

Research Article

Thermodynamic Phase Transition of Generalized Ayon-Beato Garcia Black Holes

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In this work, we study thermodynamics of generalized Ayon-Beato and Garcia (ABG) black hole metric which contains three parameters named as mass m , magnetic charge q , and dimensionless coupling constant of nonlinear electrodynamics interacting field γ . We showed that central regions of this black hole behaves as dS (AdS) vacuum space by setting $q < 2m$ ($q > 2m$) and in the case $q = 2m$ reaches to a flat Minkowski space. In the large distances, this black hole behaves as a Reissner-Nordstrom BH. However, an important role of the charge q appeared in the production of a formal variable cosmological parameter which will support pressure coordinate in the thermodynamic perspective of this black hole in our setup. We should point that this formal variable cosmological parameter is different with cosmological constant which comes from AdS/CFT correspondence, and it is effective at large distances as AdS space pressure. In our setup, the assumed pressure originated from the internal material of the black hole say q and m here. By calculating the Hawking temperature of this black hole, we obtain equation of state. Then, we plotted isothermal P-V curves and heat capacity at constant pressure. They show that the system participates in the small to large phase transition of the black hole or the Hawking-Page phase transition which is similar to the van der Waals phase transition in the ordinary thermodynamics systems. In fact in the Hawking-Page phase transition disequilibrium, evaporating generalized ABG black hole reaches to a vacuum AdS space finally.

1. Introduction

The Einstein general theory of relativity is the most efficient theory of the gravity, in which its validity has been approved through its correspondence to the observations and experiments in many years [1]. But in some cases, this theory is not practical, for instance, the spacetime singularity, which is an example of the failure of general relativity. The gravitational causal singularity is the extreme density and as a result, so intense gravity in a point of spacetime where the spacetime breaks down. These causal singularities appeared in metric solutions of Einstein's gravitational field equations, and they can not be omitted by usual coordinate transformations. Penrose-Hawking singularity theorems show by holding Einstein's metric equations under some circumstances that the existence of space time singularities is unavoidable [2–4]. In order to avoid the central singularity of black holes, Penrose suggested his cosmic supervision hypothesis which

says: the singularity of black hole is always hidden behind its event horizon [5]. Nevertheless, many agree that the singularity is generated by classical gravity theories, while they are neither physical nor exist in the universe [6]. Sakharov [7] and Gliner [8] firstly showed by considering the effects of quantum that the spacetime singularity is avoidable. Bardeen [9] inspired by Sakharov's idea, proposed the first singularity free solution of black holes which are called as regular black holes now. He suggested a static spherically symmetric solution without considering a known physical source. Later, numerous different kinds of regular black holes were suggested [10–14]. Among them, Ayon-Beato and Garcia (ABG) [15] considered a nonlinear electromagnetic field as a physical source to produce regular black holes. In this way, they obtained ABG regular black holes by solving Einstein's metric equations which are coupled with suitable nonlinear electromagnetic fields. By following this method, other authors also confirmed ABG regular

black holes [16–18]. Cai and Miao [19] achieved a kind of generalized ABG-related black hole solutions which are dependent on five parameters named mass, charge, and three parameters related to nonlinear electrodynamic fields. This kind of black hole returns to regular black hole under special conditions. Also, ABG black hole [15] and its other generalization [20] are obtained under some assumptions. In [19], a new family of ABG black holes have been focused which have three parameters named mass, charge, and dimensionless parameter γ . Cai and Miao [19] studied quasinormal modes and shadows radius for this new family of ABG black hole and also analyzed the effects of charge and γ parameter on event horizon radius and Hawking temperature. Hawking by considering quantum effects showed black holes radiate like black bodies with particular temperature [21], related to surface gravity of black holes horizon and Bekenstein attributed entropy to black holes which is related to area of surface of the black holes horizon as $S = A/4$ [22]. These two discoveries lead us to investigate the thermodynamic behavior of black holes. Black hole thermodynamics is the consequence of relation between general relativity and quantum field theory which guide us to the unknown quantum gravity. By considering black holes as thermodynamic systems, Bardeen et al. [23] rewrote the four laws of thermodynamic for black holes. In this way, Davies studied the phase transition of the Kerr black hole in [24]. In a study of thermodynamics of the black holes in the usual way, we need pressure thermodynamic coordinate which is brought from a cosmological constant in extended phase space with negative value. In fact, this originated from CFT/AdS correspondence where one can investigate to study the thermodynamics of the black holes. In this way, the Hawking and Page discovered a first-order phase transition for black holes in anti-de Sitter (AdS) spacetime [25]. Other types of phase transitions have been followed in other works [26–31]. Since the cosmological constant has been suggested as thermodynamic pressure of AdS space background which affects evaporating quantum black hole for settings of its thermodynamic equation of state [32–35], the attentions have been attracted to black hole thermodynamics in extended phase space [36–42]. However, we will use other idea to bring a pressure coordinate in studying the generalized ABG black hole in this work. In fact, central region of the ABG black hole behaves as dS/AdS space where the black hole magnetic charge plays an important role to produce a formal cosmological parameter, and so to study the thermodynamics of this kind of black hole, we do not need to add cosmological constant which comes from AdS space. In other words, this kind of black hole is self-contained while for some well-known black holes, for instance, Schwarzschild or Reissner-Nordstrom which are singular at central regions; we must add an additional cosmological constant parameter to produce pressure term. The layout of this work is as follows.

