

Research Article **Reformulating the Quantum Uncertainty Relation through Geometric Illustrations**

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The uncertainty principle stands as a fundamental tenet within the realm of quantum theory. In this study, we embark on a reexamination of an emerging variant of the uncertainty relation within both pure and mixed quantum systems, leveraging a geometric elucidation. Subsequently, an enhancement to this relation is achieved by the incorporation of a surface angle denoted as θ , thereby transforming it from an inequality into an equation. Notably, this surface angle encapsulates the dynamics inherent in quantum state transitions. Complementing our analysis, a series of calculations are conducted, yielding results that offer an intuitive elucidation of the uncertainty relation across distinct quantum states. Consequently, this method bears significance as a pivotal visual insight within the domain of quantum information and measurement.

1. Introduction

Uncertainty relations, often referred to as Heisenberg's uncertainty principle, are a fundamental concept in quantum mechanics that describes the inherent limits on our ability to simultaneously measure certain pairs of physical properties of a particle, such as position and momentum [1, 2], yielding the venerable inequality $\Delta_x \Delta_p \ge \hbar/2$. Subsequent to this, Robertson extended the notion of the uncertainty relation [3] by considering two arbitrary observables, denoted as A and B. This extension results in the generalized form $\Delta_A \Delta_B \ge |\langle C \rangle|$, where *C* denotes the commutator (2iC = [A, B]). Within the quantum information domain, the uncertainty relation also finds expression through the lens of the Shannon entropies [4, 5]. An improved formulation takes the following shape: $H(A) + H(B) \ge -2 \log c_{ab}$, where the Shannon entropy is defined as $H(A) = -\sum_{i} P_{i} \ln P_{i}$ P_j , $c_{ab} = \max_{j,k} |\langle a_j | b_k \rangle|, |a_j \rangle$ with $j = 1, \dots, N$ be the corresponding complete sets of normalized eigenvectors [6].

Advancements in the exploration of the uncertainty relation hold substantial implications for the structure of quantum mechanics, while also yielding extensive repercussions within the realm of quantum information sciences [1]. Noteworthy contributions include the derivation of quantum separability criteria [7] and the determination of quantum nonlocality [8, 9], as exemplified in a recent review such as Ref. [10]. As such, the uncertainty principle remains an enduring and indispensable cornerstone in contemporary physics.

While advances have been achieved in enhancing the variance-based uncertainty relation [11, 12], the persistent issue of lower bound reliance on the quantum state remains. Furthermore, these advancements are not exempt from the challenge known as the "triviality" problem [13]. By amalgamating the methodologies of entropic measures and variance analyses, a nearly flawless lower bound was successfully derived [14]. Consequently, the pursuit of an optimal trade-off uncertainty relation for variances of physical observables that remain independent of the quantum state continues to stand as an important and unresolved inquiry.

Evidently, the majority of quantum uncertainty relation theories adopt a bounded form. This indicates that the variance of an observable is equal to or greater than a specific value; however, the means to effectively manipulate measurement deviation have remained elusive.

The motivation for using geometric illustrations to represent the uncertainty principle comes from the desire to visualize and intuitively grasp these abstract and probabilistic concepts. While the uncertainty principle can be mathematically expressed using operators in quantum mechanics, it can be difficult to conceptualize in purely mathematical terms. Geometric representations, such as the Heisenberg uncertainty principle's phase space diagram, offer a way to visualize the trade-off between precision in position and momentum measurements. By representing uncertainty geometrically, it becomes more accessible and helps physicists and students gain a deeper insight into the fundamental limitations of our knowledge at the quantum level.

In the present study, through a detailed derivation process, a significant revelation emerges. By employing the geometric framework encompassing both the quantum state and observable state, an uncertainty relation emerges that hinges upon a surface angle. Notably, this relation transitions from an inequality to an equation. The crux of this surface angle lies in its correlation with the dynamics inherent in quantum state transitions. To elucidate these findings, several illustrative visual representations are employed, offering a collection of examples and applications.

