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Research Article

Contributions of $K_0^*(1430)$ and $K_0^*(1950)$ in the Charmed Three-Body B Meson Decays

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In this work, we investigate the resonant contributions of $K_0^*(1430)$ and $K_0^*(1950)$ in the three-body $B_{(s)} \longrightarrow D_{(s)}K\pi$ within the perturbative QCD approach. The form factor $F_{k\pi}(s)$ is adopted to describe the nonperturbative dynamics of the S-wave $K\pi$ system. The branching ratios of all concerned decays are calculated and predicted to be in the order of 10^{-10} to 10^{-5}. The ratio R of branching fractions between $B^0 \longrightarrow \bar{D}^0 K_0^{*0}(1430) \longrightarrow \bar{D}^0 K^+\pi^-$ and $B_s^0 \longrightarrow \bar{D}^0 \bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0 K^-\pi^+$ is predicted to be 0.0552, which implies the discrepancy for the LHCb measurements. We expect that the predictions in this work can be tested by the future experiments, especially, to resolve R ratio discrepancy.

1. Introduction

Decays of the type $B \longrightarrow Dhh'$, where a B meson decays to a charmed meson and two light pseudoscalar mesons, have attracted people's attention in recent years. On the one hand, the studies of these three-body processes have shown the potential to constrain the parameters of the unitarity triangle. For instance, the decay $B^0 \longrightarrow \bar{D}^0\pi^+\pi^-$ is sensitive to measure the CKM angle β [1, 2], while the Dalitz plot analysis of the decays $B^0 \longrightarrow \bar{D}^0 K^+\pi^-$ and $B_s^0 \longrightarrow \bar{D}^0 K^+K^-$can further improve the determination of the CKM angle γ [3]. On the other hand, the $B \longrightarrow Dhh'$ decays provide opportunities for probing the rich resonant structure in the final states, including the spectroscopy of charmed mesons and the components in two light meson systems. A series of results in this area have been acquired from the measurements performed by the Belle [4], BaBar [5, 6], and LHCb [3, 7–9] Collaborations.

In theory, a direct analysis of the three-body B decays is particularly difficult on account of the entangled resonant and nonresonant contributions, the complex interplay between the weak processes and the low-energy strong interactions [10], and other possible final state interactions [11, 12]. Fortunately, most of the three-body hadronic B meson decay processes are considered to be dominated by the low-energy S-, P-, and D-wave resonant states, which could be treated in the quasi-two-body framework. By neglecting the interactions between the meson pair originated from the resonant states and the bachelor particle in the final states, the factorization theorem is still valid as in the two-body case [13, 14], and substantial theoretical efforts for different quasi-two-body B meson decays have been made within different theoretical approaches [15, 16]. As well, the contributions from various intermediate resonant state for the three-body decays $B \longrightarrow Dhh'$ have been investigated in Ref. [17–20].

The understanding of the scalar mesons is a difficult and long-standing issue [21]. The scalar resonances usually have large decay widths which make them overlap strongly with the background. In the specific regions, such as the $K\bar{K}$ and $\eta\eta$ thresholds, cusps in the line shapes of the nearby resonances will appear due to the contraction of the phase space. Moreover, the inner natures of scalars are still not completely clear. Part of them, especially the ones below 1 GeV, have also been interpreted as glueballs, meson-meson bound states, or multiquark states, besides the

traditional quark-antiquark configurations [22, 23]. The $K_0^*(1430)$ is perhaps the least controversial of the light scalar mesons and generally believed to be a $q\bar{q}$ state [24]. It predominantly couples to the $K\pi$ channel and has been studied experimentally in many charmless three-body B meson decays [25–27]. Recently, measurements of the charmed three-body decays $B^0 \longrightarrow \bar{D}^0 K^+\pi^-$ and $B_s^0 \longrightarrow \bar{D}^0 K^+\pi^-$ involving the resonant state $K_0^*(1430)$ were also presented by LHCb [3, 8]. In addition, the subprocess $K_0^*(1950) \longrightarrow K\pi$ which often ignored in literatures has also been considered in Ref. [8].

In the framework of the PQCD approach [28–30], the investigation of S-wave $K\pi$ contributions to the $B_{(s)}^0 \longrightarrow \psi K\pi$ decays was carried out in Ref. [31]. In a more recent work [32], contributions of the resonant states $K_0^*(1430)$ and $K_0^*(1950)$ in the three-body decays $B \longrightarrow K\pi h$ $(h = K, \pi)$ were studied systematically within the same method. The $K_0^*(1430)$ is treated as the lowest lying $q\bar{q}$ state in view of the controversy for $K_0^*(700)$, and the scalar $K\pi$ timelike form factor $F_{K\pi}(s)$ was also discussed in detail. Motivated by the related results measured by LHCb [3, 8], we shall extent the previous work [32] to the study of the charmed three-body B decays and analyse the contributions of the resonances $K_0^*(1430)$ and $K_0^*(1950)$ in the $B \longrightarrow DK\pi$ decays in this work.

The rest of this article is structured as follows. In Section 2, we give a brief review of the framework of the PQCD approach. The numerical results and phenomenological discussions are presented in Section 3, and a short summary is given in Section 4, respectively. Finally, the relevant factorization formulae for the decay amplitudes are collected in the appendix.

2. Framework

In the light-cone coordinate system, the B meson momentum p_B, the total momentum of the $K\pi$ pair, and the D meson momentum p_3 under the rest frame of B meson can be written as

$$\begin{aligned} p_B &= \frac{m_B}{\sqrt{2}}(1, 1, 0_T), \\ p &= \frac{m_B}{\sqrt{2}}\left(1 - r^2, \eta, 0_T\right), \\ p_3 &= \frac{m_B}{\sqrt{2}}\left(r^2, 1 - \eta, 0_T\right), \end{aligned} \tag{1}$$

with m_B being the B meson mass and the mass ratio $r = m_D/m_B$. The variable η equals to $s/(m_B^2 - m_D^2)$, where s is the invariant mass squared of $K\pi$ pair in the range from $(m_K + m_\pi)^2$ to $(m_B - m_D)^2$. We also set the momenta of the light quarks in the B meson, the $K\pi$ pair, and the D meson as K_B, K, and K_3 and have the definitions as follows:

$$\begin{aligned} k_B &= \left(0, \frac{m_B}{\sqrt{2}}x_B, k_{BT}\right), \\ k &= \left(\frac{m_B}{\sqrt{2}}\left(1 - r^2\right)z, 0, k_T\right), \\ k_3 &= \left(0, \frac{m_B}{\sqrt{2}}(1 - \eta)x_3, k_{3T}\right), \end{aligned} \tag{2}$$

where x_B, z, and x_3 are the momentum fractions and run from zero to unity.

In the PQCD approach, the decay amplitude for the quasi-two-body decay $B_{(s)} \longrightarrow D_{(s)}K_0^*(1430, 1950) \longrightarrow D_{(s)} K\pi$ can be expressed as the convolution [33]

$$\mathscr{A} = \phi_B \otimes H \otimes \phi_D \otimes \phi_{K\pi}, \tag{3}$$

where the symbol H represents the hard kernel with single hard gluon exchange. ϕ_B and ϕ_D are the distribution amplitudes for the B and D mesons, respectively. $\phi_{K\pi}$ denotes the distribution amplitude for the $K\pi$ pair with certain spin in the resonant region. In this work, we use the same distribution amplitudes for the $B_{(s)}$ and $D_{(s)}$ mesons as in Ref. [18] where one can easily find their expressions and the relevant parameters. Inspired by generalized distribution amplitude [34–37], the generalized LCDA for two-meson system are introduced [33, 38] for three-body B-meson decay in the framework of PQCD approach and the heavy-to-light transition form factor in light-cone sum rules, respectively. The nonlocal matrix elements of vacuum to $K\pi$ with various spin projector can be written as

$$\begin{aligned} \left\langle K\pi \middle| \bar{s}(x)\gamma_\mu q(0) \middle| 0 \right\rangle &= p_\mu \int_0^1 dz\, e^{iz p\cdot x}\, \phi^0(z, s), \\ \langle K\pi | \bar{s}(x) q(0) | 0 \rangle &= \sqrt{s} \int_0^1 dz\, e^{iz p\cdot x}\, \phi^s(z, s), \\ \left\langle K\pi \middle| \bar{s}(x)\sigma_{\mu\nu} q(0) \middle| 0 \right\rangle &= -\frac{\sqrt{s}}{6}\left(p_\mu x_\nu - p_\nu x_\mu\right) \int_0^1 dz\, e^{iz p\cdot x}\, \phi^t(z, s). \end{aligned} \tag{4}$$

