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Research Article

Nilpotent Symmetries of a Model of 2D Diffeomorphism Invariant Theory: BRST Approach

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Within the framework of the Becchi-Rouet-Stora-Tyutin (BRST) formalism, we discuss the full set of proper BRST and anti-BRST transformations for a 2D diffeomorphism invariant theory which is described by the Lagrangian density of a standard bosonic string. The above (anti-)BRST transformations are off-shell nilpotent and absolutely anticommuting. The latter property is valid on a submanifold of the space of quantum fields where the 2D version of the universal (anti-)BRST invariant Curci-Ferrari (CF) type of restrictions is satisfied. We derive the precise forms of the BRST and anti-BRST invariant Lagrangian densities as well as the exact expressions for the conserved (anti-)BRST and ghost charges. The lucid derivation of the proper anti-BRST symmetry transformations and the emergence of the CF-type restrictions are completely novel results for our present bosonic string which has already been discussed earlier in literature where only the BRST symmetry transformations have been pointed out. We briefly mention the derivation of the CF-type restrictions from the modified version of the Bonora-Tonin superfield approach, too.

1. Introduction

One of the most exciting and captivating areas of research in theoretical high energy physics (THEP), over the last few decades, has been the subject of (super)strings and related extended objects (see, e.g., [1–4] for details). This is due to the fact that, in one stroke, these theories provide a possible scenario of unification of all the fundamental interactions of nature and a promising candidate for the precise theory of quantum gravity. The modern developments in the realm of (super)strings have influenced many other areas of research in THEP, e.g., noncommutative field theories, higher p-form ($p = 2, 3, 4, \cdots$) gauge theories, higher spin gauge theories, supersymmetric gauge theories and related mathematics, gauge-gravity duality, and AdS/CFT correspondence. The quantization of these (super)string theories have led us to imagine a higher dimensional view of the physical world we live in. It has been established that one cannot consistently quantize the dual-string theory [5] unless the space-time dimension is $D = 26$ and the intercept (α_0) of the leading Regge trajectory is $\alpha_0 = 1$. These results have been obtained and formally established from many different considerations like the requirement of the validity of proper Lorentz algebra, unitarity requirements of these string theories, nilpotency of the Becchi-Rouet-Stora-Tyutin (BRST) charge, etc. In this context, one of the earliest attempts to covariantly quantize a bosonic string theory, within the framework of BRST formalism, was undertaken by Kato and Ogawa [6] where the 2D diffeomorphism symmetry of this theory was exploited.

In the above work [6], it is precisely the infinitesimal version of the 2D classical diffeomorphism symmetry invariance of the theory that has been primarily exploited to perform the BRST quantization where only the BRST symmetries have been discussed. However, there is no discussion about the anti-BRST symmetries and related Curci-Ferrari-(CF-) type restrictions which are the hallmarks of a properly BRST quantized theory. In this work [6], the inverse of the metric tensor has been taken in such a manner that the conformal anomaly does not spoil the BRST analysis. In fact, the inverse of the metric tensor has been decomposed such that it has three independent degrees of freedom to begin with. A Lagrange multiplier field density has been incorporated into the 2D Lagrangian density so that the equation of motion w.r.t. it puts a restriction on the determinant of the above metric tensor. The latter condition reduces the independent degrees of freedom of the metric tensor from three to two. The BRST charge has been calculated in the flat limit where the metric tensor becomes Minkowskian in nature (see, e.g., [6] for more details). The nilpotency requirement of this BRST charge leads to the derivation of $D = 26$ and $\alpha_0 = 1$. One of the central theme of our present investigation is to focus on the existence of (i) the proper anti-BRST symmetries (corresponding to the BRST transformations taken in [6]) and (ii) the (anti-)BRST invariant CF-type of restrictions which are responsible for the absolute anticommutativity of the nilpotent (anti-)BRST symmetry transformations. We have taken a very modest step in this direction, in our present endeavor.

We have performed the full BRST analysis of the above theory [6] in the sense that we have derived the proper anti-BRST symmetry transformations corresponding to the BRST symmetry transformations that have been taken into account in [6]. The BRST and anti-BRST symmetry transformations are found to be off-shell nilpotent and absolutely anticommuting in nature. The latter property has been shown to be true on a submanifold of the quantum Hilbert space of quantum fields where the CF-type restrictions (20) (see below) are satisfied. We observe that these restrictions are BRST as well as anti-BRST invariant thereby implying that these are physical restrictions (which can be imposed from outside on our present theory). We have derived, in our present endeavor, the BRST and anti-BRST invariant Lagrangian densities and have shown explicitly their BRST and anti-BRST invariance. The conserved charges of the theory have been computed in the flat limit where $A_0 = A_1 = 0$, $A_2 = 1$ (compare Equation (4)). In fact, the latter conditions imply that the metric tensor of the theory transforms as $\tilde{g}^{ab} \longrightarrow \eta^{ab}$ where η^{ab} is the flat metric of the 2D Minkowski space (which is nothing but the 2D surface traced out by the propagation of the bosonic string). We have also established that the standard algebra between the ghost charge and BRST charge (as well as between the ghost charge and anti-BRST charge) is satisfied. We have commented, very briefly, on the nilpotency properties of the BRST and anti-BRST charges which are true at the quantum level only when $D = 26$ and $\alpha_0 = 1$ provided we take into account the normal mode expansions of the fields (consistent with the appropriate boundary conditions) and substitute them in the computation and proof of nilpotency: $Q_B^2 = 1/2\,\{Q_B, Q_B\} = 0$ and $\bar{Q}_B^2 = 1/2\,\{\bar{Q}_B, \bar{Q}_B\} = 0$.

The main motivating factors behind our present investigation are as follows. First, in the BRST description [6] of the present bosonic string, only the BRST transformations have been discussed corresponding to the infinitesimal diffeomorphism symmetry transformation of the theory. The nilpotent anti-BRST symmetry transformations have remained untouched in [6]. Thus, it is important for us to discuss the BRST as well as anti-BRST symmetry transformations together for the complete BRST analysis of our present theory. We have accomplished this goal in our present endeavor. Second, in the BRST description of Kato and Ogawa [6], the auxiliary fields have been modified/redefined in a very complicated fashion to simplify the theoretical analysis of the present theory. There are, however, no basic physical arguments to support such kinds of modifications/redefinitions. We have, in our present endeavor, not invoked any such kind of modifications/redefinitions as our analysis is very straightforward. Third, the hallmark of a quantum theory, discussed within the framework of BRST formalism, is the existence of the (non)trivial Curci-Ferrari- (CF-) type restrictions. We have derived such restrictions in our present endeavor which ensure the absolute anticommutativity of the (anti-)BRST symmetry transformations. Finally, our present work is important because, for this model, the recently developed superfield approach [7] would be very useful as our theory is diffeomorphism invariant. We hope that the application of this superfield formalism [7] would shed some new light on some specific aspects of our present theory (as far as the symmetries are concerned). In Appendix D, we have briefly discussed the applications of this superfield approach which has been christened by us as the modified version of the Bonora-Tonin superfield approach (MBTSA).

Our present paper is organized as follows. To set up the notations and convention, we discuss very briefly the diffeomorphism symmetry as well as the corresponding BRST approach in Section 2 which has been performed in [6]. Section 3 is devoted to the discussion of BRST and anti-BRST symmetries where we also point out the existence of the 2D version of the universal CF-type restrictions. We prove the (anti-)BRST invariance of this restriction, and we demonstrate the nilpotency as well as the absolute anticommutativity of the (anti-)BRST symmetry transformations. We derive the explicit form of the BRST as well as anti-BRST invariant Lagrangian densities in Section 4. The conserved charges, corresponding to the continuous internal symmetries of the theory, are derived in Section 5, in the flat limit. Finally, we make some concluding remarks on our present investigation in Section 6 and point out a few future directions for further investigations.

In Appendices A, B, and C, we incorporate some of the algebraic expressions as well as equations that have been used in the main body of our text. In Appendix D, we concisely discuss the derivation of the CF-type restrictions by using the modified version of the Bonora-Tonin superfield approach (MBTSA) to BRST formalism.

2. Preliminary: Diffeomorphism and BRST Invariance

We begin with the Lagrangian density of a bosonic string theory as (see, e.g., [6] for details):

$$L_0 = -\frac{1}{2k}\,\tilde{g}^{ab}\partial_a X^\mu\,\partial_b X_\mu + E(\det\tilde{g} + 1), \tag{1}$$

where $\tilde{g}^{ab} = \sqrt{-g}\, g^{ab}$ has two independent degrees of freedom because $\det \tilde{g} = -1$ due to the equation of motion w.r.t. the Lagrange multiplier field E which happens to be a scalar density (compare Equation (3)). Here, the 2D surface, traced out by the propagation of the bosonic string, is parameterized by $\xi^a = (\xi^0, \xi^1) \equiv (\tau, \sigma)$ where $a = 0, 1$, and component parameters (τ, σ) satisfy $-\infty < \tau < +\infty$ and $0 \leq \sigma \leq \pi$. The string coordinates $X^\mu(\xi)$ (with $\mu = 0, 1, 2, 3, \cdots, D-1$) are in the D-dimensional flat Minkowskian spacetime manifold and $\tilde{g}^{ab} = \sqrt{-g}\, g^{ab}$ is the metric tensor constructed with the determinant ($g = \det g_{ab}$) and inverse (g^{ab}) of the metric tensor g_{ab} of the 2D parameter space. Under the infinitesimal diffeomorphism transformations $\xi^a \longrightarrow \xi^a - \varepsilon^a(\xi)$, we have the following transformations (δ_ε) on the relevant fields of our present bosonic string theory, namely

$$\delta_\varepsilon X^\mu = \varepsilon^a \partial_a X^\mu, \delta_\varepsilon E = \partial_a(\varepsilon^a E), \delta_\varepsilon(\det \tilde{g}) = \varepsilon^a \partial_a(\det \tilde{g}), \quad (2)$$

$$\delta_\varepsilon \tilde{g}^{ab} = \partial_m\left(\varepsilon^m \tilde{g}^{ab}\right) - (\partial_m \varepsilon^a)\tilde{g}^{mb} - \left(\partial_m \varepsilon^b\right)\tilde{g}^{am}, \quad (3)$$

where $\varepsilon^a(\xi)$ are the infinitesimal diffeomorphism transformation parameters. The above transformations leave the Lagrangian density (1) quasi-invariant (i.e., $\delta_\varepsilon L_0 = \partial_a(\varepsilon^a L_0)$). This demonstrates that the action integral $S = \int d^2\xi\, L_0 \equiv \int_{-\infty}^{+\infty} d\tau \int_0^\pi d\sigma\, L_0$ remains invariant under the diffeomorphism transformations (2) provided the boundary conditions $\varepsilon^a(\xi) = 0$ at $\sigma = 0$ and $\sigma = \pi$ are imposed on the diffeomorphism parameter $\varepsilon^a(\xi)$. It will be noted that we differ from [6] by an overall sign factor in the diffeomorphism transformations (2) and BRST transformations (11) (see below) because we have chosen the infinitesimal diffeomorphism transformation $\xi^a \longrightarrow \xi^a - \varepsilon^a(\xi)$, whereas the same transformation has been taken as $\xi^a \longrightarrow \xi^a + \varepsilon^a(\xi)$ in [6]. We choose the Latin indices $a, b, c, \cdots, l, m, n, \cdots = 0, 1$ to denote τ and σ directions on the 2D surface (traced out by the propagation of the bosonic string), and the Greek indices $\mu, \nu, \lambda, \cdots = 0, 1, 2, \cdots, D-1$ stand for the spacetime directions of the D-dimensional flat Minkowskian space-time manifold corresponding to the target space. The above 2D surface is embedded in the D-dimensional Minkowskian flat target space (which turns out to be 26 at the quantum level). Throughout the whole body of our text, we denote the BRST and anti-BRST symmetry transformations by the symbols s_B and $\bar{s}_B$, respectively. We adopt the convention of left-derivative w.r.t. the fermionic fields ($C^a, \bar{C}^a$, etc.) of our present theory. Consistent with this convention, the Noether conserved currents in Equations (25) and (26) are defined (see below).