2. Reformulating the Uncertainty Relation

Quantum theory characterizes systems through density matrices, and their physical observables take the form of operators. Typically, these observable operators exhibit Hermitian properties. In the context of a specific physical system described by the density matrix ρ , the uncertainty variance pertaining to the measurement of an observable A is articulated as follows:

$$\Delta_A^2 = Tr[A^2\rho] - (Tr[A\rho])^2.$$
⁽¹⁾

Broadly speaking, we contemplate a unitary matrix of dimensions $N \times N$, symbolized as SU(N). Within this context, let λ_j represent the generators of the SU(N) group, where the index *j* spans from 1 to $N^2 - 1$. The generators follow a commutation relation.

$$\left[\lambda_{j},\lambda_{k}\right] = 2i\sum_{l=1}^{N^{2}-1} f_{jkl}\lambda_{l}, \left\{\lambda_{j},\lambda_{k}\right\} = \frac{4}{N}\delta_{ij}I + \sum_{l=1}^{N^{2}-1} d_{jkl}\lambda_{l}.$$
 (2)

In this context, *I* represents the identity matrix, while d_{jkl} denotes the symmetric structure constants and f_{jkl} stands for the antisymmetric structure constants pertaining to SU(*N*). The quantum states and observable quantity of the system can be denoted in $N \times N$ Hermitian matrix [15] as

$$\rho = \frac{1}{N}I + \frac{1}{2}\sum_{j=1}^{N^2 - 1} p_j \lambda_j, A = \frac{1}{2}\sum_{j=1}^{N^2 - 1} a_j \lambda_j.$$
(3)

In this context, we have $p_j = Tr[\rho\lambda_j]$ and $a_j = Tr[A\lambda_j]$. This pertains to the Bloch vector representation of the Hermitian matrix, as expounded in [16].

Several well-established relations hold true; λ_j exhibit tracelessness; $Tr[\lambda_j\lambda_l] = 2\delta_{ij}$; $Tr[\lambda_j\lambda_k\lambda_l] = 2d_{jkl} - 2if_{kjl}$; it is pertinent to note that a_ja_k features a symmetrical structure, whereas the antisymmetry structure constant f_{kjl} dictates that $a_ja_kf_{kjl}$ must necessarily be nullified [1].

Upon substituting Eq. (2) and Eq. (3) into Eq. (1), the variance associated with observable A within the quantum state characterized by density ρ can be expressed as [1]

$$\begin{aligned} \Delta_A^2 &= Tr[A^2\rho] - Tr[A\rho]^2 = \frac{1}{2N}\vec{a}^2 + \frac{1}{4}\vec{a'}\cdot\vec{p} - \frac{1}{4}\left|\vec{a}\cdot\vec{p}\right|^2 \\ &= \frac{1}{2N}\vec{a}^2 + \frac{1}{4}\left|\vec{a'}\right|\left|\vec{p}\right|\cos\theta_{\mathrm{pa'}} - \frac{1}{4}\left|\vec{a}\right|^2\left|\vec{p}\right|^2\cos^2\theta_{\mathrm{pa}}, \end{aligned}$$

$$\tag{4}$$

where $\overrightarrow{a} = a_i, a'_l = a_i a_k d_{ikl}$.

This variance is partly related by the angles $\cos \theta_{\text{pa}} = \vec{a} \cdot \vec{p} / |\vec{a}| |\vec{p}|$ and $\cos \theta_{\text{pa}'} = \vec{a'} \cdot \vec{p} / |\vec{a'}| |\vec{p}|$.

3. Uncertainty Relation and Relevant Equation

For the sake of convenience, let us begin by briefly revisiting an insightful reference, Ref. [1]. We adopt the utilization of 3-dimensional vectors \vec{p} , \vec{a} , and \vec{b} for the representation of the quantum state ρ and the two observables A and Bwithin a 2-dimensional Hilbert space. By defining $a_A = (1/2)$ $Tr[A^2]$, $b_B = (1/2)Tr[B^2]$, and g = (1/2)Tr[AB] and identifying the generators λ_j of SU(2) as the Pauli matrices σ_i (where i = 1, 2, 3), we proceed. Referring to Eq. (4), the variances are transformed into

$$4\Delta_A^2 = a^2 - a^2 p^2 \cos^2 \theta_{\rm pa},\tag{5}$$

$$4\Delta_B^2 = b^2 - b^2 p^2 \cos^2\theta_{\rm pb}.$$
 (6)

Given that N = 2 and $\{\sigma_j, \sigma_k\} = 2\delta_{jk}$, coupled with the application of Eq. (5) and Eq. (6), we deduce

$$\cos^2 \theta_{\rm pa} = \frac{a^2 - 4\Delta_A^2}{a^2 p^2}, \, \sin^2 \theta_{\rm pa} = \frac{a^2 p^2 - a^2 + 4\Delta_A^2}{a^2 p^2}, \tag{7}$$

$$\cos^2\theta_{\rm pb} = \frac{b^2 - 4\Delta_B^2}{b^2 p^2}, \\ \sin^2\theta_{\rm pb} = \frac{b^2 p^2 - b^2 + 4\Delta_B^2}{b^2 p^2}.$$
(8)

