Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2009, Article ID 483079, 11 pages
doi:10.1155/2009/483079

Research Article

Rotations in the Space of Split Octonions

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Received 27 March 2009; Accepted 11 May 2009

Recommended by Fedele Lizzi

The geometrical application of split octonions is considered. The new representation of products of the basis units of split octonionic having David's star shape (instead of the Fano triangle) is presented. It is shown that active and passive transformations of coordinates in octonionic "eight-space" are not equivalent. The group of passive transformations that leave invariant the pseudonorm of split octonions is $SO(4,4)$, while active rotations are done by the direct product of $O(3,4)$-boosts and real noncompact form of the exceptional group G_2. In classical limit, these transformations reduce to the standard Lorentz group.

1. Introduction

Nonassociative algebras may surely be called beautiful mathematical entities. However, they have never been systematically utilized in physics, only some attempts have been made toward this goal. Nevertheless, there are some intriguing hints that nonassociative algebras may play essential role in the ultimate theory, yet to be discovered.

Octonions are one example of a nonassociative algebra. It is known that they form the largest normed algebra after the algebras of real numbers, complex numbers, and quaternions [1–3]. Since their discovery in 1844/1845 by Graves and Cayley there have been various attempts to find appropriate uses for octonions in physics (see reviews [4–7]). One can point to the possible impact of octonions on: Color symmetry [8–11]; GUTs [12–15]; Representation of Clifford algebras [16–19]; Quantum mechanics [20–24]; Space-time symmetries [25, 26]; Field theory [27–29]; Formulations of wave equations [30–32]; Quantum Hall effect [33]; Kaluza-Klein program without extra dimensions [34–36]; Strings and M-theory [37–40]; and so forth.

In this paper we study rotations in the model, where geometry is described by the split octonions [41–43].

2. Octonionic Geometry

Let us review the main ideas behind the geometrical application of split octonions presented in our previous papers [41–43]. In our model some characteristics of physical world (such as dimension, causality, maximal velocities, and quantum behavior) can be naturally described by the properties of split octonions. Interesting feature of the geometrical interpretation of the split octonions is that their pseudonorms, in addition to some other terms, already contain the ordinary Minkowski metric. This property is equivalent to the existence of local Lorentz invariance in classical physics.

To any physical signal we correspond eight-dimensional number, the element of split octonions,

$$s = ct + x^n J_n + \hbar\lambda^n j_n + c\hbar\omega I \quad (n = 1, 2, 3). \tag{2.1}$$

Here we have one scalar basis unit (denoted as 1), the three vector-like objects J_n, the three pseudovector-like elements j_n, and one pseudoscalar-like unit I. The eight real parameters that multiply basis elements we treat as the time t, the special coordinates x^n, some quantities λ^n with the dimensions momentum^{-1}, and the quantity ω having the dimension energy^{-1}. We suppose also that (2.1) contains two fundamental constants of physics: the velocity of light c and the Planck constant $\hbar$.

The squares of basis units of split octonions are inner product resulting unit element, but with the opposite signs,

$$J_n^2 = 1, \qquad j_n^2 = -1, \qquad I^2 = 1. \tag{2.2}$$

Multiplications of different hypercomplex basis units are defined as skew products $(n, m, k = 1, 2, 3)$,

$$\begin{aligned}
J_n J_m &= -J_m J_n = \epsilon_{nmk} j^k, \\
j_n j_m &= -j_m j_n = \epsilon_{nmk} j^k, \\
J_n j_m &= -j_m J_n = -\epsilon_{nmk} J^k, \\
J_n I &= -I J_n = j_n, \\
j_n I &= -I j_n = J_n,
\end{aligned} \tag{2.3}$$

where ϵ_{nmk} is the fully antisymmetric tensor.

From (2.3) we notice that to generate complete basis of split octonions the multiplication and distribution laws of only three vector-like elements J_n are needed. In geometrical application this can explain why classical space has three dimensions. The three pseudovector-like basis units j_n can be defined as the binary product

$$j_n = \frac{1}{2}\epsilon_{nmk} J^m J^k, \tag{2.4}$$

and thus can describe oriented orthogonal planes spanned by two vector-like elements J_n. The seventh basic unit I (the oriented volume) is formed by the products of all three fundamental basis elements J_n and has three equivalent representation:

$$I = J_1 j_1 = J_2 j_2 = J_3 j_3. \tag{2.5}$$

The multiplication table of octonionic units is most transparent in graphical form. To visualize the products of ordinary octonions the Fano triangle is used [1–3], where the seventh basic unit I is place at the center of the graph. In the algebra of split octonions we have less symmetry, and for a proper description of the products (2.3) the Fano graph should be modified by shifting I from the center of the Fano triangle. Also we will use three equivalent representations of I, (2.5), and, instead of the Fano triangle, we arrive at David's star shaped duality plane for products of the split octonionic basis elements.