The $K\pi S$-wave distribution amplitude is chosen as [32]

$$\Phi_{K\pi}(z, s) = \frac{1}{\sqrt{2N_c}}\left[p\!\!/\phi^0(z, s) + \sqrt{s}\phi^s(z, s) + \sqrt{s}(n\!\!/v\!\!/ - 1)\phi^t(z, s)\right], \tag{5}$$

where $n = (1, 0, \mathbf{0}_T)$ and $v = (0, 1, \mathbf{0}_T)$ are the dimensionless lightlike unit vectors. The twist-2 and twist-3 light-cone distribution amplitudes have the form

$$\begin{aligned} \phi^0(z, s) &= \frac{F_{K\pi}(s)}{2\sqrt{2N_c}} 6z(1 - z)\left[a_0(\mu) + \sum_{m=1}^{\infty} a_m(\mu) C_m^{3/2}(2z - 1)\right], \\ \phi^s(z, s) &= \frac{F_{K\pi}(s)}{2\sqrt{2N_c}}, \\ \phi^t(z, s) &= \frac{F_{K\pi}(s)}{2\sqrt{2N_c}}(1 - 2z). \end{aligned} \tag{6}$$

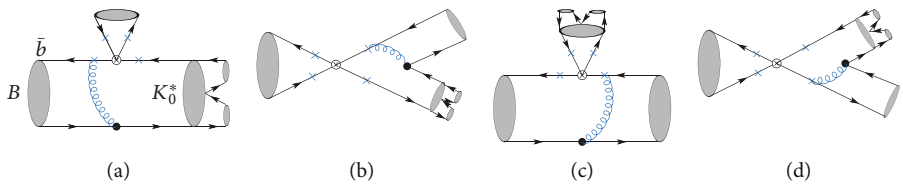

Figure 1: Typical diagrams for the quasi-two-body decays $B_{(s)} \longrightarrow D_{(s)}K_0^*(1430, 1950) \longrightarrow D_{(s)}K\pi$ including the emission diagram (a) with the $B \longrightarrow K_0^*(1430, 1950)$ transition, the emission diagram (c) with the $B \longrightarrow D$ transition, and the annihilation diagrams (b) and (d). The symbol $\otimes$ stands for the weak vertex, and $\times$ denotes possible attachments of hard gluons.

Here, $C_m^{3/2}$ are the Gegenbauer polynomials, $a_m(\mu)$ are the Gegenbauer moments, and $F_{K\pi}(s)$ is the scalar form factor for the $K\pi$ pair. In this work, we adopt the same formulae and parameters for the $K\pi S$-wave distribution amplitude as them in Ref. [32].

According to the typical Feynman diagrams as shown in Figure 1 and the quark currents for each decays, the decay amplitudes for the considered quasi-two-body decays $B \longrightarrow DK_0^*(1430, 1950) \longrightarrow DK\pi$ are given as

$$\begin{aligned}
\mathscr{A}(B^+ \longrightarrow D^0[K_0^{*+} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}\{a_2 F_{TK} + C_2 M_{TK} + a_1 F_{AD} + C_1 M_{AD}\},\\
\mathscr{A}\left(B^+ \longrightarrow \bar{D}^0[K_0^{*+} \longrightarrow]K\pi\right) &= \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us}\left\{a_2 F_{TK} + C_2 M_{TK}' + a_1 F_{TD} + C_1 M_{TD}\right\},\\
\mathscr{A}\left(B^+ \longrightarrow D^+\left[K_0^{*0} \longrightarrow\right]K\pi\right) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}\{a_1 F_{AD} + C_1 M_{AD}\},\\
\mathscr{A}\left(B^+ \longrightarrow D_s^+\left[\bar{K}_0^{*0} \longrightarrow\right]K\pi\right) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}\{a_1 F_{AD} + C_1 M_{AD}\},\\
\mathscr{A}(B^0 \longrightarrow D^0[K_0^{*0} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}\{a_2 F_{TK} + C_2 M_{TK}\},\\
\mathscr{A}\left(B^0 \longrightarrow \bar{D}^0[K_0^{*0} \longrightarrow]K\pi\right) &= \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us}\left\{a_2 F_{TK} + C_2 M_{TK}'\right\},\\
\mathscr{A}(B^0 \longrightarrow D^-[K_0^{*+} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us}\{a_1 F_{TD} + C_1 M_{TD}\},\\
\mathscr{A}(B^0 \longrightarrow D_s^-[K_0^{*+} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud}\{a_2 F_{AK} + C_2 M_{AK}\},\\
\mathscr{A}(B^0 \longrightarrow D_s^+[K_0^{*-} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}\{a_2 F_{AD} + C_2 M_{AD}\},\\
\mathscr{A}\left(B_s \longrightarrow D^0\left[\bar{K}_0^{*0} \longrightarrow\right]K\pi\right) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}\{a_2 F_{TK} + C_2 M_{TK}\},\\
\mathscr{A}\left(B_s \longrightarrow \bar{D}^0\left[\bar{K}_0^{*0} \longrightarrow\right]K\pi\right) &= \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud}\left\{a_2 F_{TK} + C_2 M_{TK}'\right\},\\
\mathscr{A}(B_s \longrightarrow D^+[K_0^{*-} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}\{a_1 F_{TK} + C_1 M_{TK}\},\\
\mathscr{A}(B_s \longrightarrow D_s^-[K_0^{*+} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us}\{a_1 F_{TD} + C_1 M_{TD} + a_2 F_{AK} + C_2 M_{AK}\},\\
\mathscr{A}(B_s \longrightarrow D_s^+[K_0^{*-} \longrightarrow]K\pi) &= \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}\{a_1 F_{TK} + C_1 M_{TK} + a_2 F_{AD} + C_2 M_{AD}\},
\end{aligned} \tag{7}$$

where G_F is the Fermi constant, V_{ij} is the CKM matrix element, and the combinations of the Wilson coefficients $a_{1,2}$ are defined as $a_1 = C_1/3 + C_2$ and $a_2 = C_2/3 + C_1$. The expressions of individual amplitudes F_{TK}, F_{TD}, F_{AK}, F_{AD}, $M_{TK}^{(\prime)}$, M_{TD}, M_{AK}, and M_{AD} from different subdiagrams in Figure 1 are collected in the appendix.

At last, we give the definition of the differential branching ratio for the considered quasi-two-body decays

$$\frac{d\mathscr{B}}{ds} = \tau_B \frac{\left|\vec{p}_1\right|\left|\vec{p}_3\right|}{64\pi^3 m_B^3}\left|\mathscr{A}\right|^2. \tag{8}$$

In the center-of-mass frame of $K\pi$ system, the magnitudes of the momenta $|\vec{p}_1|$ and $|\vec{p}_3|$ can be expressed as

$$\begin{aligned}
\left|\vec{p}_1\right| &= \frac{1}{2}\sqrt{\frac{\left[\left(m_K^2 + m_\pi^2\right)^2 - 2\left(m_K^2 + m_\pi^2\right)s + s^2\right]}{s}},\\
\left|\vec{p}_3\right| &= \frac{1}{2}\sqrt{\frac{\left[\left(m_B^2 + m_D^2\right)^2 - 2\left(m_B^2 + m_D^2\right)s + s^2\right]}{s}}.
\end{aligned} \tag{9}$$

3. Results

In the numerical calculations, the masses of the involved mesons (GeV), the lifetime of the B mesons (ps), the resonance decay widths (GeV), and the Wolfenstein parameters are taken from the Review of Particle Physics [21]

$$\begin{aligned}
m_{B^\pm} &= 5.279,\\
m_{B^0} &= 5.280,\\
m_{B_s^0} &= 5.367,\\
m_{D^0/\bar{D}^0} &= 1.865,\\
m_{D^\pm} &= 1.870,\\
m_{D_s^\pm} &= 1.968,\\
m_{K^\pm} &= 0.494,\\
m_{K^0/\bar{K}^0} &= 0.498,
\end{aligned}$$

Table 1: PQCD predictions for the branching fractions of the quasi-two-body decays $B \longrightarrow D_{(s)} K_0(1430)^* \longrightarrow D_{(s)} K\pi$ together with the available experimental data.