Examining Figure 1 reveals that $\theta_{ab} \leq \theta_{pa} + \theta_{pb}$, where the ranges satisfy $0 \leq \theta_{ab}, \theta_{pa}, \theta_{pb} \leq (\pi/2)$. This deduction yields

$$\cos \theta_{ab} \ge \cos \left(\theta_{pa} + \theta_{pb} \right) \ge \cos \theta_{pa} \cos \theta_{pb} - \sin \theta_{pa} \sin \theta_{pb}.$$
(9)

By substituting Eq. (7) and Eq. (8) into Eq. (9), while considering a pure state $(|\vec{p}| = 1)$ and adhering to the initial conditions $a_A = (1/2)Tr[A^2]$, $b_B = (1/2)Tr[B^2]$, g = (1/2)Tr[AB], and $\cos \theta_{ab} = \vec{a} \cdot \vec{b}/|a||b|$, the resultant expression for the uncertainty relation concerning arbitrary observables *A* and *B* is established [1].

$$\Delta_A \Delta_B \ge \sqrt{a_A - \Delta_A^2} \sqrt{b_B - \Delta_B^2} - g. \tag{10}$$

An intriguing point surfaces here. Upon revisiting Figure 1, it is notable that the three angles conform to the inequality $|\theta_{pa} - \theta_{pb}| \le \theta_{ab} \le \theta_{pa} + \theta_{pb}$. This relationship becomes evident as we visualize the surface (\vec{b}, \vec{p}) enveloping the \vec{p} axis. This prompts us to investigate the exact relationships among these three angles.

Pursuing the guidance of Figure 2, we unveil the correlation among the angles. To streamline the procedure, we establish $\vec{a} \longrightarrow \overrightarrow{OA}$, $\vec{b} \longrightarrow \overrightarrow{OB}$, and $\vec{p} \longrightarrow \overrightarrow{OP}$, transitioning from Figures 1 and 2. Accordingly, $\theta_{ab} \longrightarrow \theta_{AOB}$, θ_{pb} $\longrightarrow \theta_{POB}$, and $\theta_{pa} \longrightarrow \theta_{POA}$ once more come into play.

In Figure 2, we observe three real vectors: \overrightarrow{OA} , \overrightarrow{OP} , and \overrightarrow{OB} . Rescaling the lengths of these vectors offers a convenient approach to determine the interrelation of angles. To achieve this, we set the length of vector \overrightarrow{OP} such that \overrightarrow{AP} is perpendicular to \overrightarrow{OA} and \overrightarrow{BP} is also perpendicular to \overrightarrow{OP} . Consequently, both θ_{APO} and θ_{BPO} become right angles, and θ_{APB} is denoted as θ . This leads us to ascertain that $|\overrightarrow{AP}| = |\overrightarrow{OA}| \sin \theta_{AOP}$, $|\overrightarrow{OP}| = |\overrightarrow{OA}| \cos \theta_{AOP}$, $|\overrightarrow{BP}| = |\overrightarrow{OP}|$ tan $\theta_{BOP} = |\overrightarrow{OA}| \cos \theta_{AOP} \tan \theta_{BOP}$, and $|\overrightarrow{OB}| = |\overrightarrow{OP}|/\cos \theta_{BOP} = |\overrightarrow{OA}| \cos \theta_{AOP}/\cos \theta_{BOP}$.

Then, it follows from $\triangle APB$ that

$$\left|\overrightarrow{AB}\right|^{2} = \left|\overrightarrow{AP}\right|^{2} + \left|\overrightarrow{BP}\right|^{2} - 2\left|\overrightarrow{AP}\right| \left|\overrightarrow{BP}\right| \cos \theta$$
$$= \left|\overrightarrow{OA}\right|^{2} \sin^{2} \theta_{AOP} + \left|\overrightarrow{OA}\right|^{2} \cos^{2} \theta_{AOP} \tan^{2} \theta_{BOP}$$
$$- 2\left|\overrightarrow{OA}\right|^{2} \sin \theta_{AOP} \cos \theta_{AOP} \tan \theta_{BOP} \cos \theta.$$
(11)



FIGURE 1: The geometric correlation existing within the 3dimensional real space between the quantum state \vec{p} and various observables like \vec{a} , \vec{b} , and \vec{c} is depicted. This figure is copied from Ref. [1].