On this graph the product of two basis units is determined by following the oriented solid line connecting the corresponding nodes. Moving opposite to the orientation of the line contributes a minus sign to the result. Dashed lines just show that the corners of the triangle with I nodes are identified.

Conjugation, which can be understand as a reflection of the vector-like basis units J_n, reverses the order of octonionic basis elements in any given expression, thus

$$\begin{gathered} J_n^+ = -J_n, \\ j_n^+ = \frac{1}{2}\epsilon_{nmk}\left(J^m J^k\right)^+ = \frac{1}{2}\epsilon_{nmk} J^{k+} J^{m+} = -j_n, \\ I^+ = (J_n j_n)^+ = j_n^+ J_n^+ = -I, \end{gathered} \tag{2.6}$$

there is no summing in the last formula. So the conjugation of (2.1) gives

$$s^+ = ct - x_n J^n - \hbar\lambda_n j^n - c\hbar\omega I. \tag{2.7}$$

Using (2.2) one can find that the pseudonorm of (2.1),

$$s^2 = ss^+ = s^+ s = c^2 t^2 - x_n x^n + \hbar^2 \lambda_n \lambda^n - c^2 \hbar^2 \omega^2, \tag{2.8}$$

has (4 + 4) signature. If we consider s as the interval between two octonionic signals we see that (2.8) reduces to the classical formula of Minkowski space-time in the limit $\hbar \to 0$.

Using the algebra of basis elements (2.3) the octonion (2.1) can be written in the equivalent form

$$s = c(t + \hbar\omega I) + J^n(x_n + \hbar\lambda_n I). \tag{2.9}$$

We notice that the pseudoscalar-like element I introduces the "quantum" term corresponding to some kind of uncertainty of space-time coordinates. For the differential form of (2.9) the invariance of the pseudonorm (2.8) gives the relation:

$$\frac{d\sqrt{s^2}}{cdt} = \sqrt{1 - \frac{v^2}{c^2}\left(1 - \hbar^2\frac{d\lambda^n}{dx^m}\frac{d\lambda_n}{dx_m}\right) - \left(\hbar\frac{d\omega}{dt}\right)^2}, \tag{2.10}$$

where $v_n = dx_n/dt$ denotes 3-dimensional velocity measured in the frame (2.1). The generalized Lorentz factor (2.10) contains extra terms that vanish in the limit $\hbar \to 0$. So the dispersion relation in our model has a form similar to that of double-special relativity models [44, 45].

From the requirement to have the positive pseudonorm (2.8) from (2.10) we obtain several relations

$$v^2 \le c^2, \qquad \frac{dx^n}{d\lambda^n} \ge \hbar, \qquad \frac{dt}{d\omega} \ge \hbar. \tag{2.11}$$

Recalling that λ and ω have dimensions of momentum^{-1} and energy^{-1}, respectively, we conclude that the Heisenberg uncertainty principle in our model has the same geometrical meaning as the existence of the maximal velocity in Minkowski space-time.

3. Rotations

To describe rotations in 8-dimensional octonionic space (2.1) with the interval (2.8) we need to define exponential maps for the basis units of split octonions.

Since the squares of the pseudovector-like elements j_n are negative, $j_n^2 = -1$, we can define

$$e^{j_n\theta_n} = \cos\theta_n + j_n \sin\theta_n, \tag{3.1}$$

where θ_n are some real angles.

At the same time for the other basis elements J_n, I, which have the positive squares $J_n^2 = I^2 = 1$, we have

$$\begin{aligned} e^{J_n m_n} &= \cosh m_n + j_n \sinh m_n, \\ e^{I\sigma} &= \cosh\sigma + I\sinh\sigma, \end{aligned} \tag{3.2}$$

where m_n and σ are real numbers.