In Section 2, we define metric of generalized ABG black holes briefly. In Section 3, we investigate thermodynamic perspective of the model. In Section 4, we study possibility of the black hole phase transition. Section 5 is dedicated to summary and conclusion.

2. Generalized ABG Black Hole

Let us we start with the following nonlinear Einstein Maxwell action functional [15]:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi} - \frac{L(P)}{4\pi} \right], \quad (1)$$

in which, $R = g_{\mu\nu} R^{\mu\nu}$ is Ricci scalar and $g = |\det g_{\mu\nu}|$ is absolute value of determinant of metric tensor field. Nonlinear electromagnetic field lagrangian density $L(P) = 2PH_p - H(P)$ is coupled as minimally with the gravity where $P \equiv (1/4)P_{\mu\nu}P^{\mu\nu}$ is a gauge invariant scalar. $P_{\mu\nu} \equiv (F_{\mu\nu}/H_p)$ is nonlinear antisymmetric tensor versus the electromagnetic tensor field $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Here, A_μ is electromagnetic potential and $H_p = dH(P)/dP$ in which $H(P)$ is a structure function of nonlinear electrodynamic field given by [19].

$$H(P) = \frac{P \left[1 - ((\beta\gamma/2) - 1)(-2Pq^2)^{\gamma/4} \right]}{\left[1 + (-2Pq^2)^{\gamma/4} \right]^{1+(\beta/2)}} - \frac{\alpha\gamma m (-2Pq^2)^{(3+\gamma)/4}}{2q^3 \left[1 + (-2Pq^2)^{\gamma/4} \right]^{1+(\alpha/2)}}. \quad (2)$$

By looking at the ref. [19], one can infer that the above model has a spherically symmetric static black hole metric field as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (3)$$

in which

$$f(r) = 1 - \frac{2mr^{(\alpha\gamma/2)-1}}{(q^\gamma + r^\gamma)^{\alpha/2}} + \frac{q^2 r^{(\beta\gamma/2)-2}}{(q^\gamma + r^\gamma)^{\beta/2}}. \quad (4)$$

This is called generalized ABG black hole metric potential for which m is ADM mass parameter and q is magnetic charge and three different dimensionless parameters α , β , and γ are associated to nonlinear electrodynamic fields. By choosing different values for these parameters, one can show that the metric (4) may become nonsingular or naked singular at central regions [9]. For particular choice $(\alpha, \beta, \gamma) = (3, 4, 2)$, the generalized ABG metric field (4) returns to original ABG black hole solution [15] and it goes to other generalized ABG black hole solutions [20] by setting $\gamma = 2$. For simplicity, we set here a particular choice of the above mentioned parameters as $\alpha\gamma = 6$ and $\beta\gamma = 8$ for which the generalized ABG black hole metric given by ref. [19] reduces to the following form:

$$f(r) = 1 - \frac{2mr^2}{(r^\gamma + q^\gamma)^{3/\gamma}} + \frac{q^2 r^2}{(r^\gamma + q^\gamma)^{4/\gamma}}. \quad (5)$$

It is easy to check that the above metric solution reduces to the well-known Reissner-Nordström form $f(r) \sim 1 -$

$(2m/r) + (q^2/r^2)$ for limits $r/q \rightarrow \infty$ and for central regions of the ABG black hole $r/q \rightarrow 0$, we will have a flat Minkowski form for $q = 2m$ and $dS(\text{AdS}) f(r) \sim 1 + (q - 2m)(r^2/q^3)$ for $q < 2m (q > 2m)$, respectively. In other words one can infer that for particular choice $q = 2m$, the ABG black hole reads to a flat Mikowski form at central regions $r < <|q|$, while it reaches to an extremal RN form at large distance $r > >|q|$. However, by assuming that the cosmological parameter in this model is originated just from ABG magnetic charge q not an AdS background which surrounds the ABG black hole; therefore, we can choose components of the metric potential as follows.

