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Research Article

Note on the Corrected Hawking Temperature of a Parametric Deformed Black Hole with Influence of Quantum Gravity

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In this paper, using the Hamilton-Jacobi method, we discuss the tunnelling of fermions when the dual influence of quantum gravity and the deformation of a parameterized black hole are taken into account. With the influence of the generalized uncertainty principle, there exists an offset around the standard Hawking temperature. We investigate a parametric deformed black hole and find that the corrected temperature is lower than the standard one, so there exists a remnant of the black hole, and the correction is not only determined by the mass and the energy of the emitted fermion but also determined by the mass of the black hole and the deformation parameter. Under the dual influence of quantum gravity and deformation, the correction effect of quantum gravity is the main influencing factor, while the correction effect of the deformation parameter is secondary. For both the massive and massless cases, the quantum gravity correction factor is only determined by the energy of the emitted fermion, while the deformation correction factor is only determined by the mass of the black hole.

1. Introduction

The Hawking radiation near the event horizon of the black holes was found in the last century. To analyze this phenomenon, researchers have made extensive studies. The usual method is by adopting the WKB approximation to calculate the imaginary part of the emitted particle's action and the tunnelling rate [1–6]. The Hamilton-Jacobi method was first proposed in [7, 8]. In this method, the action of the emitted particles satisfies the Hamilton-Jacobi equation. Taking into account the properties of the spacetime, one can carry out a separation of variables with the action $I = -\varepsilon t + W(r) + \Phi(\theta, \varphi)$. Then, inserting the separated variables into the Hamilton-Jacobi equation and solving it, one can obtain the imaginary part. Extending this work to the tunnelling radiation of fermions, the standard Hawking temperatures of spherically symmetric and charged black holes were discussed in [9]. Other work about fermions' tunnelling radiation is referred to [10–15].

According to the theory of quantum gravity, there is a minimal observable length [16–19]. This length can be used in the model of the generalized uncertainty principle (GUP):

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 + \beta \Delta p^2\right], \tag{1}$$

where $\beta = \beta_0 (l_p^2/\hbar^2)$, β_0 is a dimensionless parameter, and l_p is the Planck length. The derivation of the GUP is based on the modified fundamental commutation relations. Kempf et al. first modified commutation relations [20] and got $[x_i, p_j] = i\hbar \delta_{ij}(1 + \beta p^2)$, where x_i and p_i are operators of position and momentum defined by

$$\begin{aligned} x_i &= x_{0i}, \\ p_i &= p_{0i}\left(1 + \beta p^2\right). \end{aligned} \tag{2}$$

And here, x_{0i} and p_{0i} satisfy the basic quantum mechanics commutation relations $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$.

This generalization of the uncertainty principle based on the modified commutation relations plays an important role in quantum gravity. With consideration of the modifications, the cosmological constant problem was discussed, and the finiteness of the constant was derived in [21]. Using a new form of GUP, the Unruh effect has been analyzed in [22]. The quantum dynamics of the Friedmann-Robertson-Walker universe was gotten in [23]. The related predictions on postinflation preheating in cosmology were derived in [24]. Based on the modifications, the thermodynamics of the black holes were discussed again in [25–32], and the tunnelling radiation of scalar particles was studied in [33, 34].

Alternative theories of gravity and the parameterized deviation approach allow black hole solutions to have additional parameters beyond mass, charge, and angular momentum [35]. Some extensions of general relativity have been proposed as alternative theories of gravity, with their corresponding black hole solutions. Recently, Johannsen and Psaltis introduced a parametric deviation approach [36]. This method can avoid some limitations of the original bumpy black hole approach [37–39]. Among the parametric deformed solutions that emerged, Konoplya and Zhidenko proposed a Kerr-like solution, introducing a parametric deformation in the mass term, keeping the asymptotic behavior of the Kerr spacetime, but changing how the mass of the black hole influences the event horizon vicinity [40].

In this paper, we focus on the tunnelling radiation of fermions from a parametric deformed black hole, where the effects of quantum gravity are taken into account. That is, we investigate the correction effect of the Hawking temperature when the dual influence of quantum gravity and deformation of a parameterized black hole is taken into account. To incorporate the effects of quantum gravity, we first modify the Dirac equation in curved spacetime by the operators of position and momentum defined in [20] and then adopt the Hamilton-Jacobi method to get the imaginary parts of the action. By calculating, we want to know the double effects of quantum gravity and deformation for a parameterized black hole and which one is the main influencing factor.