The original Lagrangian density, $-(1/2k)\sqrt{-g}\, g^{ab}\, \partial_a X^\mu \partial_b X_\mu$ (with k as the string tension parameter) is endowed with the local conformal invariance. However, this conformal invariance is broken by the conformal anomaly [8, 9] if we regularize the system in a gauge-invariant manner. We have avoided this problem by taking $\tilde{g}^{ab} = \sqrt{-g}\, g^{ab}$ as the metric tensor of our present theory [6] with three independent degrees of freedom to start with. The EoM w.r.t. E (i.e., $\det \tilde{g} = -1$) reduce the independent degrees of freedom of the above specifically defined metric tensor from three to two. For the BRST quantization of the Lagrangian density (1), we have to invoke the gauge-fixing conditions. This can be achieved if we take the following decomposition for the metric tensor $\tilde{g}^{ab}$ (see, e.g., [6])

$$\tilde{g}^{ab} = \begin{pmatrix} A_1 + A_2 & A_0 \\ A_0 & A_1 - A_2 \end{pmatrix}, \quad (4)$$

and set the gauge-fixing conditions $A_0 = A_1 = 0$ so that we obtain $\det \tilde{g} = -A_2{}^2 = -1$. This shows that, for the choice $A_2 = 1$, we obtain the flatness condition $\tilde{g}^{ab} \longrightarrow \eta^{ab}$ with the signatures (+1, -1). By exploiting the standard techniques of the BRST formalism [10, 11], we obtain the gauge-fixing and Faddeev-Poppov ghost terms for the theory, in the language of the nilpotent ($s_B^2 = 0$) BRST transformations s_B, as (see, e.g., [10, 11] for details)

$$L_{GF} + L_{FP} = s_B\left[-i\,\bar{C}_0 A_0 - i\,\bar{C}_1 A_1\right], \quad (5)$$

where $\bar{C}_0$ and $\bar{C}_1$ are the antighost fields with ghost number (-1). It will be noted that the transformations $s_B\bar{C}_0 = i\,B_0$ and $s_B\bar{C}_1 = i\,B_1$ lead to the emergence of the Nakanishi-Lautrup-type auxiliary fields of the theory as B_0 and B_1 and the nilpotency requirements produce $s_B B_0 = s_B B_1 = 0$. A close look at the transformations (2) and decomposition (3) leads to the following BRST symmetry transformations for the component gauge fields:

$$s_B A_0 = C^a\,\partial_a\,A_0 - \left(\partial_0\,C^1 + \partial_1\,C^0\right)A_1 - \left(\partial_0\,C^1 - \partial_1\,C^0\right)A_2, \quad (6)$$

$$s_B A_1 = C^a\,\partial_a\,A_1 - \left(\partial_0\,C^0 - \partial_1\,C^1\right)A_2 - \left(\partial_1\,C^0 + \partial_0\,C^1\right)A_0, \quad (7)$$

$$s_B A_2 = C^a\,\partial_a\,A_2 - \left(\partial_0\,C^0 - \partial_1\,C^1\right)A_1 - \left(\partial_1\,C^0 - \partial_0\,C^1\right)A_0, \quad (8)$$

where we have taken the replacement ($\varepsilon^a \longrightarrow C^a$) which implies that the infinitesimal diffeomorphism parameters ($\varepsilon^a, a = 0, 1$) have been replaced by the fermionic $[(C^a)^2 = 0, C^a\,C^b + C^b\,C^a = 0]$ ghost fields C^a. As a consequence of this replacement, we have the following BRST symmetry transformations vis-à-vis the transformations (2), namely

$$s_B X^\mu = C^a \partial_a X^\mu, s_B E = \partial_a(C^a E), s_B(\det \tilde{g}) = \varepsilon^a \partial_a(\det \tilde{g}), \quad (9)$$

$$s_B C^a = C^b \partial_b C^a, s_B \bar{C}^a = iB^a, s_B B^a = 0, \quad (10)$$

$$s_B \tilde{g}^{ab} = \partial_m\left(C^m \tilde{g}^{ab}\right) - (\partial_m C^a)\tilde{g}^{mb} - \left(\partial_m C^b\right)\tilde{g}^{am}, \quad (11)$$

where the transformation $s_B\,C^a = C^b\,\partial_b\,C^a$ has been derived from the requirement of the nilpotency condition ($s_B^2\,X^\mu = 0$).

With the inputs from (6) and (11), we obtain the BRST invariant Lagrangian density (L_B) from (5) and (1), modulo some total derivatives, as

$$\begin{aligned}L_B &= L_0 + B_0 A_0 + B_1 A_1 + i\left[C^a \partial_a \bar{C}_0 - \bar{C}_0(\partial_a C^a)\right. \\ &\quad \left. - \bar{C}_1\left(\partial_0 C^1 + \partial_1 C^0\right)\right] A_0 + i\left[C^a \partial_a \bar{C}_1 - \bar{C}_1\left(\partial_a C^a\right)\right. \\ &\quad \left. - \bar{C}_0\left(\partial_0 C^1 + \partial_1 C^0\right)\right] A_1 - i\left[\bar{C}_0\left(\partial_0 C^1 - \partial_1 C^0\right)\right. \\ &\quad \left. + \bar{C}_1\left(\partial_0 C^0 - \partial_1 C^1\right)\right] A_2.\end{aligned} \tag{12}$$

The Lagrangian density L_B has been written, modulo some total derivatives, in such a manner that BRST transformations (6) could be implemented in a simple and straightforward manner. The above full Lagrangian density, under the flatness limit $A_0 = A_1 = 0, A_2 = 1$, reduces to the following Lagrangian density:

$$\begin{aligned}L_B \longrightarrow L_B^{(0)} &= -\frac{1}{2\kappa}\eta^{ab}\partial_a X^\mu \partial_b X_\mu + E\left(1 - A_2^2\right) + B_0 A_0 \\ &\quad + B_1 A_1 - i\left[\bar{C}_0\left(\partial_0 C^1 - \partial_1 C^0\right) + \bar{C}_1\left(\partial_0 C^0 - \partial_1 C^1\right)\right],\end{aligned} \tag{13}$$

which has been obtained in [6] after taking the help of the redefinitions of the auxiliary fields in a complicated fashion. In fact, these redefinitions are mathematical in nature, and there are no physical arguments to support the specific choices that have been made in [6] for the simplification of the Lagrangian density in the flat space. We have obtained (13) from (12) in a straightforward manner (without any redefinitions/modifications, etc.). We would like to point out that the flatness limit (i.e., $A_0 = A_1 = 0, A_2 = 1$) has been taken in all the terms of (12) except the gauge-fixing terms (i.e., $B_0 A_0 + B_1 A_1$) and the Lagrange multiplier term (i.e., $E(1 - A_2^2)$) because the EoM w.r.t. B_0, B_1, and E imply the same thing (i.e., $A_0 = A_1 = 0, A_2 = 1$) in the straightforward fashion.

3. BRST and Anti-BRST Symmetries: Key Features

It can be checked, in a straightforward fashion, that the BRST symmetry transformations, quoted in (6) and (11), are nilpotent of order two (i.e., $s_B^2 = 0$). The proper anti-BRST symmetry transformations, corresponding to the BRST transformations (11), are

$$\bar{s}_B X^\mu = \bar{C}^a \partial_a X^\mu, \bar{s}_B \bar{C}^a = \bar{C}^b \partial_b \bar{C}^a, \bar{s}_B C^a = i\bar{B}^a, \tag{14}$$

$$\bar{s}_B E = \partial_a\left(\bar{C}^a E\right), \bar{s}_B(\det \tilde{g}) = \bar{C}^a \partial_a(\det \tilde{g}), \bar{s}_B \bar{B}^a = 0, \tag{15}$$

$$\bar{s}_B \tilde{g}^{ab} = \partial_m\left(\bar{C}^m \tilde{g}^{ab}\right) - \left(\partial_m \bar{C}^a\right)\tilde{g}^{mb} - \left(\partial_m \bar{C}^b\right)\tilde{g}^{am}, \tag{16}$$

which are off-shell nilpotent ($\bar{s}_B^2 = 0$) of order two. It will be noted that we have invoked a new Nakanishi-Lautrup type of auxiliary field $\bar{B}^a(\xi)$ in our theory. Thus, we observe that the symmetry transformations (16), (11), and (6) satisfy one (i.e., nilpotency) of the two sacrosanct properties (i.e., nilpotency and absolute anticommutativity) that have to be satisfied by any proper (anti-)BRST symmetry transformations. We further note that the last entry of (9) can be written in terms of A_0, A_1, A_2 in the following form, namely

$$\bar{s}_B A_0 = \bar{C}^a \partial_a A_0 - \left(\partial_0 \bar{C}^1 + \partial_1 \bar{C}^0\right) A_1 - \left(\partial_0 \bar{C}^1 - \partial_1 \bar{C}^0\right) A_2, \tag{17}$$

$$\bar{s}_B A_1 = \bar{C}^a \partial_a A_1 - \left(\partial_0 \bar{C}^0 - \partial_1 \bar{C}^1\right) A_2 - \left(\partial_1 \bar{C}^0 + \partial_0 \bar{C}^1\right) A_0, \tag{18}$$

$$\bar{s}_B A_2 = \bar{C}^a \partial_a A_2 - \left(\partial_0 \bar{C}^0 - \partial_1 \bar{C}^1\right) A_1 - \left(\partial_1 \bar{C}^0 - \partial_0 \bar{C}^1\right) A_0. \tag{19}$$

Thus, it is clear that the anti-BRST transformations for A_0, A_1, A_2 are exactly the same as Equation (6) with the replacement $C^a \longrightarrow \bar{C}^a$.

We dwell a bit now on the absolute anticommutativity property (i.e., $\{s_B, \bar{s}_B\} = 0$) of the (anti-)BRST symmetry transformations (19), (16), (11), and (6). It turns out that the requirement of $\{s_B, \bar{s}_B\} X^\mu = 0$ lead to the existence of the following Curci-Ferrari- (CF-) type restrictions (which are primarily two in numbers), namely

$$B^a + \bar{B}^a + i\left(C^b \partial_b \bar{C}^a + \bar{C}^b \partial_b C^a\right) = 0, (a, b = 0, 1). \tag{20}$$

It turns out that the above conditions (20) have to be imposed to obtain the absolute anticommutativity (i.e., $\{s_B, \bar{s}_B\} = 0$) property when all the relevant fields of the whole theory are taken into account. For instance, it can be checked that the requirement of $\{s_B, \bar{s}_B\} E = 0$ also requires the validity of the CF-type restrictions (11). Furthermore, we obtain the following (anti-)BRST symmetry transformations on the Nakanishi-Lautrup auxiliary fields $B^a(\xi)$ and $\bar{B}^a(\xi)$ due to the requirement of the absolute anticommutativity property (e.g., $\{s_B, \bar{s}_B\} C^a = 0$ and $\{s_B, \bar{s}_B\} \bar{C}^a = 0$), namely

$$s_B \bar{B}^a = C^b \partial_b \bar{B}^a - \bar{B}^b \partial_b C^a, \bar{s}_B B^a = \bar{C}^b \partial_b B^a - B^b \partial_b \bar{C}^a. \tag{21}$$

Interestingly, the above transformations also satisfy the off-shell nilpotency property (i.e., $s_B^2 = 0, \bar{s}_B^2 = 0$) which is one of the key requirements of a proper set of (anti-)BRST symmetry transformations. Thus, we note that the (anti-)BRST symmetry transformations (21), (19), (16), (11), and (6) satisfy the off-shell nilpotency ($s_B^2 = \bar{s}_B^2 = 0$) and absolute anticommutativity ($\{s_B, \bar{s}_B\} = 0$) on a submanifold in the 2D Hilbert space of quantum fields where the CF-restrictions (20) are satisfied.

