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Research Article

Determination of the Energy Eigenvalues of the Varshni-Hellmann Potential

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In this paper, we solve the bound state problem for the Varshni-Hellmann potential via a useful technique. In our technique, we obtain the bound state solution of the Schrödinger equation for the Varshni-Hellmann potential via ansatz method. We obtain the energy eigenvalues and the corresponding eigenfunctions. Also, the behavior of the energy spectra for both the ground and the excited state of the two body systems is illustrated graphically. The similarity of our results to the accurate numerical values is indicative of the efficiency of our technique.

1. Introduction

Exact solution of the Schrödinger equation in the D-dimensional coordinate system has been a focus of study in miscellaneous works of quantum physics and quantum chemistry. The energy eigenvalues and wave function, which are capable of showing the behavior of a quantum mechanical system, can be obtained from the Schrödinger equation. The Schrödinger equation is a second-order differential equation used to solve quantum mechanics problems. The exact and approximate solutions of the Schrödinger wave equation in nonrelativistic quantum mechanics have many features because the wave functions and their equivalent eigenvalues provide a lot of information for the description of various quantum systems, including atomic structure theory, quantum chemistry, and quantum electrodynamics. Using the experimental proof of the Schrödinger wave equation, researchers are motivated to solve the radial Schrödinger equation via different analytical methods. Attempts have been made to solve the Schrödinger and Klein-Gordon equations through different potentials. For example, William et al. studied the Hulthen potential together with the Hellmann potential [1], and Hellmann investigated the Schrödinger equation with a linear combination of Coulomb and Yukawa potentials, which is known as the Hellmann potential [2]. The Hellman potential has been applied to several branches of physics such as atomic physics, plasma physics, and solid state physics [3, 4], and it has been used in the study of electron nucleus [5] and electron ion [6].

This study seeks to obtain the eigenvalues and wave function of the 3D Schrödinger equation through the sum of Varshni and Hellmann potentials. The paper is organized as follows: in Section 2, the exact solution of the Schrödinger equation for the Varshni-Hellmann potential is derived, and we obtain the analytical expressions for energy levels and the corresponding wave functions for n and l quantum numbers. In Section 3, the results are discussed. In Section 4, summary and conclusion are presented.

2. Formulation of the Approach

The Schrödinger equation has been solved exactly by using various potentials, and it has been employed in different atomic, molecular, and nuclear fields. The Schrödinger equation is a second-order differential equation which serves to solve quantum mechanics problems. We have attempted to solve the Schrödinger and Klein-Gordon equations by using different potentials for few-quark systems [7–11]. In

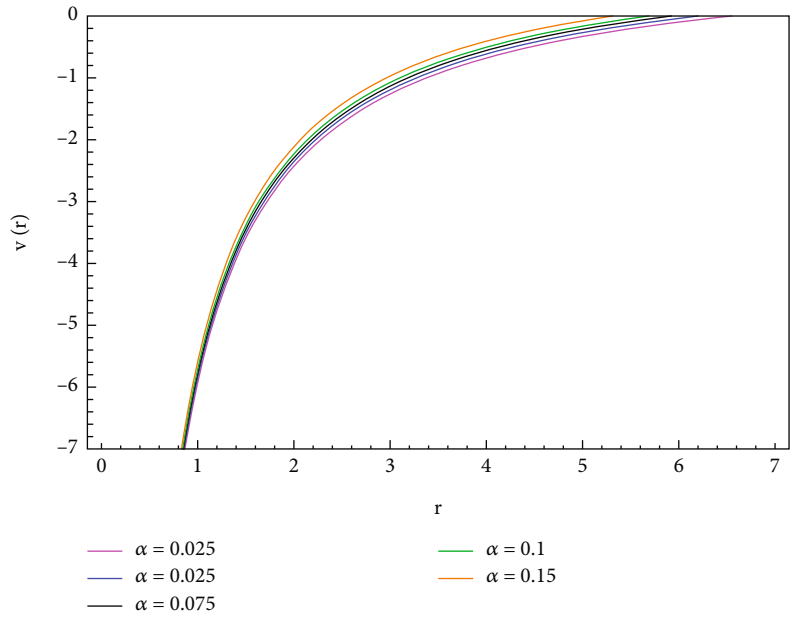

Figure 1: VHP variations in terms of different values of α.