Mode	Unit	$\mathscr{B}$	Data
$B^+ \longrightarrow D^0 K_0^{*+}(1430) \longrightarrow D^0 K^0 \pi^+$	(10^{-6})	$4.15 \pm 0.30(\omega_B) \pm 0.16(B_3) \pm 0.15(B_1) \pm 0.25\left(\Gamma_{K_0^*}\right) \pm 0.02(C_D)$	—
$B^+ \longrightarrow \bar{D}^0 K_0^{*+}(1430) \longrightarrow \bar{D}^0 K^0 \pi^+$	(10^{-5})	$2.50 \pm 0.17(\omega_B) \pm 0.28(B_3) \pm 0.04(B_1) \pm 0.20\left(\Gamma_{K_0^*}\right) \pm 0.03(C_D)$	—
$B^+ \longrightarrow D^+ K_0^{*0}(1430) \longrightarrow D^+ K^+ \pi^-$	(10^{-8})	$2.14 \pm 0.87(\omega_B) \pm 0.55(B_3) \pm 0.30(B_1) \pm 0.07\left(\Gamma_{K_0^*}\right) \pm 0.05(C_D)$	—
$B^+ \longrightarrow D_s^+ \bar{K}_0^{*0}(1430) \longrightarrow D_s^+ K^- \pi^+$	(10^{-9})	$2.75 \pm 0.70(\omega_B) \pm 1.18(B_3) \pm 0.68(B_1) \pm 0.06\left(\Gamma_{K_0^*}\right) \pm 0.22(C_D)$	—
$B^0 \longrightarrow D^0 K_0^{*0}(1430) \longrightarrow D^0 K^+ \pi^-$	(10^{-6})	$3.90 \pm 0.28(\omega_B) \pm 0.01(B_3) \pm 0.13(B_1) \pm 0.23\left(\Gamma_{K_0^*}\right) \pm 0.01(C_D)$	—
$B^0 \longrightarrow \bar{D}^0 K_0^{*0}(1430) \longrightarrow \bar{D}^0 K^+ \pi^-$	(10^{-5})	$2.23 \pm 0.18(\omega_B) \pm 0.24(B_3) \pm 0.06(B_1) \pm 0.15\left(\Gamma_{K_0^*}\right) \pm 0.05(C_D)$	0.71 [3]
$B^0 \longrightarrow D^- K_0^{*+}(1430) \longrightarrow D^- K^0 \pi^+$	(10^{-7})	$1.08 \pm 0.17(\omega_B) \pm 0.34(B_3) \pm 0.12(B_1) \pm 0.06\left(\Gamma_{K_0^*}\right) \pm 0.02(C_D)$	—
$B^0 \longrightarrow D_s^- K_0^{*+}(1430) \longrightarrow D_s^- K^0 \pi^+$	(10^{-6})	$2.17 \pm 1.08(\omega_B) \pm 1.20(B_3) \pm 0.75(B_1) \pm 0.10\left(\Gamma_{K_0^*}\right) \pm 0.10(C_D)$	—
$B^0 \longrightarrow D_s^+ K_0^{*-}(1430) \longrightarrow D_s^+ \bar{K}^0 \pi^-$	(10^{-9})	$4.24 \pm 1.99(\omega_B) \pm 1.97(B_3) \pm 0.61(B_1) \pm 0.14\left(\Gamma_{K_0^*}\right) \pm 0.09(C_D)$	—
$B_s^0 \longrightarrow D^0 \bar{K}_0^{*0}(1430) \longrightarrow D^0 K^- \pi^+$	(10^{-7})	$2.07 \pm 0.17(\omega_B) \pm 0.20(B_3) \pm 0.11(B_1) \pm 0.13\left(\Gamma_{K_0^*}\right) \pm 0.01(C_D)$	—
$B_s^0 \longrightarrow \bar{D}^0 \bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0 K^- \pi^+$	(10^{-4})	$3.76 \pm 0.16(\omega_B) \pm 0.43(B_3) \pm 0.08(B_1) \pm 0.23\left(\Gamma_{K_0^*}\right) \pm 0.03(C_D)$	3.00 [8]
$B_s^0 \longrightarrow D^+ K_0^{*-}(1430) \longrightarrow D^+ \bar{K}^0 \pi^-$	(10^{-6})	$7.67 \pm 1.71(\omega_B) \pm 0.43(B_3) \pm 0.32(B_1) \pm 0.48\left(\Gamma_{K_0^*}\right) \pm 0.01(C_D)$	—
$B_s^0 \longrightarrow D_s^- K_0^{*+}(1430) \longrightarrow D_s^- K^0 \pi^+$	(10^{-7})	$1.65 \pm 0.28(\omega_B) \pm 0.25(B_3) \pm 0.16(B_1) \pm 0.26\left(\Gamma_{K_0^*}\right) \pm 0.26(C_D)$	—
$B_s^0 \longrightarrow D_s^+ K_0^{*-}(1430) \longrightarrow D_s^+ \bar{K}^0 \pi^-$	(10^{-4})	$1.96 \pm 0.44(\omega_B) \pm 0.16(B_3) \pm 0.08(B_1) \pm 0.12\left(\Gamma_{K_0^*}\right) \pm 0.01(C_D)$	—

$$
\begin{aligned}
m_{\pi^0} &= 0.135, \\
m_{\pi^\pm} &= 0.140, \\
m_{K_0^*(1430)} &= 1.425, \\
m_{K_0^*(1950)} &= 1.945, \\
\tau_{B^0} &= 1.519, \\
\tau_{B^\pm} &= 1.638, \\
\tau_{B_s^0} &= 1.515, \\
\Gamma_{K_0^*(1430)} &= 0.270 \pm 0.080, \\
\Gamma_{K_0^*(1950)} &= 0.201 \pm 0.090, \\
A &= 0.790^{+0.017}_{-0.012}, \\
\lambda &= 0.22650 \pm 0.00048, \\
\bar{\rho} &= 0.141^{+0.016}_{-0.017}, \\
\bar{\eta} &= 0.357 \pm 0.011.
\end{aligned} \tag{10}
$$

The decay constants of the $B_{(s)}$ and $D_{(s)}$ mesons are set to the values $f_{B_{(s)}} = 0.190(0.230)\,\text{GeV}$ and $f_{D_{(s)}} = 0.212(0.250)$ GeV [39].

By integrating the differential branching ratio in Equation (8), we obtain the branching ratios for the considered quasi-two-body processes with the intermediate resonances $K_0^*(1430)$ and $K_0^*(1950)$ in Tables 1 and 2, respectively. The first error is induced by the shape parameters $\omega_{B_{(s)}} = 0.40 \pm 0.04(0.50 \pm 0.05)\,\text{GeV}$ in the distribution amplitude for the $B_{(s)}$ meson. The second and third errors come from the Gegenbauer moments $a_3 = -0.42 \pm 0.22$ and $a_1 = -0.57 \pm 0.13$ in the $K\pi S$-wave distribution amplitude, respectively. The decay widths $\Gamma_{K_0^*(1430)} = 0.270 \pm 0.080\,\text{GeV}$ and $\Gamma_{K_0^*(1950)} = 0.201 \pm 0.090\,\text{GeV}$ contribute the fourth error. The last one is due to the parameter $C_{D_{(s)}} = 0.5 \pm 0.1(0.4 \pm 0.1)$ in the distribution amplitude for $D_{(s)}$ meson. The uncertainties from other parameters are comparatively small and have been neglected.

From the numerical results as listed in Tables 1 and 2, we have the following comments:

(1) In the $B \longrightarrow DR \longrightarrow DK\pi$ decays, we can extract the two-body branching fractions $\mathscr{B}(B \longrightarrow DR)$ by using the relation under the quasi-two-body approximation

Table 2: PQCD predictions for the branching fractions of the quasi-two-body decays $B \longrightarrow D_{(s)}K_0(1950)^* \longrightarrow DK\pi$ together with the available experimental data.