We would enumerate here some of the subtle features associated with the CF-type restrictions (20) which are at the heart of the absolute anticommutativity property of our BRST and anti-BRST symmetry transformations. We note that these 2D restrictions on the auxiliary and (anti-)ghost fields are (anti-)BRST invariant quantity, namely

$$\bar{s}_B\left[B^a+\bar{B}^a+i\left(C^b\,\partial_b\,\bar{C}^a+\bar{C}^b\,\partial_b\,C^a\right)\right]=0, \tag{22}$$

$$s_B\left[B^a+\bar{B}^a+i\left(C^b\,\partial_b\,\bar{C}^a+\bar{C}^b\,\partial_b\,C^a\right)\right]=0. \tag{23}$$

This demonstrates that the CF-type restrictions of our present theory are physical (in some sense) and the submanifold of the quantum fields in the Hilbert space defined by it is physically relevant. This demonstrates that our (anti-)BRST invariant theory is consistently defined on the submanifold where the CF-type restrictions (20) are always valid. In fact, on this submanifold alone, the BRST and anti-BRST symmetry transformations have their own identities as they are linearly independent of each other (due to their absolute anticommutativity). In the proof of (23), it is obvious that we have taken into account the (anti-)BRST transformations (21), (16), and (11) as well as the above specific submanifold.

We end this section with the following remarks on the nilpotency properties (i.e., $s_B^2=0, \bar{s}_B^2=0$) associated with the BRST and anti-BRST symmetry transformations s_B and $\bar{s}_B$. First of all, we note that $s_B^2\,X^\mu=0$ leads to the derivation of $s_B\,C^a=C^b\,\partial_b\,C^a$ where $s_B^2\,C^a=0$ is also satisfied. In exactly similar fashion, we obtain $\bar{s}_B\,\bar{C}^a=\bar{C}^b\,\partial_b\,\bar{C}^a$ (where $\bar{s}_B^2\,\bar{C}^a=0$) from the requirement of nilpotency of the anti-BRST symmetry transformation on X^μ field ($\bar{s}_B^2\,X^\mu=0$). The proof of the nilpotency (i.e., $s_B^2=0, \bar{s}_B^2=0$) of the transformations $s_B\,\tilde{g}^{ab}$ and $\bar{s}_B\,\tilde{g}^{ab}$ (compare Equations (11) and (16)) is algebraically more involved. We have collected some of the crucial expressions (as well as equations) in Appendix A which establish the nilpotency ($s_B^2\,\tilde{g}^{ab}=0$) of the BRST transformations when they act on the metric tensor $\tilde{g}^{ab}$. Ultimately, it turns out that $s_B^2\,\tilde{g}^{ab}=0$ and $\bar{s}_B^2\,\tilde{g}^{ab}=0$ are indeed true. This proof, in turn, implies that the (anti-)BRST transformations (compare Equations (6) and (19)) of the component gauge fields (i.e., A_0, A_1, A_2) of the metric tensor are automatically nilpotent (i.e., $s_B^2=0, \bar{s}_B^2=0$) of order two (compare Equation (6)).

4. (Anti-)BRST Invariant Lagrangian Densities

We have already mentioned the BRST invariant Lagrangian densities (12) and (13) in the (non)flat limits. In our present section, we establish their BRST invariance. The analogue of the Lagrangian density (12) that remains invariant, under the anti-BRST symmetry transformations (16), (19), and (21), is as follows [10, 11]:

$$L_{\bar{B}}=L_0+\bar{s}_B\left[i\,C_0\,A_0+i\,C_1\,A_1\right]. \tag{24}$$

Using the explicit anti-BRST symmetry transformations (16), (19), and (21), we obtain the following explicit form of the anti-BRST invariant Lagrangian density $L_{\bar{B}}$ as

$$\begin{aligned}L_{\bar{B}}&=L_0-\bar{B}_0A_0-\bar{B}_1A_1+i\left[C_0\left(\partial_a\bar{C}^a\right)+\left(\partial_aC_0\right)\bar{C}^a+C_1\left(\partial_0\bar{C}^1+\partial_1\bar{C}^0\right)\right]A_0\\&\quad+i\left[C_1\left(\partial_a\bar{C}^a\right)+\left(\partial_aC_1\right)\bar{C}^a+C_0\left(\partial_0\bar{C}^1+\partial_1\bar{C}^0\right)\right]A_1\\&\quad+i\left[C_0\left(\partial_0\bar{C}^1-\partial_1\bar{C}^0\right)+C_1\left(\partial_0\bar{C}^0-\partial_1\bar{C}^1\right)\right]A_2,\end{aligned} \tag{25}$$

where some total derivative terms have been dropped as they do not affect the dynamics of the theory. We shall take the flatness condition $\tilde{g}^{ab}\longrightarrow\eta^{ab}$ in the language of restrictions on the component gauge fields $A_0=A_1=0, A_2=1$ for the full discussion of our theory within the framework of BRST formalism. In the flat limit (i.e., $A_0=A_1=0, A_2=1$), the above Lagrangian density (i.e., (56)), in its full blaze of glory, is as follows:

$$\begin{aligned}L_{\bar{B}}\longrightarrow L_{\bar{B}}^{(0)}&=-\frac{1}{2\kappa}\,\eta^{ab}\partial_aX^\mu\partial_bX_\mu+E\left(1-A_2^2\right)-\bar{B}_0\,A_0-\bar{B}_1\,A_1\\&\quad+i\left[C_0\left(\partial_0\,\bar{C}^1-\partial_1\,\bar{C}^0\right)+C_1\left(\partial_0\,\bar{C}^0-\partial_1\,\bar{C}^1\right)\right],\end{aligned} \tag{26}$$

where the above limit has not been imposed on the gauge-fixing terms $(-\bar{B}_0\,A_0-\bar{B}_1\,A_1)$ and the term $(E\,(1-A_2^2))$ with the Lagrange multiplier field. We shall be calculating the conserved charges of the theory from the Lagrangian densities (13) and (26) which are quoted in the flat limits (cf. Section 5 for details).

To establish the explicit (anti-)BRST invariance of the Lagrangian densities (12), (13), (25), and (26), we have to apply the (anti-)BRST transformations on every term of the above Lagrangian densities. This exercise is algebraically more involved as one has to collect the terms containing A_0, A_1, A_2, B_0, B_1, separately and independently. In Appendices B and C, we have collected these terms which appear due to the applications of s_B and $\bar{s}_B$ on the Lagrangian densities L_B and $L_{\bar{B}}$, respectively. The explicit form of the BRST transformations on the BRST invariant Lagrangian density (7) (i.e., L_B) is

$$\begin{aligned}s_B\,L_B=\partial_a&\left[C^a(L_0+B_0\,A_0+B_1\,A_1)+i\,\bar{C}_1\,C^a\left(\partial_0\,C^1+\partial_1\,C^0\right)A_0\right.\\&+i\,\bar{C}_0C^b\,\partial_b\left(C^a\,A_0\right)+i\,\bar{C}_0\,C^a\left(\partial_0\,C^1+\partial_1\,C^0\right)A_1\\&+i\,\bar{C}_1C^b\,\partial_b\left(C^a\,A_1\right)+i\,\bar{C}_0\,C^a\left(\partial_0\,C^1-\partial_1\,C^0\right)A_2\\&\left.+i\,\bar{C}_1\,C^a\left(\partial_0\,C^0-\partial_1\,C^1\right)A_2\right].\end{aligned} \tag{27}$$

In exactly similar fashion, the anti-BRST transformation acting on the anti-BRST invariant Lagrangian density $L_{\bar{B}}$ produces the following explicit transformation:

$$\begin{aligned}\bar{s}_B L_{\bar{B}} = \partial_a \Big[&\bar{C}^a \left(L_0 - \bar{B}_0 A_0 - \bar{B}_1 A_1\right) - i C_1 \bar{C}^a \left(\partial_0 \bar{C}^1 + \partial_1 \bar{C}^0\right) A_0 \\ &- i C_0 \bar{C}^b \partial_b \left(\bar{C}^a A_0\right) - i C_0 \bar{C}^a \left(\partial_0 \bar{C}^1 + \partial_1 \bar{C}^0\right) A_1 \\ &- i C_1 \bar{C}^b \partial_b \left(\bar{C}^a A_1\right) - i C_0 \bar{C}^a \left(\partial_0 \bar{C}^1 - \partial_1 \bar{C}^0\right) A_2 \\ &- i C_1 \bar{C}^a \left(\partial_0 \bar{C}^0 - \partial_1 \bar{C}^1\right) A_2\Big]. \end{aligned} \tag{28}$$

A close and careful look at (27) and (28) shows that we can obtain (28) from (27) provided we make the replacements $B_0 \longrightarrow \bar{B}_0, B_1 \longrightarrow \bar{B}_1, A_0 \longrightarrow -A_0, A_1 \longrightarrow -A_1, A_2 \longrightarrow -A_2, C_0 \leftrightarrow \bar{C}_0, C_1 \leftrightarrow \bar{C}_1$. Now it is obvious that, in the flat limit $A_0 = A_1 = 0, A_2 = 1$ of the full Lagrangian densities L_B and $L_{\bar{B}}$, we obtain the following BRST and anti-BRST symmetry invariances for the Lagrangian densities $L_B^{(0)}$ and $L_{\bar{B}}^{(0)}$, namely

$$s_B L_B^{(0)} = \partial_a \left[C^a (L_0) + i \bar{C}_0 C^a \left(\partial_0 C^1 - \partial_1 C^0\right) + i \bar{C}_1 C^a \left(\partial_0 C^0 - \partial_1 C^1\right)\right], \tag{29}$$

$$\bar{s}_B L_{\bar{B}}^{(0)} = \partial_a \left[\bar{C}^a (L_0) - i C_0 \bar{C}^a \left(\partial_0 \bar{C}^1 - \partial_1 \bar{C}^0\right) - i C_1 \bar{C}^a \left(\partial_0 \bar{C}^0 - \partial_1 \bar{C}^1\right)\right]. \tag{30}$$

The total derivatives in (27), (28), (29), and (30) establish that the (anti-)BRST transformations (21), (19), (16), (11), and (6) are the symmetries of the action integrals $S = \int d^2\xi\, L_B$, $S = \int d^2\xi\, L_{\bar{B}}$, $S = \int d^2\xi\, L_B^{(0)}$, and $S = \int d^2\xi\, L_{\bar{B}}^{(0)}$ provided we use the proper boundary conditions on the fields (and their derivatives) of the theory at $\sigma = 0$ and $\sigma = \pi$ [6].