$$\Lambda(r) = \frac{-3q^2}{(r^\gamma + q^\gamma)^{4/\gamma}}. \quad (6)$$

This is in fact a formal variable cosmological parameter and

$$M(r) = \frac{mr^3}{(r^\gamma + q^\gamma)^{3/\gamma}}, \quad (7)$$

as distribution of mass function where the metric potential (5) appears similar to the Schwarzschild AdS black hole form such that

$$f(r) = 1 - \frac{2M(r)}{r} - \frac{1}{3}\Lambda(r)r^2. \quad (8)$$

However, if we want to study thermodynamics of this generalized ABG black hole just near the central regions, it is enough to extract a variable cosmological parameter such as above form or other alternative which is proposed in the next section from metric potential itself (5) because it behaves as dS/AdS form just near the central regions of the nonsingular ABG black hole. As we mentioned in the abstract of this paper, this approach is different with respect to the usual way where a black hole is surrounded by an AdS vacuum space with constant pressure related to a negative cosmological constant. This latter proposal comes from AdS/CFT correspondence and the used cosmological constant comes from geometrical approach, and it should describe inflation of the universe in the Λ CDM model (see for instance [43]). In other words, this is effective at large distances while (6) is vanishing at large distances and has not any relation to the AdS vacuum space. In fact, our assumed variable pressure (or formal variable cosmological parameter) originates from internal materials of the black hole under consideration namely magnetic ABG charge q and the ADM mass m here. By looking at the above definitions, one can obtain easily $M(0) = 0$ and $M(\infty) = m$ with corresponding values for the cosmological parameter as $\Lambda(0) = -3/q^2$ and $\Lambda(\infty) = 0$ which by copping to proposal of the AdS/CFT correspondence we can relate charge of the ABG nonlinear electromagnetic field q to an assumed central AdS space radius ℓ_{AdS} such that $|q|\sqrt{3} = \ell_{\text{AdS}}$. In fact, these boundary values for Λ show that central region of the ABG black hole behaves as AdS space while

at large distances the metric is asymptotically flat. In other words, the central region of the ABG black hole can be assumed to be filled with some dark matter with a repeller force which prevents the central area of the black hole from collapsing. In this view, one can infer that the above assumed variable cosmological parameter is in fact different with which one came from proposal of the AdS/CFT correspondence in the large scales of bulk spacetime. Because the ansatz cosmological constant which is used in the usual AdS/CFT correspondence generated from symmetry group between gauge theory on the boundary and bulk gravity theory should be vanishing at large cosmological scales, while in our approach, it reaches to some small values at central regions of the ABG black hole. From this point of view, our idea differs with usual proposal about the unknown cosmological constant which is used in the general theory of relativity to describe the inflation with positive value or to adjust the pressure component in the study of thermodynamics of black holes with a negative sign (AdS black holes). In the subsequent section, we study thermodynamics of the above ABG black hole but with the abovementioned perspective about the negative cosmological parameter.

3. Thermodynamics of Generalized ABG Black Hole

By regarding the previous perspective about a variable formal cosmological parameter to study thermodynamics of the ABG black hole, it is useful to assume central region of the ABG black hole metric behaves as alone AdS form as $f(r) = 1 + \Lambda(r)r^2/3$ in which $\Lambda(r)$ is an alternative variable cosmological parameter such that

$$\Lambda(r) = -8\pi P(r) = 3 \left[\frac{q^2}{(r^\gamma + q^\gamma)^{4/\gamma}} - \frac{2m}{(r^\gamma + q^\gamma)^{3/\gamma}} \right], \quad (9)$$

where $P(r)$ is the variable pressure of central AdS space. It is easy to check that the horizon equation $f(r_+) = 0$ reads

$$1 - \frac{2mr_+^2}{(r_+^\gamma + q^\gamma)^{3/\gamma}} + \frac{q^2 r_+^2}{(r_+^\gamma + q^\gamma)^{4/\gamma}} = 0, \quad (10)$$

in which r_+ is exterior horizon radius and it can be shown that it asymptotically reaches to the following solutions

$$\begin{aligned} r_+ > > |q|, (r_+)_{1,2} &= m \pm \sqrt{m^2 - q^2}, \\ r_+ < < |q|, (r_+)_{1,2} &= |q| \sqrt{\frac{q}{2m - q}}. \end{aligned} \quad (11)$$

The horizon equation (10) can be rewritten as the enthalpy equation of the ABG black hole such that

$$m = H = U + PV. \quad (12)$$

TABLE 1: Critical pressure and temperature for large (P_c^+ , T_c^+) and small (P_c^- , T_c^-) generalized ABG black holes.