Mode	Unit	$\mathscr{B}$	Data
$B^+ \longrightarrow D^0K_0^{*+}(1950) \longrightarrow D^0K^0\pi^+$	(10^{-7})	$1.39 \pm 0.60(\omega_B) \pm 0.13(B_3) \pm 0.12(B_1) \pm 0.04\left(\Gamma_{K_0^*}\right) \pm 0.04(C_D)$	—
$B^+ \longrightarrow \bar{D}^0K_0^{*+}(1950) \longrightarrow \bar{D}^0K^0\pi^+$	(10^{-7})	$5.52 \pm 2.70(\omega_B) \pm 0.57(B_3) \pm 0.08(B_1) \pm 0.34\left(\Gamma_{K_0^*}\right) \pm 0.12(C_D)$	—
$B^+ \longrightarrow D^+K_0^{*0}(1950) \longrightarrow D^+K^+\pi^-$	(10^{-9})	$1.31 \pm 0.37(\omega_B) \pm 0.17(B_3) \pm 0.10(B_1) \pm 0.10\left(\Gamma_{K_0^*}\right) \pm 0.12(C_D)$	—
$B^+ \longrightarrow D_s^+\bar{K}_0^{*0}(1950) \longrightarrow D_s^+K^-\pi^+$	(10^{-10})	$1.94 \pm 0.44(\omega_B) \pm 0.82(B_3) \pm 0.44(B_1) \pm 0.02\left(\Gamma_{K_0^*}\right) \pm 0.21(C_D)$	—
$B^0 \longrightarrow D^0K_0^{*0}(1950) \longrightarrow D^0K^+\pi^-$	(10^{-7})	$1.30 \pm 0.57(\omega_B) \pm 0.16(B_3) \pm 0.12(B_1) \pm 0.07\left(\Gamma_{K_0^*}\right) \pm 0.01(C_D)$	—
$B^0 \longrightarrow \bar{D}^0K_0^{*0}(1950) \longrightarrow \bar{D}^0K^+\pi^-$	(10^{-7})	$5.23 \pm 2.14(\omega_B) \pm 0.23(B_3) \pm 0.06(B_1) \pm 0.24\left(\Gamma_{K_0^*}\right) \pm 0.14(C_D)$	—
$B^0 \longrightarrow D^-K_0^{*+}(1950) \longrightarrow D^-K^0\pi^+$	(10^{-9})	$4.32 \pm 0.38(\omega_B) \pm 1.21(B_3) \pm 0.42(B_1) \pm 0.24\left(\Gamma_{K_0^*}\right) \pm 0.43(C_D)$	—
$B^0 \longrightarrow D_s^-K_0^{*+}(1950) \longrightarrow D_s^-K^0\pi^+$	(10^{-7})	$1.04 \pm 0.57(\omega_B) \pm 0.70(B_3) \pm 0.29(B_1) \pm 0.03\left(\Gamma_{K_0^*}\right) \pm 0.05(C_D)$	—
$B^0 \longrightarrow D_s^+K_0^{*-}(1950) \longrightarrow D_s^+\bar{K}^0\pi^-$	(10^{-10})	$2.59 \pm 1.18(\omega_B) \pm 1.05(B_3) \pm 0.49(B_1) \pm 0.10\left(\Gamma_{K_0^*}\right) \pm 0.07(C_D)$	—
$B_s^0 \longrightarrow D^0\bar{K}_0^{*0}(1950) \longrightarrow D^0K^-\pi^+$	(10^{-9})	$9.77 \pm 2.68(\omega_B) \pm 0.69(B_3) \pm 0.42(B_1) \pm 0.38\left(\Gamma_{K_0^*}\right) \pm 0.08(C_D)$	—
$B_s^0 \longrightarrow \bar{D}^0\bar{K}_0^{*0}(1950) \longrightarrow \bar{D}^0K^-\pi^+$	(10^{-5})	$1.72 \pm 0.44(\omega_B) \pm 0.18(B_3) \pm 0.04(B_1) \pm 0.05\left(\Gamma_{K_0^*}\right) \pm 0.01(C_D)$	<11 [8]
$B_s^0 \longrightarrow D^+K_0^{*-}(1950) \longrightarrow D^+\bar{K}^0\pi^-$	(10^{-7})	$3.48 \pm 1.50(\omega_B) \pm 0.10(B_3) \pm 0.07(B_1) \pm 0.16\left(\Gamma_{K_0^*}\right) \pm 0.01(C_D)$	—
$B_s^0 \longrightarrow D_s^-K_0^{*+}(1950) \longrightarrow D_s^-K^0\pi^+$	(10^{-9})	$1.12 \pm 0.28(\omega_B) \pm 0.11(B_3) \pm 0.07(B_1) \pm 0.8\left(\Gamma_{K_0^*}\right) \pm 0.03(C_D)$	—
$B_s^0 \longrightarrow D_s^+K_0^{*-}(1950) \longrightarrow D_s^+\bar{K}^0\pi^-$	(10^{-6})	$8.37 \pm 3.71(\omega_B) \pm 0.22(B_3) \pm 0.12(B_1) \pm 0.39\left(\Gamma_{K_0^*}\right) \pm 0.02(C_D)$	—

$$\mathscr{B}(B \longrightarrow DR \longrightarrow DK\pi) = \mathscr{B}(B \longrightarrow DR) \cdot \mathscr{B}(R \longrightarrow K\pi). \tag{11}$$

For the branching fractions of two-body decays with $K_0^*(1430)$ and $K_0^*(1950)$, we shall apply

$$\begin{aligned}\mathscr{B}\left(K_0^{*0} \longrightarrow K^+\pi^-\right) &= \mathscr{B}\left(\bar{K}_0^{*0} \longrightarrow K^-\pi^+\right) = \mathscr{B}\left(K_0^{*+} \longrightarrow K^0\pi^+\right) \\ &= \mathscr{B}\left(K_0^{*-} \longrightarrow \bar{K}^0\pi^-\right) = \frac{2}{3}\mathscr{B}(K_0^* \longrightarrow K\pi).\end{aligned} \tag{12}$$

The values

$$\begin{aligned}\mathscr{B}(K_0^*(1430) \longrightarrow K\pi) &= (93 \pm 10)\%, \\ \mathscr{B}\left(\bar{K}_0^*(1950) \longrightarrow K^-\pi^+\right) &= (52 \pm 14)\%.\end{aligned} \tag{13}$$

Combined with the results listed in Tables 1 and 2, one can obtain the related two-body branching fractions; for example, $B_s^0 \longrightarrow \bar{D}^0\bar{K}_0^{*0}(1430) = 6.06 \pm 0.65 \times 10^{-4}$ and $B_s^0 \longrightarrow \bar{D}^0\bar{K}_0^{*0}(1950) = 3.31 \pm 0.89 \times 10^{-5}$, where the errors are propagated from Equation (13)

(2) The PQCD prediction for the branching fraction $\mathscr{B}(B_s^0 \longrightarrow \bar{D}^0\bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0K^-\pi^+)$ agrees with LHCb's data $(3.00 \pm 0.24 \pm 0.11 \pm 0.50 \pm 0.44) \times 10^{-4}$ [8] within errors, while the PQCD predicted $\mathscr{B}(B^0 \longrightarrow \bar{D}^0K_0^{*0}(1430) \longrightarrow \bar{D}^0K^+\pi^-)$ is much larger than the value $(0.71 \pm 0.27 \pm 0.33 \pm 0.47 \pm 0.08) \times 10^{-5}$ measured by LHCb [3] with significant uncertainties. By comparison, one can find that the decay modes $B_s^0 \longrightarrow \bar{D}^0\bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0K^-\pi^+$ and $B^0 \longrightarrow \bar{D}^0K_0^{*0}(1430) \longrightarrow \bar{D}^0K^+\pi^-$ contain the same decay topology when neglecting the differences of hadronic parameters between B^0 and B_s^0. Then, we evaluate the ratio

$$R = \frac{\mathscr{B}\left(B^0 \longrightarrow \bar{D}^0K_0^{*0}(1430) \longrightarrow \bar{D}^0K^+\pi^-\right)}{\mathscr{B}\left(B_s^0 \longrightarrow \bar{D}^0\bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0K^-\pi^+\right)} \approx \left|\frac{V_{us}}{V_{ud}}\right|^2 \cdot \frac{\tau_{B^0}}{\tau_{B_s^0}} = 0.0534, \tag{14}$$

which is close to the PQCD prediction 0.0593 by using the results listed in Table 1, but different from the value 0.0237 acquired from the central values of the measured branching ratio by LHCb [3, 8]. One can find that in the Ref. [8], the $K_0^*(1430)$ component receives 20% fit fraction of total $\mathscr{B}(B_s^0 \longrightarrow \bar{D}^0K^-\pi^+)$, but in Ref. [3, 8], $K_0^*(1430)$ component receives only 5.1% of total $\mathscr{B}(B^0 \longrightarrow \bar{D}^0K^+\pi^-)$.