this section, we solve the Schrödinger equation by using the Hellmann potential. The Hellmann potential is of the following form [12, 13]:

$$V(r) = -\frac{c}{r} + \frac{d}{r}e^{-\alpha r}, \tag{1}$$

where r is the internuclear interval and c and d stand for the strong points of Coulomb and Yukawa potentials. The Varshni potential is of the following form [14]:

$$V(r) = a - \frac{ab}{r}e^{-\alpha r}, \tag{2}$$

where a and b denote the strong points of the Varshni potential. The Varshni potential is a function of repulsive short-range potential energy, which has been studied in the formalism of the Schrödinger equation and contributed greatly to chemical and nuclear physics [15, 16]. In this article, we study the Schrödinger radial equation with a new proposed potential obtained from the sum of the Varshni and Hellmann potential (VHP). The potential is

$$V(r) = a + \frac{d-ab}{r}e^{-\alpha r} - \frac{c}{r}. \tag{3}$$

In Figure 1, we show the VHP variations in terms of different values of α. We expand the exponential part of the potential as follows:

$$V(r) = a + \frac{d-ab}{r}\left(1 - \alpha r + \frac{\alpha^2 r^2}{2} - \frac{\alpha^3 r^3}{6}\right) - \frac{c}{r}. \tag{4}$$

Table 1: Energy eigenvalue for VHP in terms of different values of α ($a = 1$, $b = -1$, $c = 4$, $d = -4$, and $\hbar = 2\mu = 1$).

State	α	E (ev)
1s	0.025	19.175401
	0.050	-19.101607
	0.075	-19.028616
2s	0.025	-4.058816
	0.050	-4.041517
	0.075	-4.034352
2p	0.025	-4.028798
	0.050	-4.021861
	0.075	-3.952986
3s	0.025	-1.247048
	0.050	-1.234861
	0.075	-1.221445
3p	0.025	-1.232069
	0.050	-1.227473
	0.075	-1.143767
3d	0.025	-1.215805
	0.050	-1.172321
	0.075	-1.139522

And we write the potential in a simpler form as follows:

$$V(r) = a + \frac{d-ab}{r} + (ab-d)\alpha - \frac{(ab-d)\alpha^2 r}{2} + \frac{(ab-d)\alpha^3 r^2}{6} - \frac{c}{r}. \tag{5}$$

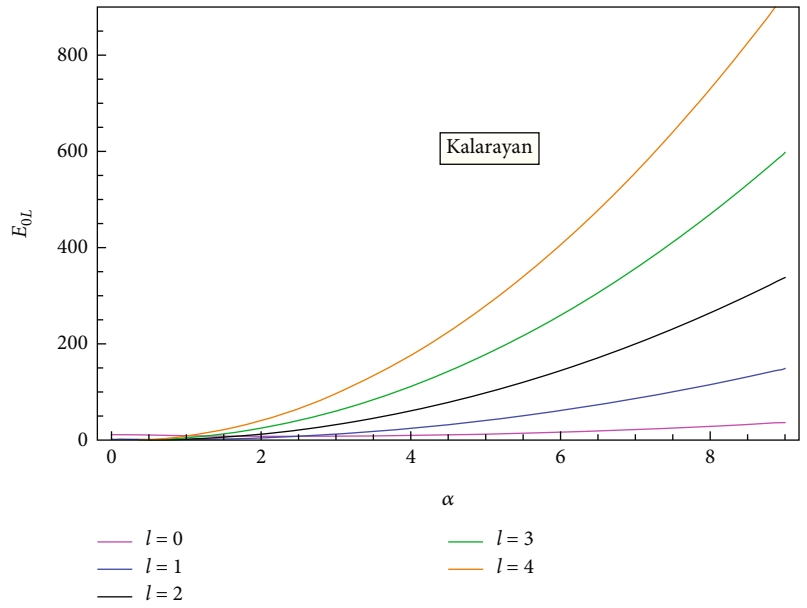

FIGURE 2: E in terms of α for different l.

