = \frac{1}{8mc^2 \sin^2(\theta/2)} \left(-\frac{qQ}{4\pi\epsilon_0} - GmM \right). \quad (225)$$

It is interesting to observe that the leading relativistic correction to the scattering amplitude due to the gravitational field has a different sign when compared to the nonrelativistic term ($\tilde{\alpha} \rightarrow -\tilde{\alpha}$). Loosely speaking, the relativistic correction looks like a “negative” mass interacting with a positive one; i.e., we have an effective gravitational repulsive force.

It is important to notice that the pieces of equation (223) proportional to $\tilde{\alpha}$ are relevant only if $|\alpha| \approx |\tilde{\alpha}|$. This implies that

$$M \approx \frac{1}{4\pi\epsilon_0 G} \frac{qQ}{m}. \quad (226)$$

For a beam of electrons incident on a target composed of particles of mass M and with the same charge of the electron, equation (226) leads to $M \approx 3.79 \times 10^{12} \text{ kg} \approx 6.35 \times 10^{-13} M_{\oplus}$, where M_{\oplus} is the mass of the Earth. Looking at equation (226), we see that we can decrease the mass M by increasing the mass of the incident particles. For instance, for incident particles with $m = 1 \mu\text{g}$ (one microgram), we get $M \approx 3.45 \mu\text{g}$ if both particles have the same charge of the electron. Moreover, by tuning the values of the charges q and Q , we can also obtain manageable values for the masses m and M that might lead to an experimental test of equation (223) if we build on state-of-the-art experimental techniques that can detect the gravitational attraction between millimeter-sized particles with masses of the order of 100 mg [25].

Finally, if we look at equation (220), we see that it is possible to completely suppress the first-order contribution to the scattering amplitude if

$$E\alpha + mc^2\tilde{\alpha} = 0. \quad (227)$$

Solving the previous equation by noting that $E = \gamma mc^2$, with $\gamma = 1/\sqrt{1 - v^2/c^2}$ and v being the speed of the incident particle, we get

$$\frac{mM}{qQ} = \frac{\gamma}{4\pi\epsilon_0 G}, \quad (228)$$

after inserting the definitions of α and $\tilde{\alpha}$.

This means that by properly setting m , M , q , and Q such that equation (228) is satisfied, only second-order effects will be present in the scattering amplitude. This feature can be employed to indirectly test the influence of gravitation at the quantum level. If after the proper tuning of the masses and charges no first-order scattering effect is seen, this can only be attributed to the concomitantly action of the static gravitational and electric fields between the incident and target particles. And any observed second-order scattering can be compared to (220) and the ones coming from the second Born approximation, two of which are proportional to $\tilde{\alpha}^2$ and $\alpha\tilde{\alpha}$, genuine quantum contributions to the scattering amplitude related to the presence of a gravitational field.

5.8. Justifying the Way We Modeled the Gravitational Interaction. We start assuming that we are dealing only with stationary electromagnetic fields and in the minimal coupling scenario. Specifically, we restrict ourselves to the Coulomb potential and set $V = 0$ in equation (167). In this case, the four-vector potential is given by equation (172) and equation (167) can be written as

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + \mathcal{V}\Psi = i\hbar\frac{\partial\Psi}{\partial t} - \frac{\hbar^2}{2mc^2}\frac{\partial^2\Psi}{\partial t^2} + \frac{i\hbar\mathcal{V}}{mc^2}\frac{\partial\Psi}{\partial t} + \frac{\mathcal{V}^2}{2mc^2}\Psi. \quad (229)$$

Using the ansatz (148), it is not difficult to see that $\partial\Psi/\partial t = -iK\Psi/\hbar$ and that $\partial^2\Psi/\partial t^2 = -K^2\Psi/\hbar^2$, where K is the bound energy of the system. Using experimental data, as

Schrödinger probably did, or relying, for example, on the Bohr model, we know that

$$|K| \sim \frac{mc^2\alpha^2}{2}. \quad (230)$$

This leads to

$$\left|\frac{\partial\Psi}{\partial t}\right| \sim \frac{mc^2\alpha^2}{2\hbar}|\Psi|, \quad (231)$$

$$\left|\frac{\partial^2\Psi}{\partial t^2}\right| \sim \frac{m^2c^4\alpha^4}{4\hbar^2}|\Psi|.$$

Moreover, using the Bohr radius r_0 , where

$$\frac{1}{r_0} = \frac{mca}{\hbar}, \quad (232)$$

we have

$$|\mathcal{V}'| \sim mc^2\alpha^2. \quad (233)$$

Thus, using equations (231) and (233), the dominant order of magnitude for the four terms on the right-hand side of equation (229) is

$$\left|i\hbar\frac{\partial\Psi}{\partial t}\right| \sim \frac{mc^2\alpha^2}{2}|\Psi|,$$

$$\left|\frac{\hbar^2}{2mc^2}\frac{\partial^2\Psi}{\partial t^2}\right| \sim \frac{mc^2\alpha^4}{8}|\Psi|, \quad (234)$$

$$\left|\frac{i\hbar\mathcal{V}}{mc^2}\frac{\partial\Psi}{\partial t}\right| \sim \frac{mc^2\alpha^4}{2}|\Psi|,$$

$$\left|\frac{\mathcal{V}^2}{2mc^2}\frac{\partial\Psi}{\partial t}\right| \sim \frac{mc^2\alpha^4}{2}|\Psi|.$$

Of the four terms above, only the first one is of order α^2 while the remaining three are of order α^4 . This fact together with equation (233) fully justifies why we can neglect as a first approximation all but the first term in the right-hand side of equation (229). Proceeding in such a way, we obtain the nonrelativistic Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + \mathcal{V}\Psi = i\hbar\frac{\partial\Psi}{\partial t} + \mathcal{O}(mc^2\alpha^4|\Psi|). \quad (235)$$

Furthermore, the term $\mathcal{V}\Psi$ is the dominant term when it comes to the presence of a static electric field acting on a charged particle. Thus, it is not unreasonable to assume that $V\Psi$ should be the dominant term when we have other static fields acting on the particle, the Newtonian gravitational field being an example (see Sections 5.6 and 5.7). And in this case, since the ‘‘gravitational fine structure constant’’ $\tilde{\alpha}$ is much smaller than α^2 , we must work with all terms appearing in equation (229) when investigating the simultaneous

action of electromagnetic and gravitational fields on a scalar charged particle, similarly to what we have done in Sections 5.6 and 5.7. (As we show in this work, antiparticles apparently possess negative masses and we can adjust the present theory such that particles and antiparticles repel or attract each other gravitationally. Therefore, all the results above where we considered two particles interacting gravitationally can be readily extended to the case where we have a particle and an antiparticle repelling each other by properly choosing the sign of $\tilde{\alpha}$.)

6. Lagrangian Formulation

We now start the second part of this paper. In this section, we will show the Lagrangian formalism associated with the Lorentz covariant Schrödinger equation. The wave function Ψ and its complex conjugate Ψ^* will be considered two independent “classical” fields. Our goal will be to develop the main features of the classical field theory associated with the Lagrangian that leads to the Lorentz covariant Schrödinger equation, in particular those features needed to prepare the ground for Section 7. In that section, we will implement the “second” quantization of the classical fields here studied and develop the quantum field theory of the Lorentz covariant Schrödinger equation. We also show in Section 6 how smoothly one goes from the relativistic conserved quantities to the ones derived from the nonrelativistic Schrödinger equation. By simply taking the $c \rightarrow \infty$ limit, we promptly recover the nonrelativistic quantities from the relativistic ones.

Treating Ψ and Ψ^* as two independent fields, we define the Lagrangian describing these two fields as

$$L = \int d^3x \mathcal{L}(\Psi, \Psi^*, \partial_\mu \Psi, \partial_\mu \Psi^*). \quad (236)$$

Here, \mathcal{L} is the Lagrangian density, assumed to depend on the fields and at most on their first derivatives; $d^3x = d^3x^1 dx^2 dx^3$ is the infinitesimal spatial volume; and, unless stated otherwise, the integration is taken over all space. We assume that the fields and their derivatives vanish at the boundaries of integration. Note that \mathcal{L} is such that $\int d^3x \mathcal{L}$ has the dimension of energy.

We define the action as

$$S = \int dt L = \frac{1}{c} \int d^4x \mathcal{L}(\Psi, \Psi^*, \partial_\mu \Psi, \partial_\mu \Psi^*), \quad (237)$$

where $d^4x = dx^0 d^3x$ is the infinitesimal four-volume. Applying the variational principle to the action, i.e., demanding that the infinitesimal variation of the action vanishes,

$$\delta S = 0, \quad (238)$$

we get the following Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \Psi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \right), \quad (239)$$

$$\frac{\partial \mathcal{L}}{\partial \Psi^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi^*)} \right), \quad (240)$$

where the Einstein summation convention is implied for repeated indexes. In obtaining equations (239) and (240), we varied the fields Ψ and Ψ^* as two independent variables and assumed that they vanished at the boundaries.

When inserted into equations (239) and (240), the simplest Lagrangian density leading to the Lorentz covariant Schrödinger equation and its complex conjugate is

$$\mathcal{L}_{\text{asym}} = \frac{\hbar^2}{2m} \partial_\mu \Psi \partial^\mu \Psi^* + i\hbar c \Psi^* \partial_0 \Psi - V |\Psi|^2, \quad (241)$$

where $|\Psi|^2 = \Psi \Psi^*$ and for simplicity, we write $(\partial_\mu \Psi)(\partial^\mu \Psi^*) = \partial_\mu \Psi \partial^\mu \Psi^*$. If we take the nonrelativistic limit, we recover the Lagrangian density that gives the Schrödinger equation, i.e., $\lim_{c \rightarrow \infty} \mathcal{L}_{\text{asym}} = \mathcal{L}_{\text{Schr}}$, where

$$\begin{aligned} \mathcal{L}_{\text{Schr}} &= \frac{\hbar^2}{2m} \partial_j \Psi \partial^j \Psi^* + i\hbar \Psi^* \partial_t \Psi - V |\Psi|^2 \\ &= -\frac{\hbar^2}{2m} \nabla \Psi \cdot \nabla \Psi^* + i\hbar \Psi^* \frac{\partial \Psi}{\partial t} - V |\Psi|^2. \end{aligned} \quad (242)$$

Note that Lagrangian density (241) is not symmetric in the fields Ψ and Ψ^* due to the second term on the right-hand side. This is solved by working with the following Hermitian Lagrangian density:

$$\mathcal{L}_{\text{sym}} = \frac{\hbar^2}{2m} \partial_\mu \Psi \partial^\mu \Psi^* + \frac{i\hbar c}{2} (\Psi^* \partial_0 \Psi - \Psi \partial_0 \Psi^*) - V |\Psi|^2. \quad (243)$$

The Lagrangian densities (241) and (243) are connected by a four-divergence and as such are equivalent. Specifically, $\mathcal{L}_{\text{sym}} = \mathcal{L}_{\text{asym}} + \partial_\mu f^\mu$, where $f^0 = -i\hbar c |\Psi|^2 / 2$ and $f^j = 0$.

The advantage of working with the symmetric form is related to the fact that all Noether currents inherit that symmetry. This is not always the case with the asymmetric Lagrangian density although, as expected, the conserved Noether charges are the same working with either Lagrangian density. Furthermore, after a long but straightforward calculation, we can show that the symmetric Lagrangian density, equation (243), is Lorentz invariant if Ψ transforms according to equation (18) and ∂_μ transforms as a covariant vector. The asymmetric Lagrangian density, on the other hand, is not Lorentz invariant, although the action $S_{\text{asym}} = \int d^4x \mathcal{L}_{\text{asym}}$ is. Obviously, the action for the symmetric Lagrangian is also Lorentz invariant since we already have Lorentz invariance at the level of its Lagrangian density. For all these reasons, we will only work with the symmetric Lagrangian density in the rest of this work.

Specifically, we will employ the following symmetric Lagrangian density to describe the Lorentz covariant Schrödinger fields, akin to the usual way one writes the complex

field Klein-Gordon Lagrangian density,

$$\begin{aligned}\mathcal{L} &= \partial_\mu \Psi \partial^\mu \Psi^* + \frac{imc}{\hbar} (\Psi^* \partial_0 \Psi - \Psi \partial_0 \Psi^*) - \frac{2mV}{\hbar^2} |\Psi|^2 \\ &= \partial_\mu \Psi \partial^\mu \Psi^* + \frac{imc}{\hbar} \Psi^* \partial_0 \Psi - \frac{2mV}{\hbar^2} |\Psi|^2.\end{aligned}\quad (244)$$

Note that $\mathcal{L} = (2m/\hbar^2) \mathcal{L}_{\text{sym}}$. Also, if in equation (244) we set $V = 0$ and use equation (108), we obtain the complex field Klein-Gordon Lagrangian density.

6.1. The Hamiltonian Density. The Hamiltonian density of the Lorentz covariant Schrödinger equation is obtained from the Lagrangian density (244) by the following Legendre transformation;

$$\mathcal{H} = \Pi_\Psi \partial_t \Psi + \Pi_{\Psi^*} \partial_t \Psi^* - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} \partial_0 \Psi + \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi^*)} \partial_0 \Psi^* - \mathcal{L}, \quad (245)$$

where $\partial_t = c \partial_0$. Here,

$$\begin{aligned}\Pi_\Psi &= \frac{\partial \mathcal{L}}{\partial(\partial_t \Psi)} = \frac{1}{c} \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)}, \\ \Pi_{\Psi^*} &= \frac{\partial \mathcal{L}}{\partial(\partial_t \Psi^*)} = \frac{1}{c} \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi^*)}\end{aligned}\quad (246)$$

are, respectively, the conjugate momenta to the fields Ψ and Ψ^* . Due to the symmetry of the Lagrangian density in those fields, it is not difficult to see that $\Pi_{\Psi^*} = \Pi_\Psi^*$. A direct calculation gives

$$\Pi_\Psi = \frac{1}{c} \partial_0 \Psi^* + \frac{im}{\hbar} \Psi^*, \quad (247)$$

$$\Pi_{\Psi^*} = \frac{1}{c} \partial_0 \Psi - \frac{im}{\hbar} \Psi. \quad (248)$$

Now, inserting equations (244), (247), and (248) into (245), we get

$$\mathcal{H} = \partial_0 \Psi \partial_0 \Psi^* + \nabla \Psi \cdot \nabla \Psi^* + \frac{2mV}{\hbar^2} |\Psi|^2. \quad (249)$$

Looking at equation (249), we see that it is clearly positive for the free-field case ($V = 0$) as well as whenever $mV > 0$. For $mV < 0$, the positiveness of equation (249) can be broken for a sufficiently high value of $|mV|$. Also, using equations (247) and (248), we can express the temporal derivatives appearing in equation (249) as functions of the fields Ψ and Ψ^* and their conjugate momenta. This leads to the following way of writing the Hamiltonian density:

$$\mathcal{H} = c^2 \Pi_\Psi \Pi_{\Psi^*} + \nabla \Psi \cdot \nabla \Psi^* + \frac{imc^2}{\hbar} (\Psi \Pi_{\Psi^*} - \Psi^* \Pi_\Psi) + \frac{m^2 c^2}{\hbar^2} \left(1 + \frac{2V}{mc^2}\right) |\Psi|^2. \quad (250)$$

Noting that $\Pi_{\Psi^*} = \Pi_\Psi^*$, equation (250) becomes

$$\begin{aligned}\mathcal{H} &= c^2 \Pi_\Psi \Pi_\Psi^* + \nabla \Psi \cdot \nabla \Psi^* + \frac{imc^2}{\hbar} (\Psi \Pi_\Psi - \Psi^* \Pi_\Psi^*) \\ &+ \frac{m^2 c^2}{\hbar^2} \left(1 + \frac{2V}{mc^2}\right) |\Psi|^2 = c^2 |\Pi_\Psi|^2 + |\nabla \Psi|^2 \\ &- \frac{2mc^2}{\hbar} \text{Im} (\Psi \Pi_\Psi) + \frac{m^2 c^2}{\hbar^2} \left(1 + \frac{2V}{mc^2}\right) |\Psi|^2,\end{aligned}\quad (251)$$

where $\text{Im}(z)$ stands for the imaginary part of the complex number z . Setting $V = 0$, we can write the Hamiltonian density as

$$\mathcal{H} = c^2 \Pi_\Psi \Pi_\Psi^* + \nabla \Psi \cdot \nabla \Psi^* + \frac{m^2 c^2}{\hbar^2} |\Psi|^2 + \frac{imc^2}{\hbar} (\Psi \Pi_\Psi - \Psi^* \Pi_\Psi^*). \quad (252)$$

Note that the first three terms at the right-hand side are formally the same as those appearing in the complex field Klein-Gordon Hamiltonian density while the last one is particular to the Hamiltonian density of the Lorentz covariant Schrödinger equation. This last term couples the conjugate momenta with the fields.

6.2. The Noether Currents. Here, we assume that the external potential V is constant. This means that the Lagrangian density (244) does not depend explicitly on the space-time coordinates, only on the fields and their derivatives. As such, whenever the Lagrangian density [4], and more generally the action [5], is invariant under a continuous one-parameter set of transformations, or a symmetry transformation for short, we get a local conserved current. This is the essence of Noether's theorem [4, 5].

Calling δx_μ an infinitesimal variation of the space-time coordinates, $\delta \Psi_T = \Psi'(x') - \Psi(x)$ and $\delta \Psi_T^* = \Psi'^*(x') - \Psi^*(x)$ the total variations of the fields, the Noether four-current density of the Lagrangian density (244) reads

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \delta \Psi_T + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)} \delta \Psi_T^* - \mathcal{F}^{\mu\nu} \delta x_\nu, \quad (253)$$

where the canonical energy-momentum tensor is

$$\mathcal{F}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \partial^\nu \Psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)} \partial^\nu \Psi^* - g^{\mu\nu} \mathcal{L}. \quad (254)$$

Note that to first order in the infinitesimal variation δx_μ , we have

$$\delta \Psi_T = \Psi'(x) - \Psi(x) + \delta x^\mu \partial_\mu \Psi(x) = \delta \Psi(x) + \delta x^\mu \partial_\mu \Psi(x), \quad (255)$$

with a similar expression for $\delta \Psi_T^*$.

If under a symmetry transformation the Lagrangian density is invariant ($\delta \mathcal{L} = 0$), we obtain the continuity equation

$\partial_\mu j^\mu = 0$. Thus, the following quantity is conserved:

$$\int d^3x j^0(x). \quad (256)$$

6.2.1. Invariance under Space-Time Translations. Under a space-time translation, $\Psi'(x') = \Psi(x)$ and thus, $\delta\Psi_T = \delta\Psi_T^* = 0$. Using this fact and assuming an infinitesimal constant translation $\delta x^\mu = a^\mu$ of the coordinates, it is not difficult to see that the Lagrangian density (244) is invariant under such operation.

In this case, the Noether current (253) becomes

$$j^\mu = -\mathcal{F}^{\mu\nu} a_\nu, \quad (257)$$

where

$$\mathcal{F}^{\mu\nu} = \partial^\mu \Psi^* \partial^\nu \Psi + \partial^\mu \Psi \partial^\nu \Psi^* + \frac{imc}{\hbar} (\Psi^* \partial^\nu \Psi - \Psi \partial^\nu \Psi^*) \delta^{\mu 0} - g^{\mu\nu} \mathcal{L}. \quad (258)$$

Since each component of a^μ is arbitrary and independent of each other, we obtain that

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0, \quad \nu = 0, 1, 2, 3. \quad (259)$$

The corresponding conserved ‘‘charges’’ are

$$H = \int d^3x \mathcal{T}^{00} = \int d^3x \mathcal{H}, \quad (260)$$

$$cP^j = \int d^3x \mathcal{T}^{0j} = \int d^3x (c\mathcal{P}^j), \quad j = 1, 2, 3. \quad (261)$$

Using equations (244) and (258), a direct computation gives

$$\mathcal{H} = \partial_0 \Psi \partial_0 \Psi^* + \nabla \Psi \cdot \nabla \Psi^* + \frac{2mV}{\hbar^2} |\Psi|^2, \quad (262)$$

$$\begin{aligned} c\mathcal{P}^j &= \partial^0 \Psi^* \partial^j \Psi + \partial^0 \Psi \partial^j \Psi^* + \frac{imc}{\hbar} (\Psi^* \partial^j \Psi - \Psi \partial^j \Psi^*) \\ &= 2 \operatorname{Re} (\partial^0 \Psi^* \partial^j \Psi) - \frac{2mc}{\hbar} \operatorname{Im} (\Psi^* \partial^j \Psi), \end{aligned} \quad (263)$$

where $\operatorname{Re}(z)$ is the real part of the complex number z .

Comparing equation (262) with (249), we see that, as expected, they are identical. Equation (262) is the Hamiltonian density of the Lorentz covariant Schrödinger equation, and its associated conserved charge, equation (260), is the Hamiltonian H giving the total energy of the system. The three quantities \mathcal{P}^j , $j = 1, 2, 3$, are interpreted as the momentum density associated with the fields Ψ and Ψ^* along three orthogonal spatial directions. This is true because they are derived from the invariance of the Lagrangian density under spatial translations. The associated conserved charges, P^j , are the total momentum of the fields projected along three orthogonal spatial directions (cf. equation (261)). Note also

that for $c \rightarrow \infty$, the first term on the right-hand side of equation (263) goes to zero and $\mathcal{P}^j \rightarrow (2m/\hbar) \operatorname{Im} (\Psi^* \partial^j \Psi)$. This latter term is proportional to the momentum density of the nonrelativistic Schrödinger Lagrangian, which is given by $-i\hbar \Psi^* \partial_j \Psi$ if we use the appropriate nonrelativistic normalization for the wave function.

We can also write the momentum density vector $\mathcal{P} = (\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3)$ as

$$c\mathcal{P} = -2 \operatorname{Re} (\partial^0 \Psi^* \nabla \Psi) + \frac{2mc}{\hbar} \operatorname{Im} (\Psi^* \nabla \Psi). \quad (264)$$

Similarly to what we did with the Hamiltonian density, if we use equations (247) and (248), we can express the momentum density as a function of the conjugate momenta,

$$\mathcal{P}^j = \Pi_\Psi \partial^j \Psi + \Pi_{\Psi^*} \partial^j \Psi^* = 2 \operatorname{Re} (\Pi_\Psi \partial^j \Psi), \quad (265)$$

where the last term comes from the fact that $\Pi_{\Psi^*} = \Pi_\Psi^*$. And noting that $\partial^j = -\partial_j$, we have

$$\mathcal{P} = -2 \operatorname{Re} (\Pi_\Psi \nabla \Psi). \quad (266)$$

6.2.2. Invariance under Spatial Rotations. Under a spatial rotation, we also have $\delta\Psi_T = \delta\Psi_T^* = 0$ (see equation (21)). Now, however, an infinitesimal spatial rotation leads to $\delta x^\mu = \omega_\nu^\mu x^\nu$, where the tensor ω_ν^μ is antisymmetric. Using the notation $[\omega_\nu^\mu]_k$ to label about which axis we are rotating, rotations about the x -, y -, and z -axes are, respectively, given by the tensors $[\omega_\nu^\mu]_1$, $[\omega_\nu^\mu]_2$, and $[\omega_\nu^\mu]_3$. Calling ε the infinitesimal angle of rotation, we have $[\omega_3^1]_1 = -[\omega_3^2]_1 = [\omega_3^1]_2 = -[\omega_1^2]_2 = [\omega_1^3]_2 = -[\omega_1^1]_3 = \varepsilon$, with all the other $[\omega_\nu^\mu]_k$ being zero. Putting together all these results and noting that the Lagrangian density (244) is unchanged by this symmetry operation, the Noether current (253) becomes

$$[j^\mu]_k = -\mathcal{F}^{\mu\nu} [\omega_{\nu\alpha}]_k x^\alpha, \quad (267)$$

where $\omega_{\nu\alpha} = -\omega_\alpha^\nu$ and $\partial_\mu [j^\mu]_k = 0$.

Focusing our attention to a rotation about the z -axis and noting that ε is an arbitrary constant, we have,

$$[j^0]_3 = (\mathcal{T}^{01} x^2 - \mathcal{T}^{02} x^1), \quad (268)$$

with the corresponding conserved charge

$$cM^3 = \int d^3x [j^0]_3 = \int d^3x (cL^3). \quad (269)$$

Computing explicitly \mathcal{T}^{01} and \mathcal{T}^{02} using equation (254), we can express the angular momentum density along the z -direction as

$$cl^3 = \left(\partial^0 \Psi - \frac{imc}{\hbar} \Psi \right) (x^2 \partial^1 \Psi^* - x^1 \partial^2 \Psi^*) + \left(\partial^0 \Psi^* + \frac{imc}{\hbar} \Psi^* \right) (x^2 \partial^1 \Psi - x^1 \partial^2 \Psi). \quad (270)$$

If we repeat the above calculation for rotations about the x - and y -axis, we get similar results, showing that the angular momentum densities along the x - and y -directions, l^1 and l^2 , are obtained from l^3 by cyclic permutations of the spatial indexes. The angular momentum density vector $\mathbf{l} = (l^1, l^2, l^3)$ can be written as

$$\begin{aligned} c\mathbf{l} &= \left(\partial^0 \Psi - \frac{imc}{\hbar} \Psi \right) \mathbf{r} \times \nabla \Psi^* + \left(\partial^0 \Psi^* + \frac{imc}{\hbar} \Psi^* \right) \mathbf{r} \times \nabla \Psi \\ &= 2 \operatorname{Re} \left[\left(\partial^0 \Psi^* + \frac{imc}{\hbar} \Psi^* \right) \mathbf{r} \times \nabla \Psi \right], \end{aligned} \quad (271)$$

where the symbol \times above stands for the cross product.

Using the three-dimensional Levi-Civita tensor ε^{ijk} , we can rewrite the angular momentum density vector as

$$cl^k = 2 \operatorname{Re} \left[\left(\partial^0 \Psi^* + \frac{imc}{\hbar} \Psi^* \right) \varepsilon^{ijk} x_i \partial_j \Psi \right]. \quad (272)$$

Here, $\varepsilon^{ijk} = g^{ia} g^{jb} g^{kc} \varepsilon_{abc}$, where $\varepsilon_{abc} = 1$ for $abc = 123$ and cyclic permutations, $\varepsilon_{abc} = -1$ for $abc = 213$ and cyclic permutations, and $\varepsilon_{abc} = 0$ whenever at least two indexes are equal.