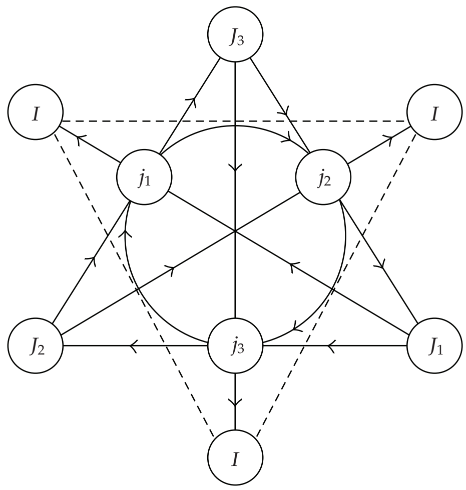

Figure 1: Split octonion multiplication as David's star.

In 8-dimensional octonionic "space-time" (2.1) there is no unique plane orthogonal to a given axis. Therefore for the operators (3.1) and (3.2) it is not sufficient to specify a single rotation axis and an angle of rotation. It can be shown that the left multiplication of the octonion s by one of the operators (3.1), (3.2) (e.g., $e^{j_1\theta_1}$) yields four simultaneous rotations in four mutually orthogonal planes. For simplicity we consider only the left products since it is known that one side multiplications generate the whole symmetry group that leaves the octonionic norms invariant [46].

So rotations naturally provide splitting of an octonion in four orthogonal planes. To define these planes note that one of them is formed by the hypercomplex element that we chose to define the rotation (j_1 in our example), together with the scalar unit element of the octonion. The rest orthogonal planes are given by the three pairs of other basis elements that lie with the considered basis unit (j_1 in the example) on the lines emerged it in David's star (see Figure 1). Thus the pairs of basis units that are rotated into each other are the pairs that form associative triplets with the considered basis unit. For example, the basis unit j_1, according to Figure 1, has three different representations in the octonionic algebra:

$$j_1 = J_2 J_3 = j_2 j_3 = J_1 I. \tag{3.3}$$

So the planes orthogonal to $(1 - j_1)$ are $(J_2 - J_3)$, $(j_2 - j_3)$, and $(J_1 - I)$. Using (3.3) and the representation (3.1) it is possible to "rotate out" the four octonionic axes, and (2.1) can be written in the equivalent form

$$s = N_t e^{j_1\theta_t} + N_x e^{j_1\theta_x} J_3 + N_\lambda e^{j_1\theta_\lambda} j_2 + N_\omega e^{j_1\theta_\omega} I, \tag{3.4}$$

where

$$N_t = \sqrt{c^2t^2 + \hbar^2\lambda_1^2}, \qquad N_x = \sqrt{x_2^2 + x_3^2},$$
$$N_\lambda = \hbar\sqrt{\lambda_2^2 + \lambda_3^2}, \qquad N_\omega = \sqrt{x_1^2 + c^2\hbar^2\omega^2} \tag{3.5}$$

are the norms in four orthogonal octonionic planes. The corresponding angles are given by

$$\theta_t = \arccos(t/N_t), \qquad \theta_x = \arccos(x_3/N_x),$$
$$\theta_\lambda = \arccos(\hbar\lambda_2/N_\lambda), \qquad \theta_\omega = \arccos(c\hbar\omega/N_\omega). \tag{3.6}$$

This decomposition of split octonion is valid only if the full pseudonorm of the octonion (2.8) is positive, that is,

$$s^2 = N_t^2 - N_x^2 + N_\lambda^2 - N_\omega^2 > 0. \tag{3.7}$$

A decomposition similar to (3.4) exists if the another pseudovector-like basis unit, j_2 or j_3, is fixed.

In contrast with uniform rotations giving by the operators j_n we have limited rotations in the planes orthogonal to $(1-J_n)$ and $(1-I)$. However, we can still perform a decomposition similar to (3.4) of s using expressions of the exponential maps (3.2). But now, unlike on (3.5), the norms of the corresponding planes are not positively defined and, instead of the condition (3.7), we should require positiveness of the norms of each four planes. For example, the pseudoscalar-like basis unit I has three different representations (2.5), and it can provide the hyperbolic rotations (3.2) in the orthogonal planes $(1-I)$, $(J_1 - j_1)$, $(J_2 - j_2)$, and $(J_3 - j_3)$. The expressions for the 2 norms (3.5) in this case are: $c\sqrt{t^2 - \hbar^2\omega^2}$, $\sqrt{x_1^2 - \hbar^2\lambda_1^2}$, $\sqrt{x_2^2 - \hbar^2\lambda_2^2}$, and $\sqrt{x_3^2 - \hbar^2\lambda_3^2}$.

Now let us consider active and passive transformations of coordinates in 8-dimensional space of signals (2.1). With a passive transformation we mean a change of the coordinates t, x_n, λ_n, and ω, as opposed to an active transformation which changes the basis 1, J_n, j_n, and I.