5. Conserved Charges: Continuous Symmetries

The BRST charge Q_B that has been computed in [6] is in the flat limit ($A_0 = A_1 = 0, A_2 = 1$) where the Lagrangian density $L_B^{(0)}$ (compare Equation (13)) plays a pivotal role. First of all, we note that the Lagrangian densities $L_B^{(0)}$ and $L_{\bar{B}}^{(0)}$ (compare Equations (13) and (26)) respect the global ghost-scale symmetry transformations

$$C_0 \longrightarrow e^{\Omega} C_0, \bar{C}_0 \longrightarrow e^{-\Omega} \bar{C}_0, C_1 \longrightarrow e^{\Omega} C_1, \bar{C}_1 \longrightarrow e^{-\Omega} \bar{C}_1, \tag{31}$$

where Ω is a global scale transformation parameter. For the sake of brevity, we set $\Omega = 1$ so that the infinitesimal version (s_g) of the above global scale symmetry transformations reduce to the following transformations on the (anti)ghost fields, namely

$$s_g C_0 = C_0,\ s_g \bar{C}_0 = -\bar{C}_0, s_g C_1 = C_1, s_g \bar{C}_1 = -\bar{C}_1. \tag{32}$$

Here, the subscript g denotes the infinitesimal ghost-scale transformations (where $\Omega = 1$). The ghost charges, computed from $L_B^{(0)}$ and $L_{\bar{B}}^{(0)}$, are as follows:

$$Q_g = \int_0^\pi d\sigma\, J_g^{(0)} \equiv -i \int_0^\pi d\sigma \left(\bar{C}_0 C_1 - \bar{C}_1 C_0\right), \tag{33}$$

$$\bar{Q}_g = \int_0^\pi d\sigma\, \bar{J}_g^{(0)} \equiv -i \int_0^\pi d\sigma \left(\bar{C}_1 C_0 - \bar{C}_0 C_1\right), \tag{34}$$

where $J_g^{(0)}$ and $\bar{J}_g^{(0)}$ are the zeroth component of the Noether conserved currents (corresponding to the infinitesimal ghost transformations (32)) that have been derived from $L_B^{(0)}$ and $L_{\bar{B}}^{(0)}$, respectively. However, the above charges are not independent of each other. Rather, they differ by a sign factor only (i.e., $Q_g = -\bar{Q}_g$). Using the following Euler-Lagrange equations of motion that emerge out from $L_B^{(0)}$, namely

$$\blacksquare X^\mu = 0, A_0 = A_1 = B_0 = B_1 = 0, A_2 = 1, E = 0, \partial_0 \bar{C}^0 + \partial_1 \bar{C}^1 = 0, \tag{35}$$

$$\partial_0 C^0 - \partial_1 C^1 = 0, \partial_0 C^1 - \partial_1 C^0 = 0, \partial_0 \bar{C}^1 + \partial_1 \bar{C}^0 = 0, \tag{36}$$

we observe that $\dot{Q}_g = i \int_0^\pi (\partial/\partial\sigma) [\bar{C}_1 C_0 - \bar{C}_0 C_1] = 0$ due to the boundary conditions. This shows that the ghost charge is conserved (i.e., $\dot{Q}_g = 0$).

We now concentrate on the derivation of the BRST charge Q_B and anti-BRST charge $\bar{Q}_{\bar{B}}$ from the Lagrangian densities $L_B^{(0)}$ and $L_{\bar{B}}^{(0)}$, respectively. Taking into account the basic concepts behind the Noether theorem, we note that $Q_B = \int_0^\pi d\sigma\, J_B^{(0)}, \bar{Q}_B = \int_0^\pi d\sigma\, \bar{J}_B^{(0)}$ where $J_B^{(0)}$ and $\bar{J}_B^{(0)}$ are the zeroth components of the Noether conserved currents (corresponding to the BRST and anti-BRST symmetry transformations) computed from the Lagrangian densities $L_B^{(0)}$ and $L_{\bar{B}}^{(0)}$, respectively. The explicit expressions for these currents, derived from the above Lagrangian densities, are

$$\begin{aligned} J_B^{(0)} &= \left(s_B X^\mu\right) \frac{\partial L_B^{(0)}}{\partial(\partial_0 X^\mu)} + \left(s_B C^0\right) \frac{\partial L_B^{(0)}}{\partial\left(\partial_0 C^0\right)} + \left(s_B C^1\right) \frac{\partial L_B^{(0)}}{\partial\left(\partial_0 C^1\right)} \\ &\quad + \left(s_B \bar{C}_0\right) \frac{\partial L_B^{(0)}}{\partial\left(\partial_0 \bar{C}_0\right)} + \left(s_B \bar{C}_1\right) \frac{\partial L_B^{(0)}}{\partial\left(\partial_0 \bar{C}_1\right)} - X^0, \\ \bar{J}_B^{(0)} &= \left(\bar{s}_B X^\mu\right) \frac{\partial L_{\bar{B}}^{(0)}}{\partial(\partial_0 X^\mu)} + \left(\bar{s}_B C^0\right) \frac{\partial L_{\bar{B}}^{(0)}}{\partial\left(\partial_0 C^0\right)} + \left(\bar{s}_B C^1\right) \frac{\partial L_{\bar{B}}^{(0)}}{\partial\left(\partial_0 C^1\right)} \\ &\quad + \left(\bar{s}_B \bar{C}_0\right) \frac{\partial L_{\bar{B}}^{(0)}}{\partial\left(\partial_0 \bar{C}_0\right)} + \left(\bar{s}_B \bar{C}_1\right) \frac{\partial L_{\bar{B}}^{(0)}}{\partial\left(\partial_0 \bar{C}_1\right)} - Y^0, \end{aligned} \tag{37}$$

where the explicit expressions for X^0 and Y^0 are

$$
\begin{aligned}
X^{(0)} &= C^0 L_0 + i\,\bar{C}_0\, C^0 \left(\partial_0 C^1 - \partial_1 C^0\right) + i\,\bar{C}_1\, C^0 \left(\partial_0 C^0 - \partial_1 C^1\right),\\
Y^{(0)} &= \bar{C}^0 L_0 - i\, C_0\, \bar{C}^0 \left(\partial_0 \bar{C}^1 - \partial_1 \bar{C}^0\right) - i\, C_1\, \bar{C}^0 \left(\partial_0 \bar{C}^0 - \partial_1 \bar{C}^1\right).
\end{aligned}
\tag{38}
$$

The above expressions are derived from Equations (29) and (30) which are nothing but the zeroth components of the expressions that have been quoted in the square brackets. Finally, we obtain the following expressions for the conserved BRST and anti-BRST charges (Q_B and $\bar{Q}_B$) from the Lagrangian densities $L_B^{(0)}$ and $L_{\bar{B}}^{(0)}$, namely

$$
\begin{aligned}
Q_B = -\int_0^\pi d\sigma \Bigg[&\frac{C^0}{2\kappa} \left(\partial_0 X^\mu\, \partial_0 X_\mu + \partial_1 X^\mu\, \partial_1 X_\mu\right)\\
&+ \frac{C^1}{2\kappa} \left(\partial_0 X^\mu\, \partial_1 X_\mu + \partial_1 X^\mu\, \partial_0 X_\mu\right)\\
&+ i\,\bar{C}_0 \left(C^a\, \partial_a C^1\right) + i\,\bar{C}_1 \left(C^a\, \partial_a C^0\right) \Bigg],
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\bar{Q}_B = -\int_0^\pi d\sigma \Bigg[&\frac{\bar{C}^0}{2\kappa} \left(\partial_0 X^\mu\, \partial_0 X_\mu + \partial_1 X^\mu\, \partial_1 X_\mu\right)\\
&+ \frac{\bar{C}^1}{2\kappa} \left(\partial_0 X^\mu\, \partial_1 X_\mu + \partial_1 X^\mu\, \partial_0 X_\mu\right)\\
&+ i\, C_0 \left(\bar{C}^a \partial_a \bar{C}^1\right) + i\, C_1 \left(\bar{C}^a \partial_a \bar{C}^0\right) \Bigg],
\end{aligned}
\tag{40}
$$

where we have used the Euler-Lagrange (EL) equations of motion (EoM) (36) derived from the Lagrangian density $L_B^{(0)}$ and the following EL-EoM that emerge from the Lagrangian density $L_{\bar{B}}^{(0)}$, namely

$$
\Box X^\mu = 0,\, A_0 = A_1 = A_2 - 1 = \bar{B}_0 = \bar{B}_1 = E = 0,\, \partial_0 \bar{C}^0 - \partial_1 \bar{C}^1 = 0,
\tag{41}
$$

$$
\partial_0 \bar{C}^1 - \partial_1 \bar{C}^0 = 0,\, \partial_0 C^1 + \partial_1 \bar{C}^0 = 0,\, \partial_0 \bar{C}^0 + \partial_1 \bar{C}^1 = 0.
\tag{42}
$$

In fact, a close and careful look at the EL-EoM (36) and (42) establishes the fact that $X^0 = Y^0 = 0$ on the on-shell (because we substitute the EL-EoM into them).

The above charges Q_B and $\bar{Q}_B$ are conserved. This can be checked by exploiting the strength of the EL-EoM (36) and (42) while we take into account the direct "time" derivative of the above charges, namely

$$
\begin{aligned}
\dot{Q}_B = -\int_0^\pi d\sigma\, \frac{\partial}{\partial\sigma} \Bigg[&\frac{C^0}{2\kappa} \left(\partial_0 X^\mu\, \partial_0 X_\mu + \partial_1 X^\mu\, \partial_1 X_\mu\right)\\
&+ \frac{C^1}{2\kappa} \left(\partial_0 X^\mu\, \partial_1 X_\mu + \partial_1 X^\mu\, \partial_0 X_\mu\right)\\
&+ i\,\bar{C}_0 \left(C^a\, \partial_a C^0\right) + i\,\bar{C}_1 \left(C^a\, \partial_a C^1\right) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\dot{\bar{Q}}_B = -\int_0^\pi d\sigma\, \frac{\partial}{\partial\sigma} \Bigg[&\frac{\bar{C}^0}{2\kappa} \left(\partial_0 X^\mu\, \partial_0 X_\mu + \partial_1 X^\mu\, \partial_1 X_\mu\right)\\
&+ \frac{\bar{C}^1}{2\kappa} \left(\partial_0 X^\mu\, \partial_1 X_\mu + \partial_1 X^\mu\, \partial_0 X_\mu\right)\\
&+ i\, C_0 \left(\bar{C}^a\, \partial_a \bar{C}^0\right) + i\, C_1 \left(\bar{C}^a\, \partial_a \bar{C}^1\right) \Bigg].
\end{aligned}
\tag{43}
$$

The above expressions demonstrate that the BRST and anti-BRST charges are conserved when we use the boundary conditions at $\sigma = 0$ and $\sigma = \pi$ on the appropriate fields and their derivatives (see, e.g., [6] for details). Thus, we have noted that there are three conserved charges (which correspond to three continuous symmetries that are present) in the theory. One can check, in a straightforward manner, that the ghost charge obeys the standard algebra with the BRST and anti-BRST charges. This can be checked in a simple manner by computing the left hand side of the following from (32), (34), (39), and (40), namely

$$
\begin{aligned}
s_g Q_g &= -i\left[Q_g, Q_g\right] = 0,\, s_g Q_B = -i\left[Q_B, Q_g\right]\\
&= Q_B,\, s_g \bar{Q}_B = -i\left[Q_g, \bar{Q}_B\right] = -\bar{Q}_B,
\end{aligned}
\tag{44}
$$

which demonstrates that we have $i\left[Q_g, Q_B\right] = +Q_B$ and $i\left[Q_g, \bar{Q}_B\right] = -\bar{Q}_B$. However, the proof of nilpotency of the BRST and anti-BRST charges requires very careful computations at the quantum level where the normal mode expansions of the fields of our theory play very important roles. In the paper by Kato and Ogawa [6], this exercise has been performed, and it turns out that the nilpotency of the BRST charge is true only when $D = 26$ and $\alpha_0 = 1$. It is obvious that we shall get the same result if we check the nilpotency of the anti-BRST charge at the quantum level with the proper boundary conditions.