Note that the angular momentum density tends to $\mathbf{l} \rightarrow 2 \operatorname{Re} [(im/\hbar)\Psi^* \mathbf{r} \times \nabla \Psi]$ when $c \rightarrow \infty$, which is proportional, up to a normalization constant, to the angular momentum density of the nonrelativistic Schrödinger Lagrangian.

Finally, using equations (247) and (248), we can write the angular momentum density as a function of the conjugate momenta,

$$\mathbf{l} = \mathbf{r} \times \Pi_{\Psi^*} \nabla \Psi^* + \mathbf{r} \times \Pi_{\Psi} \nabla \Psi = 2\mathbf{r} \times \operatorname{Re} [\Pi_{\Psi} \nabla \Psi]. \quad (273)$$

It is worth mentioning that the “correct” sign for the angular momentum density does not follow from Noether’s theorem and in order to have an angular momentum density agreeing with the standard definition of angular momentum, we need to insert a minus sign [2]. Using equation (266), this gives

$$\mathbf{l} = -2\mathbf{r} \times \operatorname{Re} [\Pi_{\Psi} \nabla \Psi] = \mathbf{r} \times \mathcal{P} \quad (274)$$

the expected standard relation between vector momentum and vector angular momentum densities. Observe that the total angular momentum is given by (we are running out of letters to denote the several quantities in this work. We used the letter M for the angular momentum instead of L since the latter is already used to denote the Lagrangian. Sometimes, we will even use the same letter to denote different quantities in order to comply with the usual notation for

those quantities. The context will make it clear which meaning to ascribe to a given notation)

$$M = \int d^3 x \mathbf{l}. \quad (275)$$

6.2.3. Invariance under Global Phase Transformations. The Lagrangian density (244) is clearly invariant under the following transformation of the field:

$$\Psi'(x) = e^{-i\theta} \Psi(x), \quad (276)$$

where θ is a real constant. For an infinitesimal θ , we get that $\Psi'(x) \approx (1 - i\theta)\Psi(x)$. This leads to

$$\delta \Psi_T = -i\theta \Psi(x) \quad (277)$$

and analogously to

$$\delta \Psi_T^* = i\theta \Psi^*(x). \quad (278)$$

Since now there is no change in the coordinates (we are dealing with an internal symmetry), we have that $\delta x^\mu = 0$. The Noether current (253) is thus

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \delta \Psi_T + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi^*)} \delta \Psi_T^*. \quad (279)$$

Inserting equations (244), (277), and (278) into (279) and dropping the arbitrary constant θ , we get

$$\mathbf{j} = (j^1, j^2, j^3) = i(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) \quad (280)$$

and

$$j^0 = -i(\Psi \partial_0 \Psi^* - \Psi^* \partial_0 \Psi) + \frac{2mc}{\hbar} |\Psi|^2, \quad (281)$$

where we used that $\partial_0 = \partial^0$ and $\partial_j = -\partial^j$. The conserved charge is (we use \tilde{q} and \tilde{Q} instead of q and Q since we save the latter to represent the exact SI value of a given particle’s electric charge)

$$c\tilde{Q} = \int d^3 x j^0 = \int d^3 x (c\tilde{q}). \quad (282)$$

Comparing equations (280) and (281) with equations (82) and (83), we see that, up to a multiplicative constant equal to $-\hbar/(2m)$, they are the same. We have thus recovered the probability four-current density we obtained before using a different method.

As we did to the other conserved quantities, we can express the charge density \tilde{q} in terms of the conjugate momenta of the fields. Inserting equations (247) and (248) into (281), we get

$$\tilde{q} = -i(\Psi \Pi_{\Psi} - \Psi^* \Pi_{\Psi}^*) = 2 \operatorname{Im} (\Psi \Pi_{\Psi}). \quad (283)$$

6.3. *Discrete Symmetries.* We now briefly discuss the three major discrete symmetries one usually encounters in any field theory, namely, space inversion (parity), time reversal, and charge conjugation. We save for later, after we second quantize the Lorentz covariant Schrödinger equation, a more thorough discussion on this subject, specially the aspects related to charge conjugation. However, even at the first quantization level, we already see that the charge conjugation operator requires a different definition in order to “save” the CPT theorem. Actually, it will become clear after second quantization that the Lagrangian density (244) satisfies the CPT theorem only if we define the charge conjugation operator such that it also implements what we will call the “mass conjugation operation.” Simply put, we must change the sign of the mass m when applying the charge conjugation operation. Only in this way can we properly exchange the roles of particles with antiparticles and, at the same time, satisfy the CPT theorem.

6.3.1. *Space Inversion or Parity.* Space inversion is implemented by changing the sign of the space coordinates of the system under investigation,

$$x = (\mathbf{r}, t) \longrightarrow x' = (\mathbf{r}', t') = (-\mathbf{r}, t). \quad (284)$$

Following the same steps detailed in ref. [5], it is not difficult to see that the parity operation leads to the following transformation rule for the second quantized field associated with the Lorentz covariant Schrödinger equation:

$$\hat{\mathcal{P}}\hat{\Psi}(\mathbf{r}, t)\hat{\mathcal{P}}^\dagger = \hat{\Psi}(-\mathbf{r}, t). \quad (285)$$

Here, $\hat{\mathcal{P}}$ is a unitary operator denoting the space inversion operation. (The most general parity operation is $\hat{\mathcal{P}}\hat{\Psi}(\mathbf{r}, t)\hat{\mathcal{P}}^\dagger = \eta_p \hat{\Psi}(-\mathbf{r}, t)$, with η_p a complex number such that $|\eta_p| = 1$. However, for charged scalar fields, one can always set $\eta_p = 1$ without losing in generality. This argument is still valid for the time reversal and charge conjugation operations. See ref. [5] for details.) Note that we are using the symbol “ \wedge ” to distinguish the second quantized fields from the classical ones. In Section 7, we will drop the hat symbol when dealing with operators to simplify notation.

Using equation (285) and its complex conjugated version, as well as equation (284) and the fact that under space inversion $(\partial_0, \partial_j) \longrightarrow (\partial_0, -\partial_j)$, we can show that

$$\hat{\mathcal{P}}\hat{\mathcal{L}}(\mathbf{r}, t)\hat{\mathcal{P}}^\dagger = \hat{\mathcal{L}}(-\mathbf{r}, t), \quad (286)$$

$$\hat{\mathcal{P}}\hat{H}\hat{\mathcal{P}}^\dagger = \hat{H}, \quad (287)$$

$$\hat{\mathcal{P}}\hat{\mathbf{P}}\hat{\mathcal{P}}^\dagger = -\hat{\mathbf{P}}, \quad (288)$$

$$\hat{\mathcal{P}}\hat{\mathbf{M}}\hat{\mathcal{P}}^\dagger = \hat{\mathbf{M}}, \quad (289)$$

where $\hat{\mathcal{L}}$, \hat{H} , $\hat{\mathbf{P}}$, and $\hat{\mathbf{M}}$ are the second quantized versions of the Lagrangian density, the Hamiltonian, the linear momen-

tum vector, and angular momentum vector of the Lorentz covariant Schrödinger equation (see equations (244), (260), (261), and (275)). The relations given by equations (286)-(289) are the ones expected from our classical intuition about space inversion.

6.3.2. *Time Reversal.* Time reversal is given by changing the sign of the time coordinate only,

$$x = (\mathbf{r}, t) \longrightarrow x' = (\mathbf{r}', t') = (\mathbf{r}, -t). \quad (290)$$

As before, the same techniques given in ref. [5] to build the time reversal operator for the Klein-Gordon field remain valid here. Repeating those steps, we get an antiunitary operator $\hat{\mathcal{T}}$ that implements the time reversal operation. Its action on the quantum field $\hat{\Psi}$ is

$$\hat{\mathcal{T}}\hat{\Psi}(\mathbf{r}, t)\hat{\mathcal{T}}^\dagger = \hat{\Psi}(\mathbf{r}, -t). \quad (291)$$

Since $\hat{\mathcal{T}}$ is antiunitary, we have that

$$\hat{\mathcal{T}}z\hat{\mathcal{T}}^\dagger = z^*, \quad (292)$$

where z is a complex number.

Using equations (290)-(292) and that $(\partial_0, \partial_j) \longrightarrow (-\partial_0, \partial_j)$ under time reversal, we now have

$$\hat{\mathcal{T}}\hat{\mathcal{L}}(\mathbf{r}, t)\hat{\mathcal{T}}^\dagger = \hat{\mathcal{L}}(\mathbf{r}, -t), \quad (293)$$

$$\hat{\mathcal{T}}\hat{H}\hat{\mathcal{T}}^\dagger = \hat{H}, \quad (294)$$

$$\hat{\mathcal{T}}\hat{\mathbf{P}}\hat{\mathcal{T}}^\dagger = -\hat{\mathbf{P}}, \quad (295)$$

$$\hat{\mathcal{T}}\hat{\mathbf{M}}\hat{\mathcal{T}}^\dagger = -\hat{\mathbf{M}}. \quad (296)$$

Note that due to the explicit presence of the imaginary i in the expression for the Lagrangian density, the antiunitarity of $\hat{\mathcal{T}}$ is crucial to obtain the above relations.

6.3.3. *Charge Conjugation.* Charge conjugation is not related to space-time coordinate transformations, and since it exchanges particles with antiparticles, it has no immediate classical analog. In fact, it is defined such that the creation and annihilation operators for particles are transformed to the ones associated with antiparticles.

If we follow the prescription defining the charge conjugation operator for the Klein-Gordon fields, we see that it is given by a unitary operator $\hat{\mathcal{C}}$ such that

$$\hat{\mathcal{C}}\hat{\Psi}(x)\hat{\mathcal{C}}^\dagger = \hat{\Psi}^\dagger(x). \quad (297)$$

However, when applying it to the Lagrangian density (244), we realize that, due to the presence of the term proportional to the imaginary number i , the charge conjugation operation (297) does not leave the Lagrangian density invariant (normal ordering implied). This fact, together with how the parity and time reversal operations $\hat{\mathcal{P}}$ and $\hat{\mathcal{T}}$ affect

the Lagrangian density, would lead to a violation of the CPT theorem since we are dealing with a Lorentz invariant Lagrangian. A possible fix would be to define $\hat{\mathcal{C}}$ as an antiunitary operator. This will change the sign of i in the Lagrangian density, leaving it invariant under charge conjugation.

Although an antiunitary charge conjugation operator solves the CPT theorem violation problem, we will see in Section 7 that either a unitary or an antiunitary $\hat{\mathcal{C}}$ does not properly exchange the roles of particles with antiparticles. The only acceptable solution is to extend the charge conjugation operation such that it anticommutes with any function of the mass. Putting it simply, we must change the sign of the mass m when applying the extended charge conjugation operator.

If we define the ‘‘mass conjugation’’ operator $\hat{\mathcal{M}}$ such that

$$\hat{\mathcal{M}}f(m)\hat{\mathcal{M}}^\dagger = f(-m), \quad (298)$$

where $f(m)$ is an arbitrary function of the mass m , the extended charge conjugation operator we need is given by

$$\hat{\mathcal{C}}_m = \hat{\mathcal{M}}\hat{\mathcal{C}}, \quad (299)$$

with $\hat{\mathcal{C}}$ given by equation (297). It is not difficult to see that $\hat{\mathcal{C}}_m$ leaves the Lagrangian density (244) invariant and, as we will see in Section 7, it properly changes the roles of particles with antiparticles. (We implicitly assumed that we were dealing with the free field ($V = 0$). However, looking at equation (244), everything that was said above also applies to $V \neq 0$ provided that $\hat{\mathcal{C}}_m V \hat{\mathcal{C}}_m^\dagger = -V$. If V satisfies the previous transformation property, we get that equations (300)-(303) are also satisfied. Moreover, it is not difficult to see that the Lagrangian density (244) is invariant if we simultaneously apply the following three transformations: $m \rightarrow -m$, $V \rightarrow -V$, and $\Psi \rightarrow \exp[-i2mc^2 t/\hbar]\Psi$.)

Using equation (299), we can show that

$$\hat{\mathcal{C}}_m \hat{\mathcal{L}}(\mathbf{r}, t) \hat{\mathcal{C}}_m^\dagger = \hat{\mathcal{L}}(\mathbf{r}, t), \quad (300)$$

$$\hat{\mathcal{C}}_m \hat{H} \hat{\mathcal{C}}_m^\dagger = \hat{H}, \quad (301)$$

$$\hat{\mathcal{C}}_m \hat{\mathbf{P}} \hat{\mathcal{C}}_m^\dagger = \hat{\mathbf{P}}, \quad (302)$$

$$\hat{\mathcal{C}}_m \hat{\mathbf{M}} \hat{\mathcal{C}}_m^\dagger = \hat{\mathbf{M}}. \quad (303)$$

6.3.4. The CPT Theorem. The CPT theorem applies to the Lorentz covariant Schrödinger Lagrangian if we use the extended charge conjugation operator $\hat{\mathcal{C}}_m$ defined above. Strictly speaking, we have the CPTM theorem here since not only the charge but the mass must change sign to leave the action invariant.

If we define the CPT operation by

$$\hat{\Theta} = \hat{\mathcal{C}}_m \hat{\mathcal{P}} \hat{\mathcal{T}} = \hat{\mathcal{M}} \hat{\mathcal{C}} \hat{\mathcal{P}} \hat{\mathcal{T}}, \quad (304)$$

it is not difficult to show that

$$\hat{\Theta} \hat{\mathcal{L}}(x) \hat{\Theta}^\dagger = \hat{\mathcal{L}}(-x), \quad (305)$$

$$\hat{\Theta} \hat{H} \hat{\Theta}^\dagger = \hat{H}, \quad (306)$$

$$\hat{\Theta} \hat{\mathbf{P}} \hat{\Theta}^\dagger = \hat{\mathbf{P}}, \quad (307)$$

$$\hat{\Theta} \hat{\mathbf{M}} \hat{\Theta}^\dagger = -\hat{\mathbf{M}}. \quad (308)$$

Using equation (305) and equation (237) that defines the action, we get after changing the variables $x \rightarrow -x$

$$\hat{\Theta} \hat{S} \hat{\Theta}^\dagger = \hat{S}. \quad (309)$$

In other words, the action is invariant under the CPT operation (304) defined here.

6.3.5. More on the New Charge Conjugation Operation. In a static external gravitational field, we can as a first approximation model the interaction of a particle with inertial mass m by setting $V = m_g \varphi(x)$, where $\varphi(x)$ represents the gravitational potential associated with the field acting on the particle and m_g the particle’s gravitational mass (‘‘gravitational charge’’). The corresponding interaction Lagrangian density is given by (see equation (244))

$$\hat{\mathcal{L}}_{\text{int}} = \frac{-2m m_g \varphi \hat{\Psi} \hat{\Psi}^\dagger}{\hbar^2}. \quad (310)$$

So far, we have not said how the charge conjugation operator affects the gravitational mass. Assuming the latter is invariant under the action of $\hat{\mathcal{C}}_m$, we have that

$$\hat{\mathcal{C}}_m \hat{\mathcal{L}}_{\text{int}} \hat{\mathcal{C}}_m^\dagger = \frac{2m m_g \varphi \hat{\Psi} \hat{\Psi}^\dagger}{\hbar^2} = -\hat{\mathcal{L}}_{\text{int}}, \quad (311)$$

where normal ordering is implied and we have used equations (298), (299), and that $\hat{\mathcal{C}}_m m_g \hat{\mathcal{C}}_m^\dagger = m_g$. Note that $\hat{\mathcal{C}}_m$ commutes with φ since the former does not act on the external source of the gravitational field.

Since the free-field part of equation (244) does not change under the action of $\hat{\mathcal{C}}_m$, the minus sign in equation (311) obtained after applying the charge conjugation operation implies that antiparticles will react to the external gravitational field differently when compared to the response of particles to the same field. If we had an attractive interaction when dealing with particles, we would now get a repulsive one for antiparticles. Note that we can make particles and antiparticles respond in the same way to a gravitational field if we postulate that the charge conjugation operation anticommutes with the gravitational mass m_g . This is equivalent to assuming that $m_g = m$ in equation (311). In other words, if the gravitational mass and inertial mass are equal, we will always have particles and particles, antiparticles and

antiparticles, and particles and antiparticles attracting each other gravitationally.

What happens if we now have an external static electric field? After the minimal coupling prescription, the interacting part of the Lagrangian density becomes (see equations (172) and (B.12))

$$\widehat{\mathcal{L}}_{\text{int}} = -\frac{iq}{\hbar} \left(\widehat{\Psi}^\dagger \partial_0 \widehat{\Psi} \right) A^0 + \frac{q^2}{\hbar^2} \widehat{\Psi} \widehat{\Psi}^\dagger A_0 A^0 - \frac{2q\mu}{\hbar} \widehat{\Psi} \widehat{\Psi}^\dagger A^0, \quad (312)$$

where A^0 is the source of the external electromagnetic field. Applying the charge conjugation operator, we now get (normal ordering implied)

$$\widehat{\mathcal{C}}_m \widehat{\mathcal{L}}_{\text{int}} \widehat{\mathcal{C}}_m^\dagger = \frac{iq}{\hbar} \left(\widehat{\Psi}^\dagger \partial_0 \widehat{\Psi} \right) A^0 + \frac{q^2}{\hbar^2} \widehat{\Psi} \widehat{\Psi}^\dagger A_0 A^0 + \frac{2q\mu}{\hbar} \widehat{\Psi} \widehat{\Psi}^\dagger A^0, \quad (313)$$

where we used that $\mu = mc/\hbar$ changes sign under the action of $\widehat{\mathcal{C}}_m$ since it depends on m ; the external potential A^0 is unaffected by the action of the charge conjugation operator. Comparing equations (312) and (313), we see that we can go from one to the other by simply changing the sign of the charge q . This is the expected result when we apply the charge conjugation operation: the antiparticle behaves like a particle with opposite electric charge. Note that we would have obtained the same conclusion if instead of working with the minimal coupling prescription, we had used equation (244) with the electrostatic interaction modeled by setting $V = q\varphi(x)$, with $\varphi(x)$ denoting the electrostatic potential of the external electric field.

7. Second Quantization of the Lorentz Covariant Schrödinger Equation

We now start the canonical quantization of the Lorentz covariant Schrödinger fields. We will try to stay as close as possible to the notation and traditional ways of dealing with the canonical quantization of a classical field theory [4, 5]. After finishing the canonical quantization of the free field and the computation of the Feynman propagator, we will present two applications of the present formalism. We will analyze the scattering cross section for two particles whose interaction is given by $(\lambda/4)[\Psi(x)\Psi^\dagger(x)]^2$, and we will also develop the scalar electrodynamics for the Lorentz covariant Schrödinger fields. Comparisons with the predictions coming from the complex Klein-Gordon fields subjected to these same interactions will also be made as well as with interactions that break Lorentz invariance.

7.1. General Solution to the Lorentz Covariant Schrödinger Equation. We can write the most general free-field solution ($V=0$) of equation (63) as (to simplify notation, we will not use the symbol “ \wedge ” to denote the second quantized field operators)

$$\Psi(x) = \int \widetilde{d}\mathbf{k}_+ a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}^+ t} e^{i\mathbf{k}\cdot\mathbf{r}} + \int \widetilde{d}\mathbf{k}_- b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}^- t} e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad (314)$$

which is equivalent to

$$\Psi(x) = \int \widetilde{d}\mathbf{k}_+ a_{\mathbf{k}} e^{i\mu x^0} e^{-i\mathbf{k}x} + \int \widetilde{d}\mathbf{k}_- b_{\mathbf{k}}^\dagger e^{i\mu x^0} e^{i\mathbf{k}x} = \Psi^+(x) + \Psi^-(x). \quad (315)$$

Note that we must multiply $b_{\mathbf{k}}^\dagger$ as well as $a_{\mathbf{k}}$ by $e^{i\mu x^0}$. Here, the expansion coefficients $a_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$, which depend on the wave number $\mathbf{k} = (k^1, k^2, k^3)$, were promoted to operators. Also, $kx = k_\mu x^\mu$ and

$$k^0 = \frac{\omega_{\mathbf{k}}}{c} = \frac{E_{\mathbf{k}}}{(\hbar c)}, \quad (316)$$

$$E_{\mathbf{k}} = \sqrt{m^2 c^4 + \hbar^2 c^2 |\mathbf{k}|^2}, \quad (317)$$

$$\omega_{\mathbf{k}}^\pm = \mp \mu c + \omega_{\mathbf{k}}, \quad (318)$$

$$\mu = \frac{mc}{\hbar}, \quad (319)$$

$$\widetilde{d}\mathbf{k}_\pm = f_\pm(|\mathbf{k}|) d^3k, \quad (320)$$

with $f_\pm(|\mathbf{k}|)$ real functions that depend only on the magnitude of the wave number. The integration in d^3k are carried out from $-\infty$ to ∞ and the fields are usually assumed to be zero at the boundaries of integration.

If we want $a_{\mathbf{k}}(b_{\mathbf{k}})$ and $a_{\mathbf{k}}^\dagger(b_{\mathbf{k}}^\dagger)$ to be bona fide particle (antiparticle) annihilation and creation operators, we should impose the following commutation relations [4]:

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger \right] = \left[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger \right] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad (321)$$

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'} \right] = \left[b_{\mathbf{k}}, b_{\mathbf{k}'} \right] = 0, \quad (322)$$

$$\left[a_{\mathbf{k}}, b_{\mathbf{k}'} \right] = \left[a_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger \right] = 0, \quad (323)$$

where $\delta^{(3)}(\mathbf{k} - \mathbf{k}') = \delta(k^1 - k'^1) \delta(k^2 - k'^2) \delta(k^3 - k'^3)$ is the three-dimensional Dirac delta function.

With these assumptions, we must choose the functions $f_\pm(|\mathbf{k}|)$ such that the standard equal-time commutation relations among the fields $\Psi(x)$ and $\Psi^\dagger(x)$ and their conjugate momenta are satisfied. In other words, $f_\pm(|\mathbf{k}|)$ is fixed in order to have the following set of commutation relations valid:

$$\left[\Psi(t, \mathbf{r}), \Pi_\Psi(t, \mathbf{r}') \right] = i\hbar \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad (324)$$

$$\left[\Psi^\dagger(t, \mathbf{r}), \Pi_{\Psi^\dagger}(t, \mathbf{r}') \right] = i\hbar \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \quad (325)$$

We also require that the remaining commutators involving any other combinations among the Ψ -fields, among their

conjugate momenta, and among them and their conjugate momenta are zero.

As we show in Appendix A, using equations (321)-(323), we can only satisfy equations (324) and (325), together with the other field commutation relations that must be zero, if

$$f_+(\|\mathbf{k}\|) = f_-(\|\mathbf{k}\|) = \left(\frac{\hbar c^2}{(2\pi)^3 2\omega_{\mathbf{k}}} \right)^{1/2}, \quad (326)$$

where $\omega_{\mathbf{k}}$ is given by equation (316). Using equation (326), we can write equation (315) as

$$\Psi(x) = e^{i\mu x^0} \int d^3k \left(\frac{\hbar c^2}{(2\pi)^3 2\omega_{\mathbf{k}}} \right)^{1/2} \left(a_{\mathbf{k}} e^{-ikx} + b_{\mathbf{k}}^\dagger e^{ikx} \right). \quad (327)$$

Note that the standard complex Klein-Gordon field expansion is given by equation (327) without the term $e^{i\mu x^0}$ [4].

Inserting equation (327) into the normal ordered expressions for the free-field Hamiltonian, linear momentum, and conserved charge, equations (260), (261), and (282), respectively, we get (see Appendix A)

$$H = \int d^3k \left(\hbar\omega_{\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \hbar\omega_{\mathbf{k}}^- b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right), \quad (328)$$

$$P^j = \int d^3k \left[\hbar k^j \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right) \right], \quad (329)$$

$$\tilde{Q} = \int d^3k \left[\hbar \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right) \right], \quad (330)$$

where

$$E_{\mathbf{k}}^\pm = \hbar\omega_{\mathbf{k}}^\pm = \mp mc^2 + E_{\mathbf{k}}. \quad (331)$$

Looking at equations (328) and (331), we see that the energies associated with particles and antiparticles are not the same, contrary to what one gets when dealing with the Klein-Gordon Hamiltonian. These energies are, nevertheless, always nonnegative, solving the “negative energy problem” that we faced when dealing with the first quantized fields (Section 5.1). Moreover, if $m \rightarrow -m$, we get that $E_{\mathbf{k}}^+ \rightarrow E_{\mathbf{k}}^-$. In this sense, when it comes to their energies, the particles and antiparticles can be regarded as effectively differing by the sign of its mass. However, as can be seen looking at equations (329) and (330), the linear momentum and conserved charge do not depend explicitly on the sign of the mass m . We can also understand the meaning of $E_{\mathbf{k}}^\pm$ by noting that $E_{\mathbf{k}}^+$ is actually the kinetic energy associated with a classical relativistic particle. In this way, $E_{\mathbf{k}}^-$ is the “kinetic energy” of a particle with an effective negative mass and it is in this sense that the discussion carried out in Section 5.1 about negative masses should be understood. (Note that in the present theory, the mass of a particle cannot be proportional to the expectation value of the Hamiltonian in its rest frame, where the expectation value is computed using

a single particle state $a_{\mathbf{k}}^\dagger|0\rangle = |\mathbf{k}\rangle$, with $\mathbf{k}=0$ since we are in the particle’s rest frame [5]. This is true because equations (328) and (331) give $\langle \mathbf{k}|H|\mathbf{k}\rangle = 0$. This implies that the usual proof stating that particles and antiparticles must have the same mass as defined by the previous expectation value cannot be applied here (see pages 331 and 332 of [5]).)

One might wonder if it is not possible to force the energies of the particles and antiparticles to be the same, in particular equal to $E_{\mathbf{k}}$, by properly adjusting the values of $f_\pm(\|\mathbf{k}\|)$ in equation (315). As we show in Appendix A, this is possible. But the price to pay is a theory that is apparently no longer local. This feature shows up since we can only make particles and antiparticles have the same energy if $[\Psi(t, \mathbf{r}), \Psi^\dagger(t, \mathbf{r}')] \neq 0$, which leads to a violation of the micro-causality condition. Also, the fact that the latter commutator is not null implies a modification to equations (324) and (325); i.e., we no longer have a canonically second quantized theory (see Appendix A). Unless stated otherwise, we will not deal with this scenario in the remaining of this section.

7.2. Continuous and Discrete Symmetries. The conserved Noether “charges” obtained for the classical fields in Section 6.2 manifest themselves here, for instance, in the following commutators being zero (see equations (328)-(330)):

$$[H, H] = [H, P^j] = [H, \tilde{Q}] = 0. \quad (332)$$

This means that the energy, the momentum, and the charge are all conserved for the free field.