The $K_0^*(1430)$ component is playing a such different role in two different process; however, on the theoretical side, the decay amplitudes are exactly same for $B^0 \longrightarrow \bar{D}^0 K_0^{*0}(1430) \longrightarrow \bar{D}^0 K^+\pi^-$ and $B_s^0 \longrightarrow \bar{D}^0 \bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0 K^-\pi^+$; if we neglect the SU(3) symmetry breaking effect, R ratio will be independent of theoretical framework. More precise measurements and more proper partial wave analysis are needed to resolve the discrepancy

(3) For the CKM suppressed decay modes $B_s^0 \longrightarrow D^0 \bar{K}_0^{*0}(1430) \longrightarrow D^0 K^-\pi^+$, their branching ratios are much smaller than the corresponding results of $B_s^0 \longrightarrow D^0 \bar{K}_0^{*0}(1430) \longrightarrow D^0 K^-\pi^+$ decays as predicted by PQCD in this work. The major reason comes from the strong CKM suppression factor

$$R_{\mathrm{CKM}} = \left|\frac{V_{ub}^* V_{cd}}{V_{cb}^* V_{ud}}\right|^2 \approx \lambda^4\left(\bar{\rho}^2 + \bar{\eta}^2\right) \approx 3 \times 10^{-4}, \qquad (15)$$

as discussed in Ref. [40]. The nonvanishing charm quark mass in the fermion propagator generates the main differences between the $B_s^0 \longrightarrow D^0 \bar{K}_0^{*0}(1430) \longrightarrow D^0 K^-\pi^+ / B_s^0 \longrightarrow \bar{D}^0 \bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0 K^-\pi^+$ and R_{CKM}. Similarly, for the $B^+ \longrightarrow D^0 K_0^{*+}(1430) \longrightarrow D^0 K^0\pi^+$ decay and $B^+ \longrightarrow \bar{D}^0 K_0^{*+}(1430) \longrightarrow \bar{D}^0 K^0\pi^+$ decay, there still exists the CKM suppression but much moderate than the previous cases:

$$R_{\mathrm{CKM}}^s = \left|\frac{V_{ub}^* V_{cs}}{V_{cb}^* V_{us}}\right|^2 \approx \left(\bar{\rho}^2 + \bar{\eta}^2\right) \approx 0.147. \qquad (16)$$

From Table 1, we have

$$R_{\mathrm{CKM}}^{s1} = \frac{\mathscr{B}\left(B^+ \longrightarrow D^0 K_0^{*+}(1430) \longrightarrow D^0 K^0\pi^+\right)}{\mathscr{B}\left(B^+ \longrightarrow \bar{D}^0 K_0^{*+}(1430) \longrightarrow \bar{D}^0 K^0\pi^+\right)} \approx 0.166,$$
$$R_{\mathrm{CKM}}^{s2} = \frac{\mathscr{B}\left(B^0 \longrightarrow D^0 K_0^{*0}(1430) \longrightarrow D^0 K^+\pi^-\right)}{\mathscr{B}\left(B^0 \longrightarrow \bar{D}^0 K_0^{*0}(1430) \longrightarrow \bar{D}^0 K^+\pi^-\right)} \approx 0.175. \qquad (17)$$

The main differences between the R_{CKM}^s and $R_{\mathrm{CKM}}^{s1,s2}$ come from the nonvanishing charm quark mass contributions in the nonfactorizable $B \longrightarrow K_0^*(1430)$ emission diagram. We also suggest more study on the decay mode $B_s^0 \longrightarrow D_s^+ K_0^{*-}(1430) \longrightarrow D_s^+ \bar{K}^0\pi^-$ because it has a large branching ratio and can be found in future experiments

(4) $K_0^*(1430)$ was often parameterized by LASS lineshape [41] in partial wave analysis, which incorporate both cusp resonance and slowly varying nonresonance contribution, and it was applied in LHCb measurements [3, 8]. However, rigorous theoretical calculation for nonresonance contribution in the context of PQCD framework is still absent [32], the comparisons between theoretical calculations and experiment measurements focus only on the S-wave $K_0^*(1430)$ contribution. More attempts can be made in future study to parameterize the nonresonance contribution for sake of giving a more reliable result

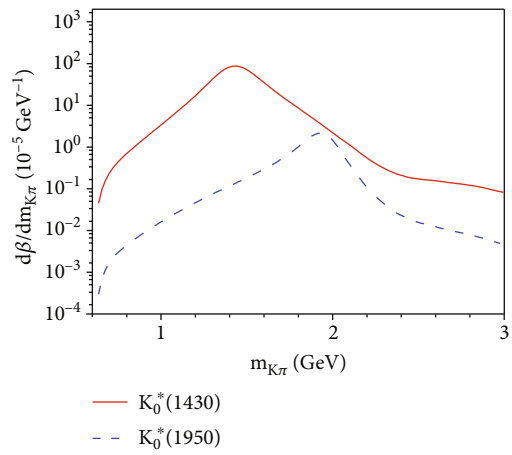

Figure 2: The $K\pi$ invariant mass-dependent differential branching fraction for $B_s^0 \longrightarrow \bar{D}_0^{*0} \bar{K}_0^{*0}(1430) \longrightarrow \bar{D}_0^{*0} K^-\pi^+$ (solid line) and $B_s^0 \longrightarrow \bar{D}_0^{*0} \bar{K}_0^{*0}(1950) \longrightarrow \bar{D}_0^{*0} K^-\pi^+$ (dashed line).

(5) The CP-averaged branching fraction of the charmless quasi-two-body decay involving the intermediate state $K_0^*(1950)$ is predicted to be about one magnitude smaller than the corresponding process containing $K_0^*(1430)$ in [32]. In quasi-two-body charmed decays, the ratio of branching fractions between Tables 1 and 2 is about few percentages, which are smaller than that of charmless cases mainly due to the absence of $(S-P)(S+P)$ amplitude, which receive resonance pole mass enhancement as discussed in [32]. And the more compact phase space can also reduce the branching fractions for the decay mode involving $K_0^*(1950)$. From the partial wave analysis in [8], the $K_0^*(1950)$ mode is measured to be about 1.5% than that of $K_0^*(1430)$ mode, which is about one-third of our prediction, i.e., 4.6%; more precise measurements and more reliable theoretical predictions are needed in the future study

(6) In Figure 2, we show the $K\pi$ invariant mass-dependent differential branching fraction for the quasi-two-body decays $B_s^0 \longrightarrow \bar{D}_0^{*0} \bar{K}_0^{*0}(1430) \longrightarrow \bar{D}_0^{*0} K^-\pi^+$ (solid line) and $B_s^0 \longrightarrow \bar{D}_0^{*0} \bar{K}_0^{*0}(1950) \longrightarrow \bar{D}_0^{*0} K^-\pi^+$ (dashed line). One can easily find that the main portion of the branching fraction comes from the region around the pole mass of the corresponding resonant states; the contribution from the $m_{K\pi}$ mass region greater than 3 GeV is evaluated

about 0.4% compared with the whole kinematic region (i.e., $[m_K + m_\pi, m_B - m_D]$) in this work and can be safely neglected