6. Conclusions

In our present investigation, we have been able to derive the proper anti-BRST symmetry transformations corresponding to the BRST transformations (that have been shown to be present for the model of 2D diffeomorphism invariant bosonic string theory [6]). The BRST and anti-BRST symmetry transformations are proved to be off-shell nilpotent of order two. However, these symmetries are found to be absolutely anticommuting only on a submanifold of the Hilbert space of quantum fields that is characterized by the 2D field Equation (20). These latter equations are nothing but the CF-type restrictions which are the hallmark of the quantum diffeomorphism/gauge-invariant theories when these theories are discussed within the framework of BRST formalism. In fact, it is the existence of the CF-type restrictions that primarily imply that the BRST and anti-BRST symmetries (and the corresponding conserved charges) have their own identities. In the language of mathematics, they are linearly independent of each other (on a submanifold of the quantum Hilbert space of fields) that is defined by the CF-type restrictions (20).

We have derived, in our present endeavor, the explicit and separate forms of BRST and anti-BRST invariant Lagrangian densities, and we have demonstrated clearly their transformation properties under the BRST and anti-BRST symmetry transformations. Using the Noether theorem, we have computed the conserved BRST, anti-BRST, and ghost charges of the theory in the flat limit. In fact, in the latter limit, the BRST charge has also been derived by Kato and Ogawa [6]. We have shown that the standard algebra is obeyed between the ghost charge and BRST charge (as well as the ghost charge and anti-BRST charge). The nilpotency ($Q_B^2 = 0, \bar{Q}_B^2 = 0$) of the BRST ($Q_B$) and anti-BRST ($\bar{Q}_B$) charges has not been derived in our present investigation as this requires the normal mode expansion of the fields and their substitution in the expressions for Q_B and $\bar{Q}_B$. In fact, the requirement of the nilpotency of the BRST charge has led to the derivation of $D = 26$ and $\alpha_0 = 1$ where D is the dimensionality of the target spacetime manifold and α_0 is the intercept in the Regge trajectory that is generated due to the concept of strings (see, e.g., [6] for details).

We would like to comment a bit on the boundary conditions that are to be imposed on the fields (and the derivatives on them) in our present theory when we demand the BRST as well as anti-BRST invariance of the Lagrangian densities (12) and (25). For the BRST invariance of the theory, the boundary conditions that have been obtained in the work by Kato and Ogawa [6] are $\partial_1 X^\mu = 0, \bar{C}_0 = 0, C^1 = 0$ at $\sigma = 0$ and $\sigma = \pi$. The BRST invariance of the boundary condition $C^1 = 0$ (at $\sigma = 0$ and $\sigma = \pi$) leads to the further boundary condition as $\partial_0 C^1 = 0$ at $\sigma = 0$ and $\sigma = \pi$. The anti-BRST invariance, in exactly similar manner, would lead to the boundary conditions $\partial_1 X^\mu = 0, C^0 = 0, \bar{C}_1 = 0$ at $\sigma = 0$ and $\sigma = \pi$. The anti-BRST invariance of the condition $\bar{C}_1 = 0$ at $\sigma = 0$ and $\sigma = \pi$ implies that $\partial_0 \bar{C}_1 = 0$ (at $\sigma = 0$ and $\sigma = \pi$). Thus, the normal mode expansions of the fields $X^\mu(\tau, \sigma), C^0(\tau, \sigma), C^1(\tau, \sigma), \bar{C}_0(\tau, \sigma), \bar{C}_1(\tau, \sigma)$ can be found in the same manner as has been obtained in the work by Kato and Ogawa [6]. We have to be just careful that for the anti-BRST invariance; the mode expansions in the ghost sector should be such that the expansions are exchanged, namely $C_a \longleftrightarrow \bar{C}_a$. The requirements of the nilpotency of Q_B and $\bar{Q}_B$ would obviously produce the results $D = 26$ and $\alpha_0 = 1$.

We would like to mention that the BRST and anti-BRST invariant Lagrangian densities (12) and (25) have been derived in a straightforward manner by utilizing the gauge-fixing $A_0 = A_1 = 0$ and the (anti-)ghost fields (compare Equations (5) and (24)). However, if we compute the Lagrangian densities in the Curci-Ferrari gauge [12, 13], that would give due respect to the CF-type conditions that have been derived in (20). We wish to devote time on the computation of the coupled Lagrangian densities (like the 4D non-Abelian gauge theory [10–13]) which produce the CF-type condition as the equations of motion. Furthermore, the coupled Lagrangian densities should respect both the BRST and anti-BRST symmetry transformations on the submanifold of the quantum Hilbert space of fields where the specific quantum fields obey the CF-type restrictions (20). At present, we are working in this direction, and our results would be reported elsewhere.

In a very recent work [7], the superfield approach to derive the proper (anti-)BRST symmetry transformations for any general D-dimensional diffeomorphism invariant theory has been developed (corresponding to its diffeomorphism symmetry invariance). It would be a very nice future endeavor for us to apply the theoretical arsenals of this superfield formalism [7] to our present bosonic string model which is also a diffeomorphism invariant theory. In fact, we hope that this superfield formalism would be able to shed more light on the geometrical origin and interpretation of the (anti-)BRST symmetries and the CF-type restrictions (20) which we have obtained for our present theory. In our earlier works on the Abelian 2-form and 3-form gauge theories [14, 15], we have established the geometrical origins and interpretations for the CF-type restrictions and their intimate connections with the geometrical objects called gerbes. It would be a challenging future endeavor for us to establish the same type of connections for our present 2D diffeomorphism invariant theory where the nontrivial CF-type restrictions exist. We are presently involved with this problem, and we plan to report about our progress in our future publication(s) [16].

It is gratifying to state that we have already exploited the beauty and strength of MBTSA in the cases of 1D diffeomorphism (i.e., reparametrization) invariant interesting models of nonrelativistic free particle [17], scalar relativistic particle [18], and spinning (i.e., SUSY) relativistic particle [19] and established the universality of the 1D CF-type restriction $B + \bar{B} + i(\bar{C}\dot{C} - \dot{\bar{C}}C) = 0$ in all the above non-SUSY and SUSY systems of interest and obtained the proper (anti-)BRST symmetry transformations corresponding to the classical 1D diffeomorphism (i.e., reparameterization) symmetry transformation. Here the (anti)ghost variables $(\bar{C})C$ are the generalization of the infinitesimal reparametrization symmetry transformation parameter $\varepsilon(\tau)$ in the transformation $\tau \longrightarrow \tau - \varepsilon(\tau)$ where τ is an evolution parameter (see, e.g., [17–19] for details). In our present 2D diffeomorphism invariant bosonic string theory, we have found the CF-type restrictions as $B^a + \bar{B}^a + i(\bar{C}^b \partial_b C^a + C^b \partial_b \bar{C}^a) = 0$ (with $a, b, = 0, 1$) which are, once again, the limiting case of the MBTSA to D-dimensional diffeomorphism invariant theory where it has been shown [7, 20] that the CF-type restrictions $B^\mu + \bar{B}^\mu + i(\bar{C}^\rho \partial_\rho C^\mu + C^\rho \partial_\rho \bar{C}^\mu) = 0$ (with $\mu, \nu, \rho \cdots = 0, 1, 2, \cdots D - 1$) are universal for the SUSY and non-SUSY systems in any arbitrary dimension of space-time where the (anti)ghost fields $(\bar{C}_\mu)C_\mu$ are the quantum generalizations of the infinitesimal D-dimensional diffeomorphism transformation parameters $\varepsilon_\mu(x)$ in the transformations $x_\mu \longrightarrow x_\mu - \varepsilon_\mu(x)$. In the above discussions, all the $(\bar{B})B$ fields, with appropriate index, are the Nakanishi-Lautrup-type auxiliary fields.

Appendix

A.1. Appendix A: On the Nilpotency Property $s_B^2 \tilde{g}^{ab} = 0$

We briefly sketch here a few essential steps that are needed in the proof of $s_B^2 \tilde{g}^{ab} = 0$. In this connection, we observe the following:

$$\begin{aligned} s_B^2 \tilde{g}^{ab} &= s_B \left[(\partial_m C^m) \tilde{g}^{ab} \right] + s_B \left[C^m \partial_m \tilde{g}^{ab} \right] \\ &\quad - s_B \left[(\partial_m C^a) \tilde{g}^{mb} \right] - s_B \left[\left(\partial_m C^b \right) \tilde{g}^{am} \right]. \end{aligned} \tag{45}$$

The first term, after the application of the BRST transformations, looks in its full glory as

$$\begin{aligned} &(\partial_m C^n)(\partial_n C^m) \tilde{g}^{ab} + C^m (\partial_m \partial_n C^n) \tilde{g}^{ab} \\ &\quad - (\partial_m C^m)(\partial_n C^n) \tilde{g}^{ab} - (\partial_m C^m) C^n \left(\partial_n \tilde{g}^{ab} \right) \\ &\quad + (\partial_m C^m)(\partial_n C^a) \tilde{g}^{nb} + (\partial_m C^m) \left(\partial_n C^b \right) \tilde{g}^{\mathrm{an}}. \end{aligned} \tag{46}$$

In exactly similar fashion, the second term turns out to be

$$\begin{aligned} &C^n (\partial_n C^m) \left(\partial_m \tilde{g}^{ab} \right) - C^m (\partial_m \partial_n C^n) \tilde{g}^{ab} - C^m (\partial_n C^n) \left(\partial_m \tilde{g}^{ab} \right) \\ &\quad - C^m (\partial_m C^n) \left(\partial_n \tilde{g}^{ab} \right) + C^n (\partial_n C^a) \left(\partial_m \tilde{g}^{mb} \right) \\ &\quad + C^n \left(\partial_n C^b \right) (\partial_m \tilde{g}^{\mathrm{an}}) + C^m (\partial_m \partial_n C^a) \tilde{g}^{mb}, \end{aligned} \tag{47}$$

where we have taken into account the fact that $C^m C^n (\partial_m \partial_n \tilde{g}^{ab}) = 0$. The third term, after the application of the BRST transformations (6), (11), and (21), looks in the following exact mathematical form

$$\begin{aligned} &-(\partial_m C^n)(\partial_n C^a) \tilde{g}^{mb} - C^n (\partial_m \partial_n C^a) \tilde{g}^{mb} \\ &\quad + (\partial_m C^a)(\partial_n C^n) \tilde{g}^{mb} + (\partial_m C^a) C^n \left(\partial_n \tilde{g}^{mb} \right) \\ &\quad - (\partial_m C^a)(\partial_n C^m) \tilde{g}^{nb} - (\partial_m C^a) \left(\partial_n C^b \right) \tilde{g}^{mn}. \end{aligned} \tag{48}$$

Finally, the fourth term can be explicitly expressed, after the application of BRST transformations (6), (11), and (21), as

$$\begin{aligned} &-(\partial_m C^n) \left(\partial_n C^b \right) \tilde{g}^{am} - C^n \left(\partial_m \partial_n C^b \right) \tilde{g}^{am} \\ &\quad + \left(\partial_m C^b \right) (\partial_n C^n) \tilde{g}^{am} + \left(\partial_m C^b \right) C^n (\partial_n \tilde{g}^{am}) \\ &\quad - \left(\partial_m C^b \right) (\partial_n C^m) \tilde{g}^{\mathrm{an}} - \left(\partial_m C^b \right) (\partial_n C^a) \tilde{g}^{mn}. \end{aligned} \tag{49}$$

It is evident that the following terms from (46), (47), (48), and (49), namely

$$\begin{aligned} &C^n (\partial_m \partial_n C^m) \tilde{g}^{ab} - C^m (\partial_m \partial_n C^n) \tilde{g}^{ab} \\ &\quad + C^m (\partial_m \partial_n C^a) \tilde{g}^{mb} C^m \left(\partial_m \partial_n C^b \right) \tilde{g}^{\mathrm{an}} \\ &\quad - C^n (\partial_m \partial_n C^a) \tilde{g}^{mb} - C^m \left(\partial_m \partial_n C^b \right) \tilde{g}^{\mathrm{an}}, \end{aligned} \tag{50}$$

cancel out with one another. Furthermore, the following terms from (48) and (49)

$$-(\partial_m C^a) \left(\partial_n C^b \right) \tilde{g}^{mn} - \left(\partial_m C^b \right) (\partial_n C^a) \tilde{g}^{mn}, \tag{51}$$

cancel out with each other because of the antisymmetric nature ($C^a C^b + C^b C^a = 0$) of the ghost fields (C^a) and the symmetric nature ($\tilde{g}^{mn} = \tilde{g}^{nm}$) of the metric tensor $\tilde{g}^{mn}$. The rest of the terms also cancel out by taking the help of the exchange of dummy indices $m \longleftrightarrow n$ and the anticommuting nature of the ghost fields. Finally, we find that the following terms, from the sum of (46), (47), (48), and (49), remain left out at the end, namely

$$[(\partial_n C^m)(\partial_m C^n) - (\partial_m C^m)(\partial_n C^n)] \tilde{g}^{ab}. \tag{52}$$

The terms in the square bracket turn out to be individually equal to zero when we take the sum over $m, n = 0, 1$. This establishes the nilpotency ($s_B^2 = 0$) of s_B when it acts on $\tilde{g}^{ab}$.