γ	P_c^-	P_c^+	T_c^-	T_c^+
$\pm\infty$	0.053	0.053	0.066	0.066
4.86	-0.009	0.005	0.007	-0.006
4.8	-0.00894586	0.00466126	5.21225×10^{-6}	2.40448×10^{-6}
4.7	-0.00873715	0.00455079	4.74355×10^{-6}	2.1582×10^{-6}
4.6	-0.00853071	0.00444403	4.29998×10^{-6}	1.92837×10^{-6}
4.5	-0.00832665	0.00432984	3.88149×10^{-6}	1.71469×10^{-6}
4.4	-0.00812505	0.00421943	3.48802×10^{-6}	1.5168×10^{-6}
4.3	-0.00792601	0.00410912	3.11942×10^{-6}	1.33433×10^{-6}
4.2	-0.00772965	0.00399893	2.77545×10^{-6}	1.16684×10^{-6}
4.1	-0.00753608	0.00388892	2.45584×10^{-6}	1.01386×10^{-6}
4.0	-0.00734544	0.00377911	2.16019×10^{-6}	8.74879×10^{-7}
3.9	-0.00715786	0.00366957	1.88807×10^{-6}	7.49348×10^{-7}
3.8	-0.0069735	0.00356033	1.63892×10^{-6}	6.36675×10^{-7}
3.7	-0.00679254	0.00345145	1.41213×10^{-6}	5.36231×10^{-7}
3.6	-0.00661518	0.00334299	1.20698×10^{-6}	4.47353×10^{-7}
3.5	-0.00644162	0.00323501	1.02268×10^{-6}	3.69348×10^{-7}
3.4	-0.00627213	0.00312758	8.58331×10^{-7}	3.01494×10^{-7}
3.2	-0.00594647	0.00291466	5.85581×10^{-7}	1.93243×10^{-7}
3.1	-0.005791	0.00280933	4.75014×10^{-7}	1.51311×10^{-7}
3.0	-0.00564099	0.00270487	3.801×10^{-7}	1.16475×10^{-7}
2.9	-0.00549696	0.00260139	2.99601×10^{-7}	8.79622×10^{-8}
2.8	-0.0053595	0.002499	2.32241×10^{-7}	6.50117×10^{-8}
2.7	-0.00522933	0.00239781	1.76714×10^{-7}	4.68846×10^{-8}
2.6	-0.00510734	0.00229797	1.31704×10^{-7}	3.28717×10^{-8}
2.5	-0.00499461	0.00219962	9.59015×10^{-8}	2.23033×10^{-8}
2.4	-0.0048925	0.00210292	6.80237×10^{-8}	1.4557×10^{-8}
2.3	-0.0048027	0.00200808	4.6836×10^{-8}	9.06631×10^{-9}
2.2	-0.00472744	0.00191531	3.11721×10^{-8}	5.32651×10^{-9}
2.1	-0.00466961	0.00182485	1.99541×10^{-8}	2.89986×10^{-9}
2.0	-0.00463312	0.001737	1.22099×10^{-8}	1.41809×10^{-9}
1.9	-0.00462331	0.00165211	7.08794×10^{-9}	5.8254×10^{-10}
1.8	-0.00464778	0.00157061	3.86669×10^{-9}	1.61493×10^{-10}
1.7	-0.00471762	0.001493	1.95847×10^{-9}	-1.52291×10^{-11}
1.6	-0.00484969	0.00141997	9.06514×10^{-10}	-6.42181×10^{-11}
1.5	-0.00507089	0.00135239	3.75326×10^{-10}	-5.82997×10^{-11}
1.4	-0.00542679	0.00129145	1.34828×10^{-10}	-3.73485×10^{-11}
1.3	-0.00600098	0.00123884	4.00938×10^{-11}	-1.8947×10^{-11}
1.2	-0.00696523	0.00119706	9.07189×10^{-12}	-7.7284×10^{-12}
1.1	-0.00874552	0.00117005	1.26329×10^{-12}	-2.48105×10^{-12}
0.5	-0.0289341	0.00321305	-6.99671×10^{-20}	-3.26406×10^{-20}
0.1	169.214	4.73581×10^9	-3.67174×10^{-82}	-4.52954×10^{-80}