The system Schrödinger equation is

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi_{v,l}(r)}{\partial r}\right)+\left(V(r)-E_{v,l}+\frac{l(l+1)\hbar^2}{2\mu r^2}\right)\psi_{v,l}(r)=0, \tag{6}$$

where μ is the reduced mass and v and l are group number and the orbital quantum number of one particle relative to another, respectively. By choosing $\psi(r)=(1/r)\phi(r)$, eq. (5) appears as

$$\phi''(r)+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{l(l+1)\hbar^2}{2\mu r^2}\right]\phi(r)=0. \tag{7}$$

$\phi(r)$ can be derived from eq. (6). Assuming $\hbar=c=1$ and $\phi(r)=f(r)\exp[g(r)]$, we can pursue the calculations related to $\phi(r)$ function, and $f(r)$ and $g(r)$ functions are presented as

$$f_j(r)=\prod_{c=1}^{j}\left(r-\alpha_i^j\right)\quad j=1,2,\cdots. \tag{8}$$

where $n=1,2,3,\cdots$ and $f_0(r)=1$ and the polynomial $g(r)$ is defined based on the type of potential. In this case, it is defined based on potential 3.

$$g(r)=\frac{-1}{2}Ar^2+Br+\delta\ln r. \tag{9}$$

From the above equations, we have

$$\phi''(r)=\left[g''(r)+g'2(r)+\frac{f''(r)+2g'(r)f'(r)}{f(r)}\right]\phi(r). \tag{10}$$

And by introducing into eq. (4),

$$\frac{-2\mu}{\hbar^2}\left[E-V(r)-\frac{l(l+1)\hbar^2}{2Mr^2}\right]=\left[g''(r)+g'2(r)+\frac{f''(r)+2g'(r)f'(r)}{f(r)}\right]. \tag{11}$$

We expand the exponential part of the potential and rewrite the potential as

$$V(r)=a+\frac{d-ab}{r}\left(1-\frac{\alpha}{r}-\frac{\alpha^2r^2}{2}-\frac{\alpha^3r^3}{6}\right)-\frac{c}{r}. \tag{12}$$

By introducing the potential quantity and the derivatives into eq. (10), we solve the equation for $n=0$ and angular momentum L, and the following equation is obtained:

$$\begin{aligned}-2\mu\Bigg(E-a+\frac{ab-d}{r}-(ab-d)\alpha+\frac{ab\alpha^2r}{2}\\+\frac{(d-ab)\alpha^3r^2}{6}+\frac{c}{r}-\frac{l(l+1)}{2\mu r^2}\Bigg)=-A+B^2\\-2A\delta+\frac{2B\delta}{r}-2ABr+A^2r^2+\frac{\delta^2}{r^2}-\frac{\delta}{r^2}.\end{aligned} \tag{13}$$

With a simple calculation and considering that the exponents of r are linearly independent, it is possible to set the coefficients of different powers of r equal to each

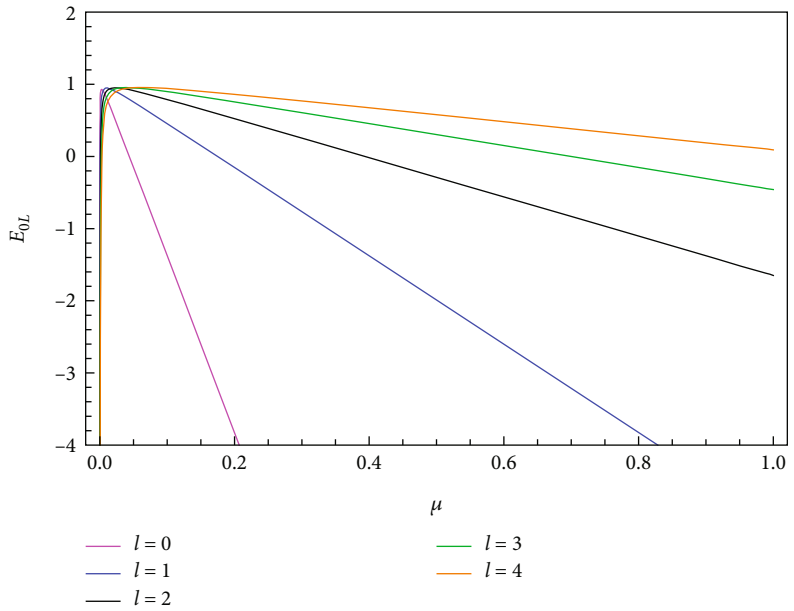

Figure 3: E in terms of μ for different l.