Moving on to the discrete symmetries, we can now finish the discussion initiated in Section 6.3. Starting with space inversion (parity), we can now insert equation (327) into (285) to obtain the following transformation rules for the creation (annihilation) operators:

$$\begin{aligned} \hat{\mathcal{P}} a_{\mathbf{k}} \hat{\mathcal{P}}^\dagger &= a_{-\mathbf{k}}, & \hat{\mathcal{P}} a_{\mathbf{k}}^\dagger \hat{\mathcal{P}}^\dagger &= a_{-\mathbf{k}}^\dagger, \\ \hat{\mathcal{P}} b_{\mathbf{k}} \hat{\mathcal{P}}^\dagger &= b_{-\mathbf{k}}, & \hat{\mathcal{P}} b_{\mathbf{k}}^\dagger \hat{\mathcal{P}}^\dagger &= b_{-\mathbf{k}}^\dagger. \end{aligned} \quad (333)$$

Proceeding similarly with the time reversal operator, we get

$$\hat{\mathcal{T}} a_{\mathbf{k}} \hat{\mathcal{T}}^\dagger = a_{-\mathbf{k}}, \quad \hat{\mathcal{T}} a_{\mathbf{k}}^\dagger \hat{\mathcal{T}}^\dagger = a_{-\mathbf{k}}^\dagger, \quad (334)$$

$$\hat{\mathcal{T}} b_{\mathbf{k}} \hat{\mathcal{T}}^\dagger = b_{-\mathbf{k}}, \quad \hat{\mathcal{T}} b_{\mathbf{k}}^\dagger \hat{\mathcal{T}}^\dagger = b_{-\mathbf{k}}^\dagger. \quad (335)$$

The analysis of charge conjugation, as anticipated in Section 6.3, is a little more involved. If we apply the standard charge conjugation operator $\hat{\mathcal{C}}$, i.e. equation (297), to equation (327), we get

$$\hat{\mathcal{C}}\Psi(x)\hat{\mathcal{C}}^\dagger = \int \tilde{d}k \left(\hat{\mathcal{C}} a_{\mathbf{k}} \hat{\mathcal{C}}^\dagger e^{i\mu x^0} e^{-ikx} + \hat{\mathcal{C}} b_{\mathbf{k}}^\dagger \hat{\mathcal{C}}^\dagger e^{i\mu x^0} e^{ikx} \right). \quad (336)$$

where $\tilde{d}k = \sqrt{\hbar c^2 / [(2\pi)^3 2\omega_{\mathbf{k}}]} d^3k$. Comparing equation (336)

with the complex conjugate of equation (327),

$$\Psi^\dagger(x) = \int \tilde{d}\mathbf{k} \left(b_{\mathbf{k}} e^{-i\mu x^0} e^{-i\mathbf{k}x} + a_{\mathbf{k}}^\dagger e^{-i\mu x^0} e^{i\mathbf{k}x} \right), \quad (337)$$

we realize that is not possible to identify $\widehat{\mathcal{C}} a_{\mathbf{k}} \widehat{\mathcal{C}}^\dagger$ with $b_{\mathbf{k}}$ and $\widehat{\mathcal{C}} b_{\mathbf{k}}^\dagger \widehat{\mathcal{C}}^\dagger$ with $a_{\mathbf{k}}^\dagger$ due to the presence of the term $e^{i\mu x^0}$ in equation (336) and $e^{-i\mu x^0}$ in equation (337). Defining $\widehat{\mathcal{C}}$ antiunitary will not do the job either since in this case, equation (336) becomes

$$\widehat{\mathcal{C}} \Psi(x) \widehat{\mathcal{C}}^\dagger = \int \tilde{d}\mathbf{k} \left(\widehat{\mathcal{C}} a_{\mathbf{k}} \widehat{\mathcal{C}}^\dagger e^{-i\mu x^0} e^{i\mathbf{k}x} + \widehat{\mathcal{C}} b_{\mathbf{k}}^\dagger \widehat{\mathcal{C}}^\dagger e^{-i\mu x^0} e^{-i\mathbf{k}x} \right). \quad (338)$$

Comparing equation (338) with (337), we see that we do not get the expected interchange of particles with antiparticles. Actually, we get $\widehat{\mathcal{C}} a_{\mathbf{k}} \widehat{\mathcal{C}}^\dagger = a_{\mathbf{k}}^\dagger$ and $\widehat{\mathcal{C}} b_{\mathbf{k}}^\dagger \widehat{\mathcal{C}}^\dagger = b_{\mathbf{k}}$.

However, we can get a bona fide charge conjugation operator if we use $\widehat{\mathcal{C}}_m$ as given by equation (299). The operator $\widehat{\mathcal{C}}_m$ incorporates the ‘‘mass conjugation’’ operation as defined in equation (298). This means that $\widehat{\mathcal{C}}_m$ anticommutes with any function of the mass m . In this case, we have

$$\widehat{\mathcal{C}}_m \Psi(x) \widehat{\mathcal{C}}_m^\dagger = \int \tilde{d}\mathbf{k} \left(\widehat{\mathcal{C}}_m a_{\mathbf{k}} \widehat{\mathcal{C}}_m^\dagger e^{-i\mu x^0} e^{-i\mathbf{k}x} + \widehat{\mathcal{C}}_m b_{\mathbf{k}}^\dagger \widehat{\mathcal{C}}_m^\dagger e^{-i\mu x^0} e^{i\mathbf{k}x} \right), \quad (339)$$

since μ is linear in m and k^0 and $\tilde{d}\mathbf{k}$ are functions of m^2 . Comparing equation (339) with (337), we immediately get

$$\widehat{\mathcal{C}}_m a_{\mathbf{k}} \widehat{\mathcal{C}}_m^\dagger = b_{\mathbf{k}}, \quad \widehat{\mathcal{C}}_m a_{\mathbf{k}}^\dagger \widehat{\mathcal{C}}_m^\dagger = b_{\mathbf{k}}^\dagger, \quad (340)$$

$$\widehat{\mathcal{C}}_m b_{\mathbf{k}} \widehat{\mathcal{C}}_m^\dagger = a_{\mathbf{k}}, \quad \widehat{\mathcal{C}}_m b_{\mathbf{k}}^\dagger \widehat{\mathcal{C}}_m^\dagger = a_{\mathbf{k}}^\dagger, \quad (341)$$

the desired properties of a good charge conjugation operator.

We should also mention that although we used equation (327) to arrive at the previous transformation properties for the creation and annihilation operators, all the results remain valid had we used equation (315) and the fact that $\widehat{\mathcal{C}}_m \tilde{d}\mathbf{k}_\pm \widehat{\mathcal{C}}_m^\dagger = \tilde{d}\mathbf{k}_\mp$. In particular, this means that all the previous results apply to the creation and annihilation operators of the noncanonically second quantized theory given in Appendix A.

7.3. Microcausality and the Lorentz Covariance of the Field Commutators. Our first task will be the explicit calculation of the commutator $[\Psi(x), \Psi^\dagger(y)]$ at arbitrary space-time points $x = (x^0, x^1, x^2, x^3)$ and $y = (y^0, y^1, y^2, y^3)$. This will allow us to clearly see its Lorentz covariance and prove that the Lorentz covariant Schrödinger fields respect the microcausality condition [4, 5]. Then, we will present a general argument connecting the commutators between the Lorentz covariant Schrödinger fields with the commutators of the

Klein-Gordon fields, allowing us to readily extend the commutation properties of the latter to the former fields.

Let us start introducing the following shorthand notation for the field operators,

$$\Psi(x) = \Psi_x, \quad \Psi^\dagger(x) = \Psi_x^\dagger, \quad (342)$$

$$\Pi_\Psi(x) = \Pi_x, \quad \Pi_{\Psi^\dagger}(x) = \Pi_x^\dagger, \quad (343)$$

where the last equality causes no confusion since for the Lorentz covariant Schrödinger fields, we always have $\Pi_{\Psi^\dagger}(x) = \Pi_\Psi^\dagger(x)$. Now, using equations (315), (342), and (343), we get

$$[\Psi(x), \Psi^\dagger(y)] = \left[\Psi_x^+, \left(\Psi_y^+ \right)^\dagger \right] + \left[\Psi_x^-, \left(\Psi_y^- \right)^\dagger \right], \quad (344)$$

where we used that $[\Psi_x^+, (\Psi_y^-)^\dagger] = [\Psi_x^-, (\Psi_y^+)^\dagger] = 0$ to arrive at the right-hand side of equation (344). (Note that $(\Psi_x^+)^\dagger$ is one thing and $(\Psi_x^\dagger)^+$ is another. The latter is the positive frequency part of Ψ_x^\dagger , associated with the destruction operator $b_{\mathbf{k}}$, while the former is the adjoint of Ψ_x^+ .) This is true since Ψ_x^+ is proportional to the operator $a_{\mathbf{k}}$, $(\Psi_y^-)^\dagger$ to $b_{\mathbf{k}}$, and $[a_{\mathbf{k}}, b_{\mathbf{k}'}] = 0$.

Using the explicit plane wave expansions for Ψ_x^\pm as given in equation (315) and the commutation relations (321), we obtain

$$\left[\Psi_x^\pm, \left(\Psi_y^\pm \right)^\dagger \right] = i\hbar c \Delta_{\text{LS}}^\pm(x-y), \quad (345)$$

where

$$\Delta_{\text{LS}}^\pm(x) = e^{i\mu x^0} \Delta^\pm(x), \quad (346)$$

$$\Delta^\pm(x) = \frac{1}{\pm i\hbar c} \int d^3\mathbf{k} [f^\pm(\mathbf{k})]^2 e^{\mp i\mathbf{k}x}. \quad (347)$$

The subscript ‘‘LS’’ reminds us that the quantities above are the ones for the Lorentz covariant Schrödinger fields. The same quantities without the subscript, as we will see shortly if we use equation (326), are the ones we obtain dealing with the complex Klein-Gordon fields.

Using equations (345) and (346), we can write equation (344) as

$$[\Psi(x), \Psi^\dagger(y)] = i\hbar c \Delta_{\text{LS}}(x-y), \quad (348)$$

where

$$\Delta_{\text{LS}}(x) = \Delta_{\text{LS}}^+(x) + \Delta_{\text{LS}}^-(x) = e^{i\mu x^0} [\Delta^+(x) + \Delta^-(x)] = e^{i\mu x^0} \Delta(x). \quad (349)$$

If we now use equation (326), we get

$$\Delta^\pm(x) = \frac{\mp ic}{2(2\pi)^3} \int d^3k \frac{e^{\mp ikx}}{\omega_{\mathbf{k}}} = -\Delta^\mp(-x) \quad (350)$$

and

$$\Delta(x) = \Delta^+(x) + \Delta^-(x) = \frac{-c}{(2\pi)^3} \int d^3k \frac{\sin(kx)}{\omega_{\mathbf{k}}}. \quad (351)$$

As usual, we are employing the definitions given in equations (316)-(319). Equations (350) and (351) are exactly the ones we get when working with the complex Klein-Gordon Lagrangian [4, 5]. Note that $\Delta(x)$ and $\Delta^\pm(x)$ satisfy the Klein-Gordon equation [4, 5] and a direct calculation shows that $\Delta_{\text{LS}}(x)$ and $\Delta_{\text{LS}}^\pm(x)$ satisfy the Lorentz covariant Schrödinger equation.

The Lorentz invariant form of equation (351) is [4]

$$\Delta(x) = \frac{-i}{(2\pi)^3} \int d^4k \delta(k^2 - \mu^2) \varepsilon(k^0) e^{-ikx}, \quad (352)$$

where $d^4k = dk^0 dk^1 dk^2 dk^3$, $k^2 = k_\nu k^\nu$, μ , and $kx = k_\nu x^\nu$ are Lorentz invariants and the integrals run from $-\infty$ to ∞ . Here, $\varepsilon(k^0)$ gives the sign of k^0 , which is also Lorentz invariant. (The sign of k^0 is invariant under proper Lorentz transformations for time-like vectors, which is what we have here for the four-vector k due to the mass shell condition $\delta(k^2 - \mu^2)$. By a proper Lorentz transformation, we mean spatial rotations and Lorentz boosts. Time reversal and space inversion are excluded.)

Looking at equations (348) and (349), it is clear that the commutator (348) is covariant under spatial rotations. Let us see what happens under a Lorentz boost. Assuming, without loss of generality, a Lorentz boost along the x^1 -direction, we have

$$\Delta(x) \longrightarrow \Delta(x), \quad (353)$$

$$\Psi(x) \longrightarrow e^{i\theta(x)/\hbar} \Psi(x), \quad (354)$$

$$x^0 \longrightarrow \gamma(x^0 + \beta x^1), \quad (355)$$

with $\theta(x)$ given by equation (19). Using equations (353)-(355), a direct calculation gives

$$[\Psi_x, \Psi_y^\dagger] = i\hbar c \Delta_{\text{LS}}(x-y) \longrightarrow [\Psi_{x'}, \Psi_{y'}^\dagger] = i\hbar c \Delta_{\text{LS}}(x-y). \quad (356)$$

This last result, together with the covariance of the commutator under spatial rotations, proves the covariance of $[\Psi(x), \Psi^\dagger(y)]$ under proper Lorentz transformations.

If we compute the previous commutator at equal times (see equation (A.8) of Appendix A), we get

$$[\Psi(t, \mathbf{x}), \Psi^\dagger(t, \mathbf{y})] = 0, \quad (357)$$

which implies according to equation (348) that

$$\Delta_{\text{LS}}(0, \mathbf{x} - \mathbf{y}) = 0. \quad (358)$$

When the time argument is zero, equation (349) gives

$$\Delta_{\text{LS}}(0, \mathbf{x} - \mathbf{y}) = \Delta(0, \mathbf{x} - \mathbf{y}), \quad (359)$$

and thus, according to equation (358),

$$\Delta(0, \mathbf{x} - \mathbf{y}) = 0. \quad (360)$$

But since $\Delta(x-y)$ is invariant under a proper Lorentz transformation, we have that $\Delta(x-y) = 0$ whenever $(x-y)^2 < 0$. This is true because any two space-like vectors are connected to each other via a proper Lorentz transformation. Also, the fact $\Delta(x-y) = 0$ when $(x-y)^2 < 0$ and equation (349) imply that $\Delta_{\text{LS}}(x-y) = 0$ whenever $(x-y)^2 < 0$. This last result combined with equation (348) and with equation (356), which shows the Lorentz covariance of the commutator, leads to

$$[\Psi(x), \Psi^\dagger(y)] = 0, \quad \text{if } (x-y)^2 < 0. \quad (361)$$

Equation (361), telling us that fields with space-like separation commute, is the microcausality condition. This guarantees that the second quantized Lorentz covariant Schrödinger Lagrangian leads to a local quantum field theory [4, 5].

We can also connect the field commutators of the Lorentz covariant Schrödinger theory with the ones of the Klein-Gordon theory as follows. First, if we use equation (342), i.e., the relation connecting the Lorentz covariant Schrödinger Lagrangian with the Klein-Gordon one, we get

$$[\Psi(x), \Psi^\dagger(y)] = e^{i\mu(x^0 - y^0)} [\Phi(x), \Phi^\dagger(y)], \quad (362)$$

where $\Phi(x)$ and $\Phi^\dagger(x)$ are the complex Klein-Gordon fields. Moreover, applying equation (108) to (247), we obtain

$$\Pi_\Psi(x) = e^{-i\mu x^0} \Pi_\Phi(x), \quad (363)$$

where $\Pi_\Phi(x)$ is the conjugate momentum to $\Phi(x)$. Thus, using equations (108) and (363), the remaining nontrivial (not obviously zero) commutators can be written as

$$[\Pi_\Psi(x), \Pi_{\Psi^\dagger}(y)] = e^{-i\mu(x^0 - y^0)} [\Pi_\Phi(x), \Pi_{\Phi^\dagger}(y)], \quad (364)$$

$$[\Psi(x), \Pi_\Psi(y)] = e^{i\mu(x^0 - y^0)} [\Phi(x), \Pi_\Phi(y)]. \quad (365)$$

The first thing worth noting is that the equal time commutation relations of the Lorentz covariant Schrödinger and Klein-Gordon fields are the same. Indeed, for equal times, we have $x^0 = y^0$ and according to equations (362), (364),

and (365), we immediately get

$$\begin{aligned} [\Psi(t, \mathbf{x}), \Psi^\dagger(t, \mathbf{y})] &= [\Phi(t, \mathbf{x}), \Phi^\dagger(t, \mathbf{y})], \\ [\Pi_\Psi(t, \mathbf{x}), \Pi_{\Psi^\dagger}(t, \mathbf{y})] &= [\Pi_\Phi(t, \mathbf{x}), \Pi_{\Phi^\dagger}(t, \mathbf{y})], \\ [\Psi(t, \mathbf{x}), \Pi_\Psi(t, \mathbf{y})] &= [\Phi(t, \mathbf{x}), \Pi_\Phi(t, \mathbf{y})]. \end{aligned} \quad (366)$$

As such, we can understand equations (108) and (363) as a canonical transformation since they preserve the commutation relations as we move to the new “variables,” namely, fields. It is also important to remark that equations (362), (364), and (365) are only valid if equation (326) is true, since this is the only way to guarantee a canonical quantization of the Lorentz covariant Schrödinger fields. Therefore, equations (362), (364), and (365) do not apply to the noncanonically quantized Lorentz covariant Schrödinger fields given in Appendix A, where, for example, $[\Psi(t, \mathbf{x}), \Psi^\dagger(t, \mathbf{y})] \neq 0$, in clear contradiction to equation (362) and the fact that $[\Phi(t, \mathbf{x}), \Phi^\dagger(t, \mathbf{y})] = 0$.

Before we move on to the computation of the Feynman propagator, we note that we can write $\Delta^\pm(x)$ and $\Delta(x)$ as the following integrals, where k^0 is considered a complex variable [4]:

$$\Delta^\pm(x) = -\frac{1}{(2\pi)^4} \int_{C^\pm} d^4k \frac{e^{-ikx}}{k^2 - \mu^2}, \quad (367)$$

$$\Delta(x) = -\frac{1}{(2\pi)^4} \int_C d^4k \frac{e^{-ikx}}{k^2 - \mu^2}. \quad (368)$$

The complex contour integral in k_0 is such that C^\pm is any counterclockwise closed path encircling only $\pm\omega_{\mathbf{k}}/c$ and C is any counterclockwise closed path encircling $\omega_{\mathbf{k}}/c$ and $-\omega_{\mathbf{k}}/c$, the two simple poles of the integrand.

7.4. The Feynman Propagator. The first step towards a perturbative treatment of any interaction involving the Lorentz covariant Schrödinger fields, or involving them and other types of fields, is the calculation of the Feynman propagator. The Feynman propagator $\Delta_{F_{\text{LS}}}(x - x')$ is defined as [4, 5]

$$i\hbar c \Delta_{F_{\text{LS}}}(x - x') = \langle 0 | T \{ \Psi(x) \Psi^\dagger(x') \} | 0 \rangle, \quad (369)$$

where $|0\rangle$ is the vacuum state and we use the subscript “LS” to differentiate the Feynman propagator associated with the Lorentz covariant Schrödinger fields from $\Delta_F(x)$, the Feynman propagator related to the Klein-Gordon fields.

In equation (369), the symbol T denotes time ordering, giving rise to the T -product (time-ordered product),

$$T \{ \Psi(x) \Psi^\dagger(x') \} = h(x^0 - x'^0) \Psi(x) \Psi^\dagger(x') + h(x'^0 - x^0) \Psi^\dagger(x') \Psi(x), \quad (370)$$

with h being the Heaviside step function,

$$\begin{aligned} h(x^0) &= 1, \quad \text{if } x^0 > 0, \\ h(x^0) &= 0, \quad \text{if } x^0 < 0. \end{aligned} \quad (371)$$

Using equation (345) and remembering that $\Psi(x) = \Psi^+(x) + \Psi^-(x)$, with $\Psi^+(x)$ and $\Psi^-(x)$, respectively, functions of annihilation and creation operators, we have

$$\langle 0 | \Psi(x) \Psi^\dagger(x') | 0 \rangle = i\hbar c \Delta_{\text{LS}}^+(x - x'), \quad (372)$$

$$\langle 0 | \Psi^\dagger(x') \Psi(x) | 0 \rangle = -i\hbar c \Delta_{\text{LS}}^-(x - x'). \quad (373)$$

If we now insert equations (370), (372), and (373) into (369), we get

$$\Delta_{F_{\text{LS}}}(x) = h(x^0) \Delta_{\text{LS}}^+(x) - h(-x^0) \Delta_{\text{LS}}^-(x) \quad (374)$$

and also

$$\Delta_{F_{\text{LS}}}(x) = e^{i\mu x^0} \Delta_F(x), \quad (375)$$

if we employ equation (346). Here, $\Delta_F(x) = h(x^0) \Delta^+(x) - h(-x^0) \Delta^-(x)$ is the Feynman propagator associated with the complex field Klein-Gordon Lagrangian, also known as the charged meson propagator [4, 5].

Under a proper Lorentz transformation, $\Psi(x)$ changes according to equations (18) and (19) while $h(x^0)$ is an invariant. The latter is true because the sign of x^0 for time-like vectors is not altered under a proper Lorentz transformation. These two facts lead to the following transformation rule for equation (369):

$$\Delta_{F_{\text{LS}}}(x) \longrightarrow e^{i\theta(x)/\hbar} \Delta_{F_{\text{LS}}}(x). \quad (376)$$

To arrive at equation (376), which tells us how the Feynman propagator of the Lorentz covariant Schrödinger fields transforms under a proper Lorentz transformation, we have also assumed that the latter changes the vacuum state $|0\rangle$ by at most a global phase. Equation (376), combined with the transformation rule for x^0 and the invariance of $\Delta_F(x)$ under a proper Lorentz transformation, allows us to show that equation (375) is covariant under a proper Lorentz transformation, i.e.,

$$\Delta_{F_{\text{LS}}}(x) = e^{i\mu x^0} \Delta_F(x) \longrightarrow \Delta_{F_{\text{LS}}}(x) = e^{i\mu x^0} \Delta_F(x). \quad (377)$$

Due to the connection between the Klein-Gordon and the Lorentz covariant Schrödinger propagators as given by equation (375), we can carry over the integral representations of the former to the latter. Following ref. [4], we have

$$\Delta_{F_{\text{LS}}}(x) = \frac{e^{i\mu x^0}}{(2\pi)^4} \int_{C_F} d^4k \frac{e^{-ikx}}{k^2 - \mu^2}, \quad (378)$$

where the contour of integration C_F is given in Figure 2 and should be understood in the following sense. For $x^0 > 0$, the contour of integration is closed in the lower half of the complex k^0 -plane, leading to a clockwise contour integration whose path we denote by C_F^+ . On the other hand, for $x^0 < 0$, we use the upper half plane to close the path of integration, which gives a counterclockwise integration around path C_F^- . In this way, computing the complex integral in k^0 using the residue theorem, we obtain $\Delta_{LS}^+(x)$ for the former contour of integration and $-\Delta_{LS}^-(x)$ for the latter, the expected results for the Feynman propagator (see equation (374)).

We can also write the Feynman propagator with all integrals being real variable integrals if we slightly displace the poles off the real axis,

$$\pm \frac{\omega_{\mathbf{k}}}{c} \longrightarrow \pm \frac{\omega_{\mathbf{k}}}{c} \mp i\eta, \quad (379)$$

with $0 < \eta \ll 1$, and use the following integral representation:

$$\begin{aligned} \Delta_{F_{LS}}(x) &= \frac{e^{i\mu x^0}}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{(k^0)^2 - ((\omega_{\mathbf{k}}/c) - i\eta)^2} \\ &= \frac{e^{i\mu x^0}}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{k^2 - \mu^2 + i\epsilon}. \end{aligned} \quad (380)$$

To arrive at the last equality, we have neglected the η^2 term since $\eta \ll 1$ and made the following identification:

$$\epsilon = \frac{2\eta\omega_{\mathbf{k}}}{c}. \quad (381)$$

With this definition, the k^0 variable is also integrated from $-\infty$ to ∞ . After the integration in k^0 , we take the limit $\eta \rightarrow 0$, or equivalently $\epsilon \rightarrow 0$, obtaining the corresponding expression for the Feynman propagator, namely, $\Delta_{LS}^+(x)$ for $x^0 > 0$ and $-\Delta_{LS}^-(x)$ for $x^0 < 0$.

Using equation (380), we can show that the Feynman propagator $\Delta_{F_{LS}}(x)$ is the Green function of the free Lorentz covariant Schrödinger equation. Setting $V = 0$ in equation (63) and replacing $\Psi(x)$ for $\Delta_{F_{LS}}(x)$ as given in equation (380), we obtain after a little calculation and after taking the limit $\epsilon \rightarrow 0$ that

$$\partial_\nu \partial^\nu \Delta_{F_{LS}}(x) - i2\mu \partial_0 \Delta_{F_{LS}}(x) = -\frac{e^{i\mu x^0}}{(2\pi)^4} \int d^4k e^{-ikx}. \quad (382)$$

Noting that the four-dimensional Dirac delta function can be written as $(2\pi)^4 \delta^{(4)}(x) = \int d^4k e^{-ikx}$, we get

$$\partial_\nu \partial^\nu \Delta_{F_{LS}}(x) - i2\mu \partial_0 \Delta_{F_{LS}}(x) = -e^{i\mu x^0} \delta^{(4)}(x) = -\delta^{(4)}(x). \quad (383)$$

We have used that $e^{i\mu x^0} \delta^{(4)}(x) = \delta^{(4)}(x) = 0$ for $x^0 \neq 0$ and that $\lim_{x^0 \rightarrow 0} e^{i\mu x^0} \delta^{(4)}(x) = \lim_{x^0 \rightarrow 0} \delta^{(4)}(x)$ to arrive at the last equality. If we equate the left-hand side of equation (383) to zero, we get the free Lorentz covariant Schrödinger equation.

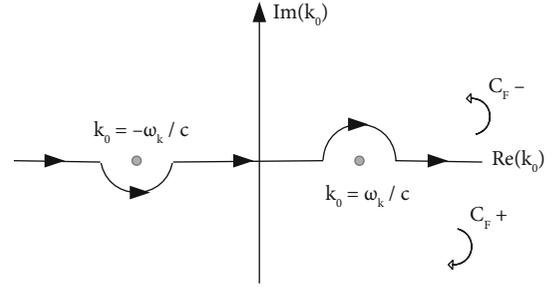


FIGURE 2: The contours of integration leading to the Feynman propagator for the Lorentz covariant Schrödinger fields.