A.2. Appendix B: On the BRST Symmetry Invariance of L_B

We collect here all the terms that are generated due to the application of BRST symmetry transformations (s_B) on L_B (compare Equation (12)). It is straightforward to note that $s_B L_0 = \partial_a (C^a L_0)$. We assemble, first of all, the terms that contain B_0 and B_1 fields due to the application of s_B on all the terms that are present in L_B. These terms with B_0 field are

$$\begin{aligned} &C^a (\partial_a B_0) A_0 + B_0 C^a (\partial_a A_0) - B_0 \left(\partial_0 C^1 + \partial_1 C^0 \right) A_1 \\ &\quad - B_0 \left(\partial_0 C^1 - \partial_1 C^0 \right) A_2 - B_0 \left(\partial_1 C^0 \right) A_2 + B_0 \left(\partial_0 C^1 \right) A_2 \\ &\quad + B_0 \left(\partial_1 C^0 \right) A_1 + B_0 \left(\partial_0 C^1 \right) A_1 + B_0 (\partial_a C^a) A_0. \end{aligned} \tag{53}$$

Similarly, the terms containing B_1 fields are as follows:

$$\begin{aligned} &B_1 C^a (\partial_a A_1) - B_1 (\partial_a C^a) A_2 + B_1 \left(\partial_1 C^1 \right) A_2 - B_1 \left(\partial_1 C^0 \right) A_0 \\ &\quad + B_1 (\partial_a C^a) A_1 - B_1 \left(\partial_0 C^1 \right) A_0 - B_1 \left(\partial_1 C^1 \right) A_2 \\ &\quad + B_1 \left(\partial_0 C^0 \right) A_2 + B_1 \left(\partial_0 C^1 \right) A_0 + B_1 \left(\partial_1 C^0 \right) A_0 \\ &\quad + C^a (\partial_a B_1) A_1. \end{aligned} \tag{54}$$

It is clear that if we sum these terms (i.e., (53) and (54)) carefully with $s_B L_0 = \partial_a (C^a L_0)$, they lead to the sum of the following total derivative:

$$\partial_a \left[C^a \left(L_0 + B_0 A_0 + B_1 A_1\right)\right]. \tag{55}$$

Thus far, we have obtained the total derivative from the original Lagrangian density (1) and terms that contain necessarily the Nakanishi-Lautrup auxiliary fields B_0 and B_1. It is straightforward to note that the original Lagrangian density (1) respects the classical 2D diffeomorphism symmetry transformations that have been pointed out in Section 2. Thus, it is crystal clear that it will respect the BRST symmetry transformation where the classical diffeomorphism transformation parameter ε^a is replaced by the quantum ghost field C^a. This is why we have $s_B L_0 = \partial_a [C^a L_0]$ which is explicitly present in Equation (55).

We now collect the terms that incorporate A_2 after the application of s_B on L_B (compare Equation (12)). These useful terms are as follows:

$$\begin{aligned}
& i\bar{C}_1 \left(\partial_1 C^1\right) C^a \left(\partial_a A_2\right) - i C^a \left(\partial_a \bar{C}_0\right)\left(\partial_0 C^1 - \partial_1 C^0\right) A_2 \\
&\quad - i\bar{C}_1 \left(\partial_0 C^0\right) C^a \partial_a A_2 + i\bar{C}_0 \left(\partial_1 C^0\right) C^a \left(\partial_a A_2\right) \\
&\quad - i\bar{C}_o \left(\partial_0 C^1\right) C^a \left(\partial_a A_2\right) + i\bar{C}_1 \left(\partial_a C^a\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_2 \\
&\quad - i C^a \left(\partial_a \bar{C}_1\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_2 + i\bar{C}_0 \left(\partial_a C^a\right)\left(\partial_0 C^1 - \partial_1 C^0\right) A_2 \\
&\quad - i\bar{C}_1 C^a \left(\partial_a \partial_1 C^1\right) A_2 + i\bar{C}_1 \left(\partial_0 C^1\right)\left(\partial_0 C^1 - \partial_1 C^0\right) A_2 \\
&\quad + i\bar{C}_1 \left(\partial_1 C^0\right)\left(\partial_0 C^1 - \partial_1 C^0\right) A_2 - i\bar{C}_1 \left(\partial_1 C^a\right)\left(\partial_a C^1\right) A_2 \\
&\quad - i\bar{C}_1 \left(\partial_0 C^a\right)\left(\partial_a C^0\right) A_2 + i\bar{C}_1 C^a \left(\partial_a \partial_0 C^0\right) A_2 \\
&\quad + i\bar{C}_0 \left(\partial_0 C^1\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_2 - i\bar{C}_0 \left(\partial_1 C^a\right)\left(\partial_a C^0\right) A_2 \\
&\quad - i\bar{C}_0 C^a \left(\partial_a \partial_1 C^0\right) A_2 + i\bar{C}_0 \left(\partial_0 C^a\right)\left(\partial_a C^1\right) A_2 \\
&\quad + i\bar{C}_0 C^a \left(\partial_a \partial_0 C^1\right) A_2 + i\bar{C}_0 \left(\partial_1 C^0\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_2.
\end{aligned} \tag{56}$$

It is very interesting to note that all these terms, after many surprising cancellations, sum up to yield a total derivative as

$$\partial_a \left[i\bar{C}_0 C^a \left(\partial_0 C^1 - \partial_1 C^0\right) A_2 + i\bar{C}_1 C^a \left(\partial_0 C^0 - \partial_1 C^1\right) A_2\right]. \tag{57}$$

We now concentrate on all the terms that contain A_0 which emerge from the application of s_B on the relevant terms of the Lagrangian density L_B. These are as follows

$$\begin{aligned}
& i\bar{C}_1 \left(\partial_0 C^0\right)\left(\partial_1 C^0 - \partial_0 C^1\right) A_0 - i\bar{C}_1 \left(\partial_1 C^1\right)\left(\partial_1 C^0 - \partial_0 C^1\right) A_0 \\
&\quad - i\bar{C}_0 \left(\partial_1 C^0\right)\left(\partial_1 C^0 - \partial_0 C^1\right) A_0 + i\bar{C}_0 \left(\partial_0 C^1\right)\left(\partial_1 C^0 - \partial_0 C^1\right) A_0 \\
&\quad + i\bar{C}_0 \left(\partial_1 C^0\right)\left(\partial_1 C^0 + \partial_0 C^1\right) A_0 + i\bar{C}_0 \left(\partial_0 C^1\right)\left(\partial_1 C^0 + \partial_0 C^1\right) A_0 \\
&\quad + i\bar{C}_1 C^a \left(\partial_1 \partial_a C^0\right) A_0 - i\bar{C}_1 \left(\partial_1 C^0\right) C^a \partial_a A_0 \\
&\quad + i\bar{C}_1 \left(\partial_0 C^a\right)\left(\partial_a C^1\right) A_0 + i\bar{C}_1 C^a \left(\partial_0 \partial_a C^1\right) A_0 \\
&\quad - i\bar{C}_1 \left(\partial_0 C^1\right) C^a \partial_a A_0 + i\bar{C}_0 \left(\partial_a C^b\right)\left(\partial_b C^a\right) A_0 \\
&\quad + i\bar{C}_0 C^b \left(\partial_b \partial_a C^a\right) A_0 - i\bar{C}_0 \left(\partial_a C^a\right) C^b \partial_b A_0 \\
&\quad + i\bar{C}_b \left(\partial_b C^a\right)\left(\partial_a \bar{C}_0\right) A_0 + i C^a \left(\partial_a \bar{C}_0\right) C^b \partial_b A_0 \\
&\quad + i\bar{C}_1 \left(\partial_1 C^a\right)\left(\partial_a C^0\right) A_0.
\end{aligned} \tag{58}$$

It is amazing to find out that the sum of the above terms, after some miraculous cancellations, yields a total derivative as

$$\partial_a \left[i\bar{C}_1 C^a \left(\partial_0 C^1 + \partial_1 C^0\right) A_0 + i\bar{C}_0 C^b \partial_b (C^a A_0)\right]. \tag{59}$$

Now, at the far end, we have only one option left out. As a consequence, ultimately, we focus on the terms that necessarily incorporate A_1 field after the application of the BRST transformation s_B on the Lagrangian density (12). These terms are

$$\begin{aligned}
& i\bar{C}_0 \left(\partial_a C^a\right)\left(\partial_0 C^1 + \partial_1 C^0\right) A_1 + i\bar{C}_1 C^b \partial_b \partial_a C^a A_1 \\
&\quad - i\bar{C}_1 \left(\partial_a C^a\right) C^b \partial_b A_1 + i C^b \partial_b C^a \left(\partial_a \bar{C}_1\right) A_1 \\
&\quad + i C^b \left(\partial_b \bar{C}_1\right) C^a \partial_a A_1 - i\bar{C}_0 \left(\partial_0 C^1\right) C^a \partial_a A_1 \\
&\quad + i\bar{C}_1 \left(\partial_a C^b\right) C^a \partial_a A_1 - i\bar{C}_0 \left(\partial_1 C^0\right) C^a \partial_a A_1 \\
&\quad + i\bar{C}_0 C^a \left(\partial_0 \partial_a C^1\right) A_1 - i\bar{C}_1 \left(\partial_1 C^1\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_1 \\
&\quad + i\bar{C}_1 \left(\partial_0 C^0\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_1 \\
&\quad - i\bar{C}_0 \left(\partial_1 C^0\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_1 \\
&\quad + i\bar{C}_0 \left(\partial_0 C^1\right)\left(\partial_0 C^0 - \partial_1 C^1\right) A_1 - i\bar{C}_0 \left(\partial_1 C^a\right)\left(\partial_a C^0\right) A_1 \\
&\quad + i\bar{C}_0 C^a \left(\partial_1 \partial_a C^0\right) A_1 + i\bar{C}_0 \left(\partial_0 C^a\right)\left(\partial_a C^1\right) A_1 \\
&\quad + i\bar{C}_1 \left(\partial_0 C^1\right)\left(\partial_0 C^1 + \partial_1 C^0\right) A_1 \\
&\quad + i\bar{C}_1 \left(\partial_1 C^0\right)\left(\partial_0 C^1 + \partial_1 C^0\right) A_1 \\
&\quad - i C^a \left(\partial_a \bar{C}_0\right)\left(\partial_0 C^1 + \partial_1 C^0\right) A_1.
\end{aligned} \tag{60}$$

The above terms add up to yield a total derivative term as

$$\partial_a \left[i\bar{C}_0 C^a \left(\partial_0 C^1 + \partial_1 C^0\right) A_1 + i\bar{C}_1 C^b \partial_b (C^a A_1)\right]. \tag{61}$$

It is interesting to point out that the terms with A_0 and that of A_1 sum up to yield exactly similar types of result in the total derivative where $A_0 \longleftrightarrow A_1, \bar{C}_0 \longleftrightarrow \bar{C}_1$. It is clear that the application of s_B on L_B produces the total derivative term which is the sum of (55), (57), (59), and (61). Thus, the BRST transformations s_B are a symmetry of the action.