FIGURE 2: Three real vectors, denoted as $\vec{l_1}$, $\vec{l_2}$, and $\vec{l_3}$, exhibit angles of θ_p , θ_a , and θ_b between $\vec{l_1}$ and $\vec{l_2}$, $\vec{l_2}$, and $\vec{l_3}$, and $\vec{l_1}$ and $\vec{l_3}$, respectively.

$$\cos \theta_{AOB} = \frac{\left| \overrightarrow{OA} \right|^2 + \left| \overrightarrow{OB} \right|^2 - \left| \overrightarrow{AB} \right|^2}{2 \left| \overrightarrow{OA} \right| \left| \overrightarrow{OB} \right|}$$
$$= \cos \theta_{BOP} \cos \theta_{AOP} + \sin \theta_{BOP} \sin \theta_{AOP} \cos \theta,$$
(12)

when $\theta = 0$, $\cos \theta_{AOB} = \cos (\theta_{BOP} - \theta_{AOP})$, and when the two surfaces are coplane $(\theta = \pi)$, $\cos \theta_{AOB} = \cos (\theta_{BOP} + \theta_{AOP})$.

4. The Surface Angle

Now, we have a special angle θ , and it is the surface angle of the surface (\vec{a}, \vec{p}) and surface (\vec{b}, \vec{p}) . From Eq. (10), we know that uncertainty $\Delta_A \Delta_B$ has the relation

$$\frac{\left|\sqrt{a_A - \Delta_A^2}\sqrt{b_B - \Delta_B^2} - g\right|}{\Delta_A \Delta_B} \le 1,$$
(13)

where both Δ_A and Δ_B are greater than zero. When we use surface angle θ to express $\cos \theta_{ab}$ in Eq. (9), we have

$$\cos \theta_{ab} = \cos \theta_{pa} \cos \theta_{pb} + \sin \theta_{pa} \sin \theta_{pb} \cos \theta.$$
(14)

For $\triangle AOB$, we have a new relevant equation

Substituting Eq. (7) and Eq. (8) into Eq. (14), we get

$$\Delta_A \Delta_B \cos \theta = \left| \sqrt{a_A - \Delta_A^2} \sqrt{b_B - \Delta_B^2} - g \right|.$$
(15)

When Δ_A and Δ_B are nonzero, we have

$$\frac{\left|\sqrt{a_A - \Delta_A^2}\sqrt{b_B - \Delta_B^2} - g\right|}{\Delta_A \Delta_B} = \cos \theta.$$
(16)

We rigorously derive an equation of equality (16), rather than an inequality (13). In other words, the choice of $\cos \theta$ is not arbitrary. $\cos \theta$ possesses a meaningful interpretation that applies to both geometrical and physical explanations of the uncertainty relation.

The angle θ represents the angular separation between two surfaces, each composed of vectors. In regard to Eq. (16), we ultimately establish that the uncertainty relation governing the observables A and B (as indicated in Eq. (10)) not only adheres to a value less than one, but it also adopts the form of $\cos \theta$, with θ being no mere arbitrary quantity; rather, it exhibits an intricate connection with the

vectors \vec{a} , b, and \vec{p} .

Assuming $A = \vec{\sigma} \cdot \vec{n_a}$ and $B = \vec{\sigma} \cdot \vec{n_b}$, with $\vec{n_a}$ and $\vec{n_b}$, denoting arbitrary unit vectors, applying Eq. (16), we obtain

$$\Delta_A \Delta_B \cos \theta = \left| \sqrt{1 - \Delta_A^2} \sqrt{1 - \Delta_B^2} - \cos \theta_{\rm ab} \right|. \tag{17}$$

In this quantum system, suppose $\Delta_A^2 = 1/4$ and the angle between observables *A* and *B* is $\pi/2$, Eq. (18) can tell us

$$\Delta_B = \frac{6}{\sqrt{5 - \cos(2\theta)}} \text{ or } \quad \Delta_B = \frac{6}{\sqrt{7 + \cos(2\theta)}}.$$
 (18)

Nonetheless, if we were to utilize the conventional relationship stated in Eq. (10), the outcome would be confined to a bound-form result.

$$\frac{\sqrt{3}}{2} \le \Delta_B \le \sqrt{\frac{3}{2}}.$$
(19)

However, within this context, we unveil a precise solution linking the uncertainty of an observable Δ_B and the surface angle θ , as exemplified in Figure 3.