The rest is organized as follows. In the next section, to facilitate further discussion, we will review the generalized Dirac equation in curved spacetime. In Section 3, we investigate the tunnelling radiation of a parametric deformed black hole. Section 4 is devoted to the conclusion and outlook. We use the natural units $G = c = \hbar = 1$ and signature $(-, +, +, +)$.

2. Generalized Dirac Equation in Curved Spacetime

Based on the modified fundamental commutation relation in [20], one can modify the Dirac equation in curved spacetime. According to Equation (2), the square of the momentum operator is gotten as

$$p^2 = p_i p^i = -\hbar^2 \left[1 - \beta\hbar^2 \left(\partial_j \partial^j\right)\right] \partial_i \cdot \left[1 - \beta\hbar^2 \left(\partial^j \partial_j\right)\right] \partial^i \approx -\hbar^2 \left[\partial_i \partial^i - 2\beta\hbar^2 \left(\partial^j \partial_j\right) \left(\partial^i \partial_i\right)\right]. \quad (3)$$

Because of the small value of β, the higher order terms of β can be neglected. According to the theory of quantum gravity, the generalized frequency is $\tilde{\omega} = E(1 - \beta E^2)$, where E is the energy operator and denoted as $E = i\hbar\partial_t$. With consideration of the energy mass shell condition $p^2 + m^2 = E^2$, the generalized expression of the energy was gotten as [33]

$$\tilde{E} = E\left[1 - \beta\left(p^2 + m^2\right)\right]. \quad (4)$$

In curved spacetime, the Dirac equation is

$$i\gamma^\mu \left(\partial_\mu + \Omega_\mu\right)\Psi + \frac{m}{\hbar}\Psi = 0, \quad (5)$$

where $\Omega_\mu = (i/2)\omega_\mu{}^{ab}\Sigma_{ab}$, $\omega_\mu{}^{ab}$ is the spin connection defined by the tetrad $e^\lambda{}_b$ and the ordinary connection:

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a. \quad (6)$$

The Latin indices live in the flat metric η_{ab} while Greek indices are raised and lowered by the curved metric $g_{\mu\nu}$. The tetrad can be constructed from

$$\begin{aligned} g_{\mu\nu} &= e_\mu{}^a e_\nu{}^b \eta_{ab}, \\ \eta_{ab} &= g_{\mu\nu} e^\mu{}_a e^\nu{}_b, \\ e^\mu{}_a e_\nu{}^a &= \delta^\mu_\nu, \\ e^\mu{}_a e_\mu{}^b &= \delta^b_a. \end{aligned} \quad (7)$$

In Equation (5), Σ_{ab} is the Lorentz spinor generator defined by

$$\begin{aligned} \Sigma_{ab} &= \frac{i}{4}\left[\gamma^a, \gamma^b\right], \\ \left\{\gamma^a, \gamma^b\right\} &= 2\eta^{ab}. \end{aligned} \quad (8)$$

Then, one can construct γ^μs in the curved spacetime as

$$\begin{aligned} \gamma^\mu &= e^\mu{}_a \gamma^a, \\ \left\{\gamma^\mu, \gamma^\nu\right\} &= 2g^{\mu\nu}. \end{aligned} \quad (9)$$

To get the generalized Dirac equation in the curved spacetime, one can rewrite Equation (5) as

$$-i\gamma^0 \partial_0 \Psi = \left(i\gamma^i \partial_i + i\gamma^\mu \Omega_\mu + \frac{m}{\hbar}\right)\Psi. \quad (10)$$

Using Equations (3), (4), and (10) and neglecting the higher order terms of β, one can get

$$-i\gamma^0\partial_0\Psi = \left(i\gamma^i\partial_i + i\gamma^\mu\Omega_\mu + \frac{m}{\hbar}\right)\cdot\left(1+\beta\hbar^2\partial_j\partial^j - \beta m^2\right)\Psi, \tag{11}$$

which is rewritten as

$$\begin{aligned}&\left[i\gamma^0\partial_0 + i\gamma^i\partial_i\left(1-\beta m^2\right) + i\gamma^i\beta\hbar^2\left(\partial_j\partial^j\right)\partial_i + \frac{m}{\hbar}\right.\\&\left.\cdot\left(1+\beta\hbar^2\partial_j\partial^j - \beta m^2\right) + i\gamma^\mu\Omega_\mu\left(1+\beta\hbar^2\partial_j\partial^j - \beta m^2\right)\right]\Psi = 0.\end{aligned} \tag{12}$$

Thus, the generalized Dirac equation is derived. In the following sections, we adopt Equation (12) to describe the fermion tunnelling of a parametric deformed black hole.