A.3. Appendix C: On the Anti-BRST Symmetry Invariance of $L_{\bar{B}}$

Here, we collect the terms that are generated after the application of the anti-BRST symmetry transformations $\bar{s}_B$ on $L_{\bar{B}}$ (compare Equation (25)). It can be readily checked that $\bar{s}_B L_0 = \partial_a (\bar{C}^a L_0)$. In addition to it, we have the following terms that contain the auxiliary field $\bar{B}_0$ after the application of $\bar{s}_B$ on $L_{\bar{B}}$, namely

$$-\bar{C}^a\,(\partial_a\,\bar{B}_0)\,A_0 - \bar{B}_0\,\bar{C}^a\,(\partial_a\,A_0) + \bar{B}_0\left(\partial_0\,\bar{C}^1 + \partial_1 C^0\right)A_1 + \bar{B}_0\left(\partial_0\,\bar{C}^1 - \partial_1\,\bar{C}^0\right)A_2 + \bar{B}_0\left(\partial_1\,\bar{C}^0\right)A_2 - \bar{B}_0\left(\partial_0\,\bar{C}^1\right)A_2 - \bar{B}_0\left(\partial_1\,\bar{C}^0\right)A_1 - \bar{B}_0\left(\partial_0\,\bar{C}^1\right)A_1 - \bar{B}_0\,(\partial_a\,\bar{C}^a)\,A_0, \tag{62}$$

which add up to yield $\partial_a\,[-\bar{C}^a\,\bar{B}_0\,A_0)]$. Similarly, the following terms containing $\bar{B}_1$ fields (that are generated after the application of $\bar{s}_B$ on $L_{\bar{B}}$), namely

$$-\bar{B}_1\,\bar{C}^a\,(\partial_a A_1) + \bar{B}_1\left(\partial_0\,\bar{C}^0\right)A_2 - \bar{B}_1\left(\partial_1\,\bar{C}^1\right)A_2 + \bar{B}_1\left(\partial_1\,\bar{C}^0\right)A_0 - \bar{B}_1\,(\partial_a\,\bar{C}^a)\,A_1 + \bar{B}_1\left(\partial_0\,\bar{C}^1\right)A_0 + \bar{B}_1\left(\partial_1\,\bar{C}^1\right)A_2 - \bar{B}_1\left(\partial_0\,\bar{C}^0\right)A_2 - \bar{B}_1\left(\partial_0\,\bar{C}^1\right)A_0 - \bar{B}_1\left(\partial_1\,\bar{C}^0\right)A_0 - \bar{C}^a\,(\partial_a\,\bar{B}_1)\,A_1, \tag{63}$$

sum up to produce $\partial_a\,[-\bar{C}^a\,\bar{B}_1\,A_0]$. Thus, it is clear that we have so far the following total derivatives: $\partial_a\,[\bar{C}^a\,(L_0 - \bar{B}_0\,A_0 - \bar{B}_1\,A_1)]$. Now, we focus on the collection of A_0 terms that are generated after the application of anti-BRST transformations $\bar{s}_B$ on $L_{\bar{B}}$. These are

$$+i\,\partial_a\,(C_0\,\bar{C}^a)\left(\bar{C}^b\,\partial_b\,A_0\right) + i\,C_1\left(\partial_1\,\bar{C}^0 + \partial_0\,\bar{C}^1\right)\left(\bar{C}^b\,\partial_b\,A_0\right) - \partial_a\left[i\,C_0\left(\bar{C}^b\,\partial_b\,\bar{C}^a\right)\right]A_0 - i\,C_1\,\partial_0\left(\bar{C}^b\,\partial_b\,\bar{C}^1\right)A_0 - i\,C_1\,\partial_1\left(\bar{C}^b\,\partial_b\,\bar{C}^0\right)A_0 - i\,\partial_a\,(C_1\,\bar{C}^a)\left(\partial_1\,\bar{C}^0 + \partial_0\,\bar{C}^1\right)A_0 - i\,C_1\left(\partial_0\,\bar{C}^0 - \partial_1\,\bar{C}^1\right)\left(\partial_1\,\bar{C}^0 - \partial_0\,\bar{C}^1\right)A_0. \tag{64}$$

It will be noted that we have collected here the A_0 terms which look completely different from the corresponding terms in the BRST symmetry invariance (compare Equation (58)). This is due to the fact we have not written each term separately and independently. However, these terms are actually similar to (58). The above terms add up to produce the following total derivative terms, namely

$$\partial_a\left[-i\,C_1\,\bar{C}^a\left(\partial_0\,\bar{C}^1 + \partial_1\,\bar{C}^0\right)A_0 - i\,C_0\,\bar{C}^b\,\partial_b\left(\bar{C}^a\,A_0\right)\right]. \tag{65}$$

We now concentrate on all the terms that are generated after the application of $\bar{s}_B$ on $L_{\bar{B}}$ and contain necessarily the A_1 field. These are as follows:

$$+i\,\partial_a\,(C_1\,\bar{C}^a)\left(\bar{C}^b\,\partial_b\,A_1\right) + i\,C_0\left(\partial_1\,\bar{C}^0 + \partial_0\,\bar{C}^1\right)\left(\bar{C}^b\,\partial_b\,A_1\right) - \partial_a\left[i\,C_1\left(\bar{C}^b\,\partial_b\,\bar{C}^a\right)\right]A_1 - i\,C_0\,\partial_0\left(\bar{C}^b\,\partial_b\,\bar{C}^1\right)A_1 - i\,C_0\,\partial_1\left(\bar{C}^b\,\partial_b\,\bar{C}^0\right)A_1 - i\,\partial_a\,(C_0\,\bar{C}^a)\left(\partial_1\,\bar{C}^0 + \partial_0\,\bar{C}^1\right)A_1 - i\,C_1\left(\partial_0\,\bar{C}^0 - \partial_1\,\bar{C}^1\right)\left(\partial_1\,\bar{C}^0 - \partial_0\,\bar{C}^1\right)A_1. \tag{66}$$

The above terms add up to produce the following total derivative:

$$\partial_a\left[-i\,C_0\,\bar{C}^a\left(\partial_0\,\bar{C}^1 + \partial_1\,\bar{C}^0\right)A_1 - i\,C_1\,\bar{C}^b\,\partial_b\left(\bar{C}^a\,A_1\right)\right]. \tag{67}$$

Finally, we have the following set of terms that contain necessarily A_2 field after the application of $\bar{s}_B$ on $L_{\bar{B}}$, namely

$$-\partial_a\left[i\,C_0\,\bar{C}^a\right]\left(\partial_0\,\bar{C}^1 - \partial_1\,\bar{C}^0\right)A_2 - i\,C_0\left(\partial_0\,\bar{C}^1 + \partial_1\,\bar{C}^0\right)\cdot\left(\partial_0\,\bar{C}^0 - \partial_1\,\bar{C}^1\right)A_2 + i\,C_0\left(\partial_0\,\bar{C}^1 - \partial_1\,\bar{C}^0\right)\left(\bar{C}^b\,\partial_b\,A_2\right) - i\,\partial_a\,(C_1\,\bar{C}^a)\left(\partial_0\,\bar{C}^0 - \partial_1\,\bar{C}^1\right)A_2 + i\,C_1\left(\partial_0\,\bar{C}^0 - \partial_1\,\bar{C}^1\right)\cdot\left(\bar{C}^b\,\partial_b\,A_2\right) - i\,C_0\,\partial_0\left(\bar{C}^b\,\partial_b\,\bar{C}^1\right)A_2 - i\,C_1\,\partial_0\left(\bar{C}^b\,\partial_b\,\bar{C}^0\right)A_2 + i\,C_a\,\partial_1\left(\bar{C}^b\,\partial_b\,\bar{C}^a\right)A_2 + 2\,i\,C_1\,\partial_0\,\bar{C}^1\right)\left(\partial_1\,\bar{C}^0\right)A_2. \tag{68}$$

The above terms produce, after their addition, the following total derivative:

$$\partial_a\left[-i\,C_0\,\bar{C}^a\left(\partial_0\,\bar{C}^1 - \partial_1\,\bar{C}^0\right)A_2 - i\,C_1\,\bar{C}^a\left(\partial_0\,\bar{C}^0 - \partial_1\,\bar{C}^1\right)A_2\right]. \tag{69}$$

The total derivatives, present in this Appendix, sum up to produce the total derivative that has been quoted in the main body of our text (compare Equation (28)).

A.4. Appendix D: On the Superfield Approach to the Derivation of the CF-Type Restrictions and Absolute Anticommutativity

In this Appendix, we exploit the theoretical tricks and strength of the modified version of the Bonora-Tonin superfield approach (MBTSA) to BRST formalism [7] to concisely derive the 2D version of the universal CF-type restrictions: $B^a + \bar{B}^a + i\,(C^m\,\partial_m\,\bar{C}^a + \bar{C}^m\,\partial_m\,C^a) = 0$. In this context, first of all, we focus on the general form of the 2D diffeomorphism transformations $\xi^a \longrightarrow \xi' a = h^a(\xi)$ where $h^a(\xi)$ is a physically well-defined function of ξ^a such that it is finite at the origin and vanishes off at $\tau \longrightarrow \pm\infty$. We take now the generalizations of function $h^a(\xi)$ onto the $(2, 2)$-dimensional supermanifold as follows:

$$h^a(\xi) \longrightarrow \tilde{h}^a\left(\xi, \theta, \bar{\theta}\right) = \xi^a - \theta\, \bar{C}^a(\xi) - \theta\, C^a(\xi) + \theta\, \bar{\theta}\, k^a(\xi), \tag{70}$$

where the (anti)ghost fields $(\bar{C}^a)C^a$ are the ones which have appeared in the (anti-)BRST symmetry transformations (11) and (16). We note, at this stage, that the infinitesimal version of the classical diffeomorphism transformations $\xi^a \longrightarrow \xi' a = h^a(\xi) \equiv \xi^a - \varepsilon^a(\xi)$ denotes that the classical diffeomorphism transformations $\delta\, \xi^a = \xi' a - \xi^a = \varepsilon^a(\xi)$ can be elevated to their counterpart quantum (anti-)BRST symmetry transformations as $s_b\, \xi^a = -C^a, s_{ab}\, \xi^a = -\bar{C}^a$. This is the reason behind the choice in (70) where the coefficients of θ and $\bar{\theta}$ are $\bar{C}^a$ and C^a, respectively. This has been done due to the standard BRST prescription where the infinitesimal bosonic parameters $\varepsilon^a(\xi)$ are replaced by the fermionic (i.e., $(C^a)^2 = (\bar{C}^a)^2 = 0, C^a\, \bar{C}^b + \bar{C}^b\, C^A = 0,$ etc.) (anti)ghost fields $(\bar{C}^a)$ C^a within the framework of BRST formalism. The secondary fields $k^a(\xi)$ (i.e., the coefficient of $\theta\bar{\theta}$ in (70)) have to be determined from the consistency conditions and basic concepts behind the BRST formalism (which we elaborate below in a very concise manner).