4. Conclusion

Motivated by the phenomenological importance of the charmed three-body hadronic B-meson decays, in the present work, we have studied the quasi-two-body decays $B_{(s)} \longrightarrow D_{(s)}K_0^*(1430, 1950) \longrightarrow D_{(s)}K\pi$ in the PQCD factorization approach with the help of the scalar form factor $F_{k\pi}(s)$ as a nonperturbative input. The branching ratios of all concerned decays are calculated and are of the order 10^{-10} to 10^{-5}; the corresponding two-body branching fractions can be obtained by using the quasi-two-body approximation relation in Equation (11). Under SU(3) flavor symmetry, we found the theoretical framework independent ratio $R = \mathscr{B}(B^0 \longrightarrow \bar{D}^0 K_0^{*0}(1430) \longrightarrow \bar{D}^0 K^+\pi^-)/\mathscr{B}(B_s^0 \longrightarrow \bar{D}^0 \bar{K}_0^{*0}(1430) \longrightarrow \bar{D}^0 K^-\pi^+) \approx |V_{us}/V_{ud}|^2 \cdot \tau_{B^0}/\tau_{B_s^0} \approx 0.0534$ by neglecting the differences of hadronic parameters between B^0 and B_s^0; this result is consistent with our PQCD prediction, but inconsistent with LHCb measurements. For the decays $B_s^0 \longrightarrow D^0\bar{K}_0^{*0}(1430) \longrightarrow D^0K^-\pi^+$ and $B_s^0 \longrightarrow D^0\bar{K}_0^{*0}(1430) \longrightarrow D^0K^-\pi^+$, the great difference in their corresponding branching fractions can be understood by a strong CKM suppression factor $R_{\mathrm{CKM}} \approx \lambda^4(\bar{\rho}^2 + \bar{\eta}^2) \approx 3 \times 10^{-4}$, while the moderate difference between $B^+ \longrightarrow D^0K_0^{*+}(1430) \longrightarrow D^0K^0\pi^+$ and $B^+ \longrightarrow \bar{D}^0 K_0^{*+}(1430) \longrightarrow \bar{D}^0K^0\pi^+$ as well as $B^0 \longrightarrow D^0K_0^{*0}(1430) \longrightarrow D^0K^+\pi^-$ and $B^0 \longrightarrow \bar{D}^0K_0^{*0}(1430) \longrightarrow \bar{D}^0K^+\pi^-$ is mainly due to the $R_{\mathrm{CKM}}^s \approx (\bar{\rho}^2 + \bar{\eta}^2) \approx 0.147$. More reliable theoretical predictions are needed in the future study for the nonresonance contribution and S-wave $K_0^*(1950)$ contribution. We hope the predictions in this work can be tested by the future experiments, especially, to resolve R ratio discrepancy.

Appendix

Decay Amplitudes

The factorization formulae for the individual amplitudes from different subdiagrams in Figure 1 are

$$
\begin{aligned}
F_{TK} = {} & 8\pi C_F m_B^4 f_D \int dx_B dz \int b_B db_B b db \phi_B(x_B, b_B) \\
& \cdot \Big\{ \Big[\sqrt{\eta(1-r^2)} \big[\phi^s \big(r^2(-2z\eta + 2z + 1) + (\eta - 1)(2z - 1) \big) \\
& - \phi^t \big(1 + \eta + r^2(2(\eta - 1)z + 1) + 2z(1 - \eta) \big) \big] \\
& - \phi^0 \big((\eta - 1) r^4 z + r^2(-2\eta(z + 1) + 2z + 1) + (\eta - 1) \\
& \cdot (z + 1) \big) \Big] E_{1ab}(t_{1a}) h_{1a}(x_B, z, b_B, b) S_t(z) \\
& + \Big[r^4 \phi^0 (\eta - x_B) + (\eta - 1) \Big(\eta\phi^0 - 2\phi^s \sqrt{\eta(1-r^2)} \Big) \\
& + r^2 \Big[2\phi^s \sqrt{\eta(1-r^2)} (2\eta - 1 - x_B) + \big(x_B - \eta^2 \big) \phi^0 \Big] \Big] \\
& \cdot E_{1ab}(t_{1b}) h_{1b}(x_B, z, b_B, b) S_t(|x_B - \eta|) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
M_{TK} = {} & \frac{32}{\sqrt{6}\pi C_F m_B^4 \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B) \phi_D} \\
& \cdot \Big\{ \big[-\phi^0 \big(1 + r^2 - \eta \big) \times \big(\eta + r^2 (\eta(2x_3 + z - 2) - x_3 - x_B + 1) \\
& - \eta(x_3 + z) + x_3 + x_B - 1 \big) + \sqrt{\eta(1-r^2)} \\
& \cdot \big[r^2 \big(\phi^s (2x_3 + x_B + z - 2 - (2x_3 + z - 2)\eta) + \phi^t (x_B + z\eta - z) \big) \\
& + (\eta - 1) z \big(\phi^s - \phi^t \big) \big] \big] E_{1cd}(t_{1c}) h_{1c}(x_B, z, x_3, b_B, b_3) \\
& + \Big[z(2\eta - 1) r^4 \phi^0 - r^3 r_c \phi^0 + r r_c \Big(\phi^0 (1 + \eta) - 4\phi^s \sqrt{\eta(1-r^2)} \Big) \\
& + (\eta - 1) \Big(z\sqrt{\eta(1-r^2)} \big(\phi^s + \phi^t \big) + \phi^0 (\eta x_3 - x_3 + x_B - z) \Big) \\
& + r^2 \Big[2(\eta - 1) x_3 \Big(\eta\phi^0 - \sqrt{\eta(1-r^2)} \phi^s \Big) \\
& + x_B \Big((2\eta - 1)\phi^0 + \sqrt{\eta(1-r^2)} \big(\phi^t - \phi^s \big) \Big) \\
& + z \Big((2 - 3\eta)\phi^0 + (\eta - 1)\sqrt{\eta(1-r^2)} \times \big(\phi^t + \phi^s \big) \Big) \Big] \Big] \\
& \cdot E_{1cd}(t_{1d}) h_{1d}(x_B, z, x_3, b_B, b_3) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
M'_{TK} = {} & \frac{32}{\sqrt{6}\pi C_F m_B^4 \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B) \phi_D} \\
& \cdot \Big\{ \Big[\sqrt{\eta(1-r^2)} \times \big[r^2 \phi^s (x_3 + z - 2)(1 - \eta) + r^2 x_B \phi^s \\
& + r^2 \phi^t (x_B + z\eta - z) + 4 r r_c \phi^s + (\eta - 1) z \big(\phi^s - \phi^t \big) \big] \\
& - \phi^0 \big(r^4 (\eta(2x_3 + z - 2) + 1 - x_3 - x_B) - r^3 r c + \eta r^2 \\
& \cdot (\eta(2 - z - x_3) + x_3 + x_B - 2) + (\eta + 1) r r_c + (\eta - 1) \\
& \cdot (\eta(x_3 + z - 1) - x_3 - x_B + 1) \big) \Big] E_{1cd}\big(t'_{1c}\big) h'_{1c}(x_B, z, x_3, b_B, b_3) \\
& + \big[\phi^0 \big(1 + r^2(2\eta - 1) - \eta \big) \big(x_B - x_3 \big(1 + r^2 - \eta \big) + z \big(r^2 - 1 \big) \big) \\
& + \sqrt{\eta(1-r^2)} \times \big[\phi^s \big(2x_3 r^2 (1 - \eta) - r^2 x_B + (\eta - 1)\big(r^2 - 1\big) z \big) \\
& + \phi^t \big(r^2 x_B + (\eta - 1)\big(r^2 - 1\big) z \big) \big] \big] \times E_{1cd}\big(t'_{1d}\big) h'_{1d}(x_B, z, x_3, b_B, b_3) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
F_{AK} = {} & 8\pi C_F m_B^4 f_B \int dx_3 dz \int b_3 db_3 b db \phi_D \\
& \cdot \Big\{ \Big[\sqrt{\eta(1-r^2)} \big[2r^3 z \big(\phi^s - \phi^t \big) + r^2 r_c \big(\phi^t - \phi^s \big) \\
& + 2zr \big(\phi^t - \phi^s \big) + 4r\phi^s + (\eta - 1)\big(\phi^t + \phi^s \big) r_c \big] \\
& + \phi^0 \big[\big(r^2 - 1 \big) \big(\big(r^2 - 1 \big) z - r r_c + 1 \big) \\
& - \eta \Big(\big(r^2 - 1 \big)^2 z + 2r^2 - 2r r_c - 1 \Big) \big] \\
& \cdot E_{1ef}(t_{1e}) h_{1e}(x_3, z, b_3, b) S_t(z) + \phi^0 \big[\eta^2 \big(r^2 - 1 \big) \\
& + \eta \big(1 - r^4 \big) + x_3 \big((\eta - 1)^2 \big(1 - r^2 \big) \big) + \eta \big(1 - r^4 \big) \big] + + \phi^s \\
& \cdot \big[2r \big(r^2 - 1 - \eta - x_3(\eta - 1) \big) \big] \sqrt{\eta(1-r^2)} \Big] \\
& \cdot E_{1ef}\big(t_{1f}\big) h_{1f}(x_3, z, b_3, b) S_t(x_3) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
M_{AK} = {} & \frac{32}{\sqrt{6}\pi C_F m_B^4 \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B) \phi_D} \\
& \cdot \Big\{ \big[r^4 \phi^0 (x_3 + x_B - 1 - \eta(x_3 + z - 2)) \\
& + r^2 \phi^0 \big(\eta^2 (x_3 + z - 2) - \eta(x_3 + x_B) + 1 \big) + r\sqrt{\eta(1-r^2)} \\
& \times \big[\phi^s (\eta - \eta x_3 + x_3 + x_B - z + 3) - \phi^t (\eta - \eta x_3 + x_3 + xB + z) \big] \\
& + r^3 \sqrt{\eta(1-r^2)} \times \big(\phi^s + \phi^t \big) - (\eta - 1)\phi^0 (\eta(x_3 + z - 1) - x_3 - x_B) \big] \\
& \cdot E_{1gh}\big(t_{1g}\big) h_{1g}(x_B, z, x_3, b_B, b_3) + \big[\phi^0 \big(r^2 - \eta - 1 \big) \\
& \cdot \big(\eta + r^2 (\eta(2x_3 + z - 2) - 2x_3 + x_B - z + 1) - \eta z + z - 1 \big) \\
& + r\sqrt{\eta(1-r^2)} \big(\phi^s \big(r^2 (1 - z) + \eta(x_3 - 1) - x_3 + x_B + z - 1 \big) \\
& + \phi^t \big(r^2 (z - 1) + \eta(x_3 - 1) - x_3 + x_B - z + 1 \big) \big) \big] \\
& \cdot E_{1gh}(t_{1h}) h_{1h}(x_B, z, x_3, b_B, b_3) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
F_{TD} = {} & \frac{8\pi C_F m_B^4 F_{K\pi}(s)}{\mu_s \int dx_B dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\phi_D} \\
& \cdot \Big\{ (1+r)\big[\eta^2(r-1)x_3 + \eta\big(2(r-1)^2 x_3 - 2r + 1\big) \\
& + r(x_3(3-2r)+r) - x3 - 1\big] E_{2ab}(t_{2a}) h_{2a}(x_B, x_3, b_B, b_3) \\
& \cdot S_t(x_3) + \big[(\eta-1)r^4 + 2r^3(1-2\eta+r_c) \\
& - r^2\big(\eta^2 - 2\eta r_c + r_c - 1\big) + (\eta-1)(\eta x_3 - r_c) \\
& - 2r(\eta(r_c - x_B - 1) + r_c + 1)\big] \\
& \cdot E_{2ab}(t_{2b}) h_{2b}(x_B, x_3, b_B, b_3) S_t(x_B) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
M_{TD} = {} & \frac{32}{\sqrt{6}\pi C_F m_B^4 \int dx_B dz dx_3 \int b_B db_B b db \phi_B(x_B, b_B)\phi_D \phi^0} \\
& \cdot \{ \big[\eta^2\big(r^2(2-z-x_3) + x_B + z - 1\big)\big(r^2-1\big) \\
& \cdot \big(r^2(x_3+z-1) - r x_3 - x_B - z + 1\big) \\
& + \eta r(r(r(\eta-1)(x_3+z-2) - x_B - z + 2) + x_3 + x_B + z - 2)\big] \\
& \times E_{2cd}(t_{2c}) h_{2c}(x_B, z, x_3, b_B, b) + [(r-1)(\eta + (2\eta-1)r - 1) \\
& \cdot \big(z\big(r^2-1\big) + x_B\big) - x_3(1-\eta)(1-\eta+r(r(2\eta+r-1)-1))] \\
& \cdot E_{2cd}(t_{2d}) h_{2d}(x_B, z, x_3, b_B, b)\},
\end{aligned}
$$