Equating it to $-\delta^{(4)}(x)$, we obtain the definition of its Green's function, proving that $\Delta_{F_{LS}}(x)$ is indeed the Green function of the free Lorentz covariant Schrödinger equation. For the Klein-Gordon equation, the analogous expression to equation (383) is [5]

$$\partial_\nu \partial^\nu \Delta_F(x) + \mu^2 \Delta_F(x) = -\delta^{(4)}(x). \quad (384)$$

7.5. Turning on the Interaction among the Fields. The techniques built to perturbatively handle scattering problems among interacting quantum fields, in particular those suited to the Klein-Gordon fields [4, 5], remain valid for the Lorentz covariant Schrödinger fields. Working in the interaction picture (in previous sections of this work, specially when dealing with the second quantization of the free fields, we were working in the Heisenberg picture), it is not difficult to see that the Lorentz covariant Schrödinger fields evolve according to the same dynamical equations and satisfy the same commutation relations of the free fields. On the other hand, the evolution of the quantum state describing the initial particle configuration evolves according to the Schrödinger equation, whose Hamiltonian is given by the interaction part of the total Hamiltonian describing the system being investigated. Moreover, a simple inspection on the proofs leading to the Dyson series and to Wick's theorem shows that they remain valid here too.

As such, the probability amplitude for a system initially in the state $|\alpha_{in}\rangle$ to be found after the collision (scattering) in the final state $|\alpha_{out}\rangle$ is

$$S_{fi} = \langle \alpha_{out} | S | \alpha_{in} \rangle. \quad (385)$$

Here, invoking the adiabatic hypothesis, $|\alpha_{in(out)}\rangle$ are eigenstates of the free-field Hamiltonian, with $|\alpha_{in}\rangle$ the initial state in the remote past ($t \rightarrow -\infty$) and $|\alpha_{out}\rangle$ the final state in the far future ($t \rightarrow \infty$). The operator describing this transition is called the S-matrix and its manifestly covariant perturbative expansion is given by the Dyson series,

$$\begin{aligned} S &= T \exp \left\{ -\frac{i}{\hbar c} \int d^4x \mathcal{H}_{int}^I(x) \right\} \\ &= 1 + \sum_{n=1}^{\infty} \frac{[-i/(\hbar c)]^n}{n!} \int d^4x_1 \cdots d^4x_n T \{ \mathcal{H}_{int}^I(x_1) \cdots \mathcal{H}_{int}^I(x_n) \}, \end{aligned} \quad (386)$$

where T is the time ordering operator and $\mathcal{H}_{\text{int}}^I$ is the Hamiltonian density describing the interaction among the fields expressed in the interaction picture (the superscript I denotes interaction picture). Similarly to the Klein-Gordon case, the time-ordered term in the Dyson series is expanded using Wick's theorem in order to carry out transition amplitude calculations. It is in this step that the Feynman propagator (369) appears, being usually called in this context a "contraction."

An important property of the Lorentz covariant Schrödinger fields, valid when they are canonically quantized, is the simple relation connecting them to the complex Klein-Gordon fields as given by equations (108) and (363). These relations, originally shown to be true for the free fields being described in the Heisenberg picture, remain valid in the interaction picture. This is easily seen either by inspecting the time evolution of these fields in the interaction picture or by noting that these transformations connecting the Lorentz covariant Schrödinger fields to the Klein-Gordon ones are given by c -numbers and, as such, commute with the unitary transformation taking us from the Heisenberg to the interaction picture. Written in the interaction picture, equations (108) and (363) become

$$\Psi^I(x) = e^{i\mu x^0} \Phi^I(x), \quad (387)$$

$$\Pi_{\Psi}^I(x) = e^{-i\mu x^0} \Pi_{\Phi}^I(x). \quad (388)$$

As can be seen explicitly in Appendix B, these relations are crucial to prove that to any order in perturbation theory and for Lorentz covariant interactions, the Klein-Gordon and the Lorentz covariant Schrödinger theories are essentially equivalent. However, for interactions violating Lorentz invariance, the predictions stemming from both theories are drastically different (the noncanonically quantized theory given in Appendix A is not equivalent to the Klein-Gordon theory either, even for Lorentz covariant interactions. This is true because, contrary to the canonically quantized case, the propagator of the noncanonically quantized theory is not trivially related to the Klein-Gordon one).

7.6. Breaking Lorentz Invariance. The equivalence between the Klein-Gordon and the Lorentz covariant Schrödinger theories established above is only valid for Lorentz invariant Lagrangian densities (actually, being more general, we just need Lorentz invariant actions). If we introduce, nevertheless, interaction terms breaking Lorentz invariance, we can have certain processes that only occur in the Lorentz covariant Schrödinger theory. See, however, Refs. [26, 27] for interesting and alternative ways of introducing interaction terms that break Lorentz invariance in the framework of the standard Klein-Gordon theory and in the Higgs sector.

For example, consider the following interaction term (as always, normal ordering is implied, and here, it is also implied an adjoint term in order to have a Hermitian Hamiltonian):

$$\mathcal{H}_{\text{int}} \propto \Psi(\Psi^\dagger)^3. \quad (389)$$

This term is not Lorentz invariant since after a proper Lorentz transformation, we have, according to equation (18),

$$\Psi(x) [\Psi^\dagger(x)]^3 \longrightarrow e^{-i2\theta(x)/\hbar} \Psi(x) [\Psi^\dagger(x)]^3. \quad (390)$$

The analogous term for the Klein-Gordon theory, namely, $\Phi(\Phi^\dagger)^3$, is invariant under a proper Lorentz transformation since in this case, $\Phi \longrightarrow \Phi$.

An interaction term such as (389) allows, among other processes, for an antiparticle at rest to decay into two particles and one antiparticle, all of them at rest too (see Figure 3(a)). This process is kinematically forbidden for the Klein-Gordon theory, violating conservation of energy, while for the Lorentz covariant Schrödinger theory, the total energy and momentum can be made equal before and after the decay.

This can be seen by noting that to first order in perturbation theory (at the tree level), this process is associated with the initial state $|\alpha_{\text{in}}\rangle = b_{\mathbf{k}}^\dagger |0\rangle$ and to the final state $|\alpha_{\text{out}}\rangle = b_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger a_{\mathbf{k}_3}^\dagger |0\rangle$. A nonnull probability amplitude is obtained using the following term coming from (389):

$$\Psi^- [(\Psi^\dagger)^-]^2 (\Psi^\dagger)^+, \quad (391)$$

where $(\Psi^\dagger)^+$ destroys the antiparticle, Ψ^- creates another antiparticle, and $[(\Psi^\dagger)^-]^2$ creates the remaining two particles. The previous field operators are defined by noting that we can write $\Psi = \Psi^+ + \Psi^-$ and $\Psi^\dagger = (\Psi^\dagger)^+ + (\Psi^\dagger)^-$, which after equation (327) can be explicitly written as

$$\Psi^+(x) = \int d^3k f_{\mathbf{k}} e^{i\mu x^0} e^{-ikx} a_{\mathbf{k}}, \quad (392)$$

$$\Psi^-(x) = \int d^3k f_{\mathbf{k}} e^{i\mu x^0} e^{ikx} b_{\mathbf{k}}^\dagger, \quad (393)$$

$$[(\Psi^\dagger)^-](x) = \int d^3k f_{\mathbf{k}} e^{-i\mu x^0} e^{ikx} a_{\mathbf{k}}^\dagger, \quad (394)$$

$$[(\Psi^\dagger)^+](x) = \int d^3k f_{\mathbf{k}} e^{-i\mu x^0} e^{-ikx} b_{\mathbf{k}}, \quad (395)$$

where

$$f_{\mathbf{k}} = \left[\frac{\hbar c^2}{(2\pi)^3 2\omega_{\mathbf{k}}} \right]^{1/2}, \quad (396)$$

$$k^0 = \frac{\omega_{\mathbf{k}}}{c} = \frac{E_{\mathbf{k}}}{(\hbar c)},$$

$$E_{\mathbf{k}} = \sqrt{m^2 c^4 + \hbar^2 c^2 |\mathbf{k}|^2}.$$

Suppressing the superscript I that indicates operators in the interaction picture, equations (385), (386), and (391)

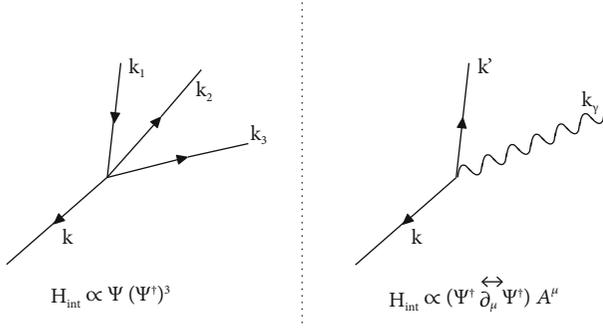


FIGURE 3: Feynman diagrams in momentum space showing two possible processes for the Lorentz covariant Schrödinger theory when we introduce interaction terms that break Lorentz invariance. These two processes are kinematically forbidden for the Klein-Gordon theory subjected to analogous interaction terms. (a) An antiparticle (b -particle) with momentum \mathbf{k} and energy $E_{\mathbf{k}}^-$ decays into another antiparticle and two particles (a -particles). The energies of the particles are $E_{\mathbf{k}_1}^+$ and $E_{\mathbf{k}_2}^+$. We can represent this process by the following equation, $b \rightarrow baa$. Note that the process (not shown in the picture) in which particles and antiparticles exchange roles, $a \rightarrow abb$, is kinematically forbidden. (b) An antiparticle decays into a photon and a particle, $b \rightarrow a\gamma$. The processes (not shown in the picture) $a \rightarrow b\gamma$, $a \rightarrow a\gamma$, and $b \rightarrow b\gamma$ are all kinematically forbidden.

give to the first order in perturbation theory

$$S_{\text{fi}}^{(1)} = \langle \alpha_{\text{out}} | S^{(1)} | \alpha_{\text{in}} \rangle, \quad (397)$$

where

$$S^{(1)} = -\frac{i}{\hbar c} \int d^4x \mathcal{H}_{\text{int}} = -\frac{i}{\hbar c} \int d^4x \Psi^\dagger(x) \{ [\Psi^\dagger(x)]^- \}^2 [\Psi^\dagger(x)]^+, \quad (398)$$

with g being the effective coupling constant associated with this process. If we now use equations (392)-(395) and the explicit expansion for $|\alpha_{\text{in(out)}}\rangle$, we get

$$S_{\text{fi}}^{(1)} = -\frac{ig}{\hbar c} \int d^4x d^3q_1 \cdots d^3q_4 f_{\mathbf{q}_1} \cdots f_{\mathbf{q}_4} e^{-i[2\mu x^0 + (q_4 - q_1 - q_2 - q_3)x]} \cdot \langle 0 | a_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{q}_2}^\dagger a_{\mathbf{q}_3}^\dagger b_{\mathbf{k}_1} b_{\mathbf{q}_1}^\dagger b_{\mathbf{q}_4} b_{\mathbf{k}}^\dagger | 0 \rangle. \quad (399)$$

A direct calculation using the commutation relations given by equation (321) leads to

$$\begin{aligned} \langle 0 | a_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{q}_2}^\dagger a_{\mathbf{q}_3}^\dagger b_{\mathbf{k}_1} b_{\mathbf{q}_1}^\dagger b_{\mathbf{q}_4} b_{\mathbf{k}}^\dagger | 0 \rangle &= \delta^{(3)}(\mathbf{k} - \mathbf{q}_4) \delta^{(3)}(\mathbf{k}_1 - \mathbf{q}_1) \delta^{(3)} \\ &\cdot (\mathbf{k}_2 - \mathbf{q}_3) \delta^{(3)}(\mathbf{k}_3 - \mathbf{q}_2) + \delta^{(3)}(\mathbf{k} - \mathbf{q}_4) \delta^{(3)}(\mathbf{k}_1 - \mathbf{q}_1) \delta^{(3)} \\ &\cdot (\mathbf{k}_2 - \mathbf{q}_2) \delta^{(3)}(\mathbf{k}_3 - \mathbf{q}_3). \end{aligned} \quad (400)$$

Inserting equation (400) into (399) and noting that by relabeling the integration variables, the two terms in equa-

tion (400) give the same contribution, we get

$$\begin{aligned} S_{\text{fi}}^{(1)} &= -\frac{2ig}{\hbar c} \int d^4x f_{\mathbf{k}} f_{\mathbf{k}_1} f_{\mathbf{k}_2} f_{\mathbf{k}_3} e^{-i(2\mu + k^0 - k_1^0 - k_2^0 - k_3^0)x^0} e^{i(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \cdot \mathbf{r}} \\ &= -\frac{2ig}{\hbar c} f_{\mathbf{k}} f_{\mathbf{k}_1} f_{\mathbf{k}_2} f_{\mathbf{k}_3} (2\pi)^4 \delta(2\mu + k^0 - k_1^0 - k_2^0 - k_3^0) \delta^{(3)} \\ &\cdot (\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3). \end{aligned} \quad (401)$$

To arrive at the last line, we employed the integral representation of the Dirac delta function.

The two Dirac delta functions in equation (401) are related to the conservation of energy and linear momentum for the present process; i.e., $S_{\text{fi}}^{(1)}$ is different from zero only if the arguments of the delta functions are zero. Considering the decaying particle at rest, we have $\hbar c k^0 = mc^2$ and $\mathbf{k} = 0$. Thus, conservation of energy leads to

$$\begin{aligned} 2\mu + k^0 - k_1^0 - k_2^0 - k_3^0 &= 0, \\ 2\hbar c \mu + \hbar c k^0 - \hbar c(k_1^0 - k_2^0 - k_3^0) &= 0, \\ 2mc^2 + mc^2 - E_{\mathbf{k}_1} - E_{\mathbf{k}_2} - E_{\mathbf{k}_3} &= 0. \end{aligned} \quad (402)$$

But $E_{\mathbf{k}_j} = \sqrt{m^2 c^4 + h^2 c^2 |\mathbf{k}_j|^2} \leq mc^2$, for $j = 1, 2, 3$. Therefore, we can only satisfy

$$E_{\mathbf{k}_1} + E_{\mathbf{k}_2} + E_{\mathbf{k}_3} = 3mc^2 \quad (403)$$

if $\mathbf{k}_j = 0$; i.e., all particles are at rest after the decay. Note that the momentum conservation equation is automatically satisfied if $\mathbf{k} = \mathbf{k}_j = 0$.

We could have anticipated the possibility of this decay process by noting that for the Lorentz covariant Schrödinger theory, the particle and antiparticles have different energies, $E_{\mathbf{k}}^+ = -mc^2 + E_{\mathbf{k}}$ and $E_{\mathbf{k}}^- = mc^2 + E_{\mathbf{k}}$, respectively (see equations (314), (328), and (331)). Hence, if we look at the Feynman diagram for this process (Figure 3(a)), energy conservation leads to

$$E_{\mathbf{k}}^- = E_{\mathbf{k}_1}^- + E_{\mathbf{k}_2}^+ + E_{\mathbf{k}_3}^+, \quad (404)$$

which is exactly what we obtain according to equation (401) if we use the definition of $E_{\mathbf{k}}^\pm$ (for the Klein-Gordon theory, the analog of equation (404) is $E_{\mathbf{k}} = E_{\mathbf{k}_1} + E_{\mathbf{k}_2} + E_{\mathbf{k}_3}$, which can never be satisfied. Indeed, in the rest frame of the antiparticle with momentum \mathbf{k} , we have $E_{\mathbf{k}} = mc^2$ while the right-hand side is always greater or equal to $3mc^2$. Also, to obtain the same predictions associated with the Lorentz covariant Schrödinger theory when the fields interact according to $\mathcal{H}_{\text{int}} \propto \Psi(\Psi^\dagger)^3$, we need for the Klein-Gordon theory an interaction given by $\mathcal{H}_{\text{int}} \propto e^{-i2\mu x^0} \Phi(\Phi^\dagger)^3$, a very unnatural interaction term).

For Lorentz invariant interactions, which we have seen must always be built with bilinear terms such as $\Psi\Psi^\dagger$, this gap of energy between particles and antiparticles gets

“averaged out.” This happens because we always have an equal number of $e^{-i\mu x^0}$ and $e^{i\mu x^0}$ when we employ the field expansions to do any calculation (see, for instance, equations (392)-(395)). However, when we have an interaction term breaking Lorentz invariance, we no longer have a balanced number of $e^{\pm i\mu x^0}$ in the field expansions. This is why the previous process does not violate energy-momentum conservation and why we have $S_{fi}^{(1)}$ as given by equation (401).

Remark 1. The above interaction that breaks Lorentz invariance should be considered with caution. It is best to understand it as a “toy model” and not as a fundamental interaction. The same should apply to the interaction we will study next. They are simple examples illustrating a kinematically forbidden interaction in the framework of the Klein-Gordon theory that is kinematically allowed in the framework of the Lorentz covariant Schrödinger theory. The interaction term given by equation (389), leading to the Feynman diagram of Figure 3(a), when taken alone brings to the table an “unstable vacuum.” Since an antiparticle decays into two particles and an antiparticle, the decay process will never cease because we will always have the same antiparticle as a product of the decay. This unstoppable proliferation of decays is not currently observed. We could have avoided this problem working with the following interaction, $\mathcal{H}_{\text{int}} \propto (\Psi^\dagger)^3$, which also breaks Lorentz invariance but leads to one antiparticle decaying into only two particles. We can alternatively understand the interaction (389) as being modulated by a time-dependent coupling constant that decreases as time goes by. In primordial times, it was relevant, but today, it is completely suppressed.

We can also have processes involving photons that are not seen in the Klein-Gordon theory but that are present in the Lorentz covariant Schrödinger theory if we introduce appropriate Lorentz invariance-breaking terms in the interaction Hamiltonian density. For instance, interactions such as

$$(\Psi^\dagger \partial_\mu \Psi^\dagger) A^\mu \text{ or } \Psi^\dagger \Psi^\dagger A^\mu \quad (405)$$

allow for the possibility of an antiparticle decaying into a particle and a photon (see Figure 3(b)).

The equation expressing the conservation of energy for this process is

$$E_{\mathbf{k}}^- = E_{\mathbf{k}'}^+ + E_{\mathbf{k}_\gamma}, \quad (406)$$

which in the rest frame of the decaying particle ($\mathbf{k} = 0$) becomes

$$mc^2 + E_{\mathbf{k}} = -mc^2 + E_{\mathbf{k}'} + E_{\mathbf{k}_\gamma}, \quad (407)$$

$$2mc^2 = -mc^2 + E_{\mathbf{k}'} + E_{\mathbf{k}_\gamma}. \quad (408)$$

Equation (407) tells us that we must have

$$E_{\mathbf{k}'} + E_{\mathbf{k}_\gamma} = 3mc^2, \quad (409)$$

which can be satisfied since any process such that $E_{\mathbf{k}'} + E_{\mathbf{k}_\gamma} > mc^2$ and that conserves momentum ($\mathbf{k}' + \mathbf{k}_\gamma = 0$) is kinematically possible. The solution to this problem is $E_{\mathbf{k}_\gamma} = 4mc^2/3$, $E_{\mathbf{k}}^+ = 2mc^2/3$, and $|\mathbf{p}_\gamma| = |\mathbf{p}'| = \hbar c|\mathbf{k}_\gamma| = \hbar c|\mathbf{k}'| = 4mc/3$.

For the Klein-Gordon theory, we would have instead of equations (406) and (407)

$$\begin{aligned} E_{\mathbf{k}} &= E_{\mathbf{k}'} + E_{\mathbf{k}_\gamma}, \\ mc^2 &= E_{\mathbf{k}'} + E_{\mathbf{k}_\gamma}, \end{aligned} \quad (410)$$

which cannot be satisfied because the right-hand side must be greater than mc^2 .

Remark 2. The two interactions that break Lorentz invariance studied in this section when taken alone lead to the violation of charge conservation as well. For the interaction given by the Feynman diagram of Figure 3(b), we have, for instance, a positive antiparticle decaying into a neutral photon and a negative particle. If we want to save charge conservation, we can either say that those processes are forbidden despite being kinematically allowed or postulate the existence of a negative antiparticle decaying in an analogous way into a positive particle and a photon. Those two processes taken together would lead to an overall charge conservation in our universe. A similar reasoning can restore overall charge conservation for the interaction (389).

Before we move on, it is worth mentioning two points concerning the previous results. First, the Lorentz covariant Schrödinger theory supplemented with appropriate interaction terms that break Lorentz invariance might lead to simple and useful effective field theories to describe condensed matter physical process. Indeed, in a condensed matter system, there is a “privileged” inertial frame, namely, the condensed matter system itself, and the introduction of Lorentz invariance-breaking interactions is not a serious threat to the modeling of the interactions among pseudoparticles within it.

Second, the asymmetry in the decay rates associated with particles and antiparticles, due to the introduction of interaction terms breaking Lorentz invariance, points to a possible way to understand the asymmetry between matter and antimatter in our universe [28]. As we previously remarked, the introduction of an interaction term such as $\Psi(\Psi^\dagger)^3$ implies that only the decay $b \rightarrow baa$ is possible (see Figure 3(a)), while the corresponding process obtained by exchanging the roles of particles with antiparticles, $a \rightarrow abb$, is kinematically forbidden. Therefore, for an initially symmetric distribution of matter and antimatter, the asymmetry in the previous two decay channels will eventually lead to an asymmetry in the matter and antimatter distribution (baryogenesis). And if this asymmetry in the decay

channels is always present and not counterbalanced by any other process, in the long run, there will be practically no antimatter in the universe.

Note that the underlying reason for the asymmetry in the two decay channels above is the gap in the mass between particle and antiparticle naturally present in the Lorentz covariant Schrödinger theory and no extra fields or interactions are needed to obtain an asymmetric behavior between matter and antimatter [26, 27, 29–33]. On the one hand, for Lorentz invariant interactions, this mass difference is effectively suppressed and no asymmetry between matter and antimatter is observed. On the other hand, for interaction terms violating Lorentz invariance, the mass gap becomes relevant, leading to an asymmetry between matter and antimatter that is testable. With that in mind, two important issues arise: (1) Is there a fundamental physical process leading to the violation of the Lorentz invariance of the Lorentz covariant Schrödinger theory? What causes it? Can it be traced back to the presence of a background gravitational field? Is this Lorentz invariance-breaking process a feature of the present-day universe or it was relevant in its beginning, being suppressed during its evolution? (2) Can we find a Lorentz invariant theory that gives *different masses* for particles and antiparticles and, at the same time, leads to an asymmetry in the decay of particles and antiparticles as described above? Can it be done without violating the CPT theorem, or at least without violating its extension, the CPTM theorem as given in this work?

8. Generalizing the Lorentz Covariant Schrödinger Equation

If we look at the second term of the Lagrangian density (244), we realize that there are no derivatives in the spatial variables and the following question naturally arises: Is it possible to come up with a Lorentz invariant Lagrangian density where the four space-time derivatives appear on an equal footing? Furthermore and similarly to the Lorentz covariant Schrödinger fields of the previous sections, can this be done in such a way that this Lagrangian density yields the same predictions of the Klein-Gordon one if we restrict ourselves to Lorentz invariant interactions? Our goal in this section is to show that the answers to those questions are affirmative.

8.1. Obtaining the Lagrangian Density and the Transformation Law for $\Psi(x)$. We start with the following Hermitian Lagrangian density ansatz:

$$\mathcal{L} = \partial_\mu \Psi \partial^\mu \Psi^* + i\kappa^\mu \Psi^* \partial_\mu \Psi - i(\kappa^\mu)^* \Psi \partial_\mu \Psi^*, \quad (411)$$

where κ^μ does not transform like a four-vector under a Lorentz transformation. It is just a shorthand notation for the four constants $\kappa^0, \kappa^1, \kappa^2, \kappa^3$. Being more specific, we postulate that $\kappa^0, \kappa^1, \kappa^2, \kappa^3$ are Lorentz invariants under proper Lorentz transformations. These constants are the analog in the present theory of the mass “ m ” that appears in the Klein-Gordon Lagrangian density. We will see later that κ^μ must be a real quantity to guarantee the Lorentz invariance

of the Lagrangian density (411) and that $\kappa^2 = \kappa_\mu \kappa^\mu = \mu^2 = m^2 c^2 / \hbar^2$, with m being the rest mass of a Klein-Gordon scalar particle, if we want the present theory to be equivalent to the Klein-Gordon theory. Also, if $\kappa^0 = \mu$ and $\kappa^j = 0$, we recover the Lorentz covariant Schrödinger Lagrangian density previously studied.

Under an infinitesimal proper Lorentz transformation (spatial rotations or boosts), the space-time coordinates in the rest frames S and S' are connected by the following relation:

$$x^\mu = x'^\mu - \varepsilon^{\mu\nu} x'_\nu, \quad (412)$$

where $|\varepsilon^{\mu\nu}| \ll 1$ and $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$. The antisymmetry of $\varepsilon^{\mu\nu}$ is a consequence of the invariance of the norm of the four-vector x^μ under a proper Lorentz transformation. The derivative transforms according to

$$\partial_\mu = \partial'_{\mu'} - \varepsilon_{\mu\nu} \partial'^{\nu'}. \quad (413)$$

We also assume that under an infinitesimal proper Lorentz transformation, the wave functions $\Psi(x)$ and $\Psi'(x')$ are connected by

$$\Psi(x) = e^{(i/\hbar) b_\mu x'^\mu} \Psi'(x'), \quad (414)$$

where b_μ is real and of the order of $\varepsilon_{\mu\nu}$.