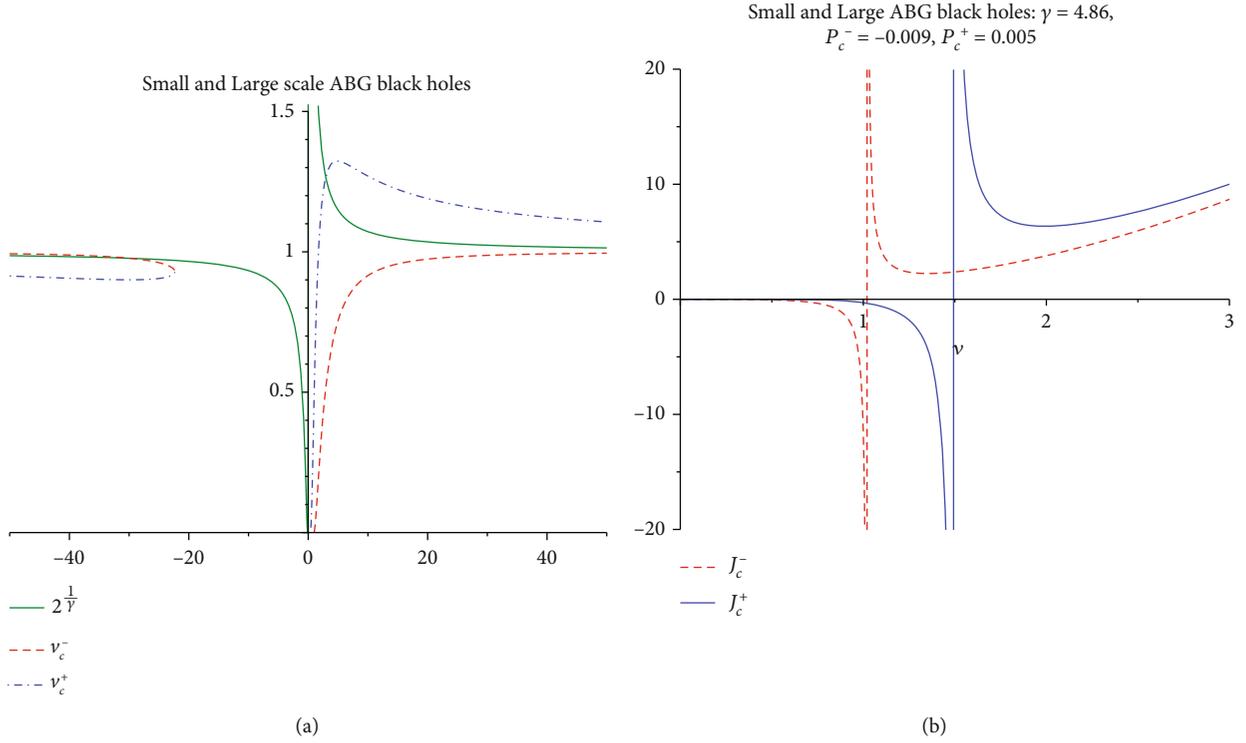


FIGURE 1: (a) Diagrams of the critical specific volume of small and large generalized ABG black holes vs γ and (b) Variation of entropy per specific volume of small $J^- = \Delta S/\Delta v^-$ and large $J^+ = \Delta S/\Delta v^+$ generalized ABG black holes.

Here, we call $m = H$ to be enthalpy and V and U are thermodynamic volume and internal energy, respectively, with the following forms.

$$V(r_+) = \frac{4\pi}{3} (r_+^\gamma + q^\gamma)^{3/\gamma}, U(r_+) = \frac{q^2}{2(r_+^\gamma + q^\gamma)^{1/\gamma}}. \quad (13)$$

By looking at equation (13), one can infer that for this black hole the thermodynamic volume has different form with respect to its geometrical volume, while for some of black hole, they have same forms. In fact, thermodynamic volume of a black hole system is conjugate quantity for the pressure in the black hole equation of state and it is determined by the first law of the black hole thermodynamics $TdS = dU + PdV$. To study thermodynamics of the ABG black hole, we need to calculate its Hawking temperature which is given versus the surface gravity on the exterior horizon r_+ . By substituting (12) and (13) into the metric potential (5) and by calculating the surface gravity $f'(r_+)$, we obtain the Hawking temperature for the ABG black hole as follows:

$$T(r_+) = vP + F(v), \quad (14)$$

where $F(v)$ is defined by specific volume v for which we defined

$$v = \frac{r_+}{3} \left(\frac{r_+^\gamma - 2q^\gamma}{r_+^\gamma + q^\gamma} \right), F(v) = -\frac{q^2 r_+^{1+\gamma}}{8\pi (r_+^\gamma + q^\gamma)^{1+(4/\gamma)}}. \quad (15)$$

In the next section, we investigate possibility of thermodynamic phase transitions.

4. Thermodynamic Phase Transitions

Equations (14) and (15) are parametric forms for the ABG black hole equation of state. To obtain a single closed form, we must separate scales of the ABG black hole to small and large black holes which by regarding positivity condition on the specific volume equation (15) gives us

$$\left| \frac{r_+}{q} \right| \leq 2^{1/\gamma} \longrightarrow v \approx -\frac{r_+}{3}, T \approx Pv - \frac{1}{8\pi q} \frac{(-3v/q)^\gamma}{[1 + (-3v/q)^\gamma]^{1+(4/\gamma)}}, \quad (16)$$

for small-scale ABG black holes and

$$\left| \frac{r_+}{q} \right| > 2^{1/\gamma} \longrightarrow v \approx \frac{r_+}{3}, T \approx Pv - \frac{1}{8\pi q} \frac{(3v/q)^\gamma}{[1 + (3v/q)^\gamma]^{1+(4/\gamma)}} \quad (17)$$

for large-scale ABG black holes. For simplicity of investigation of the phase transition, we set ansatz $q = -3$ and $q = 3$ for equations (16) and (17), respectively, such that

$$T \approx Pv + \frac{1}{24\pi} \frac{v^\gamma}{(1 + v^\gamma)^{1+(4/\gamma)}}, 0 \leq v \leq 2^{1/\gamma}, \quad (18)$$