It is the basic tenet of MBTSA that the target space coordinate fields $X^\mu(\xi)$ have to be generalized onto their counterpart superfields on the $(2, 2)$-dimensional supermanifold as

$$\begin{aligned} X^\mu(\xi) \longrightarrow \tilde{X}^\mu\left[\tilde{h}\left(\xi, \theta, \bar{\theta}\right), \theta, \bar{\theta}\right] &= X^\mu\left[\tilde{h}\left(\xi, \theta, \bar{\theta}\right)\right] \\ + \theta\, \bar{Q}^\mu\left[\tilde{h}\left(\xi, \theta, \bar{\theta}\right)\right] + \bar{\theta}\, Q^\mu\left[\tilde{h}\left(\xi, \theta, \bar{\theta}\right)\right] &+ \theta\, \bar{\theta}\, T^\mu\left[\tilde{h}\left(\xi, \theta, \bar{\theta}\right)\right]. \end{aligned} \tag{71}$$

It should be noted, at this juncture, that all the secondary superfields on the r.h.s. (right hand side) are still a function of transformation $\tilde{h}^a(\xi, \theta, \bar{\theta})$ that has been pointed out in (70). We take another Taylor expansion for all the secondary superfields on the r.h.s. as

$$\theta\, \bar{\theta}\, T^\mu\left[\xi^a - \theta\, \bar{C}^a - \bar{\theta}\, C^a + \theta\, \bar{\theta}\, k^a\right] = \theta\, \bar{\theta}\, T^\mu(\xi), \tag{72}$$

$$\bar{\theta}\, Q^\mu\left[\xi^a - \theta\, \bar{C}^a - \bar{\theta}\, C^a + \theta\, \bar{\theta}\, k^a\right] = \bar{\theta}\, Q^\mu(\xi) + \theta\, \bar{\theta}\, \bar{C}^a\, \partial_a\, Q^\mu(\xi), \tag{73}$$

$$\theta\, \bar{Q}^\mu\left[\xi^a - \theta\, \bar{C}^a - \bar{\theta}\, C^a + \theta\, \bar{\theta}\, k^a\right] = \theta\, \bar{Q}^\mu(\xi) - \theta\, \bar{\theta}\, C^a\, \partial_a\, \bar{Q}^\mu(\xi), \tag{74}$$

$$\begin{aligned} X^\mu\left[\xi^a - \theta\, \bar{C}^a - \bar{\theta}\, C^a + \theta\, \bar{\theta}\, k^a\right] &= X^\mu(\xi) - \theta\, \bar{C}^a\, \partial_a\, X^\mu \\ - \bar{\theta}\, C^a\, \partial_a\, X^\mu + \theta\, \bar{\theta}\, &\left[k^a\, \partial_a\, X^\mu - \bar{C}^a\, C^b\, \partial_a\, \partial_b\, X^\mu\right]. \end{aligned} \tag{75}$$

It is pertinent and important to point out that we have to make two-step generalizations of $X^\mu(\xi)$ to the $(2, 2)$-dimensional supermanifold in order to incorporate the diffeomorphism symmetry transformation $h^a(\xi) \longrightarrow \tilde{h}^a(\xi, \theta, \bar{\theta})$. In other words, we have the following:

$$X^\mu(\xi) \longrightarrow X^\mu\left(\xi, \theta, \bar{\theta}\right) \longrightarrow \tilde{X}^\mu\left[\tilde{h}\left(\xi, \theta, \bar{\theta}\right), \theta, \bar{\theta}\right], \tag{76}$$

where in the first-step, the target space coordinate field $[X^\mu(\xi)]$ has been generalized to $X^\mu(\xi, \theta, \bar{\theta})$ when there is no diffeomorphism transformation on ξ^a. In the next step, we incorporate $\xi^a \longrightarrow \tilde{h}^a(\xi, \theta, \bar{\theta})$, and then, we make the superexpansion as given in (71). Collecting the coefficients of $\theta, \bar{\theta}$, and $\theta\, \bar{\theta}$ from the r.h.s. (compare Equation (72)), we obtain

$$\begin{aligned} \tilde{X}^\mu\left[\tilde{h}\left(\xi, \theta, \bar{\theta}\right), \theta, \bar{\theta}\right] &= X^\mu(\xi) + \theta\left[\bar{Q}^\mu - \bar{C}^a\, \partial_a\, X^\mu\right] + \bar{\theta}\left[Q^\mu - C^a\, \partial_a\, X^\mu\right] \\ &+ \theta\, \bar{\theta}\left[T^\mu + \bar{C}^a\, \partial_a\, Q^\mu - C^a\, \partial_a\, \bar{Q}^\mu + k^a\, \partial_a\, X^\mu - \bar{C}^a\, C^b\, \partial_a\, \partial_b\, X^\mu\right]. \end{aligned} \tag{77}$$

At this stage, our key objective is to determine the values of $Q^\mu, \bar{Q}^\mu$, and T^μ so that we can obtain the quantum (anti-)BRST symmetries (corresponding to the 2D classical diffeomorphism symmetry transformations) for the target space coordinates $X^\mu(\xi)$ and the CF-type restrictions $B^a + \bar{B}^a + i\,(C^m\, \partial_m\, \bar{C}^a + \bar{C}^m\, \partial_m\, C^a) = 0$.

To accomplish the above goal, we apply the horizontality condition (HC):

$$X^\mu\left[\tilde{h}^a\left(\xi, \theta, \bar{\theta}\right)\right] = X^\mu(\xi), \tag{78}$$

which amounts to setting the coefficients of θ, $\bar{\theta}$, and $\theta\bar{\theta}$ equal to zero. Physically, this condition implies that all the pure Lorentz scalar (super)fields on the l.h.s. (left hand side) as well as on the r.h.s. should not change at all. In other words, we have the following:

$$\begin{aligned} \tilde{X}^\mu\left[\tilde{h}^a\left(\xi, \theta, \bar{\theta}\right), \theta, \bar{\theta}\right] &= X^\mu\left[\xi, \theta, \bar{\theta}\right] \equiv X^\mu(\xi), \\ \bar{Q}^\mu\left[\tilde{h}^a\left(\xi, \theta, \bar{\theta}\right)\right] = \bar{Q}^\mu(\xi), &\; Q^\mu\left[\tilde{h}^a\left(\xi, \theta, \bar{\theta}\right)\right] = Q^\mu(\xi), \\ T^\mu\left[\tilde{h}^a\left(\xi, \theta, \bar{\theta}\right)\right] &= T^\mu(\xi), \end{aligned} \tag{79}$$

so that we have the following superexpansion:

$$\begin{aligned} X^\mu\left(\xi, \theta, \bar{\theta}\right) &= X^\mu(\xi) + \theta\, \bar{Q}^\mu(\xi) + \bar{\theta}\, Q^\mu(\xi) + \theta\, \bar{\theta}\, T^\mu(\xi) \\ &\equiv X^\mu(\xi) + \theta\, (\bar{s}_B\, X^\mu) + \bar{\theta}\, (s_B\, X^\mu) + \theta\, \bar{\theta}\, (s_B\, \bar{s}_B\, X^\mu). \end{aligned} \tag{80}$$

It should be recalled that s_B and $\bar{s}_B$ are the BRST and anti-BRST symmetry transformations that have been listed in Equations (11) and (16). For the expansion of the type (81), Bonora-Tonin (BT) have found the mappings $s_B \longleftrightarrow$

$\partial_{\bar{\theta}}|_{\theta=0}, \bar{s}_B \longleftrightarrow \partial_\theta|_{\bar{\theta}=0}$ in the context of BT-superfield approach to 4D non-Abelian 1-form (see, e.g., [21–23]) gauge theory (without any interaction with matter fields) where the (anti-)BRST symmetry transformations as well as the CF-condition (36) have been found. We note that the HC (compare Equation (78)) implies, taking into account the inputs from (77), the following:

$$Q^\mu = C^a\, \partial_a\, X^\mu, \bar{Q}^\mu = \bar{C}^a\, \partial_a\, X^\mu, \tag{81}$$

$$T^\mu = C^a\, \partial_a\, \bar{Q}^\mu + \bar{C}^a\, C^b\, \partial_a\, \partial_b\, X^\mu - \bar{C}^a\, \partial_a\, Q^\mu - k^a\, \partial_a\, X^\mu. \tag{82}$$

The substitutions of the above into (80) and their interpretations in terms of the off-shell nilpotent (anti-)BRST symmetry transformations $(\bar{s}_B)s_B$ imply that

$$Q^\mu = s_B\, X^\mu = C^a\, \partial_a\, X^\mu, \bar{Q}^\mu = \bar{C}^a\, \partial_a\, X^\mu = \bar{s}_B\, X^\mu, \tag{83}$$

$$\begin{aligned} T^\mu = s_B\, \bar{s}_B\, (X^\mu) &\equiv C^a\, \partial_a \left[\bar{C}^b\, \partial_b\, X^\mu\right] + C^a\, \bar{C}^b\, \partial_a\, \partial_b\, X^\mu \\ &\quad - \bar{C}^a\, \partial_a \left[C^b\, \partial_b\, X^\mu\right] - k^a\, \partial_a\, X^\mu \\ &\equiv \left[C^a\, \partial_a\, \bar{C}^b - \bar{C}^a\, \partial_a\, C^b - k^b\right] (\partial_b\, X^\mu) - \bar{C}^a\, C^b\, \partial_a\, \partial_b\, X^\mu. \end{aligned} \tag{84}$$

At this juncture, we demand the absolute anticommutativity (i.e., $\{s_B, \bar{s}_B\}\, X^\mu = 0$) of the (anti-)BRST symmetry transformations $(\bar{s}_B)s_B$ which leads to the following:

$$\begin{aligned} s_B\bar{s}_B X^\mu = s_B\left[\bar{C}^a \partial_a X^\mu\right] &\equiv iB^a \partial_a X^\mu - \bar{C}^b C^a \partial_a \partial_b X^\mu \\ &\quad - \bar{C}^a \left(\partial_a C^b\right)(\partial_b\, X^\mu), -\bar{s}_B s_B X^\mu = -\bar{s}_B (C^a \partial_a X^\mu) \\ &\equiv -i\bar{B}^a \partial_a X^\mu - C^a \bar{C}^b \partial_a \partial_b\, X^\mu + C^a \left(\partial_a\, \bar{C}^b\right) \partial_b X^\mu, \end{aligned} \tag{85}$$

where we have taken the standard (anti-)BRST symmetry transformations: $s_B\, \bar{C}^a = i\, B^a, \bar{s}_B\, C^a = i\, \bar{B}^a, s_B\, B^a = 0, \bar{s}_B\, \bar{B}^a = 0$ and $s_B\, C^a = C^b\, \partial_b\, C^a, \bar{s}_B\, \bar{C}^a = \bar{C}^b\, \partial_b\, \bar{C}^a$. The last two symmetry transformations have been obtained from the nilpotency requirements $s_B^2 X^\mu = 0, \bar{s}_B^2 X^\mu = 0$, respectively. Each equation from (85) can be compared with the exact expression for T^μ that has been obtained in Equation (83). This comparison implies

$$k^a = -i\, B^a + C^b \partial_b\, \bar{C}^a \equiv i\, \bar{B}^a - \bar{C}^b \partial_b\, C^a, \tag{86}$$

which leads to the derivation of the CF-type restriction [24] within the framework of MBTSA. Thus, it is very interesting to note that the requirement of the absolute anticommutativity property (compare Equation (85)) of the (anti-)BRST symmetry transformations is responsible for the determination of k^a (compare Equation (86)) which, ultimately, leads to the derivation of the CF-type restrictions (20) within the ambit of MBTSA.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest

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