Inserting equations (413) and (414) into (411), we get to first order in $\varepsilon_{\mu\nu}$,

$$\begin{aligned} \mathcal{L} = & \partial_{\mu'} \Psi' \partial'^{\mu'} \Psi'^* + i\kappa^\mu \Psi'^* \partial_{\mu'} \Psi' - i(\kappa^\mu)^* \Psi' \partial_{\mu'} \Psi'^* \\ & - \frac{|\Psi'|^2}{\hbar} [\kappa^\mu + (\kappa^\mu)^*] b_\mu - i \left(\frac{b^\mu}{\hbar} + \varepsilon^{\nu\mu} \kappa_\nu \right) \Psi'^* \partial_{\mu'} \Psi' \\ & + i \left(\frac{b^\mu}{\hbar} + \varepsilon^{\nu\mu} (\kappa_\nu)^* \right) \Psi' \partial_{\mu'} \Psi'^* + \mathcal{O}(|\varepsilon_{\mu\nu}|^2). \end{aligned} \quad (415)$$

Comparing equations (411) and (415), we see that they have the same form if

$$[\kappa^\mu + (\kappa^\mu)^*] b_\mu = 0, \quad (416)$$

$$\frac{b^\mu}{\hbar} + \varepsilon^{\nu\mu} \kappa_\nu = 0, \quad (417)$$

$$\frac{b^\mu}{\hbar} + \varepsilon^{\nu\mu} (\kappa_\nu)^* = 0. \quad (418)$$

The last two equations imply that $(\kappa_\nu)^* = \kappa_\nu$, i.e., κ_ν is real, and that

$$b^\mu = \hbar \varepsilon^{\mu\nu} \kappa_\nu, \quad (419)$$

where we used the antisymmetry of $\varepsilon^{\mu\nu}$ to express the right-hand side as shown above. Inserting equation (419) into the

left-hand side of (416) and noting that κ^μ is real and that $\varepsilon_{\mu\nu}$ is antisymmetric, we get

$$2\hbar\varepsilon_{\mu\nu}\kappa^\mu\kappa^\nu = \hbar(\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu})\kappa^\mu\kappa^\nu = 0. \quad (420)$$

Thus, equations (416)-(418) are all satisfied if b^μ is given by equation (419).

This means that under an infinitesimal proper Lorentz transformation, the Lagrangian density (it is worth noting that we cannot have Lorentz invariance if $\mathcal{L} = i\kappa^\mu\Psi^*\partial_\mu\Psi - i(\kappa^\mu)^*\Psi\partial_\mu\Psi^*$. The term $\partial_\mu\Psi\partial^\mu\Psi^*$ is crucial to guarantee a Lorentz invariant Lagrangian density)

$$\mathcal{L} = \partial_\mu\Psi\partial^\mu\Psi^* + i\kappa^\mu\Psi^*\partial_\mu\Psi, \quad (421)$$

with κ^μ real and constant, is invariant if Ψ transforms as

$$\Psi(x) \longrightarrow e^{-i\varepsilon_{\mu\nu}\kappa^\mu x^\nu}\Psi(x). \quad (422)$$

To arrive at the transformation law for $\Psi(x)$ under finite proper Lorentz transformations, we follow the usual prescription of implementing N infinitesimal transformations and then letting $N \longrightarrow \infty$. For instance, a counterclockwise rotation of ϕ radians about the x^3 -axis gives (see Appendix C)

$$\begin{aligned} \Psi(x) &\longrightarrow e^{i[k^1(1-\cos\phi)-\kappa^2\sin\phi]x^1} e^{i[k^1\sin\phi+\kappa^2(1-\cos\phi)]x^2} \\ &\cdot \Psi(x) \longrightarrow e^{i[(\kappa^1x^1+\kappa^2x^2)(1-\cos\phi)+(\kappa^1x^2-\kappa^2x^1)\sin\phi]}\Psi(x). \end{aligned} \quad (423)$$

Similarly, for a rotation of ϕ about the x^1 -axis, we get

$$\Psi(x) \longrightarrow e^{i[(\kappa^2x^2+\kappa^3x^3)(1-\cos\phi)+(\kappa^2x^3-\kappa^3x^2)\sin\phi]}\Psi(x). \quad (424)$$

The transformation law for a rotation about the x^2 -axis is obtained from the previous one by relabeling the superscript indexes: $2 \longrightarrow 3$ and $3 \longrightarrow 1$ (from the above results, it is clear that the transformation law for $\Psi(x)$ when we have a rotation of ϕ about an arbitrary axis \mathbf{n} is $\Psi(x) \longrightarrow \exp\{i[\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}] \cdot \boldsymbol{\kappa}(1 - \cos\phi) + i(\boldsymbol{\kappa} \times \mathbf{r}) \cdot \hat{\mathbf{n}} \sin\phi\}\Psi(x)$).

Since from spatial rotations about the x^3 and x^1 -axes we can build an arbitrary spatial rotation, it is sufficiently general to give only how $\Psi(x)$ transforms after a boost in a specified direction. For a boost along the x^1 -axis characterized by β , we have (see Appendix C)

$$\Psi(x) \longrightarrow e^{i[(\gamma-1)\kappa^0-\gamma\beta\kappa^1]x^0+i[\gamma\beta\kappa^0-(\gamma-1)\kappa^1]x^1}\Psi(x), \quad (425)$$

where $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor. To check the consistency of the calculations given in Appendix C, a direct calculation using equations (423), (424), and (425), with the corresponding transformation laws for ∂_μ , shows that the Lagrangian (421) is indeed invariant under those finite proper Lorentz transformations.

8.2. The Wave Equation and Its Solution. The Lagrangian density (421) leads to the following wave equation if we

insert it into the Euler-Lagrange equation (240):

$$\partial_\mu\partial^\mu\Psi - 2i\kappa^\mu\partial_\mu\Psi = 0. \quad (426)$$

Inserting the ansatz

$$\Psi(x) = e^{i(\kappa_\mu x^\mu \pm k_\mu x^\mu)} = e^{i(\kappa \pm k)x} \quad (427)$$

into equation (426), we get

$$(\kappa^2 - k^2)\Psi = 0. \quad (428)$$

We can only satisfy equation (428) and thus raise the ansatz (427) to the status of a solution to the wave equation, if

$$k^2 = \kappa^2 = (\kappa^0)^2 - |\boldsymbol{\kappa}|^2. \quad (429)$$

Due to the linearity of the wave equation (426), its general solution is a linear combination of (427) comprising all values of k compatible with the boundary conditions of the problem being solved.

So far, the four real constants κ^μ are arbitrary. If we set $\kappa^j = 0$ and $\kappa^0 = \mu = mc/\hbar$, we get back to the Lorentz covariant Schrödinger equation. This shows that the latter equation is a particular case of the general wave equation we are now dealing with. As we show next, if $\kappa^2 = \mu^2$, with m interpreted as the rest mass of a scalar particle, we can connect the present wave equation with the Klein-Gordon equation in a straightforward way.

8.3. Connection with the Klein-Gordon Equation. Building on our previous experience with the Lorentz covariant Schrödinger equation, we expect that the following relation will take us from the present wave equation to the Klein-Gordon one:

$$\Psi(x) = e^{i\kappa x} \Phi(x) = e^{i\kappa_\mu x^\mu} \Phi(x). \quad (430)$$

Using equation (430), we get that

$$\begin{aligned} \partial_\mu\partial^\mu\Psi &= (\partial_\mu\partial^\mu\Phi + 2i\kappa^\mu\partial_\mu\Phi - \kappa^2\Phi)e^{i\kappa x}, \\ -2i\kappa^\mu\partial_\mu\Psi &= (2\kappa^2\Phi - 2i\kappa^\mu\partial_\mu\Phi)e^{i\kappa x}, \end{aligned} \quad (431)$$

and thus, the wave equation (426) becomes

$$\partial_\mu\partial^\mu\Phi + \kappa^2\Phi = 0. \quad (432)$$

The wave equation (432) is formally equivalent to the Klein-Gordon one, and we have an exact match if

$$\kappa^2 = \kappa_\mu\kappa^\mu = \mu^2 = (mc/\hbar)^2, \quad (433)$$

where m is the rest mass of the scalar particle described by the Klein-Gordon equation. We can also show that equation (430) when inserted into the Lagrangian density (421) leads

to the Klein-Gordon Lagrangian density if equation (433) is satisfied.

It is worth mentioning that we just need $\kappa^2 = \mu^2$ to identify the transformed wave equation with the Klein-Gordon one. In other words, out of the four real constants κ^μ , only one is fixed by equation (433). Three of them are still free to be set to any value we wish. For an isotropic three-dimensional space, we expect all κ^j to be equal. This reduces the free parameters to just one real constant. However, for condensed matter systems, where anisotropic physical systems are commonplace, the freedom to choose the values of κ^j might be an important asset [34, 35].

8.4. Canonical Quantization. We now start the canonical quantization of the generalized Lorentz covariant Schrödinger fields. We have two main goals in carrying out this task. First, we want to show that it is possible to implement this second quantization program to its logical completion without any contradiction. Then, for Lorentz covariant interactions, we want to show the equivalence between the just developed second quantized theory and Klein-Gordon's.

8.4.1. Equal Time Commutation Relations. The most general solution to the wave equation (426), now understood as an operator, can be written as

$$\Psi(x) = \Psi^+(x) + \Psi^-(x), \quad (434)$$

where

$$\begin{aligned} \Psi^+(x) &= \int \tilde{d}k_+ a_{\mathbf{k}} e^{i\kappa x} e^{-i\mathbf{k}x}, \\ \Psi^-(x) &= \int \tilde{d}k_- b_{\mathbf{k}}^\dagger e^{i\kappa x} e^{i\mathbf{k}x}, \end{aligned} \quad (435)$$

and

$$\begin{aligned} k^0 &= \sqrt{\kappa^2 + |\mathbf{k}|^2}, \\ \tilde{d}k_\pm &= f_\pm(|\mathbf{k}|) d^3k. \end{aligned} \quad (436)$$

Requiring that the commutation relations for the creation and annihilation operators are given by equations (321)-(323), the fields Ψ and Ψ^\dagger satisfy the equal time canonical commutation relations (324) and (325) if

$$f_+(|\mathbf{k}|) = f_-(|\mathbf{k}|) = f(|\mathbf{k}|) = \left(\frac{\hbar c^2}{(2\pi)^3 2\omega_{\mathbf{k}}} \right)^{1/2}, \quad (437)$$

where

$$\hbar\omega_{\mathbf{k}} = E_{\mathbf{k}} = \hbar c k^0. \quad (438)$$

The conjugate momenta to the fields are given by

$$\begin{aligned} \Pi_\Psi &= \frac{1}{c} \partial_0 \Psi^\dagger + \frac{i\kappa^0}{c} \Psi^\dagger, \\ \Pi_{\Psi^\dagger} &= \frac{1}{c} \partial_0 \Psi - \frac{i\kappa^0}{c} \Psi, \end{aligned} \quad (439)$$

and they are connected to the Klein-Gordon ones as follows:

$$\Pi_\Psi = e^{-i\kappa x} \Pi_\Phi, \quad (440)$$

$$\Pi_{\Psi^\dagger} = e^{i\kappa x} \Pi_{\Phi^\dagger}. \quad (441)$$

8.4.2. Conserved Quantities. Following the prescription given in Section 6, we now have the following expressions for the Hamiltonian, momentum, and charge densities (normal ordering always implied):

$$\mathcal{H} = \partial_0 \Psi \partial_0 \Psi^\dagger + \nabla \Psi \cdot \nabla \Psi^\dagger - i\kappa \cdot \Psi^\dagger \nabla \Psi, \quad (442)$$

$$c\mathcal{P}^j = -\partial_0 \Psi \partial_j \Psi^\dagger - \partial_0 \Psi^\dagger \partial_j \Psi - i\kappa^0 \Psi^\dagger \partial_j \Psi, \quad (443)$$

$$c\tilde{q} = i\Psi^\dagger \partial_0 \Psi + 2\kappa^0 \Psi^\dagger \Psi. \quad (444)$$

Note that now the charge current is given by

$$\tilde{\mathbf{j}} = i\Psi^\dagger \nabla \Psi + 2\kappa \Psi^\dagger \Psi. \quad (445)$$

Similarly to what we did for the Lorentz covariant Schrödinger fields, by inserting equation (434) into equations (442)-(444) and then into equations (260), (261), and (282), we obtain

$$\begin{aligned} H &= \int d^3k \left(\hbar\omega_{\mathbf{k}}^+ a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \hbar\omega_{\mathbf{k}}^- b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right), \\ P^j &= \int d^3k \left(\hbar k_+^j a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \hbar k_-^j b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right), \\ \tilde{Q} &= \int d^3k \left[\hbar \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right) \right], \end{aligned} \quad (446)$$

where

$$\begin{aligned} E_{\mathbf{k}}^\pm &= \hbar\omega_{\mathbf{k}}^\pm = \mp \hbar c k^0 + E_{\mathbf{k}} \\ p_{\pm}^j &= \hbar k_{\pm}^j = \mp \hbar k^j + \hbar k^j. \end{aligned} \quad (447)$$

In addition to the energy gap associated with a pair of particle and antiparticle characterized by the vector \mathbf{k} , we also have a momentum gap. In contrast to the Lorentz covariant Schrödinger equation, we now have a complete symmetry between energy and momentum, which allows us to define the two quantities below:

$$p_{\pm}^\mu = (E_{\mathbf{k}}^\pm, c\mathbf{p}_{\pm}), \quad (448)$$

where $\mathbf{p}_{\pm} = (p_{\pm}^1, p_{\pm}^2, p_{\pm}^3)$. Equation (448) gives the energy and momentum of a particle (p_+^μ) and antiparticle (p_-^μ) created, respectively, by $a_{\mathbf{k}}^\dagger$ and $b_{\mathbf{k}}^\dagger$ acting on the vacuum state.

When it comes to the discrete symmetries, we need to modify the operators that implement the parity, time reversal, and charge conjugation operations as follows. The parity, time reversal, and charge conjugation operators are now defined as (note that to simplify notation, we are not using the symbol “ $\hat{}$ ” to denote operators as we did in Section 6.3)

$$\mathcal{P}_\kappa \Psi(\mathbf{r}, t) \mathcal{P}_\kappa^\dagger = \Psi(-\mathbf{r}, t), \quad (449)$$

$$\mathcal{T}_\kappa \Psi(\mathbf{r}, t) \mathcal{T}_\kappa^\dagger = \Psi(\mathbf{r}, -t), \quad (450)$$

$$\mathcal{C}_\kappa \Psi(\mathbf{r}, t) \mathcal{C}_\kappa^\dagger = \Psi^\dagger(\mathbf{r}, t). \quad (451)$$

In equations (449)-(451), we have

$$\begin{aligned} \mathcal{P}_\kappa &= \mathcal{H}_1 \mathcal{P}, \\ \mathcal{T}_\kappa &= \mathcal{H}_1 \mathcal{T}, \\ \mathcal{C}_\kappa &= \mathcal{H}_2 \mathcal{C}, \end{aligned} \quad (452)$$

where

$$\mathcal{H}_1 f(\kappa^\mu) \mathcal{H}_1^\dagger = f(\kappa_\mu), \quad (453)$$

$$\mathcal{H}_2 f(\kappa^\mu) \mathcal{H}_2^\dagger = f(-\kappa^\mu). \quad (454)$$

Here, $f(\kappa^\mu)$ is an arbitrary function of κ^μ and \mathcal{P} , \mathcal{T} , and \mathcal{C} are the standard parity, time reversal, and charge conjugation operators of the Klein-Gordon theory (see also Section 6.3). We should remark that \mathcal{P} and \mathcal{C} are unitary operators while \mathcal{T} is antiunitary.

Equation (454) generalizes the “mass conjugation” operation, equation (298), which is needed to correctly define a charge conjugation operation for the Lorentz covariant Schrödinger equation. The operation defined in equation (453), on the other hand, only changes the sign of the spatial components of κ^μ , i.e., $\mathcal{H}_1 f(\kappa^0, \boldsymbol{\kappa}) \mathcal{H}_1^\dagger = f(\kappa^0, -\boldsymbol{\kappa})$.

Using equations (449)-(451), we can show that we get the expected behavior for properly defined parity, time reversal, and charge conjugation operations. Specifically, equations (286)-(289), equations (293)-(296), and equations (300)-(303) are all satisfied. Furthermore, equations (449)-(451) imply that the creation and annihilation operators satisfy the correct transformation laws, namely, equations (333)-(335) and (340)-(341).

Similarly, the CPT theorem, which in the present context should be more properly called the CPTM theorem, is once more “saved” if we define the CPT operation as

$$\Theta = \mathcal{C}_\kappa \mathcal{P}_\kappa \mathcal{T}_\kappa = \mathcal{H}_2 \mathcal{C} \mathcal{P} \mathcal{T}. \quad (455)$$

To arrive at the last equality we used that \mathcal{H}_j commutes with \mathcal{C} , \mathcal{P} , and \mathcal{T} and that \mathcal{H}_j^2 is the identity operator. With this definition for the CPT operation, equations (305)-(309) continue to hold for the generalized Lorentz covariant Schrödinger theory.

8.4.3. Arbitrary Time Commutators. A direct calculation gives

$$\begin{aligned} [\Psi(x), \Psi^\dagger(x')] &= \int d^3 k [f(|\mathbf{k}|)]^2 \left[e^{-i(k-\kappa)(x-x')} - e^{i(k+\kappa)(x-x')} \right] \\ &= \int d^3 k [f(|\mathbf{k}|)]^2 e^{i(\mathbf{k}-\kappa) \cdot (\mathbf{r}-\mathbf{r}')} \left[e^{-i\omega_{\mathbf{k}}^+(t-t')} - e^{i\omega_{\mathbf{k}}^-(t-t')} \right]. \end{aligned} \quad (456)$$

Similarly to the Lorentz covariant Schrödinger fields, we can write the previous commutator as

$$[\Psi(x), \Psi^\dagger(x')] = i\hbar c \Delta_{\text{GLS}}(x - x'), \quad (457)$$

where

$$\Delta_{\text{GLS}}(x) = e^{i\kappa x} \Delta(x). \quad (458)$$

Here, the Lorentz invariant $\Delta(x)$ is given by

$$\Delta(x) = \frac{-i}{2\pi^3} \int d^4 k \delta(k^2 - \kappa^2) \varepsilon(k^0) e^{-ikx}, \quad (459)$$

where all variables are real and integrated from $-\infty$ to ∞ . Note that contrary to equation (352), we have κ^2 instead of μ^2 inside the delta function.

The other nonnull commutators are

$$\begin{aligned} [\Pi_\Psi(x), \Pi_{\Psi^\dagger}(x')] &= e^{-i\kappa(x-x')} \int d^3 k [f(|\mathbf{k}|)]^2 \left(\frac{\omega_{\mathbf{k}}^2}{c^4} \right) \\ &\quad \cdot \left[e^{-ik(x-x')} - e^{ik(x-x')} \right] \\ &= \int d^3 k [f(|\mathbf{k}|)]^2 \left(\frac{\omega_{\mathbf{k}}^2}{c^4} \right) e^{i(\mathbf{k}+\kappa) \cdot (\mathbf{r}-\mathbf{r}')} \\ &\quad \cdot \left[e^{-i\omega_{\mathbf{k}}^-(t-t')} - e^{i\omega_{\mathbf{k}}^+(t-t')} \right], \end{aligned} \quad (460)$$

$$\begin{aligned} [\Psi(x), \Pi_\Psi(x')] &= e^{i\kappa(x-x')} \int d^3 k [f(|\mathbf{k}|)]^2 \left(\frac{i\omega_{\mathbf{k}}}{c^2} \right) \\ &\quad \cdot \left[e^{-ik(x-x')} + e^{ik(x-x')} \right] \\ &= \int d^3 k [f(|\mathbf{k}|)]^2 \left(\frac{i\omega_{\mathbf{k}}}{c^2} \right) e^{i(\mathbf{k}-\kappa) \cdot (\mathbf{r}-\mathbf{r}')} \\ &\quad \cdot \left[e^{-i\omega_{\mathbf{k}}^+(t-t')} + e^{i\omega_{\mathbf{k}}^-(t-t')} \right]. \end{aligned} \quad (461)$$

8.4.4. The Feynman Propagator. The Feynman propagator, which we call $\Delta_{\text{FGLS}}(x)$, is given by

$$\Delta_{\text{FGLS}}(x) = h(x^0) \Delta_{\text{GLS}}^+(x) - h(-x^0) \Delta_{\text{GLS}}^-(x), \quad (462)$$

where

$$\Delta_{\text{GLS}}^\pm(x) = e^{i\kappa x} \Delta^\pm(x). \quad (463)$$

Here, $\Delta^\pm(x)$ are the standard delta functions appearing in the analysis of the Klein-Gordon propagator, with all quantities that are functions of μ^2 changed to functions of κ^2 (equation (350) with k^0 and ω_k expressed as functions of κ^2 instead of μ^2).

Alternatively, using equations (462) and (463), we have

$$\Delta_{F_{\text{GLS}}}(x) = e^{i\kappa x} \Delta_F(x), \quad (464)$$

with

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{k^2 - \kappa^2 + i\epsilon}. \quad (465)$$

Here, $\eta \ll 1$, $\epsilon = 2\eta\omega_k/c$, and all variables are real and integrated from $-\infty$ to ∞ . As usual, the limit $\epsilon \rightarrow 0$ is implied after the integration is made.

8.5. Connection with the Klein-Gordon Theory. We divide the forthcoming analysis in two parts. First, we show that the S-matrix describing a given process (scattering or decay) involving the generalized Lorentz covariant Schrödinger fields is equal to the S-matrix one obtains modeling the same process using Klein-Gordon fields. This only happens for interactions that are invariant under a proper Lorentz transformation.

Second, we show that the same S-matrix leads to the same scattering cross section (or decay rate) for both theories. This apparently trivial result should be properly discussed since the generalized Lorentz covariant Schrödinger Lagrangian and the Klein-Gordon Lagrangian lead to formally different expressions for the current density vector (flux of particles), an important quantity employed in the definition of a scattering cross section.

8.5.1. Equivalence of the S-Matrices. For interactions of the type given by equation (B.7), with the $(\lambda/4)(\Psi\Psi^\dagger)^2$ interaction being a particular example, the equivalence of the S-matrices is proved using exactly the same arguments and steps employed in the corresponding proof given before for the Lorentz covariant Schrödinger fields. The only difference, a formal one that does not change the arguments used in the previous proof, is that instead of equations (387) and (388), we now have according to equations (430) and (440)

$$\Psi^I(x) = e^{i\kappa x} \Phi^I(x), \quad (466)$$

$$\Pi_\Psi^I(x) = e^{-i\kappa x} \Pi_\Phi^I(x). \quad (467)$$

Furthermore, since in the present case the conjugate momenta to the fields are the same as the ones associated with the Lorentz covariant Schrödinger fields, the proof of the equivalence of the S-matrices when we include electromagnetic interactions is almost the same as the one given in Appendix B.2. Note that the conjugate momenta do not change because the new extra terms appearing in the Lagrangian density describing the generalized Lorentz

covariant Schrödinger fields do not depend on time derivatives.

Repeating the same calculations given in Appendix B.2, we obtain in the interaction picture the following Hamiltonian density describing the electromagnetic interaction among the fields:

$$\mathcal{H}_{\text{int}}^I = -\mathcal{L}_{\text{int}}^I + \frac{q^2}{\hbar^2} (\Psi^I)^\dagger \Psi^I A_0^I (A^I)^0, \quad (468)$$

where

$$\begin{aligned} \mathcal{L}_{\text{int}}^I = & -\frac{iq}{\hbar} [(\Psi^I)^\dagger \partial_\mu \Psi^I] (A^I)^\mu + \frac{q^2}{\hbar^2} (\Psi^I)^\dagger \Psi^I A_\mu^I (A^I)^\mu \\ & - \frac{2q}{\hbar} \kappa^\mu (\Psi^I)^\dagger \Psi^I A_\mu^I. \end{aligned} \quad (469)$$

If we now use equation (466), a direct calculation shows that equation (468) is transformed to equation (B.43), the interaction Hamiltonian density for the Klein-Gordon theory. Using the same arguments given in Appendix B.2, this fact is enough to establish the equivalence between the S-matrix of the present theory and the S-matrix coming from the Klein-Gordon theory.

8.5.2. Equivalence of the Scattering Cross Sections. If we analyze the standard way to define a differential cross section for a given process [4, 5], we realize that the following three distinct steps are taken to arrive at an experimentally meaningful quantity:

- (1) Terms like $[(2\pi)^4 \delta^{(4)}(\sum k'_{\text{final}} - \sum k_{\text{initial}})]^2$ are identified with $VT(2\pi)^4 \delta^{(4)}(\sum k'_{\text{final}} - \sum k_{\text{initial}})$, where V and T represent, respectively, a finite volume (box normalization for plane waves) and the duration of the experiment. Here, $\sum k'_{\text{final}} - \sum k_{\text{initial}}$ denotes the conservation of energy and momentum written in terms of the four-wave vector. The most common situation is the one where we have k_1 and k_2 , two initial particles, and two or more final ones
- (2) The calculation of the transition probability per unit time, $w = |S_{\text{fi}}|^2/T$, where S_{fi} is the shorthand notation for the probability amplitude describing a given process
- (3) The definition of the differential scattering cross section as

$$d\sigma = \frac{w}{|\mathbf{j}|} \prod_{\text{final}} \frac{V d^3 k'_{\text{final}}}{(2\pi)^3}, \quad (470)$$

where $|\mathbf{j}|$ is the magnitude of the flux of incoming particles and $\prod_{\text{final}} V d^3 k'_{\text{final}} / (2\pi)^3$ represents a group of final particles with wave numbers in the interval $\mathbf{k}'_{\text{final}}$ and $\mathbf{k}'_{\text{final}} + d\mathbf{k}'_{\text{final}}$. Also, the normalization adopted for the wave functions is one particle per volume V

Since we have proved the equivalence of the S-matrices, items (1) and (2) above are easily seen to be the same whether we deal with the Klein-Gordon or with the generalized Lorentz covariant Schrödinger fields. Item (3), however, deserves a little more thought. To prove that it is equivalent for both theories, we have to show the equivalence of the incident particle flux \mathbf{j} .

The flux associated with the generalized Lorentz covariant Schrödinger equation, given by equation (445), can be written as

$$\mathbf{j} = -\frac{i\hbar}{2m} (\Psi^\dagger \nabla \Psi + 2i\mathbf{k} \Psi^\dagger \Psi). \quad (471)$$

In the above expression, we are using the first quantization nonrelativistic normalization $-\hbar/(2m)$ for the flux for reasons that will become clear in a moment. If we now use equation (430), equation (471) becomes

$$\mathbf{j} = -\frac{i\hbar}{2m} \Phi^\dagger \nabla \Phi, \quad (472)$$

which is the incident particle flux we obtain working directly with the Klein-Gordon equation.

To better appreciate the equivalence of both fluxes, we explicitly compute \mathbf{j} for the two cases of interest here. For simplicity, we will work at the first quantization level.

For the Klein-Gordon equation and assuming the target at rest, the plane wave normalized to one particle per volume representing an incident flux of particles with four-wave number k^μ is given by

$$\Phi(x) = \frac{1}{\sqrt{V}} e^{-ikx}. \quad (473)$$

A direct calculation using equation (472) gives

$$\mathbf{j} = \frac{\hbar\mathbf{k}}{m} |\Phi|^2 = \frac{\mathbf{p}}{m} \frac{1}{V} = \frac{\mathbf{v}}{V}, \quad (474)$$

where we have made the identification of $\hbar\mathbf{k}$ with the Klein-Gordon particle's momentum and \mathbf{p}/m with its velocity \mathbf{v} .