$$
\begin{aligned}
F_{AD} = {} & 8\pi C_F m_B^4 f_B \int dz dx_3 \int b db b_3 db_3 \phi_D \\
& \cdot \Big\{ \Big[2r\phi^s \sqrt{\eta(1-r^2)}(x_3(\eta-1) - 2) \\
& + \phi^0\big(\eta + -r^2\big(2\eta + (\eta-1)^2 x_3 - 1\big) + (\eta-2)\eta x_3 + x_3 - 1\big)\Big] \\
& \cdot E_{2ef}(t_{2e}) h_{2e}(z, x_3, b_3, b) S_t(x_3) \\
& + \big[r^4\phi^0(\eta - \eta z + z - 1) + r^2\phi^0(\eta-1)(2z-1-\eta) \\
& + 2r\sqrt{\eta(1-r^2)}\big(\phi^s(1+z-\eta) + \phi^t(\eta+z-1)\big) \\
& - 2r^3(z-1)\sqrt{\eta(1-r^2)}\big(\phi^s + \phi^t\big) - (\eta-1)z\phi^0 \\
& - r^2 r_c \sqrt{\eta(1-r^2)}\big(\phi^s + \phi^t\big) - 2r(1+\eta) r_c \phi^0 \\
& + (1-\eta) r_c \sqrt{\eta(1-r^2)}\big(\phi^t - \phi^s\big)\big] \\
& \cdot E_{2ef}\big(t_{2f}\big) h_{2f}(z, x_3, b_3, b) S_t(z) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
M_{AD} = {} & \frac{32}{\sqrt{6}\pi C_F m_B^4 \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\phi_D} \\
& \cdot \Big\{ \big[r^4\phi^0(2(\eta-1)x_3 + \eta(z-2) - z + 1) \\
& - r^2\phi^0\big(\eta^2(x_3+z-2) - x_3 + \eta(x_B+z) - x_B - 2z + 1\big) \\
& - r\sqrt{\eta(1-r^2)}(\phi^s(\eta(x_3-1) - x_3 + x_B + z + 3) \\
& + \phi^t(\eta(1-x_3) + x_3 + x_B + z - 1)) + r^3\sqrt{\eta(1-r^2)}(z-1) \\
& \cdot \big(\phi^s + \phi^t\big) + (\eta-1)\phi^0(\eta(x_B+z-1) + x_B + z)\big] \\
& \times E_{2gh}\big(t_{2g}\big) h_{2g}(x_B, z, x_3, b_B, b) \\
& + \big[\phi^0\big(\eta - r^2 - 1\big)(\eta + x_3 - 1 - \eta(x_3 - x_B + z) \\
& + r^2(\eta(2x_3+z-2) - x_3 + 1)) + r\sqrt{\eta(1-r^2)} \\
& \cdot \big(\phi^t\big(r^2(z-1) + \eta(x_3-1) - x_3 + x_B - z + 1\big) \\
& - \phi^s\big(\eta + r^2(z-1) - \eta x_3 + x_3 + x_B - z - 1\big)\big)\big] \\
& \times E_{2gh}(t_{2h}) h_{2h}(x_B, z, x_3, b_B, b) \Big\},
\end{aligned}
\tag{A.1}
$$

where the hard functions are written as

$$
\begin{aligned}
& h_i(x_1, x_2, (x_3,)b_1, b_2) = h_1(\beta, b_2) \times h_2(\alpha, b_1, b_2), \\
& h_1(\beta, b_2) = \begin{cases} K_0\big(\sqrt{\beta} b_2\big), & \beta \geq 0, \\ \dfrac{i\pi}{2} H_0^{(1)}\big(\sqrt{-\beta} b_2\big), & \beta < 0, \end{cases} \\
& h_2(\alpha, b1, b_2) = \begin{cases} \theta(b_2 - b_1) K_0\big(\sqrt{\alpha} b_2\big) I_0\big(\sqrt{\alpha} b_1\big), & \alpha \geq 0, \\ \theta(b_2 - b_1) \dfrac{i\pi}{2} H_0^{(1)}\big(\sqrt{-\alpha} b_2\big) J_0\big(\sqrt{-\alpha} b_1\big), & \alpha < 0, \end{cases}
\end{aligned}
\tag{A.2}
$$