for small ABG black holes and

$$T \approx Pv - \frac{1}{24\pi} \frac{v^\gamma}{(1+v^\gamma)^{1+(4/\gamma)}}, v > 2^{1/\gamma}, \quad (19)$$

for large ABG black holes, respectively. Now, we are in position to find the critical thermodynamic variables of the system through solving below equations:

$$\left. \frac{\partial T}{\partial v} \right|_P = 0, \quad \left. \frac{\partial^2 T}{\partial v^2} \right|_P = 0, \quad (20)$$

which by substituting (18) and (19), we obtain parametric forms of the critical points (v_c, P_c, T_c) in phase space such that

$$\begin{aligned} (v_c^\pm)_{\text{large}} &= (v_c^\pm)_{\text{small}} \\ &= \left[\frac{\gamma^2 + 13\gamma - 4 \pm \sqrt{\gamma^4 + 26\gamma^3 + 81\gamma^2 - 24\gamma + 16}}{40} \right]^{1/\gamma}, \end{aligned} \quad (21)$$

$$-(P_c^\pm)_{\text{large}} = (P_c^\pm)_{\text{small}} = \frac{(v_c^\pm)^{\gamma-1} [4(v_c^\pm)^\gamma - \gamma]}{24\pi [1 + (v_c^\pm)^\gamma]^{2+(4/\gamma)}}, \quad (22)$$

$$-(T_c^\pm)_{\text{large}} = (T_c^\pm)_{\text{small}} = \frac{(v_c^\pm)^{\gamma-1} [5(v_c^\pm)^\gamma + 1 - \gamma]}{24\pi [1 + (v_c^\pm)^\gamma]^{2+(4/\gamma)}}, \quad (23)$$

$$P_\gamma = P(2^{1/\gamma}), T_\gamma = T(2^{1/\gamma}). \quad (24)$$

We collected some numeric values for the critical pressures and critical temperatures for different values of the γ parameter in Table 1. At the first step, we should determine numeric values for γ parameter for which the phase transitions may possible happens for small and large ABG black holes. To do so, we look diagrams of the critical specific volumes (21) given in Figure 1 for which we see a local maximum point just for diagram of v_c^+ but not for v_c^- . Furthermore, we can obtain

$$\lim_{\gamma \rightarrow \pm\infty} v_c^\pm(\gamma) = 1, v_c^-(0) \rightarrow 0. \quad (25)$$

To study the phase transition of this ABG black hole, we must select a suitable numeric value for γ parameter given in Figure 1(a). As a suitable sample, we select numeric value for γ by solving the equation $dv_c^+/d\gamma = 0$ for large ABG black hole such that $\gamma = 4.858260939$. By substituting $\gamma = 4.858260939$, one can obtain $v_c^+ = 1.323138639$ for specific volume of largest ABG black hole and $v_c^- = 0.7457595729$ for specific volume of smaller one ABG black hole. We say these are large and small ABG black holes because for this particular value of γ the corresponding limit specific volume reads $2^{1/\gamma} = 1.153353666$ for which the inequality condition $v_c^- < 2^{1/\gamma}$ and $v_c^+ > 2^{1/\gamma}$ is established. We collect all components of the critical points given by (21), (22), and (23) for this particular γ value in Table 2, and corresponding critical pres-

TABLE 2: Critical specific volume for large v_c^+ and small v_c^- generalized ABG black holes.

γ	$2^{1/\gamma}$	v_c^-	v_c^+
$\pm\infty$	1	1	1
4.86	1.15	0.75	1.32
4.8	1.15535	0.741427	1.32311
4.7	1.15891	0.733726	1.32295
4.6	1.16263	0.725675	1.32263
4.5	1.16653	0.717254	1.32212
4.4	1.17062	0.708439	1.32141
4.3	1.17492	0.699207	1.32048
4.2	1.17943	0.689532	1.3193
4.1	1.18419	0.679386	1.31784
4.0	1.18921	0.66874	1.31607
3.9	1.1945	0.657562	1.31397
3.8	1.2001	0.645819	1.31148
3.7	1.20603	0.633474	1.30856
3.6	1.21233	0.620488	1.30517
3.5	1.21901	0.60682	1.30125
3.4	1.22613	0.592427	1.29674
3.2	1.24186	0.561278	1.28563
3.1	1.25057	0.54442	1.27886
3.0	1.25992	0.526637	1.27115
2.9	1.27	0.507872	1.26236
2.8	1.28089	0.48807	1.25237
2.7	1.29268	0.467174	1.24101
2.6	1.30551	0.445129	1.22809
2.5	1.31951	0.421882	1.21341
2.4	1.33484	0.39739	1.19673
2.3	1.35171	0.371616	1.17775
2.2	1.37035	0.344542	1.15616
2.1	1.39107	0.316172	1.13158
2.0	1.41421	0.286547	1.10358
1.9	1.44025	0.255752	1.07166
1.8	1.46973	0.223943	1.03526
1.7	1.50341	0.191364	0.993727
1.6	1.54221	0.158386	0.946364
1.5	1.5874	0.125544	0.892396
1.4	1.64067	0.0935867	0.831023
1.3	1.70436	0.0635304	0.761483
1.2	1.7818	0.0367104	0.683169
1.1	1.87786	0.0148133	0.595845
0.5	4.0	0.00390625	0.04
0.1	1024.	1.26329×10^{-8}	2.69539×10^{-16}