On the other hand, the solution to the generalized Lorentz covariant Schrödinger equation representing a particle with four-wave number k^μ is

$$\Psi(x) = \frac{1}{\sqrt{V}} e^{ixx} e^{-ikx}. \quad (475)$$

Using equation (471), we get

$$\mathbf{j} = \frac{\hbar\mathbf{k}}{m} |\Psi|^2 = \frac{\hbar\mathbf{k}}{m} |\Phi|^2 = \frac{\hbar\mathbf{k}}{mV}, \quad (476)$$

which is clearly the same flux \mathbf{j} we obtain working with the Klein-Gordon fields, proving thus the complete equivalence between both theories.

Before closing this section, we would like to reinforce once more that the above analysis, in particular the equivalence

of the S-matrices, is only true for a Lorentz invariant Lagrangian. The same arguments given in Section 7.6 apply here. Therefore, for interaction terms breaking Lorentz invariance, we can have certain types of decay and scattering processes that are impossible to happen in the Klein-Gordon theory (note also that the previous equivalence proof between the two theories assumed two types of possible interactions: self-interactions and electromagnetic interactions. It is possible that whenever we have a consistent quantum field theory of gravitation, the Klein-Gordon and the Lorentz covariant Schrödinger theories will show different predictions even if a fully Lorentz invariant interaction is present. The reason for such a guess is related to the fact that the mass (energy) and momentum gap may not be trivially "averaged out" if a consistent quantum gravity theory is built. More intuitively, "negative" masses will reduce the usual contribution of positive masses in generating a gravitational field. However, as we showed in Section 6.3.5, the mathematical framework of the Lorentz covariant Schrödinger Lagrangian can be adjusted to accommodate particles and antiparticles attracting or repelling each other gravitationally, at least at the level of Newtonian static gravitational fields. Therefore, at our present level of understanding, the issue of how particles and antiparticles interact gravitationally in the framework of the Lorentz covariant Schrödinger theory is left as an open problem).

9. Conclusion

The original motivation leading to this work, on the one hand, stems from the fact that the great majority of physical observables in the framework of quantum field theory are bilinear functions of the fields. In nonrelativistic quantum mechanics, we also have that the experimentally relevant quantities are almost always functions of the probability density $\Psi(x)\Psi^*(x)$, a bilinear function of the wave function. Therefore, working either with $\Psi(x)$ or $e^{(i\hbar)f(x)}\Psi(x)$, where $f(x)$ is an arbitrary function of the spacetime coordinates, we obtain the same predictions.

On the other hand, in classical and quantum field theories, one usually defines a complex scalar field as a quantity that is itself invariant under a given symmetry transformation: $\Psi(x) \rightarrow \Psi(x)$. However, it is clear that whenever bilinear functions of the fields are associated with observable quantities, we only need those bilinears to be invariant under that symmetry operation to obtain invariant physical results. In other words, by assuming the more general transformation rule for the fields, $\Psi(x) \rightarrow e^{(i\hbar)f(x)}\Psi(x)$, we should get in principle an equivalent description for a physical system whose observables are bilinear functions of those fields. Therefore, we revisited the classical and quantum field theories of complex scalar fields assuming from the start the more general transformation law for the fields when they are subjected to a symmetry operation. We wanted to check if we could consistently develop those field theories to their logical conclusion, comparing the final products with current complex scalar field theories based on the more simple transformation law, where it is assumed that $f(x)$ is either zero or a nonnull constant.

As we showed in this work, we can indeed build logically consistent classical and quantum field theories assuming the more general transformation law for the fields under a symmetry operation. In particular, we showed that it is possible to have a Lorentz covariant theory if we assume that the complex scalar field $\Psi(x)$ transforms according to the more general prescription above under proper Lorentz transformations (boosts or spatial rotations). With the aid of very natural auxiliary assumptions, namely, linearity and a wave equation with at most second-order derivatives, we obtained the Lagrangian, the corresponding wave equation, and the function $f(x)$ that give the most general Lorentz covariant theory in this scenario. We also determined under what conditions the complex scalar field theories here developed match the Klein-Gordon theory and under what conditions we may have different predictions. It turned out that for Lorentz invariant self-interactions and for electromagnetic interactions, we can make the present theories equivalent to the Klein-Gordon theory. For interaction terms that violate Lorentz invariance, however, we showed by giving explicit examples that the present theories are no longer equivalent to Klein-Gordon's.

In addition to the formal development of the present complex scalar field theories, we applied them to describe several physical systems. This helped us to become familiarized with the new concepts introduced along the logical construction of the theory. Of the many physical systems we studied, we would like to call attention to our investigations about the bound states and the scattering cross sections associated with two interacting charged particles, where both the electromagnetic and gravitational interactions were simultaneously included to model the interaction between them. The electromagnetic interaction entered via the minimal coupling prescription while gravity was introduced via an external potential. We obtained the exact solution for the bound state problem and computed perturbatively the differential scattering cross section when dealing with the corresponding scattering problem. We also estimated the order of magnitude of the charges and masses of the particles in which gravitational effects can no longer be discarded.

During our studies, a few unexpected results emerged. The first one showed up already at the first quantization level. We observed that the eigenenergies associated with the plane wave solutions to the Lorentz covariant Schrödinger equation implied that particles and antiparticles no longer had degenerated relativistic energies and, intriguingly, we observed that we could only exchange the roles of particles with antiparticles if, in addition to changing the sign of the charge, we also changed the sign of the mass for all physical quantities depending on it. This suggested that we could assign "negative" masses to the antiparticles. A full understanding of this fact was possible when we second quantized the theory, where it was noted that this interpretation for the mass sign of the antiparticle is actually necessary for the logical consistency of the theory. This became apparent when we dealt with the charge conjugation operation. It could only be consistently defined if, and only if, we extended its standard definition such that the charge conjugation operator anticommuted with functions of the mass of the particle.

In other words, the charge conjugation operation has to change the sign of the masses when acting upon a given field operator to be properly defined.

Despite the "negative" masses for antiparticles, we showed that both particles and antiparticles have nonnegative energies and the same momentum for a given wave number \mathbf{k} . Later, at the last part of this work, when we generalized the Lorentz covariant Schrödinger equation, we showed that it is also possible to break the degeneracy in the value of the momentum. For a given value of \mathbf{k} , particles and antiparticles also have different momenta.

As we already remarked above, the present theories were shown to be equivalent to the Klein-Gordon one for Lorentz invariant self-interactions and electromagnetic interactions. However, by including interaction terms in the Lagrangian density that are not Lorentz invariant, we showed that this equivalence with the Klein-Gordon theory is no longer valid. We showed that certain types of interactions that violate Lorentz invariance imply that an antiparticle can decay into two particles and one antiparticle or into a particle and a photon. The equivalent processes for particles decaying into more antiparticles than particles were shown to be kinematically forbidden. This pointed to a second unexpected result, namely, that very simple Lorentz invariance-breaking interactions could explain the asymmetry in the abundance of matter and antimatter in the present-day universe.

In order to better understand the nondegeneracy of the energies of particles and antiparticles, we also second quantized the Lorentz covariant Schrödinger Lagrangian imposing that the energies of particles and antiparticles were the same and given by the standard relativistic expression. We observed that this could only be achieved if the fields did not satisfy the canonical commutation relations anymore. This led to the violation of the microcausality condition as well. Albeit this nonlocal character of the noncanonically quantized theory, we noted that the specific way in which the microcausality condition was violated resulted in the emergence of an instantaneous gravitational-like interaction between the particles of the theory. In other words, particles and antiparticles could be brought to have the same relativistic energy if they interacted similarly to what Newton's gravitational law prescribes. The emergence of this gravitational-like potential may be just a particularity of the present theory but it suggests that the assumption that antiparticles have negative masses with positive energies might shed a new light in our quest for a consistent quantum theory of gravity.

Summing up, let us distill and write down the two major messages that we tried to convey by writing this work. First, it is possible to build logically consistent and Lorentz covariant classical and quantum field theories assuming a more general transformation law for complex scalar fields under a symmetry operation. These theories can be adjusted to reproduce exactly the Klein-Gordon theory when we have Lorentz invariant self-interactions and when we introduce electromagnetic interactions via the minimal coupling prescription. Second, for logical consistency, we have to assume that antiparticles possess negative masses while still having positive energies. These two points when analyzed together

tell us that it is perfectly legitimate, at least at the level of electromagnetic interactions, to assume that particles and antiparticles have masses with different signs without contradicting any known experimental fact. When we include gravitational interactions, it remains an open problem to rigorously analyze whether matter and antimatter attract or repel each other within the present framework (at the level of static Newtonian fields, we showed that it can be adjusted to accommodate both possibilities). To solve this issue for ordinary matter and antimatter, we need to measure with high accuracy how a particle and an antiparticle interact gravitationally. We need to know, for example, how an anti-hydrogen atom responds to the Earth's gravitational field [36]. If they repel each other, the Lorentz covariant Schrödinger Lagrangian is another possible starting point to theoretically understand more fully what is going on [37–40]. If they attract each other, the present theory is as good as the Klein-Gordon one to describe this experimental result.

Appendix

A. Proof of the Main Results of Section 7

A.1. Canonical versus Noncanonical Quantization. Our main goal here is to show how equations (324) and (325) follow from equations (321)-(323) with the appropriate choice of $f_{\pm}(|\mathbf{k}|)$ for equation (315). We also want to determine what values for $f_{\pm}(|\mathbf{k}|)$ make particles and antiparticles have the same energies, and, then, we want to explore the main consequences of the theory built on this particular choice for $f_{\pm}(|\mathbf{k}|)$. From now on, we assume we are dealing with the free field ($V = 0$).

Let us start listing the mathematical identities employed routinely in all calculations that follow. In the calculations of either the Hamiltonian, linear momentum, and conserved charge or in the computations of the several commutators below, the following representation of the Dirac delta function is useful:

$$\delta^{(3)}(\mathbf{k} - \mathbf{k}') = \frac{1}{(2\pi)^3} \int d^3x e^{\pm i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}}. \quad (\text{A.1})$$

With the help of equation (A.1), it is not difficult to arrive at the following identities:

$$\begin{aligned} \frac{1}{(2\pi)^3} \int d^3x e^{\pm i(k-k')x} &= \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ \frac{1}{(2\pi)^3} \int d^3x e^{\pm i(k+k')x} &= e^{\pm i(k^0+k'^0)x^0} \delta^{(3)}(\mathbf{k} + \mathbf{k}'), \end{aligned} \quad (\text{A.2})$$

where k^0 and k'^0 are given by equation (316). Another pair of useful identities is

$$\begin{aligned} \omega_{\mathbf{k}}^- \omega_{\mathbf{k}}^+ &= c^2 |\mathbf{k}|^2, \\ \sqrt{\frac{\omega_{\mathbf{k}}^-}{\omega_{\mathbf{k}}^+}} - \sqrt{\frac{\omega_{\mathbf{k}}^+}{\omega_{\mathbf{k}}^-}} &= \frac{2\mu}{|\mathbf{k}|}, \end{aligned} \quad (\text{A.3})$$

where μ is given in equation (319) and $\omega_{\mathbf{k}}^{\pm}$ are defined in equation (331).

Inserting equation (315) into the normal ordered expressions of equations (260), (261), and (282), we obtain with the help of equations (A.1)-(A.3)

$$H = \int d^3k \frac{2(2\pi)^3}{c^2} \left\{ [f_+(\mathbf{k})]^2 \omega_{\mathbf{k}}^+ \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + [f_-(\mathbf{k})]^2 \omega_{\mathbf{k}}^- \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right\}, \quad (\text{A.4})$$

$$P^j = \int d^3k \frac{2(2\pi)^3}{c^2} \left\{ [f_+(\mathbf{k})]^2 \omega_{\mathbf{k}} k^j a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + [f_-(\mathbf{k})]^2 \omega_{\mathbf{k}} k^j b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right\}, \quad (\text{A.5})$$

$$\tilde{Q} = \int d^3k \frac{2(2\pi)^3}{c^2} \left\{ [f_+(\mathbf{k})]^2 \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - [f_-(\mathbf{k})]^2 \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right\}. \quad (\text{A.6})$$

Similarly, using the commutation relations for the creation and annihilation operators given by equations (321)-(323), the equal time commutation relations involving the fields and their conjugate momenta (equations (247) and (315)) become

$$\begin{aligned} [\Psi(t, \mathbf{r}), \Psi(t, \mathbf{r}')] &= [\Psi^\dagger(t, \mathbf{r}), \Psi^\dagger(t, \mathbf{r}')] \\ &= [\Pi_\Psi(t, \mathbf{r}), \Pi_\Psi(t, \mathbf{r}')] = 0, \\ [\Pi_{\Psi^\dagger}(t, \mathbf{r}), \Pi_{\Psi^\dagger}(t, \mathbf{r}')] &= [\Psi(t, \mathbf{r}), \Pi_{\Psi^\dagger}(t, \mathbf{r}')] \\ &= [\Psi^\dagger(t, \mathbf{r}), \Pi_\Psi(t, \mathbf{r}')] = 0, \end{aligned} \quad (\text{A.7})$$

with the nontrivial ones being

$$\begin{aligned} [\Psi(t, \mathbf{r}), \Psi^\dagger(t, \mathbf{r}')] &= -[\Psi^\dagger(t, \mathbf{r}), \Psi(t, \mathbf{r}')]^\dagger \\ &= \int d^3k e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \{ [f_+(\mathbf{k})]^2 - [f_-(\mathbf{k})]^2 \}, \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} [\Pi_\Psi(t, \mathbf{r}), \Pi_{\Psi^\dagger}(t, \mathbf{r}')] &= -[\Pi_{\Psi^\dagger}(t, \mathbf{r}), \Pi_\Psi(t, \mathbf{r}')]^\dagger \\ &= \int d^3k e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \left(\frac{\omega_{\mathbf{k}}}{c^2} \right) \{ [f_-(\mathbf{k})]^2 - [f_+(\mathbf{k})]^2 \}, \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} [\Psi(t, \mathbf{r}), \Pi_\Psi(t, \mathbf{r}')] &= -[\Psi^\dagger(t, \mathbf{r}), \Pi_{\Psi^\dagger}(t, \mathbf{r}')]^\dagger \\ &= \int d^3k e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \left(\frac{i\omega_{\mathbf{k}}}{c^2} \right) \{ [f_+(\mathbf{k})]^2 + [f_-(\mathbf{k})]^2 \}. \end{aligned} \quad (\text{A.10})$$

Repeating the above calculations for two arbitrary space-

time points x and x' and using that $x^0 = ct$, we get

$$\begin{aligned} [\Psi(x), \Psi(x')] &= [\Psi^\dagger(x), \Psi^\dagger(x')] = [\Pi_\Psi(x), \Pi_\Psi(x')] = 0, \\ [\Pi_{\Psi^\dagger}(x), \Pi_{\Psi^\dagger}(x')] &= [\Psi(x), \Pi_{\Psi^\dagger}(x')] = [\Psi^\dagger(x), \Pi_\Psi(x')] = 0, \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} [\Psi(x), \Psi^\dagger(x')] &= -[\Psi^\dagger(x), \Psi(x')]^\dagger \\ &= \int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \left\{ e^{-i\omega_{\mathbf{k}}^+(t-t')} [f_+(\mathbf{k})]^2 - e^{i\omega_{\mathbf{k}}^-(t-t')} [f_-(\mathbf{k})]^2 \right\}, \\ [\Pi_\Psi(x), \Pi_{\Psi^\dagger}(x')] &= -[\Pi_{\Psi^\dagger}(x), \Pi_\Psi(x')]^\dagger \\ &= \int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \left(\frac{\omega_{\mathbf{k}}^2}{c^4} \right) \\ &\quad \cdot \left\{ e^{-i\omega_{\mathbf{k}}^-(t-t')} [f_-(\mathbf{k})]^2 - e^{i\omega_{\mathbf{k}}^+(t-t')} [f_+(\mathbf{k})]^2 \right\}, \\ [\Psi(x), \Pi_\Psi(x')] &= -[\Psi^\dagger(x), \Pi_{\Psi^\dagger}(x')]^\dagger \\ &= \int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \left(\frac{i\omega_{\mathbf{k}}}{c^2} \right) \\ &\quad \cdot \left\{ e^{-i\omega_{\mathbf{k}}^+(t-t')} [f_+(\mathbf{k})]^2 + e^{i\omega_{\mathbf{k}}^-(t-t')} [f_-(\mathbf{k})]^2 \right\}. \end{aligned} \quad (\text{A.12})$$

Looking at equations (A.8) and (A.9), we see that they can only be zero if $f_+(|\mathbf{k}|) = \pm f_- (|\mathbf{k}|)$. The only remaining nonzero commutators with this choice are given by equation (A.10), and they become equal to $i\hbar\delta^{(3)}(\mathbf{r}-\mathbf{r}') = (i\hbar)/(2\pi)^3 \int d^3k e^{i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{k}}$ if

$$\left(\frac{i2\omega_{\mathbf{k}}}{c^2} \right) [f_\pm(\mathbf{k})]^2 = \frac{i\hbar}{(2\pi)^3}. \quad (\text{A.13})$$

Solving for $f_\pm(\mathbf{k})$, we obtain equation (326), and inserting it into equation (A.10), we recover equations (324) and (325) of the main text. Note that strictly speaking, $f_+(\mathbf{k}) = -f_-(\mathbf{k})$ is also a possible solution. However, all relevant quantities depend only on the square of these functions and we choose $f_+(\mathbf{k}) = f_-(\mathbf{k})$, equation (326), for simplicity. Moreover, inserting equation (326) into equations (A.4)-(A.6) we obtain equations (328)-(330) of the main text.

We have just seen that the choice $f_+(\mathbf{k}) = f_-(\mathbf{k}) = [(\hbar c^2)/(2\pi)^3 2\omega_{\mathbf{k}}]^{1/2}$ leads to the canonical commutation relations for the Lorentz covariant Schrödinger fields. In this scenario, the energies of the particles and antiparticles are not the same. We now look for the values of $f_+(\mathbf{k})$ and $f_-(\mathbf{k})$ that lead to particles and antiparticles having the same energy. As we will see, this can only be achieved if the fields no longer satisfy the canonical commutation relations and if the microcausality condition [4] is violated.

If we set

$$[f_\pm(\mathbf{k})]^2 = \frac{\hbar c^2}{(2\pi)^3 2\omega_{\mathbf{k}}^\pm}, \quad (\text{A.14})$$

we can write the normal ordered equations (A.4)-(A.6) as follows:

$$H = \int d^3k \hbar \omega_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}}), \quad (\text{A.15})$$

$$P^j = \int d^3k \hbar k^j \left(\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^+} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^-} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right), \quad (\text{A.16})$$

$$\tilde{Q} = \int d^3k \hbar \left(\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^+} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - \frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^-} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right). \quad (\text{A.17})$$

Looking at equation (A.15), we see that the particle and the antiparticle have the same energy $E_{\mathbf{k}} = \hbar\omega_{\mathbf{k}}$. If we use equations (316) and (331), we get

$$\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^\pm} = \left(1 \pm \frac{mc^2}{\hbar\omega_{\mathbf{k}}^\pm} \right), \quad (\text{A.18})$$

which gives us the correction to the momentum and charge as compared to what one traditionally expects for these quantities. For particles created by $a_{\mathbf{k}}^\dagger$, their momentum and charge are increased by $(mc^2)/(\hbar\omega_{\mathbf{k}}^+)$ while for antiparticles created by $b_{\mathbf{k}}^\dagger$, the same quantities are reduced by $(mc^2)/(\hbar\omega_{\mathbf{k}}^-)$. This is consistent with considering particles with mass m and antiparticles with mass $-m$. Also, we can understand $(mc^2)/(\hbar\omega_{\mathbf{k}}^+)$ as the ratio between the rest energy of a particle with mass m to its kinetic energy $\hbar\omega_{\mathbf{k}}^+$. Now, if $m \rightarrow -m$, we get $(mc^2)/(\hbar\omega_{\mathbf{k}}^+) \rightarrow (-mc^2)/(\hbar\omega_{\mathbf{k}}^-)$. This means that we can interpret the latter as the ratio between the rest energy of an antiparticle with mass $-m$ and kinetic energy $\hbar\omega_{\mathbf{k}}^-$. Note that we can interchange the roles of particles with antiparticles in the expressions for the momentum and charge, leaving them invariant, if $a_{\mathbf{k}} \leftrightarrow b_{\mathbf{k}}$ and $m \rightarrow -m$, where the latter operation implies $\omega_{\mathbf{k}}^\pm \rightarrow \omega_{\mathbf{k}}^\mp$.

Let us see now what happens to the equal time commutation relations of the fields. Inserting equation (A.14) into equations (A.8)-(A.10), the non-trivial commutation relations become

$$[\Psi(t, \mathbf{r}), \Psi^\dagger(t, \mathbf{r}')] = -\frac{mc^2}{\sqrt{r}} \delta^{(3)}(\mathbf{r}-\mathbf{r}'), \quad (\text{A.19})$$

$$[\Pi_\Psi(t, \mathbf{r}), \Pi_{\Psi^\dagger}(t, \mathbf{r}')] = -m \left(1 - \frac{\mu^2}{\sqrt{r}} \right) \delta^{(3)}(\mathbf{r}-\mathbf{r}'), \quad (\text{A.20})$$

$$[\Psi(t, \mathbf{r}), \Pi_\Psi(t, \mathbf{r}')] = i\hbar \left(1 - \frac{\mu^2}{\sqrt{r}} \right) \delta^{(3)}(\mathbf{r}-\mathbf{r}'), \quad (\text{A.21})$$

where $1/\nabla_{\mathbf{r}}^2$ is the inverse Laplacian,

$$\frac{1}{\nabla_{\mathbf{r}}^2} e^{i\mathbf{k}\cdot\mathbf{r}} = -\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2}. \quad (\text{A.22})$$

Note that for this particular choice of $f_{\pm}(\mathbf{k})$, we have that $[\Pi_{\Psi}(t, \mathbf{r}), \Pi_{\Psi^{\dagger}}(t, \mathbf{r}')] = (im/\hbar)[\Psi(t, \mathbf{r}), \Pi_{\Psi}(t, \mathbf{r}')]$.

If we use the integral representation of the Dirac delta function, equation (A.1), and apply the inverse Laplacian, we get the following integral during the calculations leading to the commutators (A.19)-(A.21):

$$\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{|\mathbf{k}|^2} = \frac{4\pi}{|\mathbf{r}-\mathbf{r}'|} \int_0^{\infty} du \frac{\sin u}{u} = \frac{2\pi^2}{|\mathbf{r}-\mathbf{r}'|}. \quad (\text{A.23})$$

With the aid of equation (A.23), we get for equations (A.19)-(A.21)

$$[\Psi(t, \mathbf{r}), \Psi^{\dagger}(t, \mathbf{r}')] = \frac{mc^2}{4\pi} \frac{1}{|\mathbf{r}-\mathbf{r}'|}, \quad (\text{A.24})$$

$$[\Pi_{\Psi}(t, \mathbf{r}), \Pi_{\Psi^{\dagger}}(t, \mathbf{r}')] = -m \left(\delta^{(3)}(\mathbf{r}-\mathbf{r}') + \frac{\mu^2}{4\pi|\mathbf{r}-\mathbf{r}'|} \right), \quad (\text{A.25})$$

$$[\Psi(t, \mathbf{r}), \Pi_{\Psi}(t, \mathbf{r}')] = i\hbar \left(\delta^{(3)}(\mathbf{r}-\mathbf{r}') + \frac{\mu^2}{4\pi|\mathbf{r}-\mathbf{r}'|} \right). \quad (\text{A.26})$$

The important messages are contained in equations (A.24) and (A.26). First, equation (A.24) tells us that $[\Psi(t, \mathbf{r}), \Psi^{\dagger}(t, \mathbf{r}')] \neq 0$, which implies the violation of microcausality [4, 5]. On the other hand, equation (A.26) shows that the field and its conjugate momentum do not satisfy the canonical commutation relation. In addition to the canonical term, we have a correction given by $i\hbar\mu^2/(4\pi|\mathbf{r}-\mathbf{r}'|)$. This term, whose origin stems from equation (A.24), can be seen as giving origin to an instantaneous Coulomb-like potential. Contrary to the instantaneous Coulomb potential coming from the photon propagator [4], here, the ‘‘charges’’ associated with this interaction are the masses of the particles. What is intriguing here, however, is that we are dealing with scalar matter fields not vector fields like the photons. The emergence of this Coulomb-like potential may be just a coincidence but, pushing further the speculation, it may be that a more general quantum field theory incorporating positive and negative masses might naturally lead to the emergence of the gravitational interaction.

A.2. More on the Commutators for the Noncanonically Quantized Fields. We can repeat the steps followed in Section 7.3 using equations (A.13) and (A.14) to obtain that

$$[\Psi(x), \Psi^{\dagger}(y)] = i\hbar c \tilde{\Delta}_{\text{LS}}(x-y), \quad (\text{A.27})$$

where

$$\begin{aligned} \tilde{\Delta}_{\text{LS}}(x) &= e^{i\mu x^0} \tilde{\Delta}(x), \\ \tilde{\Delta}(x) &= \tilde{\Delta}^+(x) + \tilde{\Delta}^-(x), \\ \tilde{\Delta}^+(x) &= \frac{-ic}{2(2\pi)^3} \int d^3k \frac{e^{-ikx}}{\omega_{\mathbf{k}}^+}, \\ \tilde{\Delta}^-(x) &= \frac{ic}{2(2\pi)^3} \int d^3k \frac{e^{ikx}}{\omega_{\mathbf{k}}^-}, \end{aligned} \quad (\text{A.28})$$

with $k^0 = \omega_{\mathbf{k}}/c$. We are putting the tilde ‘‘~’’ on top of any quantity coming from the noncanonically quantized theory in order to differentiate it from the ones associated with the canonically quantized theory in the main text.

If we use the Heaviside step function $h(k^0)$, in which $h(k^0) = 1$ for $k^0 > 0$ and $h(k^0) = 0$ for $k^0 < 0$, we have

$$\begin{aligned} \tilde{\Delta}^{\pm}(x) &= \frac{-i}{(2\pi)^3} \int d^4k h(\pm k^0) \varepsilon(k^0) \delta(k^2 \pm 2\mu\omega_{\mathbf{k}}^{\pm}/c) e^{-ikx} \\ &= \frac{-i}{(2\pi)^3} \int d^4k \frac{k^0}{k^0 - \mu} h(\pm k^0) \varepsilon(k^0) \delta(k^2 - \mu^2) e^{-ikx}, \end{aligned} \quad (\text{A.29})$$

where the other quantities above were defined in Section 7.