other. In this case, the following relations are obtained between the potential coefficients and the energy can be obtained:

$$\begin{aligned}
&-2\mu(E - a + (d - ab)\alpha) = -A + B^2 - 2A\delta,\\
&-2M\left(\frac{ab - d}{r} + \frac{c}{r}\right) = \frac{2B\delta}{r},\\
&-2\mu\left(\frac{(ab - d)\alpha^2 r}{2}\right) = -2ABr,\\
&-2\mu\left(-\frac{(ab - d)\alpha^3 r^2}{6}\right) = -A^2 r^2,\\
&l(l+1) = \frac{\delta^2}{r^2} - \frac{\delta}{r^2}.
\end{aligned} \tag{14}$$

By solving the above equations, the special relation of the energy values for the state $n = 0$ is obtained as follows.

$$\begin{aligned}
E_{0L} = a + (ab - d)\alpha - \frac{(l+1)\left(\left((ab - d)\alpha^2\right)/2\right)}{2\mu(ab + c - d)}(2l + 3)\\
- \frac{2\mu(ab + c - d)^2}{4(l+1)^2},
\end{aligned} \tag{15}$$

$$\psi_{0L} = N_1 r^{L+1} \exp\left(\frac{-1}{2}Ar^2 + Br\right). \tag{16}$$

The energy for the first excited state $n = 1$ and the angular momentum L is equal to

Table 2: Energy eigenvalue for the Hellman potential $a = b = 0$, $c = 2$, $d = -1$,$(\hbar = 2\mu = 1)$.

State	α	E (ev)	Ref. [17]	Ref. [12]
1s	0.001	-2.238 00	-2.24900	-2.24898
	0.005	-2.244 01	-2.24501	-2.24499
	0.01	-2.2413	-2.24005	-2.24003
2s	0.001	-0.5606	-0.56150	-0.56150
	0.005	-0.5569	-0.55755	-0.55754
	0.01	-0.55198	-0.55269	-0.55269
2p	0.001	-0.5601	-0.56150	-0.56150
	0.005	-0.5562	-0.55754	-0.55754
	0.01	-0.5516	-0.55266	-0.55266
3s	0.001	-0.2381	-0.24900	-0.24900
	0.005	-0.24378	-0.24511	-0.245 11
	0.01	-0.2399	-0.24043	-0.24043
3p	0.001	-0.2478	-0.24900	-0.24900
	0.005	-0.2436	-0.24510	-0.24510
	0.01	-0.2390	-0.24040	-0.24040
3d	0.001	-0.2468	-0.24900	-0.24900
	0.005	-0.2446	-0.24508	-0.24508
	0.01	-0.2389	-0.24034	-0.24034

$$\begin{aligned}
E_{1L} = a + (ab - d)\alpha - \frac{(l+1)\left(\left(ab\alpha^2/2\right) - \left(d\alpha^2/2\right)\right)}{2\mu(ab + c - d)}(2l + 5)\\
- \frac{2\mu(ab + c - d)^2}{4(l+2)^2}.
\end{aligned} \tag{17}$$

Table 3: Energy eigenvalue for the Varshni potential $a = b = -1$, $(\hbar = 2\mu = 1)$.

State	α	E (ev)	[18]
1s	0.001	-1.249001	—
	0.050	-1.203750	—
	0.100	-1.165000	—
2s	0.001	-1.0615025	—
	0.050	-1.0187500	—
	0.100	-0.9875000	—
2p	0.001	-1.026784	-1.061750
	0.050	-0.995277	-1.0256250
	0.100	-0.997777	-0.990000
3s	0.001	-1.026781	—
	0.05	-0.9865277	—
	0.1	-0.9627777	—
3p	0.001	-1.014634	—
	0.050	-0.988125	—
	0.100	-1.005625	—
3d	0.001	-1.0090165	-1.026944
	0.05	-1.001250	-0.986736
	0.1	-1.075000	-0.946944
4s	0.001	-1.014629	—
	0.050	-0.976875	—
	0.100	-0.960625	—
4p	0.001	-1.009011	-1.01506
	0.05	-0.987500	-0.99515
	0.1	-1.020000	-0.990000
4d	0.01	-0.992075	-1.01493
	0.050	-0.991805	-0.98515
	0.100	-1.088055	-0.96250
4f	0.01	-1.004132	-1.01475
	0.050	-1.030102	-0.97250
	0.100	-1.205102	-0.97250

3. Numerical Results

The detailed analysis of the results in terms of various domains of parameters a, b, c, and α of the VHP reveals a few important facts concerning the application of the perturbed formalism. In the present study, the discrete energy eigenvalues for the VHP have been calculated as functions of the strength a, b, c, and the screening parameter α of the VHP.