ures and critical temperatures are collected in Table 1. Also, we use (18) and (19) to plot the P-V diagrams at constant temperatures in Figures 2(a) and 2(b) for small (large) ABG black hole with specific volume $v_c^-(v_c^+)$ which are given in Table 2. By looking at these diagrams, one can infer that Figure 2(b) shows a van der Waals phase transition at

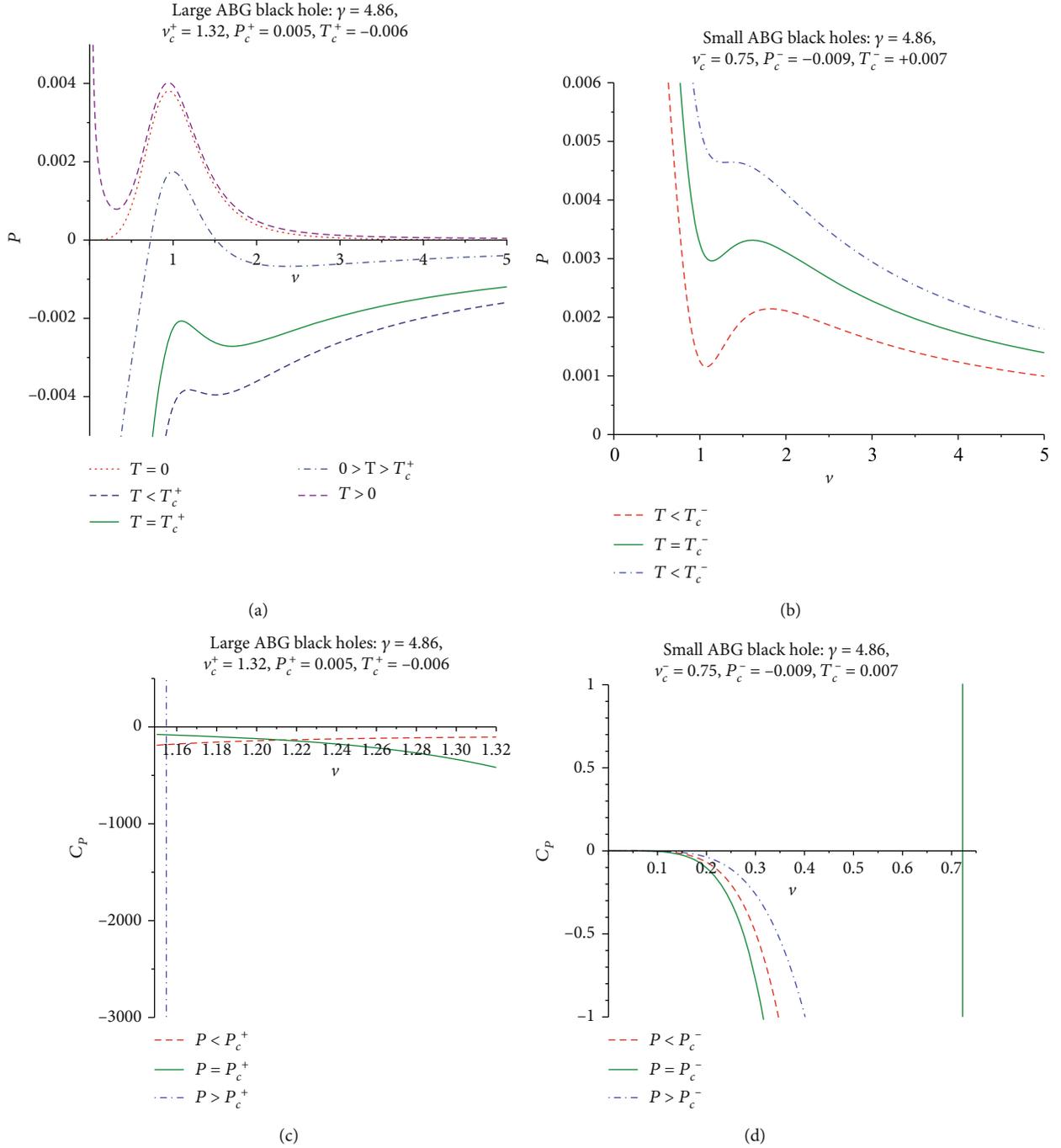


FIGURE 2: (a, b) P-V diagrams at constant temperatures for large and small generalized ABG black hole, respectively. (c, d) Heat capacity diagrams at constant pressure for large and small generalized ABG black holes, respectively.