where $E_{1mn}, E_{2mn} (m = a, c, e, g$ and $n = b, d, f, h)$ are the evolution factors, which are given by

$$
\begin{aligned}
& E_{1ab}(t) = \alpha(t) \exp\left[-S_B(t) - S_K(t)\right], \\
& E_{1cd}(t) = \alpha(t) \exp\left[-S_B(t) - S_K(t) - S_D(t)\right]_{b=b_B}, \\
& E_{1ef}(t) = \alpha(t) \exp\left[-S_D(t) - S_K(t)\right], \\
& E_{1gh}(t) = \alpha(t) \exp\left[-S_B(t) - S_K(t) - S_D(t)\right]_{b=b_3}, \\
& E_{2ab}(t) = \alpha(t) \exp\left[-S_B(t) - S_D(t)\right], \\
& E_{2cd}(t) = \alpha(t) \exp\left[-S_B(t) - S_K(t) - S_D(t)\right]_{b_3=b_B}, \\
& E_{2ef}(t) = E_{1ef}(t), \\
& E_{2gh}(t) = E_{1gh}(t),
\end{aligned}
\tag{A.3}
$$

in which the Sudakov exponents $S_{(B,K,D)}(t)$ are defined as

$$
\begin{aligned}
& S_B(t) = s\left(\frac{x_B m_B}{\sqrt{2}}, b_B\right) + \frac{5}{3}\int_{1/b_B}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \\
& S_K(t) = s\left(\frac{z\big(1-r^2\big)m_B}{\sqrt{2}}, b\right) + s\left(\frac{(1-z)\big(1-r^2\big)m_B}{\sqrt{2}}, b\right) \\
& \qquad + 2\int_{1/b}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \\
& S_D(t) = s\left(\frac{x_3 m_B}{\sqrt{2}}, b_3\right) + 2\int_{1/b_3}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})),
\end{aligned}
\tag{A.4}
$$

where the quark anomalous dimension $\gamma_q = -\alpha_s/\pi$. The explicit form for $s(Q, b)$ at one loop can be found in [42]. t_{1x} and $t_{2x} (x = a, b \cdots h)$ are hard scales which are chosen to be the maximum of the virtuality of the internal momentum transition in the hard amplitudes as

$$
t_{1a} = \mathrm{Max}\left\{\sqrt{|\alpha_{1a}|}, \sqrt{|\beta_{1a}|}, \frac{1}{b_B}, \frac{1}{b}\right\},
$$

$$
t_{1b} = \mathrm{Max}\left\{\sqrt{|\alpha_{1b}|}, \sqrt{|\beta_{1b}|}, \frac{1}{b_B}, \frac{1}{b}\right\},
$$

$$t_{1c} = \mathrm{Max}\left\{\sqrt{|\alpha_{1c}|}, \sqrt{|\beta_{1c}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\},$$

$$t'_{1c} = \mathrm{Max}\left\{\sqrt{|\alpha_{1c}|}, \sqrt{|\beta'_{1c}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\},$$

$$t_{1d} = \mathrm{Max}\left\{\sqrt{|\alpha_{1d}|}, \sqrt{|\beta_{1d}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\},$$

$$t'_{1d} = \mathrm{Max}\left\{\sqrt{|\alpha_{1d}|}, \sqrt{|\beta'_{1d}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\},$$

$$t_{2a} = \mathrm{Max}\left\{\sqrt{|\alpha_{2a}|}, \sqrt{|\beta_{2a}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\},$$

$$t_{2b} = \mathrm{Max}\left\{\sqrt{|\alpha_{2b}|}, \sqrt{|\beta_{2b}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\},$$

$$t_{2c} = \mathrm{Max}\left\{\sqrt{|\alpha_{2c}|}, \sqrt{|\beta_{2c}|}, \frac{1}{b_B}, \frac{1}{b}\right\},$$

$$t_{2d} = \mathrm{Max}\left\{\sqrt{|\alpha_{2d}|}, \sqrt{|\beta_{2d}|}, \frac{1}{b_B}, \frac{1}{b}\right\},$$

$$t_{2e} = \mathrm{Max}\left\{\sqrt{|\alpha_{2e}|}, \sqrt{|\beta_{2e}|}, \frac{1}{b_3}, \frac{1}{b}\right\},$$

$$t_{2f} = \mathrm{Max}\left\{\sqrt{|\alpha_{2f}|}, \sqrt{|\beta_{2f}|}, \frac{1}{b_3}, \frac{1}{b}\right\},$$

$$t_{2g} = \mathrm{Max}\left\{\sqrt{|\alpha_{2g}|}, \sqrt{|\beta_{2g}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\},$$

$$t_{2h} = \mathrm{Max}\left\{\sqrt{|\alpha_{2h}|}, \sqrt{|\beta_{2h}|}, \frac{1}{b_B}, \frac{1}{b_3}\right\}, \tag{A.5}$$

where we have

$$\alpha_{1a} = z\left(1-r^2\right)m_B^2,$$

$$\beta_{1a} = x_B z\left(1-r^2\right)m_B^2 = \beta_{1b} = \alpha_{1c} = \alpha_{1d} = \alpha'_{1c} = \alpha'_{1d},$$

$$\alpha_{1b} = \left(1-r^2\right)(x_B-\eta)m_B^2,$$

$$\beta_{1c} = -\left[z\left(1-r^2\right)+r^2\right]\left[(1-\eta)(1-x_3)-x_B\right]m_B^2,$$

$$\beta_{1d} = \left\{r_c^2 - \left[z\left(1-r^2\right)\right]\left[(1-\eta)x_3-x_B\right]\right\}m_B^2,$$

$$\beta'_{1c} = \left\{r_c^2 - \left[z\left(1-r^2\right)+r^2\right]\left[(1-\eta)(1-x_3)-x_B\right]\right\}m_B^2,$$

$$\beta'_{1d} = -\left[z\left(1-r^2\right)\right]\left[(1-\eta)x_3-x_B\right]m_B^2,$$

$$\alpha_{1e} = -\left[1-z\left(1-r^2\right)-r_c^2\right]m_B^2,$$

$$\beta_{1e} = -\left[\left(1-r^2\right)(1-z)\right]\left[\eta+(1-\eta)x_3\right]m_B^2 = \beta_{1f} = \alpha_{1g} = \alpha_{1h},$$

$$\alpha_{1f} = -\left(1-r^2\right)(\eta+(1-\eta)x_3)m_B^2,$$

$$\beta_{1g} = \left\{1-\left[z\left(1-r^2\right)+r^2\right]\left[(1-\eta)(1-x_3)-x_B\right]\right\}m_B^2,$$

$$\beta_{1h} = -\left[(1-z)\left(1-r^2\right)\right]\left[(1-\eta)x_3+\eta-x_B\right]m_B^2,$$

$$\alpha_{2a} = x_3(1-\eta)m_B^2,$$

$$\beta_{2a} = x_3 x_B(1-\eta)m_B^2 = \beta_{2b} = \alpha_{2c} = \alpha_{2d},$$

$$\alpha_{2b} = x_B(1-\eta)m_B^2,$$

$$\beta_{2c} = -\left[\left(1-r^2\right)(1-z)-x_B\right]\left[\eta+(1-\eta)x_3\right]m_B^2,$$

$$\beta_{2d} = -(1-\eta)x_3\left[\left(1-r^2\right)z-x_B\right]m_B^2,$$

$$\alpha_{2e} = -\left[1-x_3(1-\eta)\right]m_B^2,$$

$$\beta_{2e} = -\left[r^2+z\left(1-r^2\right)\right](1-\eta)(1-x_3)m_B^2 = \beta_{2f} = \alpha_{2g} = \alpha_{2h},$$

$$\alpha_{2f} = \left\{r_c^2 - \left[r^2+z\left(1-r^2\right)\right](1-\eta)\right\}m_B^2,$$

$$\beta_{2g} = \left\{1-\left[\left(1-r^2\right)(1-z)-x_B\right]\left[\eta+(1-\eta)x_3\right]\right\}m_B^2,$$

$$\beta_{2h} = -\left[r^2+z\left(1-r^2\right)-x_B\right](1-\eta)(1-x_3)m_B^2. \tag{A.6}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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