(1) For VHP, the energy eigenvalues are given by Eqs. (15) and (17). In Table 1, we show the energy eigenvalues for VHP (with $a = 1$, $b = -1$, $c = 4$, $d = -4$, and $\hbar = 2\mu = 1$) in terms of different values of α. As alpha increases, the magnitude of the binding energy decreases. The energy values corresponding to the states $n = 2, 3, \cdots$ are also obtained by the same method. In this way, the Schrödinger equation was solved analytically and the eigenvalues of E_{nL} were obtained. In Figures 2 and 3, we show the variation of energy as a function of α and M for different l by using results of Table 1. As we can see, for $l > 0$, the energy increases with the increase of α, and for $\mu > 0.1$, the energy decreases with the increase of μ

Table 4: Energy eigenvalue for the Yukawa potential $d = \sqrt{2}$, $\alpha = gd(\hbar = \mu = 1)$.

State	g	E (ev)	Ref. [20]	Ref. [19]
1s	0.002	-1.00133	-0.99600	-0.99600
	0.005	-1.00518	-0.99004	-0.99003
	0.010	-0.9901	-0.98015	-0.98014
	0.020	-0.9519	-0.96059	-0.96059
2s	0.002	-0.2378	-0.24602	-0.24602
	0.005	-0.2396	-0.24015	-0.24014
	0.010	-0.2276	-0.23059	-0.23058
	0.020	-0.2105	-0.21230	-0.21229
2p	0.002	-0.2455	-0.24602	-0.24601
	0.005	-0.2391	-0.24012	-0.24012
	0.010	-0.22860	-0.23049	-0.23049
	0.020	-0.21101	-0.21192	-0.21192
3p	0.002	-0.1067	-0.10716	-0.10716
	0.005	-0.1009	-0.10142	-0.10141
	0.010	-0.09076	-0.09231	-0.09230
	0.020	-0.07489	-0.07570	-0.07570
3d	0.002	-0.1067	-0.10715	-0.10715
	0.005	-0.1001	-0.10140	-0.10136
	0.010	-0.0916	-0.09212	-0.09212
	0.020	-0.0747	-0.07502	-0.07503

(2) We have shown the energy eigenvalue for the Hellman potential for $a = b = 0$ in Table 2 and compared our results with [12, 17]. Hamzavi et al. in [12] have obtained the approximate analytical solutions of the radial Schrödinger equation for the Hellmann potential. By using the generalized parametric Nikiforov-Uvarov (NU) method [17], they analyze a perturbative treatment for the bound states of the Hellman potential. As we have shown in Table 2, as the value of α increases, the correlation energy decreases

(3) We have shown energy eigenvalue for the Varshni potential $a = b = -1, (\hbar = 2\mu = 1)$ in Table 3 and compared our results with Ref. [18]. Ebomwonyi et al. have studied the Schrödinger equation for the Varshni potential function with two eigensolution techniques such as the NU and the semiclassical WKB approximation methods in Ref. [18]. Our results are in good agreement with Ref. [18]

(4) Table 4 shows numerical values of the binding energies of the Yukawa potential ($a = b = c = 0$) in terms of the values of M and α. The results obtained are compared with those of [19, 20]. Karakoc and Boztosun [19] apply the asymptotic iteration method to solve the radial Schrödinger equation for the Yukawa type potentials. Accurate numerical solutions have been obtained for the Schrödinger equation through a Yukawa potential in Ref. [20]

4. Conclusions

In this research, we analyzed the Schrödinger equation with the Varshni-Hellman potential using the Ansatz method. We study that the discrete energy eigenvalues for the Varshni-Hellmann potential have been calculated as functions of the screening parameter α of the Yukawa potential. We compared our findings with other theoretical formalisms. We found that the energy eigenvalues obtained using this method are in good agreement with other works in the literature. Therefore, analytical solutions while opening a new window can be used to provide valuable information about the dynamics of quantum mechanics in molecular and atomic physics.

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

This manuscript was submitted as a preprint in the following link: https://arxiv.org/abs/2401.11151v3.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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