positive temperatures higher than the critical (negative) temperature $T > T_c^+$ while it is in unstable state thermally at below of the critical temperature $T \leq T_c^+$ and may reaches from AdS ($P > 0$) to dS ($P < 0$). However, by looking at Figure 2(a), we understand that the large ABG black hole is always in the AdS phase ($P > 0$) but it participates in the small to large black hole phase transition (a van der Waals type) at temperatures below the critical one $T \leq T_c^+$. Also, the diagram shows that by raising the specific volume v

the black hole passes from a local maximum pressure and then can participate in the Hawking-Page phase transition by making up as a gas form with infinite specific volume. Numeric values of the critical points for choices $\gamma = \{1, 0.9, 0.8, 0.7, 0.6, 0.4, 0.3, 0.2, 0\}$ are not obtained real finite numbers. Mathematica software generates as imaginary complex numbers or undetermined and so they are not shown in Tables 1 and 2. Also, we checked numerical solutions of the critical points for negative γ and obtained

some complex imaginary numbers again for $-22 < \gamma < 0$ but there are some real valued critical specific volume obtained for $\gamma < -23$ for which $1 > 2^{1/\gamma} > v_c^- > v_c^+ > 0$ and so we do not consider them again in this paper because they describe small ABG black holes which we considered via positive values for γ parameter. By substituting the internal energy and the thermodynamic volume given by equations (13) and the Hawking temperature (14) into the first law of the black hole thermodynamics $TdS = dU + PdV$, we obtain integral equation for the Bekenstein entropy of this kind of the ABG black hole such that

$$S = 12\pi q^2 \int J(v) dv, \quad (26)$$

$$C_p(v) = \frac{108\pi v^\gamma (1 + v^\gamma)^{1+(3/\gamma)} \left[24\pi P(1 + v^\gamma)^{1+(4/\gamma)} + \varepsilon v^{\gamma-1} \right] \left[72\pi P(1 + v^\gamma)^{(4/\gamma)} - 1 \right]}{\left[24\pi P(1 + v^\gamma)^{2+(4/\gamma)} + \varepsilon v^{\gamma-1}(\gamma - 4v^\gamma) \right] \left[72\pi P(v^\gamma - 2)(1 + v^\gamma)^{4/\gamma} - 3v^{1+\gamma} \right]}, \quad (28)$$

in which $\varepsilon = +1(-1)$ corresponds to small (large) black holes. Diagram of the above heat capacity is plotted vs. v for small and large ABG black holes in Figures 2(c) and 2(d), respectively, for critical point given in Table 2 for $\gamma = 4.86$.

5. Conclusion

By extracting a formal variable cosmological parameter from the generalized nonsingular ABG black hole where its central region behaves same as dS/AdS space, we can obtain a suitable equation of state in which the assumed variable cosmological parameter plays the same as the pressure thermodynamic coordinate. Positivity condition of the specific thermodynamic volume of the black hole system restricts us to separate negative (positive) sign for the magnetic charge to produce some acceptable equation of states which are corresponded to some small (large) scale generalized ABG black hole. By plotting isothermal P-V curves at constant temperature and also by plotting the heat capacity at constant pressure for both of small- and large-scale ABG black hole, we obtained situations for these diagrams where the black hole can be participate in one of two possible phase transitions called as (a) small to large phase transition or (b) Hawking-Page transition. In the former case, an unstable quantum black hole changes its scale after evaporation, but in the later one, a black hole evaporates full and reaches to a gas with infinite volume. It is same as van der Waals gas/fluid phase transition in the ordinary thermodynamic system. As extension of this work, we like to study other thermodynamic behavior of the modified ABG black hole such as heat engine and Joule-Thomson expansion, etc.

in which we defined

$$J(v) = \frac{v^{\gamma-1}(1 + v^\gamma)^{3/\gamma} \left[8\pi P q^2 (1 + v^\gamma)^{4/\gamma} - 1 \right]}{8\pi P q^2 (v^\gamma - 2)(1 + v^\gamma)^{4/\gamma} - 3v^{1+\gamma}}. \quad (27)$$

The above integral equation has not an analytic closed form solution regretfully, and so to use it in drawing the Gibbs free energy $G = m - TS$, we should use numerical calculations (diagrams for critical J_c^\pm are plotted in Figure 1(b)). But we can use variations of the entropy $J(v)$ to obtain a closed form for heat capacity at constant pressure. This is done by according to ordinary thermodynamics systems where the heat capacity at constant pressure is defined by $C_p = T(\partial S/\partial T)_P$. Thus, we substitute the above entropy equation and temperatures (18) and (19) and the charge value $q = \pm 3$ to obtain

Data Availability

All data used for this article are included within the article.

Disclosure

A preprint of this work has previously been published [44].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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