Noting that $h(k^0) + h(-k^0) = 1$ for any $k^0 \neq 0$, we get

$$\tilde{\Delta}(x) = \frac{-i}{(2\pi)^3} \int d^4k \frac{k^0}{k^0 - \mu} \varepsilon(k^0) \delta(k^2 - \mu^2) e^{-ikx}. \quad (\text{A.30})$$

If inside the integral sign we add and subtract the term $-\mu/(k^0 - \mu)$, we can write $\tilde{\Delta}(x)$ in the following illustrative way:

$$\tilde{\Delta}(x) = \Delta(x) + \tilde{\delta}(x), \quad (\text{A.31})$$

where

$$\tilde{\delta}(x) = \frac{-i}{(2\pi)^3} \int d^4k \frac{\mu}{k^0 - \mu} \varepsilon(k^0) \delta(k^2 - \mu^2) e^{-ikx} \quad (\text{A.32})$$

and $\Delta(x)$ is given by equation (352). We can thus understand $\tilde{\delta}(x)$ as the correction to $\Delta(x)$ when we deviate from the canonical quantization path of Section 7.

In analogy to equation (367), we can write $\tilde{\Delta}^{\pm}(x)$ as the following contour integral, where k^0 is considered a complex variable:

$$\tilde{\Delta}^{\pm}(x) = -\frac{1}{(2\pi)^4} \int_{\tilde{C}^{\pm}} d^4k \frac{e^{-i\mu x^0} e^{-ikx}}{k_0^2 - (\omega_{\mathbf{k}}^{\pm}/c)^2} = -\frac{1}{(2\pi)^4} \int_{\tilde{C}^{\pm}} d^4k \frac{e^{-i\mu x^0} e^{-ikx}}{k^2 \pm (2\mu\omega_{\mathbf{k}}^{\pm}/c)}. \quad (\text{A.33})$$

Here, \tilde{C}^+ is any counterclockwise closed path encircling only $\omega_{\mathbf{k}}^+/c$ and \tilde{C}^- is any counterclockwise closed path

encircling $-\omega_{\mathbf{k}}^-/c$. Obviously, $\tilde{\Delta}(x) = \tilde{\Delta}^+(x) + \tilde{\Delta}^-(x)$. We can have an integral representation in which $\tilde{\Delta}^+(x)$ and $\tilde{\Delta}^-(x)$ have the same integrand, differing only in the path of integration,

$$\tilde{\Delta}^\pm(x) = -\frac{1}{(2\pi)^4} \int_{C^\pm} d^4k \frac{k^0}{k^0 - \mu} \frac{e^{-ikx}}{k^2 - \mu^2}. \quad (\text{A.34})$$

The above paths of integration are defined in equation (367); i.e., C^\pm encircles counterclockwise only $\omega_{\mathbf{k}}^\pm/c$. Note that we cannot express $\tilde{\Delta}(x)$ with the integral representation above and using the path C , as we did for $\Delta(x)$ in equation (368), since now we have a pole in $k^0 = \mu$. If we insist using the path C of equation (368), we must subtract the contribution of the latter pole to the integral,

$$\tilde{\Delta}(x) = -\frac{1}{(2\pi)^4} \int_C d^4k \left[\frac{k^0}{k^0 - \mu} \frac{e^{-ikx}}{k^2 - \mu^2} \right] - \frac{i\mu e^{-i\mu x^0}}{2(2\pi) |\mathbf{x}|}. \quad (\text{A.35})$$

Let us return to the second line of equation (A.29), which is conveniently written to prove the invariance of $\tilde{\Delta}^\pm(x)$ under proper Lorentz transformations. Its invariance under spatial rotations is obvious. The subtle point lies in proving its invariance under a Lorentz boost and to prove that we first note that the Dirac delta guarantees that we are on mass shell and therefore $h(\pm k^0)$ and $\varepsilon(k^0)$ are invariant under a Lorentz boost. The other invariants in equation (A.29) are d^4k , k^2 , μ^2 , and kx . Thus, after a Lorentz boost from reference frame S to S' in the x^1 -direction, we have

$$\begin{aligned} [\tilde{\Delta}^\pm]'(x') &= \frac{-i}{(2\pi)^3} \int d^4k' \frac{\gamma(k^{0'} + \beta k^{1'})}{\gamma(k^{0'} + \beta k^{1'}) - \mu} h(\pm k^{0'}) \\ &\cdot \varepsilon(k^{0'}) \delta\left[(k')^2 - \mu^2\right] e^{-ik'x'}. \end{aligned} \quad (\text{A.36})$$

If we now make the following change of variables (this is not another Lorentz boost),

$$k^{0''} = \gamma(k^{0'} + \beta k^{1'}), \quad (\text{A.37})$$

$$k^{1''} = \gamma(k^{1'} + \beta k^{0'}), \quad (\text{A.38})$$

we get for the right-hand side of equation (A.36),

$$\begin{aligned} [\tilde{\Delta}^\pm]'(x') &= \frac{-i}{(2\pi)^3} \int d^4k'' \frac{k^{0''}}{k^{0''} - \mu} h(\pm k^{0''}) \varepsilon(k^{0''}) \\ &\cdot \delta\left[(k'')^2 - \mu^2\right] e^{-ik''x''} \\ &\times e^{ik^{1''}[\gamma(x^{1'} + \beta x^{0'})] + ik^{2''}x^{2'} + ik^{3''}x^{3'}}. \end{aligned} \quad (\text{A.39})$$

Note that we still have x' in equation (A.39) since we are just changing variables in the integral and not performing a Lorentz boost. Moreover, the change of variables (A.37)-(A.38) are formally equivalent to a Lorentz boost from the point of view of k' and this is why $\int d^4k' h(\pm k^0) \varepsilon(k^0) \delta[(k')^2 - \mu^2]$ is invariant when we go to k'' .

Finally, if we remember that the variables in S' are connected to the ones in S by the following relation,

$$\begin{aligned} x^0 &= \gamma(x^{0'} + \beta x^{1'}), \\ x^1 &= \gamma(x^{1'} + \beta x^{0'}), \\ x^2 &= x^{2'}, \\ x^3 &= x^{3'}, \end{aligned} \quad (\text{A.40})$$

equation (A.39) becomes after we rename the integration variable k'' to k

$$[\tilde{\Delta}^\pm]'(x') = \frac{-i}{(2\pi)^3} \int d^4k \frac{k^0}{k^0 - \mu} h(\pm k^0) \varepsilon(k^0) \delta(k^2 - \mu^2) e^{-ikx}. \quad (\text{A.41})$$

Comparing equation (A.41) with (A.29), we see that they are the same, which proves that under a proper Lorentz transformation, $\tilde{\Delta}^\pm(x)$ is invariant. The invariance of $\tilde{\Delta}^\pm(x)$ and $\Delta^\pm(x)$ also implies that $\tilde{\Delta}(x)$ and $\tilde{\delta}(x)$ are invariant under proper Lorentz transformations and that the commutator (A.27) and the Feynman propagator (A.42) are Lorentz covariant quantities.

A.3. The Feynman Propagator for the Noncanonically Quantized Fields. In the present case, equation (374) becomes

$$\tilde{\Delta}_{\text{LS}}(x) = h(x^0) \tilde{\Delta}_{\text{LS}}^+(x) - h(-x^0) \tilde{\Delta}_{\text{LS}}^-(x), \quad (\text{A.42})$$

with

$$\tilde{\Delta}_{\text{LS}}^\pm(x) = e^{i\mu x^0} \tilde{\Delta}^\pm(x). \quad (\text{A.43})$$

Using equation (A.33), we get

$$\tilde{\Delta}_{\text{LS}}^\pm(x) = -\frac{1}{(2\pi)^4} \int_{\tilde{C}^\pm} d^4k \frac{e^{-ikx}}{k^2 \pm (2\mu\omega_{\mathbf{k}}^\pm/c)}. \quad (\text{A.44})$$

It is not as simple as in the canonically quantized case to write the Feynman propagator as a single complex integral. The reason for this difficulty can be seen looking at the denominators of equation (A.33). Instead of the two simple poles we met when dealing with the canonically quantized theory, we now have four simple poles, namely, $k^0 = \pm\omega_{\mathbf{k}}^+/c$ and $k^0 = \pm\omega_{\mathbf{k}}^-/c$. This means that the path along the real axis of the complex contour integral defining the propagator is different whether we have $x^0 > 0$ or $x^0 < 0$. For $x^0 > 0$, the clockwise path \tilde{C}_F^+ must enclose only the pole $k^0 = \omega_{\mathbf{k}}^+/c$, skirting counterclockwise the other three poles. For $x^0 < 0$, the counterclockwise path \tilde{C}_F^- must enclose only the pole $k^0 = -\omega_{\mathbf{k}}^-/c$, skirting clockwise the remaining three poles. With this understanding for the path \tilde{C}_F , we have

$$\begin{aligned}\tilde{\Delta}_{F_{\text{LS}}}(x) &= \frac{1}{(2\pi)^4} \int_{\tilde{C}_F} d^4k \left[\frac{1}{k^2 + (2\mu\omega_{\mathbf{k}}^+/c)} + \frac{1}{k^2 - (2\mu\omega_{\mathbf{k}}^-/c)} \right] e^{-ikx} \\ &= \frac{1}{(2\pi)^4} \int_{\tilde{C}_F} d^4k \left[\frac{2k^2 - 4\mu^2}{k^4 - 4\mu^2(k^0)^2} \right] e^{-ikx}.\end{aligned}\quad (\text{A.45})$$

The same difficulty above manifests itself when writing the propagator considering the four variables k^ν real, with all integrals running from $-\infty$ to ∞ . In this case, we can circumvent the four-pole problem using the Heaviside step function, which allows us to write

$$\tilde{\Delta}_{F_{\text{LS}}}(x) = \frac{1}{(2\pi)^4} \int d^4k \left[\frac{h(x^0)}{k^2 + (2\mu\omega_{\mathbf{k}}^+/c) + i\varepsilon^+} + \frac{h(-x^0)}{k^2 - (2\mu\omega_{\mathbf{k}}^-/c) + i\varepsilon^-} \right] e^{-ikx}.\quad (\text{A.46})$$

In equation (A.46), the limit $\varepsilon^\pm \rightarrow 0$ is understood. Also, in analogy to the canonically quantized theory,

$$\varepsilon^\pm = \frac{2\eta\omega_{\mathbf{k}}^\pm}{c},\quad (\text{A.47})$$

with $0 < \eta \ll 1$, and

$$k^2 \pm (2\mu\omega_{\mathbf{k}}^\pm/c) + i\varepsilon^\pm = (k^0)^2 - (\omega_{\mathbf{k}}^\pm/c - i\eta)^2,\quad (\text{A.48})$$

where second-order terms in η^2 were discarded.

Using equations (A.31)-(A.32), (A.42)-(A.43), and (A.46), we have

$$\tilde{\Delta}_{F_{\text{LS}}}(x) = \Delta_{F_{\text{LS}}}(x) + \tilde{\delta}_{F_{\text{LS}}}(x),\quad (\text{A.49})$$

where $\Delta_{F_{\text{LS}}}(x)$ is the propagator for the canonically quantized theory and

$$\tilde{\delta}_{F_{\text{LS}}}(x) = \frac{-ie^{i\mu x^0}}{(2\pi)^3} \int d^4k \left[\frac{\mu}{k^0 - \mu} \frac{1 + \varepsilon(k^0)\varepsilon(x^0)}{2} \delta(k^2 - \mu^2) \right] e^{-ikx},\quad (\text{A.50})$$

where $\varepsilon(x^0)$ is the sign function (do not confuse it with the small ε appearing in the denominator of the propagator). A similar interpretation attributed to $\tilde{\delta}(x)$ applies here; namely, $\tilde{\delta}_{F_{\text{LS}}}(x)$ is the correction to the Feynman propagator due to our departure from the canonical quantization of Section 7.

Building on equation (A.35), we also have

$$\tilde{\Delta}_{F_{\text{LS}}}(x) = \frac{e^{i\mu x^0}}{(2\pi)^4} \int d^4k \left[\frac{k^0}{k^0 - (\mu - i\alpha)k^2 - \mu^2 + i\varepsilon} \right] - \frac{i\mu h(x^0)}{2(2\pi)|\mathbf{x}|},\quad (\text{A.51})$$

where $\alpha > 0$, $\varepsilon > 0$, and the limit $(\alpha, \varepsilon) \rightarrow (0, 0)$ is implied after integration in k^0 . If we use that $\lim_{x^0 \rightarrow 0} h(x^0) = 1/2$, we realize that in equation (A.51), the last term is related to the instantaneous gravitational-like potential that arises when we noncanonically quantize the Lorentz covariant Schrödinger equation.

Furthermore, if we use the integral representation for the Heaviside step function,

$$h(x^0) = -\frac{1}{2\pi i} \int dk^0 \frac{e^{-ik^0 x^0}}{k^0 + i\alpha},\quad (\text{A.52})$$

with $\alpha > 0$ and $\alpha \rightarrow 0$ implied, and that

$$\frac{1}{|\mathbf{x}|} = \frac{2}{(2\pi)^2} \int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^2},\quad (\text{A.53})$$

we have after changing the variable of integration $k^0 \rightarrow k^0 - \mu$

$$\frac{-i\mu h(x^0)}{2(2\pi)|\mathbf{x}|} = \frac{\mu e^{i\mu x^0}}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{[k^0 - (\mu - i\alpha)]|\mathbf{k}|^2}.\quad (\text{A.54})$$

Thus, using equation (A.54), we can rewrite equation (A.51) as

$$\begin{aligned}\tilde{\Delta}_{F_{\text{LS}}}(x) &= \frac{e^{i\mu x^0}}{(2\pi)^4} \int d^4k \left\{ \frac{k^0}{[k^0 - (\mu - i\alpha)][k^2 - \mu^2 + i\varepsilon]} + \frac{\mu}{[k^0 - (\mu - i\alpha)]|\mathbf{k}|^2} \right\} e^{-ikx} \\ &= \frac{e^{i\mu x^0}}{(2\pi)^4} \int d^4k e^{-ikx} \left\{ \frac{1}{k^0 - (\mu - i\alpha)} \left[\frac{k^0}{k^2 - \mu^2 + i\varepsilon} + \frac{\mu}{|\mathbf{k}|^2} \right] \right\}.\end{aligned}\quad (\text{A.55})$$

B. Equivalence with the Klein-Gordon Theory

B.1. The $(\lambda/4)[\Psi(x)\Psi^\dagger(x)]^2$ Interaction. The interaction Lagrangian density

$$\mathcal{L}_{\text{int}} = -(\lambda/4)[\Psi(x)\Psi^\dagger(x)]^2\quad (\text{B.1})$$

does not depend on time derivatives and thus [4, 5]

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = (\lambda/4)[\Psi(x)\Psi^\dagger(x)]^2.\quad (\text{B.2})$$

Note that \mathcal{H}_{int} is clearly Lorentz covariant since the phase change induced in $\Psi(x)$ by a proper Lorentz transformation is compensated by the one induced in $\Psi^\dagger(x)$ (see equation (18)). Also, since the interaction picture is connected to the Heisenberg picture via a unitary transformation, we obtain

$$\mathcal{H}_{\text{int}}^I = -\mathcal{L}_{\text{int}}^I = (\lambda/4) \left\{ \Psi^I(x) [\Psi^I(x)]^\dagger \right\}^2. \quad (\text{B.3})$$

Inserting equation (B.3) into (386), we get

$$S = T \exp \left\{ -\frac{i\lambda}{4\hbar c} \int d^4x \left\{ \Psi^I(x) [\Psi^I(x)]^\dagger \right\}^2 \right\}. \quad (\text{B.4})$$

If we now use equation (387), we arrive at

$$S = T \exp \left\{ -\frac{i\lambda}{4\hbar c} \int d^4x \left\{ \Phi^I(x) [\Phi^I(x)]^\dagger \right\}^2 \right\}. \quad (\text{B.5})$$

Looking at equation (B.5), we realize that it is the S-matrix for the complex Klein-Gordon fields if they interact according to

$$\mathcal{H}_{\text{int}}^I = (\lambda/4) \left\{ \Phi^I(x) [\Phi^I(x)]^\dagger \right\}^2. \quad (\text{B.6})$$

This tells us that the S-matrix for the Klein-Gordon and Lorentz covariant Schrödinger fields is identical if they are subjected to the same type of interaction (equations (B.3) and (B.6)). Furthermore, the flux and linear momentum associated with a particle are formally the same in both theories, as can be seen comparing equations (82) and (329) with the respective ones coming from the Klein-Gordon theory. Therefore, any scattering cross section (decay rate) computed for both theories has the same value if the colliding (decaying) particles are assumed to have the same initial linear momentum.

The preceding analysis can be easily extended to any interaction of the type

$$\mathcal{H}_{\text{int}} = g_n [\Psi(x) \Psi^\dagger(x)]^n, \quad (\text{B.7})$$

with n a real number and g_n the associated coupling constant. Since Ψ and Ψ^\dagger have the same exponent, we can use equation (387) and the same arguments above to prove the equivalence between the Klein-Gordon and the Lorentz covariant Schrödinger theories.

On the other hand, the requirement for Lorentz covariance forbids interactions given by

$$\mathcal{H}_{\text{int}} = g_{n\tilde{n}} [\Psi(x)]^n [\Psi^\dagger(x)]^{\tilde{n}}, \quad \text{with } n \neq \tilde{n}. \quad (\text{B.8})$$

Indeed, for different values of n and \tilde{n} , we have after a

proper Lorentz transformation

$$[\Psi(x)]^n [\Psi^\dagger(x)]^{\tilde{n}} \longrightarrow e^{i(n-\tilde{n})\theta(x)/\hbar} [\Psi(x)]^n [\Psi^\dagger(x)]^{\tilde{n}}, \quad (\text{B.9})$$

which clearly shows the lack of Lorentz covariance if $n \neq \tilde{n}$.

When we have two (or more) types of particles with different masses, similar arguments forbid interaction terms proportional to $\Psi_1^{n_1} (\Psi_1^\dagger)^{\tilde{n}_1} \Psi_2^{n_2} (\Psi_2^\dagger)^{\tilde{n}_2}$, for $n_j \neq \tilde{n}_j$, $j = 1, 2$. Here, Ψ_j corresponds to the field associated with the particle of mass m_j . For Klein-Gordon fields, interactions given by terms similar to those of equation (B.8) are possible since they remain invariant under a proper Lorentz transformation.

B.2. Scalar Electrodynamics. The proof of the equivalence between the Klein-Gordon and the Lorentz covariant Schrödinger quantum scalar electrodynamics is more subtle. Before presenting this proof, we need first to develop the classical scalar electrodynamics of the Lorentz covariant Schrödinger fields. After obtaining the classical Hamiltonian density describing all fields and the interactions among them, we are ready to proceed to its canonical quantization and give the proof of the equivalence of both theories.

Applying the minimal coupling prescription, equation (165), to the free Lorentz covariant Schrödinger Lagrangian density, equation (244), we get

$$\mathcal{L} = (D_\mu \Psi)(D^\mu \Psi)^* + i\mu[\Psi^*(D_0 \Psi) - \Psi(D_0 \Psi)^*] = \mathcal{L}_{\text{LS}} + \mathcal{L}_{\text{int}}. \quad (\text{B.10})$$

Here, the free field and interaction Lagrangian densities are

$$\mathcal{L}_{\text{LS}} = \partial_\mu \Psi \partial^\mu \Psi^* + i\mu \Psi^* \partial_0 \Psi, \quad (\text{B.11})$$

$$\mathcal{L}_{\text{int}} = -\frac{iq}{\hbar} (\Psi^* \partial_\mu \Psi) A^\mu + \frac{q^2}{\hbar^2} \Psi \Psi^* A_\mu A^\mu - \frac{2q\mu}{\hbar} \Psi \Psi^* A^0. \quad (\text{B.12})$$

Equation (B.10) is invariant under a gauge transformation, namely, if

$$\begin{aligned} \Psi(x) &\longrightarrow e^{(iq/\hbar)\chi(x)} \Psi(x), \\ A_\mu &\longrightarrow A_\mu - \partial_\mu \chi(x), \end{aligned} \quad (\text{B.13})$$

we have that

$$\mathcal{L} \longrightarrow \mathcal{L}. \quad (\text{B.14})$$

In equation (B.12), the third term at the right-hand side is characteristic of the Lorentz covariant Schrödinger Lagrangian, while the first and second ones are formally equivalent to what one would get by applying the minimal coupling prescription to the Klein-Gordon Lagrangian.

If we use equation (108),

$$\Psi(x) = e^{i\mu x^0} \Phi(x), \quad (\text{B.15})$$

which connects the Lorentz covariant Schrödinger and Klein-Gordon equations, we get after inserting it in equation (B.10)

$$\mathcal{L}_{\text{LS}} \longrightarrow \mathcal{L}_{\text{KG}}, \quad (\text{B.16})$$

$$\mathcal{L}_{\text{int}} \longrightarrow -\frac{iq}{\hbar} (\Phi^* \partial_\mu \Phi) A^\mu + \frac{q^2}{\hbar^2} \Phi \Phi^* A_\mu A^\mu. \quad (\text{B.17})$$

Here, \mathcal{L}_{KG} is the Klein-Gordon free-field Lagrangian density and the right-hand side of equation (B.17) is exactly the interaction term one gets by applying the minimal coupling prescription to \mathcal{L}_{KG} .

We should also mention that similarly to \mathcal{L}_{LS} , the interaction Lagrangian density \mathcal{L}_{int} , equation (B.12), is invariant under a proper Lorentz transformation. This is proved via a direct calculation using equation (18), the transformation rule for $\Psi(x)$ under a proper Lorentz transformation, and the respective covariant and contravariant transformation rules for the vectors ∂_μ and A^μ .

The complete scalar electrodynamic Lagrangian density also has the electromagnetic (EM) free-field term \mathcal{L}_{EM} , which is a function of the four-potential A^μ (the four components of A^μ are considered independent variables) [4, 5]. Therefore, the total Lagrangian density for the scalar electrodynamics (SED) becomes

$$\mathcal{L}_{\text{SED}} = \mathcal{L}_{\text{LS}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int}}. \quad (\text{B.18})$$

In order to carry out the canonical quantization of the scalar electrodynamics, we need the Hamiltonian density associated with \mathcal{L}_{SED} . Since the interaction term \mathcal{L}_{int} does not contain time derivatives of the fields A^μ , the canonically conjugate fields related to them are the same as those of the free-field case. As such, after the Legendre transformation leading from the Lagrangian to the Hamiltonian density, we have $\mathcal{L}_{\text{EM}} \longrightarrow \mathcal{H}_{\text{EM}}$, where \mathcal{H}_{EM} is the standard free-field Hamiltonian density for the electromagnetic field [4, 5]. However, \mathcal{L}_{int} contains terms involving time derivatives of the fields Ψ and Ψ^* . This means that their canonically conjugate fields are different from the free-field case, which ultimately implies that \mathcal{H}_{int} is no longer simply $-\mathcal{L}_{\text{int}}$.

The new conjugate field to Ψ is

$$\Pi_\Psi = \frac{\partial \mathcal{L}_{\text{SED}}}{\partial(\partial_t \Psi)} = \frac{\partial \mathcal{L}_{\text{LS}}}{c \partial(\partial_0 \Psi)} + \frac{\partial \mathcal{L}_{\text{int}}}{c \partial(\partial_0 \Psi)} = \frac{1}{c} \left(\partial_0 \Psi^* + i\mu \Psi^* - \frac{iq}{\hbar} \Psi^* A^0 \right). \quad (\text{B.19})$$

In a similar way, the conjugate field to Ψ^* is

$$\Pi_{\Psi^*} = \frac{1}{c} \left(\partial_0 \Psi - i\mu \Psi + \frac{iq}{\hbar} \Psi A^0 \right) = \Pi_\Psi^*. \quad (\text{B.20})$$

Using equations (B.19) and (B.20), the Hamiltonian density

$$\mathcal{H} = \Pi_\Psi \partial_t \Psi + \Pi_{\Psi^*} \partial_t \Psi^* - \mathcal{L}_{\text{LS}} - \mathcal{L}_{\text{int}} + \mathcal{H}_{\text{EM}} \quad (\text{B.21})$$

becomes

$$\mathcal{H} = \mathcal{H}_{\text{LS}} + \mathcal{H}_{\text{EM}} + \mathcal{H}_{\text{int}}, \quad (\text{B.22})$$

where \mathcal{H}_{LS} and \mathcal{H}_{EM} are, respectively, the free-field Hamiltonian densities for the Lorentz covariant Schrödinger and electromagnetic fields, while

$$\begin{aligned} \mathcal{H}_{\text{int}} = & \frac{iqc}{\hbar} (\Pi_{\Psi^*} \Psi^* - \Pi_\Psi \Psi) A^0 + \frac{iq}{\hbar} (\Psi^* \partial_j \Psi) \\ & \cdot A^j - \frac{q^2}{\hbar^2} \Psi \Psi^* A_\mu A^\mu + \frac{q^2}{\hbar^2} \Psi \Psi^* A_0 A^0. \end{aligned} \quad (\text{B.23})$$

Equation (B.23) is formally the same as the one we would get if we had worked with the Klein-Gordon Lagrangian [5]. Here, however, the expressions for Π_Ψ and Π_{Ψ^*} are given by equations (B.19) and (B.20) while in the Klein-Gordon case, we do not have the terms $i\mu \Psi^*$ and $-i\mu \Psi$ in equations (B.19) and (B.20), respectively.

In order to quantize equation (B.23), the independent field variables Ψ , Ψ^* , Π_Ψ , Π_{Ψ^*} , and A^μ are promoted to operators, satisfying the usual equal-time commutation relations. The commutation relations for the scalar fields are given in equations (324) and (325), and the commutation relations for the four-vector potential can be found in refs. [4, 5]. Also, to avoid ordering ambiguities among the operators as well as to eliminate nonphysical vacuum contributions, we normal order all products of operators.