However, in practice, this does not align with a true physics experimental procedure. The surface angle θ signifies a quantum state's position as it navigates through the Hilbert space, guided by the parameter θ . This is due to the fact that each quantum state ρ position establishes a connection with measurement states *a* and *b*. When performing measurements in an actual experi-



FIGURE 3: The changing of Δ_B with surface angle θ , where $(\sqrt{3}/2) \leq \Delta_B \leq \sqrt{(3/2)}$. In the case of $A = \vec{\sigma} \cdot \vec{n_a}$, $B = \vec{\sigma} \cdot \vec{n_b}$ and $\Delta_A^2 = 1/4$, $\theta_{ab} = \pi/2$.

ment, it is not possible to confine the quantum state within a preparatory phase. All that can be achieved beforehand is the accurate determination of the two observables' states.

Let us delve into the practical measurement process. To begin with, we revert to the scenario of a pure state, denoted by $|\vec{p}| = 1$. As an angular vector, \vec{p} entails two parameters, namely, (θ, ϕ) , where θ signifies the polar angle and ϕ represents the azimuthal angle. During an actual experiment, if we have designated two observables, *a* and *b*, the angle θ_{ab} between them must already be established. Subsequently, leveraging geometrical principles, we can deduce θ_{ap} and θ_{bp} . Thus, the surface angle θ_s is directly derived via Eq. (14),

$$\cos \theta_{\rm ap} = \sin \theta \cos \phi, \cos \theta_{\rm bp} = \sin \theta \cos (\theta_{\rm ab} - \phi), \quad (20)$$

$$\cos \theta_s = \frac{\cos \theta_{ab} - \cos \theta_{ap} \cos \theta_{bp}}{\sin \theta_{ap} \sin \theta_{bp}}.$$
 (21)

Putting Eq. (20) and Eq. (21) into Eq. (18), we reach the final result in Figure 4 that shows Δ_B versus (θ, ϕ) in the cases of $\Delta_A = 1/2$ and $\theta_{ab} = \pi/2$.

In Figure 4, it becomes evident that distinct placements of a quantum state \vec{p} yield varying degrees of uncertainty. This vividly illustrates that the dynamic nature of quantum states gives rise to the presence of uncertainty. Figure 5 demonstrates alternative scenarios depicting varied values of Δ_A and θ_{ab} .



FIGURE 4: (a) In the case of $\Delta_A^2 = 1/4$ and $\theta_{ab} = \pi/2$, we can find Δ_B has a specific scope from $(\sqrt{3}/2)$ to $\sqrt{(3/2)}$ for the (θ, ϕ) plane. (b) We can change it to polar coordinates to make Δ_B be radius; it can be understood directly that different positions of the quantum state will lead to uncertainty of measurement.



FIGURE 5: (a) When $\Delta_A^2 = 1/9$ and $\theta_{ab} = \pi/3$, Δ_B corresponds to the radius in polar coordinates. (b) When $\Delta_A^2 = 1/25$ and $\theta_{ab} = \pi/4$, Δ_B also represents the radius in polar coordinates.

5. Summary and Conclusions

We revisit Li and Qiao's method pertaining to a novel uncertainty relation [1], elaborating on certain calculation and proof specifics. We then recognize the potential for a more precise generalization of this relation through geometric interpretation, succinctly encapsulated by a surface angle θ . By incorporating the $\cos \theta$ term into the uncertainty relation, we not only transform it into an equality equation but also establish a means to determine any value of uncertainty $\Delta_A \Delta_B$ via the surface angle θ . This paper marks the inaugural discovery of this shift from inequality to equation. The intrinsic significance of this surface angle lies in capturing the dynamic behavior of quantum states. Through specialized calculations in a specific scenario, we attain results that vividly elucidate the interplay between uncertainty and distinct quantum states. This method stands as a pivotal approach and an indispensable visual outcome in the realm of quantum information and measurement. Building upon the visual outcomes observed in the Bloch vector space [16], our findings could also find application in various aspects of the uncertainty relation, such as incompatible observables [17–19].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

Hao Xu, Mengjie Shi, and Shuijing Li wrote the main manuscript text and prepared Figures 2–5. All authors reviewed the manuscript.

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