3. Fermions' Tunnelling of a Parametric Deformed Black Hole

In this section, we investigate fermions' tunnelling from the event horizon of a parametric deformed black hole. To do so, we choose a static spherically symmetric black hole metric [11].

$$ds^2 = -F(r)dt^2 + \frac{1}{G(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{13}$$

where

$$F(r) = G(r) = 1 - \frac{2M}{r} - \frac{\eta}{r^3}, \tag{14}$$

$0 \le \varphi \le 2\pi, 0 \le \theta \le \pi, 0 \le r < \infty$, η is the deformation parameter of the black hole, and M is the mass of the black hole. For the sake of simplicity in the discussion, we take $0 \le \eta \le 1$ and choose the real root of the event horizon located at [35].

$$r_+ = \frac{2M}{3} + \left(\frac{27\eta + 16M^3}{54} + A(\eta)\right)^{1/3} + \left(\frac{27\eta + 16M^3}{54} - A(\eta)\right)^{1/3}, \tag{15}$$

where

$$A(\eta) = \left(\left(\frac{27\eta + 16M^3}{54}\right)^2 - \frac{64M^6}{729}\right)^{1/2}. \tag{16}$$

If the deformation parameter $\eta = 0$, the event horizon $r_+ = 2M$, it reduces to the event horizon of the Schwarzchild black hole. Neglecting the higher order terms of η, we get

$$r_+ = 2M + \frac{3}{4M^2}\eta. \tag{17}$$

Here, we only investigate the spin-up state. Assume that the wave function of the fermion in the spin-up state is

$$\Psi = \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar}I(t,r,\theta,\varphi)\right). \tag{18}$$

The tetrad is easily constructed as

$$e_\mu{}^a = \mathrm{diag}\left(\sqrt{F}, 1/\sqrt{G}, \sqrt{g^{\theta\theta}}, \sqrt{g^{\varphi\varphi}}\right). \tag{19}$$

The gamma matrices take on the form as

$$\begin{aligned}\gamma^t &= \frac{1}{\sqrt{F(r)}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\\ \gamma^\theta &= \sqrt{g^{\theta\theta}}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},\\ \gamma^r &= \sqrt{G(r)}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},\\ \gamma^\varphi &= \sqrt{g^{\varphi\varphi}}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}.\end{aligned} \tag{20}$$

In the above equations, $g^{\theta\theta} = 1/r^2, g^{\varphi\varphi} = 1/r^2\sin^2\theta$. Inserting the wave function and the gamma matrices into the generalized Dirac equation, we get

$$\begin{aligned}&-iA\frac{1}{\sqrt{F}}\partial_t I - B\left(1-\beta m^2\right)\sqrt{G}\partial_r I\\&\quad - Am\beta\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\varphi\varphi}\left(\partial_\varphi I\right)^2\right]\\&\quad + B\beta\sqrt{G}\partial_r I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\varphi\varphi}\left(\partial_\varphi I\right)^2\right]\\&\quad + Am\left(1-\beta m^2\right) = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&iB\frac{1}{\sqrt{F}}\partial_t I - A\left(1-\beta m^2\right)\sqrt{G}\partial_r I\\&\quad - Bm\beta\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\varphi\varphi}\left(\partial_\varphi I\right)^2\right]\\&\quad + A\beta\sqrt{G}\partial_r I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\varphi\varphi}\left(\partial_\varphi I\right)^2\right]\\&\quad + Bm\left(1-\beta m^2\right) = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}&A\left\{-\left(1-\beta m^2\right)\sqrt{g^{\theta\theta}}\partial_\theta I + \beta\sqrt{g^{\theta\theta}}\partial_\theta I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2\right.\right.\\&\quad \left. + g^{\varphi\varphi}\left(\partial_\varphi I\right)^2\right] - i\left(1-\beta m^2\right)\sqrt{g^{\varphi\varphi}}\partial_\varphi I + i\beta\sqrt{g^{\varphi\varphi}}\partial_\varphi I\\&\quad \left.\cdot\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\varphi\varphi}\left(\partial_\varphi I\right)^2\right]\right\} = 0,\end{aligned} \tag{23}$$