Going to the interaction picture, we have

$$\mathcal{H}^I = \mathcal{H}_{\text{LS}}^I + \mathcal{H}_{\text{EM}}^I + \mathcal{H}_{\text{int}}^I, \quad (\text{B.24})$$

where the operator \mathcal{O} in the Heisenberg picture is connected to its representation in the interaction picture by the following unitary transformation:

$$\mathcal{O}^I = U(t) \mathcal{O} U^\dagger(t). \quad (\text{B.25})$$

The unitary operator above is [4, 5]

$$U(t) = e^{iH_0^S t/\hbar} e^{-iHt/\hbar}, \quad (\text{B.26})$$

with

$$H_0^S = H_{\text{LS}}^S + H_{\text{EM}}^S \quad (\text{B.27})$$

being the total free-field Hamiltonian in the Schrödinger picture. Note that $H_j^S = \int d^3x \mathcal{H}_j^S(t, \mathbf{x})$, $j = \text{LS}$ or EM . The quantity $H = \int d^3x \mathcal{H}(t, \mathbf{x})$ is the complete Hamiltonian, having the same form in the Schrödinger and Heisenberg pictures.

In order to apply the Dyson series, we have to write the interaction part of the Hamiltonian density in terms of operators in the interaction picture. Using equation (B.23), the unitarity of $U(t)$, and that $U(t)$ does not depend on the

spatial variables, we obtain with the help of equation (B.25)

$$\begin{aligned} \mathcal{H}_{\text{int}}^I &= \frac{iqc}{\hbar} \left[\Pi_{\Psi^\dagger}^I (\Psi^I)^\dagger - \Pi_{\Psi}^I \Psi^I \right] (A^I)^0 + \frac{iq}{\hbar} \left[(\Psi^I)^\dagger \partial_j \Psi^I \right] \\ &\cdot (A^I)^j - \frac{q^2}{\hbar^2} \Psi^I (\Psi^I)^\dagger A_\mu^I (A^I)^\mu + \frac{q^2}{\hbar^2} \Psi^I (\Psi^I)^\dagger A_0^I (A^I)^0. \end{aligned} \quad (\text{B.28})$$

We now eliminate the conjugate fields Π_{Ψ}^I and $\Pi_{\Psi^\dagger}^I$ in favor of the time derivatives of the fields $\partial_0 \Psi^I$ and $\partial_0 (\Psi^I)^\dagger$. To that end, we note first that

$$\partial_0 \Psi^I = \partial_0 (U \Psi U^\dagger) = (\partial_0 U) \Psi U^\dagger + U (\partial_0 \Psi) U^\dagger + U \Psi (\partial_0 U^\dagger). \quad (\text{B.29})$$

Using the definitions of $U(t)$ and $U^\dagger(t)$, a direct calculation leads to

$$\partial_0 U = -\frac{i}{\hbar c} U H_{\text{int}}, \quad (\text{B.30})$$

$$\partial_0 U^\dagger = \frac{i}{\hbar c} H_{\text{int}} U^\dagger. \quad (\text{B.31})$$

Moreover, $\partial_0 \Psi$ is obtained from the time evolution of Ψ in the Heisenberg picture,

$$\partial_0 \Psi = \frac{1}{i\hbar c} [\Psi, H]. \quad (\text{B.32})$$

Inserting equations (B.30), (B.31), and (B.32) into (B.29), we get

$$\partial_0 \Psi^I = \frac{i}{\hbar c} U [H_0, \Psi] U^\dagger, \quad (\text{B.33})$$

where

$$H_0 = H_{\text{LS}} + H_{\text{EM}}. \quad (\text{B.34})$$

Since H_{EM} has no dependence on the matter fields, it commutes with Ψ and equation (B.33) is reduced to

$$\partial_0 \Psi^I = \frac{i}{\hbar c} U [H_{\text{LS}}, \Psi] U^\dagger. \quad (\text{B.35})$$

The commutator above, with the aid of equations (252), (324), and (325), is given by

$$\begin{aligned} [H_{\text{LS}}, \Psi(t, \mathbf{x})] &= \int d^3 x' \left[\mathcal{H}_{\text{LS}}(t, \mathbf{x}'), \Psi(t, \mathbf{x}) \right] \\ &= \int d^3 x' c^2 \left[\Pi_{\Psi}(t, \mathbf{x}'), \Psi(t, \mathbf{x}) \right] \Pi_{\Psi^\dagger}(t, \mathbf{x}') \\ &\quad + \int d^3 x' i\mu c \Psi(t, \mathbf{x}') \left[\Pi_{\Psi}(t, \mathbf{x}'), \Psi(t, \mathbf{x}) \right] \\ &= -i\hbar c^2 \Pi_{\Psi^\dagger}(t, \mathbf{x}) + \mu \hbar \Psi(t, \mathbf{x}). \end{aligned} \quad (\text{B.36})$$

Inserting equation (B.36) into (B.35), we finally get

$$\partial_0 \Psi^I = c \Pi_{\Psi^\dagger}^I + i\mu \Psi^I. \quad (\text{B.37})$$

An analogous calculation leads to

$$\partial_0 (\Psi^I)^\dagger = c \Pi_{\Psi}^I - i\mu (\Psi^I)^\dagger. \quad (\text{B.38})$$

Equations (B.37) and (B.38) are what we need to eliminate the conjugate fields in equation (B.28). Remembering that we are adopting the normal ordering prescription (we can freely write, for example, $(\Psi^I)^\dagger \Psi^I$ instead of $\Psi^I (\Psi^I)^\dagger$), we obtain

$$\mathcal{H}_{\text{int}}^I = -\mathcal{L}_{\text{int}}^I + \frac{q^2}{\hbar^2} (\Psi^I)^\dagger \Psi^I A_0^I (A^I)^0, \quad (\text{B.39})$$

where

$$\begin{aligned} \mathcal{L}_{\text{int}}^I &= -\frac{iq}{\hbar} \left[(\Psi^I)^\dagger \partial_\mu \Psi^I \right] (A^I)^\mu + \frac{q^2}{\hbar^2} (\Psi^I)^\dagger \Psi^I A_\mu^I (A^I)^\mu \\ &\quad - \frac{2\mu q}{\hbar} (\Psi^I)^\dagger \Psi^I (A^I)^0. \end{aligned} \quad (\text{B.40})$$

Note that $\mathcal{L}_{\text{int}}^I$ is invariant under a proper Lorentz transformation while the so-called ‘‘normal-dependent term’’ $(q^2/\hbar^2)(\Psi^I)^\dagger \Psi^I A_0^I (A^I)^0$ is not [5].

Equation (B.39) differs from the one we would obtain if we worked with the Klein-Gordon fields by the last term in equation (B.40),

$$\frac{2\mu q}{\hbar} (\Psi^I)^\dagger \Psi^I (A^I)^0. \quad (\text{B.41})$$

However, the extra term (B.41), together with the rest of $\mathcal{H}_{\text{int}}^I$, will not lead to different physical processes that are not contained in the Klein-Gordon theory.

To see this, we rewrite equation (B.39) as

$$\begin{aligned} \mathcal{H}_{\text{int}}^I &= \frac{q}{\hbar} \left\{ \left[i(\Psi^I)^\dagger \partial_\mu \Psi^I \right] (A^I)^\mu + 2\mu (\Psi^I)^\dagger \Psi^I (A^I)^0 \right\} \\ &\quad + \frac{q^2}{\hbar^2} \left[(\Psi^I)^\dagger \Psi^I A_0^I (A^I)^0 - (\Psi^I)^\dagger \Psi^I A_\mu^I (A^I)^\mu \right]. \end{aligned} \quad (\text{B.42})$$

In equation (B.42), we have grouped terms proportional to the charge q and those proportional to q^2 .

If we now insert equation (B.15) into (B.42), we obtain

$$\begin{aligned} \mathcal{H}_{\text{int}}^I &= \frac{q}{\hbar} \left\{ \left[i(\Phi^I)^\dagger \partial_\mu \Phi^I \right] (A^I)^\mu \right\} + \frac{q^2}{\hbar^2} \left[(\Phi^I)^\dagger \Phi^I A_0^I (A^I)^0 \right. \\ &\quad \left. - (\Phi^I)^\dagger \Phi^I A_\mu^I (A^I)^\mu \right]. \end{aligned} \quad (\text{B.43})$$

Note that terms proportional to q and q^2 in equation

(B.42) are respectively mapped to the ones proportional to q and q^2 in equation (B.43).

Furthermore, equation (B.43) is exactly the interaction Hamiltonian density we obtain when dealing with the Klein-Gordon theory and it is not difficult to see that the free-field Hamiltonians are all transformed to the ones of the Klein-Gordon theory after equation (B.15). Therefore, the S-matrix of the Klein-Gordon and Lorentz covariant Schrödinger theories is the same. This implies, using the same arguments given in Appendix B.1, that in the framework of scalar electrodynamics, both theories lead to equivalent predictions when it comes to scattering and decay processes.

C. Transformation Law for $\Psi(\mathbf{x})$ under Finite Proper Lorentz Transformations

After an infinitesimal counterclockwise spatial rotation of a reference frame about its x^3 -axis, the original coordinates x^μ describing a four-vector are related to those in the new frame ($x^{\mu'}$) as follows:

$$x^0 = x^{0'}, \quad (\text{C.1})$$

$$x^1 = x^{1'} - \varepsilon x^{2'}, \quad (\text{C.2})$$

$$x^2 = x^{2'} + \varepsilon x^{1'}, \quad (\text{C.3})$$

$$x^3 = x^{3'}, \quad (\text{C.4})$$

where ε is an infinitesimal angle of rotation.

To simplify the notation in the following calculations, it is convenient to use ct, x, y, z to label the coordinates of the above contravariant four-vector. Also, the inertial frame before any rotation is implemented is called S_0 and the inertial frame after n infinitesimal rotations is denoted by S_n . The coordinates of a four-vector in S_n is given by ct_n, x_n, y_n, z_n . Therefore, according to equations (C.2) and (C.3), after n spatial rotations about the z -axis, we have

$$x_{n-1} = x_n - \varepsilon y_n, \quad (\text{C.5})$$

$$y_{n-1} = \varepsilon x_n + y_n. \quad (\text{C.6})$$

After N infinitesimal rotations, we end up at the frame S_N . Solving the system of recursive relations (C.5) and (C.6) with the “final conditions” x_N and y_N , we get

$$x_n = \frac{1}{2} [(1 - i\varepsilon)^{N-n} + (1 + i\varepsilon)^{N-n}] x_N - \frac{i}{2} [(1 - i\varepsilon)^{N-n} - (1 + i\varepsilon)^{N-n}] y_N, \quad (\text{C.7})$$

$$y_n = \frac{i}{2} [(1 - i\varepsilon)^{N-n} - (1 + i\varepsilon)^{N-n}] x_N + \frac{1}{2} [(1 - i\varepsilon)^{N-n} + (1 + i\varepsilon)^{N-n}] y_N. \quad (\text{C.8})$$

According to equations (414) and (419), after an infinitesimal rotation about the z -axis, the wave functions in S_{n-1} and S_n are connected to each other according to the fol-

lowing transformation law:

$$\Psi_{n-1} = e^{-i\varepsilon(\kappa_y x_n - \kappa_x y_n)} \Psi_n. \quad (\text{C.9})$$

To arrive at equation (C.9), we used that $\varepsilon^{21} = -\varepsilon^{12} = \varepsilon$, with all other $\varepsilon^{\mu\nu}$ being zero, and renamed κ^1 and κ^2 to κ_x and κ_y .

Using repeatedly equation (C.9), we obtain after N infinitesimal rotations ε about the z -axis that

$$\Psi_N = e^{i\varepsilon \left(\kappa_y \sum_{n=1}^N x_n - \kappa_x \sum_{n=1}^N y_n \right)} \Psi_0. \quad (\text{C.10})$$

If we now employ equations (C.7) and (C.8), the sums in equation (C.10) become

$$\sum_{n=1}^N x_n = -\frac{i}{2\varepsilon} [(1 + i\varepsilon)^N - (1 - i\varepsilon)^N] x_N - \frac{1}{2\varepsilon} [2 - (1 - i\varepsilon)^N - (1 + i\varepsilon)^N] y_N, \quad (\text{C.11})$$

$$\sum_{n=1}^N y_n = \frac{1}{2\varepsilon} [2 - (1 - i\varepsilon)^N - (1 + i\varepsilon)^N] x_N - \frac{i}{2\varepsilon} [(1 + i\varepsilon)^N - (1 - i\varepsilon)^N] y_N. \quad (\text{C.12})$$

Inserting equations (C.11) and (C.12) into (C.10), we obtain

$$\Psi_N = \exp \left\{ -\frac{i\kappa_x x_N}{2} f(\varepsilon) + \frac{\kappa_y x_N}{2} g(\varepsilon) - \frac{\kappa_x y_N}{2} g(\varepsilon) - \frac{i\kappa_y y_N}{2} f(\varepsilon) \right\} \Psi_0, \quad (\text{C.13})$$

where

$$f(\varepsilon) = 2 - (1 - i\varepsilon)^N - (1 + i\varepsilon)^N, \quad (\text{C.14})$$

$$g(\varepsilon) = (1 + i\varepsilon)^N - (1 - i\varepsilon)^N. \quad (\text{C.15})$$

A finite rotation ϕ can be split into N infinitesimal ones such that

$$\varepsilon = \phi/N. \quad (\text{C.16})$$

Inserting equation (C.16) into equations (C.14) and (C.15) and taking the limit for large N , we get

$$\lim_{N \rightarrow \infty} f(\phi/N) = 2(1 - \cos \phi), \quad (\text{C.17})$$

$$\lim_{N \rightarrow \infty} g(\phi/N) = 2i \sin \phi. \quad (\text{C.18})$$

If we identify Ψ_0 as the wave function in the rest frame S before the finite rotation ϕ is implemented, $\lim_{N \rightarrow \infty} \Psi_N$ as the wave function at the rest frame S' after the rotation, and if we go back to the four-vector notation, equations

(C.17) and (C.18) when inserted into (C.13) give

$$\Psi(x) = e^{i \left[(\kappa^1 x^{1'} + \kappa^2 x^{2'}) (1 - \cos \phi) + (\kappa^1 x^{2'} - \kappa^2 x^{1'}) \sin \phi \right]} \Psi'(x'). \quad (\text{C.19})$$

Equation (C.19) is nothing but the transformation law given by equation (423) for a finite rotation ϕ about the x^3 -axis. By similar calculations, we can get the transformation laws for $\Psi(x)$ when we rotate about the x^2 and x^1 -axes.

Let us now show how the transformation law for finite boosts can be derived for $\Psi(x)$. To that aim, we write a finite boost along the x^1 -axis as a rotation in a hyperbolic space. The coordinate transformation is thus

$$x^0 = x^{0'} \cosh \xi + x^{1'} \sinh \xi, \quad (\text{C.20})$$

$$x^1 = x^{0'} \sinh \xi + x^{1'} \cosh \xi, \quad (\text{C.21})$$

$$x^2 = x^{2'}, \quad (\text{C.22})$$

$$x^3 = x^{3'}, \quad (\text{C.23})$$

where the ‘‘hyperbolic angle’’ ξ is called the ‘‘rapidity’’ and is given by

$$\tanh \xi = \beta = v/c. \quad (\text{C.24})$$

Here, v is the speed of frame S' with respect to S , directed along the x^1 -axis. Note that $\cosh \xi = \gamma$ and $\sinh \xi = \beta\gamma$.

Expressed as given by equations (C.20) and (C.21), two successive boosts along the x^1 -axis with rapidity ξ and ξ' give $x^0 = x^{0''} \cosh(\xi + \xi') + x^{1''} \sinh(\xi + \xi')$ and $x^1 = x^{0''} \sinh(\xi + \xi') + x^{1''} \cosh(\xi + \xi')$, where $x^{\mu''}$ are the coordinates of the four-vector $x^{\mu'}$ in the frame S'' . This is formally equivalent to spatial rotations if the hyperbolic sines and cosines are changed to the usual trigonometric ones. Therefore, the calculations above leading to the transformation law for $\Psi(x)$ after a spatial rotation can be readily adapted to a boost if we note that for an infinitesimal boost ($\xi \ll 1$), we have

$$\begin{aligned} x^0 &= x^{0'} + \xi x^{1'}, \\ x^1 &= x^{1'} + \xi x^{0'}. \end{aligned} \quad (\text{C.25})$$

The analog to equations (C.5) and (C.6) is

$$\begin{aligned} ct_{n-1} &= ct_n + \xi x_n, \\ x_{n-1} &= \xi ct_n + x_n, \end{aligned} \quad (\text{C.26})$$

whose solution is

$$\begin{aligned} ct_n &= \frac{1}{2} \left[(1 + \xi)^{N-n} + (1 - \xi)^{N-n} \right] ct_N + \frac{1}{2} \left[(1 + \xi)^{N-n} - (1 - \xi)^{N-n} \right] x_N, \\ x_n &= \frac{1}{2} \left[(1 + \xi)^{N-n} - (1 - \xi)^{N-n} \right] ct_N + \frac{1}{2} \left[(1 + \xi)^{N-n} + (1 - \xi)^{N-n} \right] x_N. \end{aligned} \quad (\text{C.27})$$

Repeating all the steps of the previous calculation, we get

$$\Psi_N = e^{i\xi \left(\kappa_x \sum_{n=1}^N ct_n - \kappa_{ct} \sum_{n=1}^N x_n \right)} \Psi_0 \quad (\text{C.28})$$

and

$$\sum_{n=1}^N ct_n = \frac{1}{2\xi} \left[(1 + \xi)^N - (1 - \xi)^N \right] ct_N - \frac{1}{2\xi} \left[2 - (1 + \xi)^N - (1 - \xi)^N \right] x_N, \quad (\text{C.29})$$

$$\sum_{n=1}^N x_n = -\frac{1}{2\xi} \left[2 - (1 + \xi)^N - (1 - \xi)^N \right] ct_N + \frac{1}{2\xi} \left[(1 + \xi)^N - (1 - \xi)^N \right] x_N. \quad (\text{C.30})$$

Inserting equations (C.29) and (C.30) into (C.28), we arrive at

$$\Psi_N = \exp \left\{ \frac{i\kappa_{ct} ct_N}{2} f(\xi) + \frac{i\kappa_x ct_N}{2} g(\xi) - \frac{i\kappa_{ct} x_N}{2} g(\xi) - \frac{i\kappa_x x_N}{2} f(\xi) \right\} \Psi_0, \quad (\text{C.31})$$

where

$$f(\xi) = 2 - (1 + \xi)^N - (1 - \xi)^N, \quad (\text{C.32})$$

$$g(\xi) = (1 + \xi)^N - (1 - \xi)^N. \quad (\text{C.33})$$

To go from successive infinitesimal boosts to a finite one, we set

$$\xi = \Xi/N \quad (\text{C.34})$$

and take the appropriate limits in equations (C.32) and (C.33),

$$\lim_{N \rightarrow \infty} f(\Xi/N) = 2(1 - \cosh \Xi), \quad (\text{C.35})$$

$$\lim_{N \rightarrow \infty} g(\Xi/N) = 2 \sinh \Xi. \quad (\text{C.36})$$

Noting that $\cosh \Xi = \gamma$ and $\sinh \Xi = \beta\gamma$, equations (C.35) and (C.36) allow us to write (C.31) as

$$\Psi(x) = e^{i \left[(\gamma-1)\kappa^0 - \gamma\beta\kappa^1 \right] x^0 + i \left[\gamma\beta\kappa^0 - (\gamma-1)\kappa^1 \right] x^1} \Psi'(x'). \quad (\text{C.37})$$

To arrive at equation (C.37), we have reversed to the usual four-vector notation, namely, $ct_N = x^0$, $x_N = x^1$, $\kappa_{ct} = \kappa^0$, $\kappa_x = \kappa^1$, and identified Ψ_0 with $\Psi(x)$ and Ψ_N with

$\Psi'(x')$. Equation (C.37) is the transformation law for $\Psi(x)$ when it is subjected to a finite boost along the x^1 -axis, namely, equation (425) given in the main text.

Data Availability

This is a theoretical work, and no data were used in writing it.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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References

- [1] L. E. Ballentine, *Quantum Mechanics: A Modern Development*, World Scientific, Singapore, 1998.
- [2] W. Greiner, *Relativistic Quantum Mechanics: Wave Equations*, Springer-Verlag, Berlin, 2000.
- [3] D. J. Griffiths, *Introduction to Quantum Mechanics*, Prentice-Hall, Upper Saddle River, 1995.
- [4] F. Mandl and G. Shaw, *Quantum Field Theory*, John Wiley & Sons, Chichester, 1986.
- [5] W. Greiner and J. Reinhardt, *Field Quantization*, Springer-Verlag, Berlin, 1996.
- [6] E. Schrödinger, “An undulatory theory of the mechanics of atoms and molecules,” *Physical Review*, vol. 28, no. 6, pp. 1049–1070, 1926.
- [7] E. C. G. Stueckelberg, “La mécanique du point matériel en théorie de la relativité et en théorie des quanta,” *Helvetica Physica Acta*, vol. 15, p. 23, 1942.
- [8] Y. Nambu, “The use of the proper time in quantum electrodynamics I,” *Progress of Theoretical Physics*, vol. 5, p. 82, 1950.
- [9] L. P. Horwitz and C. Piron, “Relativistic dynamics,” *Helvetica Physica Acta*, vol. 46, p. 316, 1973.
- [10] J. R. Fanchi and R. E. Collins, “Quantum mechanics of relativistic spinless particles,” *Foundations of Physics*, vol. 8, p. 851, 1978.
- [11] L. L. Foldy and S. A. Wouthuysen, “On the Dirac theory of spin 1/2 particles and its non-relativistic limit,” *Physical Review*, vol. 78, p. 29, 1950.
- [12] M. A. H. Tucker and A. F. G. Wyatt, “Direct evidence for R-rotons having antiparallel momentum and velocity,” *Science*, vol. 283, p. 1150, 1999.
- [13] B. Julsgaard, A. Kozhekin, and E. S. Polzik, “Experimental long-lived entanglement of two macroscopic objects,” *Nature (London)*, vol. 413, p. 400, 2001.
- [14] M. Tsang and C. M. Caves, “Evading quantum mechanics: engineering a classical subsystem within a quantum environment,” *Physical Review X*, vol. 2, article 031016, 2012.
- [15] M. Conforti, S. Trillo, A. Mussot, and A. Kudlinski, “Parametric excitation of multiple resonant radiations from localized wavepackets,” *Scientific Reports*, vol. 5, p. 9433, 2015.
- [16] M. A. Khamehchi, K. Hossain, M. E. Mossman et al., “Negative-mass hydrodynamics in a spin-orbit coupled Bose-Einstein condensate,” *Physical Review Letters*, vol. 118, article 155301, 2017.
- [17] E. C. G. Stueckelberg, “Remarque à propos de la création de paires de particules en théorie de relativité,” *Helvetica Physica Acta*, vol. 14, p. 588, 1941.
- [18] O. Heaviside, “Electromagnetic induction and its propagation,” *Electrician*, vol. 19, p. 79, 1887.
- [19] O. Heaviside, *Electromagnetic Theory*, vol. 1, The Electrician Printing & Publishing Company, London, 1893.
- [20] P. Sancho, “The quantum telegraph equation,” *Nuovo Cimento, B*, vol. 112, p. 1437, 1997.
- [21] C. M. Will, “Gravitational red-shift measurements as tests of nonmetric theories of gravity,” *Physical Review D*, vol. 10, p. 2330, 1974.
- [22] P. Tourrenc and J. L. Grossiord, “Modification du spectre de l’hydrogène atomique sous l’influence de champs de gravitation,” *Il Nuovo Cimento B*, vol. 32, p. 163, 1976.
- [23] L. Parker, “One-electron atom in curved space-time,” *Physical Review Letters*, vol. 44, p. 1559, 1980.
- [24] E. Fischbach, B. S. Freeman, and W.-K. Cheng, “General-relativistic effects in hydrogenic systems,” *Physical Review D*, vol. 23, p. 2157, 1981.
- [25] T. Westphal, H. Hepach, J. Pfaff, and M. Aspelmeyer, “Measurement of gravitational coupling between millimeter-sized masses,” 2021, <https://arxiv.org/abs/2009.09546>.
- [26] B. R. Edwards and V. A. Kostelecký, “Riemann-Finsler geometry and Lorentz-violating scalar fields,” *Physics Letters B*, vol. 786, p. 319, 2018.
- [27] D. L. Anderson, M. Sher, and I. Turan, “Lorentz and CPT violation in the Higgs sector,” *Physical Review D*, vol. 70, article 016001, 2004.
- [28] M. Dine and A. Kusenko, “Origin of the matter-antimatter asymmetry,” *Reviews of Modern Physics*, vol. 76, p. 1, 2004.
- [29] O. Bertolami, D. Colladay, V. A. Kostelecký, and R. Potting, “CPT violation and baryogenesis,” *Physics Letters B*, vol. 395, p. 178, 1997.
- [30] D. Colladay and V. A. Kostelecký, “Lorentz-violating extension of the standard model,” *Physical Review D*, vol. 58, article 116002, 1998.
- [31] S. M. Carroll and J. Shu, “Models of baryogenesis via spontaneous Lorentz violation,” *Physical Review D*, vol. 73, article 103515, 2006.
- [32] M. de Cesare, N. E. Mavromatos, and S. Sarkar, “On the possibility of tree-level leptogenesis from Kalb-Ramond torsion background,” *The European Physical Journal C*, vol. 75, p. 514, 2015.
- [33] J. Sakstein and A. R. Solomon, “Baryogenesis in Lorentz-violating gravity theories,” *Physics Letters B*, vol. 773, p. 186, 2017.
- [34] V. I. Safonov, “Conduction electron in the anisotropic medium,” *International Journal of Modern Physics B*, vol. 7, p. 3899, 1993.
- [35] A. Zhao, J. Zhang, Q. Gu, and R. A. Klemm, “A relativistic electron in an anisotropic conduction band,” 2019, <https://arxiv.org/abs/1905.03127>.
- [36] M. S. Safronova, D. Budker, D. DeMille, D. F. J. Kimball, A. Derevianko, and C. W. Clark, “Search for new physics with atoms and molecules,” *Reviews of Modern Physics*, vol. 90, article 025008, 2018.

- [37] H. Bondi, "Negative mass in general relativity," *Reviews of Modern Physics*, vol. 29, p. 423, 1957.
- [38] M. Kowitt, "Gravitational repulsion and Dirac antimatter," *International Journal of Theoretical Physics*, vol. 35, p. 605, 1996.
- [39] M. Villata, "CPT symmetry and antimatter gravity in general relativity," *EPL (Europhysics Letters)*, vol. 94, p. 20001, 2011.
- [40] J. S. Farnes, "A unifying theory of dark energy and dark matter: negative masses and matter creation within a modified Λ CDM framework," *Astronomy & Astrophysics*, vol. 620, p. A92, 2018.