$$
\begin{aligned}
&B\Big\{-(1-\beta m^2)\sqrt{g^{\theta\theta}}\partial_\theta I+\beta\sqrt{g^{\theta\theta}}\partial_\theta I\\
&\cdot\left[g^{rr}(\partial_r I)^2+g^{\theta\theta}(\partial_\theta I)^2+g^{\varphi\varphi}(\partial_\varphi I)^2\right]\\
&-i(1-\beta m^2)\sqrt{g^{\varphi\varphi}}\partial_\varphi I+i\beta\sqrt{g^{\varphi\varphi}}\partial_\varphi I\\
&\cdot\left[g^{rr}(\partial_r I)^2+g^{\theta\theta}(\partial_\theta I)^2+g^{\varphi\varphi}(\partial_\varphi I)^2\right]\Big\}=0.
\end{aligned}\tag{24}
$$

It is difficult to solve the action I from the above equations. Considering the properties of the metric (13), we carry out the separation of variables as

$$I=-\varepsilon t+W(r)+\Phi(\theta,\varphi),\tag{25}$$

where ε is the energy of the radiation particle. We first observe Equations (23) and (24) and find that they are irrelevant to A and B and can be reduced to the same equation. Inserting Equation (25) into Equations (23) and (24) yields

$$
\begin{aligned}
&\left(\sqrt{g^{\theta\theta}}\partial_\theta\Phi+i\sqrt{g^{\varphi\varphi}}\partial_\varphi\Phi\right)\\
&\cdot\left[1-\beta m^2-\beta g^{rr}(\partial_r W)^2-\beta g^{\theta\theta}(\partial_\theta\Phi)^2-\beta g^{\varphi\varphi}(\partial_\varphi\Phi)^2\right]=0.
\end{aligned}\tag{26}
$$

In the above equation, the summation of factors in the square brackets can not be zero. Therefore, it should be

$$\sqrt{g^{\theta\theta}}\partial_\theta\Phi+i\sqrt{g^{\varphi\varphi}}\partial_\varphi\Phi=0.\tag{27}$$

This implies

$$g^{\theta\theta}(\partial_\theta\Phi)^2+g^{\varphi\varphi}(\partial_\varphi\Phi)^2=0,\tag{28}$$

which yields a complex function solution (other than the trivial constant solution) of Φ. However, this solution has no contribution to the tunnelling rate. Therefore, we will not consider its contribution in the calculation. Now, our interest is the first two equations which determine the Hawking temperature of the black hole. Inserting Equation (25) into Equations (21) and (22) and canceling A and B yield

$$A_6(\partial_r W)^6+A_4(\partial_r W)^4+A_2(\partial_r W)^2+A_0=0,\tag{29}$$

where

$$
\begin{aligned}
A_6&=\beta^2G^3F,\\
A_4&=\beta G^2F(m^2\beta-2),\\
A_2&=GF\left[(1-\beta m^2)^2+2\beta m^2(1-m^2\beta)\right],\\
A_0&=-m^2(1-\beta m^2)^2F-\varepsilon^2.
\end{aligned}\tag{30}
$$

Solving Equation (29) at the event horizon, we get the solution of W. Thus, the imaginary part of W is

$$
\begin{aligned}
\operatorname{Im} W_\pm&=\pm\int dr\sqrt{\frac{m^2F+\varepsilon^2}{GF}}\left(1+\beta m^2+\beta\frac{\varepsilon^2}{F}\right)\\
&=\pm\pi\varepsilon B_0(1+\beta\cdot B_1(1+\eta\cdot B_2)),
\end{aligned}\tag{31}
$$

where

$$
\begin{aligned}
B_0&=3r_+-4M,\\
B_1&=\frac{m^2+12\varepsilon^2}{2(1+m^2)},\\
B_2&=\frac{7m^2-72\varepsilon^2}{8M^3(m^2+12\varepsilon^2)}.
\end{aligned}\tag{32}
$$

In the above equation, the +/- sign corresponds to the outgoing/ingoing wave, $F=G=1-(2M/r)-(\eta/r^3)$. In the derivation of the above equation, one can find that there is a second-order pole at $r=r_+$ in the integrand function. So, one can use the residue method to calculate the integrals. At the same time, in order to reduce the complexity, the higher order terms of β and η are neglected in the Taylor series expansion, and the value of r_+ can be found in Equation (17). Thus, the tunnelling rate of the fermion crossing the horizon is

$$
\begin{aligned}
\Gamma&=\frac{P(\text{emission})}{P(\text{absorption})}=\frac{\exp(-2\operatorname{Im} I_+)}{\exp(-2\operatorname{Im} I_-)}\\
&=\frac{\exp(-2\operatorname{Im} W_+-2\operatorname{Im}\Phi)}{\exp(-2\operatorname{Im} W_--2\operatorname{Im}\Phi)}\\
&=\exp\left[-4\pi\varepsilon B_0(1+\beta\cdot B_1(1+\eta\cdot B_2))\right].
\end{aligned}\tag{33}
$$

Then, the Boltzmann factor with the Hawking temperature

$$T=\frac{1}{4\pi B_0(1+\beta\cdot B_1(1+\eta\cdot B_2))}=T_0(1-\beta\cdot B_1(1+\eta\cdot B_2)),\tag{34}$$

where $T_0=1/4\pi B_0$ is the standard Hawking temperature of the black hole. It is shown that the corrected temperature is lower than the standard one. This result is similar to the result in [28]. That is, there is a balance point. At this point, the evaporation stops, and the remnant is left.

The correction is not only determined by the mass and the energy of the emitted fermion (m and ε) but also determined by the mass of the black hole M and the deformation parameter η.

When $\eta=0$, $T=T_0(1-\beta\cdot B_1)$, the case of the tunnelling of fermions only in the presence of quantum gravity is recovered [28, 41].

For the massless case $m=0$, $B_1=6\varepsilon^2$, and $B_2=-3/4M^3$. For the massive case $m\neq 0$, with the Einstein mass-energy relation $\varepsilon=m$, $B_1=13\varepsilon^2/2(1+\varepsilon^2)$, and $B_2=-65/104M^3$. That is, for both the massive and massless cases, the quantum gravity correction factor B_1 is only determined by the energy of the emitted fermion ε, while the deformation correction factor B_2 is only determined by the mass of the black hole M.

Moreover, from Equation (34), we can find that under the dual influence of quantum effects and deformation, the correction effect of quantum gravity is the main influencing factor, while the correction effect of deformation parameter is secondary.

4. Conclusion and Outlook

In this paper, using the Hamilton-Jacobi method, we discussed the tunnelling of fermions when the dual influence of quantum gravity and the deformation of a parameterized black hole are taken into account. Taking into account the influence of quantum gravity, one can modify the Dirac equation in curved spacetime by the modified fundamental commutation relations. Then, the tunnelling radiation of fermions from the event horizon of a parametric deformed black hole was investigated. The corrected Hawking temperatures were gotten. From Equation (34), we found the following:

(i) The corrected temperature is lower than the standard one, and this result is similar to the findings in previous studies

(ii) The correction is not only determined by the mass and the energy of the emitted fermion (m, ε) but also determined by the mass of the black hole M and the deformation parameter η

(iii) Under the dual influence of quantum effects and deformation, the correction effect of quantum gravity is the main influencing factor, while the correction effect of the deformation parameter is secondary

(iv) For both the massive and massless cases, the quantum gravity correction factor B_1 is only determined by the energy of the emitted fermion ε, while the deformation correction factor B_2 is only determined by the mass of the black hole M

The corrected temperature is lower than the standard one, and that is, with the evaporation proceeds, the Hawking temperature decreases, and the black hole finally reaches an equilibrium state. At this state, the evaporation stops, and the remnant is produced. This is a qualitative conclusion in this paper. There were some discussions about the remnants in the final state in [42–48]. A review of this topic can be found in [49]. It is of interest to discuss quantitatively the size of the black hole remnant in the present parametric deformed mode. This issue is a bit complex, and we look forward to discussing it in another article.

Data Availability

All data used to support the findings of this study are included within the article.

Disclosure

We would like to express our gratitude for the valuable feedback received during the preprint stage of this work, which was presented as a preprint available at ARXIV ID: arXiv: 2306.09362[hep-th] [50].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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