The passive transformations of the octonionic coordinates t, x_n, λ_n, and ω, which leave invariant the norm (2.8) form $SO(4,4)$. We can represent these transformations of (2.1) by the left products

$$s' = Rs, \tag{3.8}$$

where R is one of (3.1), (3.2). The operator R simultaneously transforms four planes of s. However, in three planes R can be rotated out by the proper choice of octonionic basis. Thus

R can represent rotations separately in four orthogonal planes of s. Similarly we have some four angles for the other six operators (3.1), (3.2) and thus totally $4 \times 7 = 28$ parameters corresponding to $SO(4,4)$ group of passive coordinate transformations. For example, in the case of the decomposition (3.4) we can introduce four arbitrary angles ϕ_t, ϕ_x, ϕ_λ, and ϕ_ω, and

$$s' = N_t e^{j_1(\theta_t+\phi_t)} + N_x e^{j_1(\theta_x+\phi_x)} J_3 + N_\lambda e^{j_1(\theta_\lambda+\phi_\lambda)} j_2 + N_\omega e^{j_1(\theta_\omega+\phi_\omega)} I. \tag{3.9}$$

Obviously under these transformations the pseudonorm (2.8) is invariant. By the fine tuning of the angles in (3.9) we can define rotations in any single plane from four.

Now let us consider active coordinate transformations, or transformations of basis units 1, J_n, j_n, and I. For them, because of nonassociativity, the results of two different rotations (3.1) and (3.2) are not unique. This means that not all active octonionic transformations (3.1) and (3.2) form a group and can be considered as a real rotation. Thus in the octonionic space (2.1) not to the all passive $SO(4,4)$-transformations we can make corresponding active ones, only the transformations that have a realization as associative multiplications should be considered. It is known that associative transformations can be done by the combined rotations of special form in two octonionic planes that form a subgroup of $SO(4,4)$, known as the automorphism group of split octonions G_2^{NC} (the real noncompact form of Cartan's exceptional Lie group G_2). Some general results on G_2^{NC} and its subgroup structure can be found in [47, 16b].

Let us recall that the automorphism A of a algebra is defined as the transformations of the hypercomplex basis units x and y under which the multiplication table of the algebra is invariant, that is,

$$\begin{aligned} A(x+y) &= Ax + Ay, \\ (Ax)(Ay) &= (A(xy)). \end{aligned} \tag{3.10}$$

Associativity of these transformations is obvious from the second relation, and the set of all automorphisms of composition algebras form a group. In the case of quaternions, because of associativity, active and passive transformations, $SU(2)$ and $SO(3)$, respectively, are isomorphic and quaternions are useful to describe rotations in 3-dimensional space. One has a different situation for octonions. Each automorphism in the octonionic algebra is completely defined by the images of three elements that do not form quaternionic subalgebras, that is, they all not lie on the same David's line [49]. Consider one such set, say (j_1, j_2, J_1). Then there exists an automorphism

$$\begin{aligned} j_1' &= j_1, \\ j_2' &= j_2 \cos(\alpha_1+\beta_1)/2 + j_3 \sin(\alpha_1+\beta_1)/2, \\ J_1' &= J_1 \cos\beta_1 + I \sin\beta_1, \end{aligned} \tag{3.11}$$

where α_1 and β_1 are some independent real angles. By the definition (3.10) the automorphism does not affect unit scalar 1. The images of the other basis elements under automorphism (3.11) are determined by the conditions

$$\begin{aligned}
j_3' &= j_3 \cos(\alpha_1 + \beta_1)/2 - j_2 \sin(\alpha_1 + \beta_1)/2 = j_1' j_2',\\
I' &= I \cos\beta_1 - J_1 \sin\beta_1 = J_1' j_1',\\
J_2' &= J_2 \cos(\alpha_1 - \beta_1)/2 + J_3 \sin(\alpha_1 - \beta_1)/2 = j_2' I',\\
J_3' &= J_3 \cos(\alpha_1 - \beta_1)/2 - J_2 \sin(\alpha_1 - \beta_1)/2 = j_3' I'.
\end{aligned} \tag{3.12}$$

It can easily be checked that transformed bases J_n', j_n', I' satisfy the same multiplication rules as J_n, j_n, I.

There exist similar automorphisms with fixed j_2 and j_3 axes, which are generated by the angles α_2, β_2 and α_3, β_3, respectively.

One can define also hyperbolic automorphisms for the vector-like units J_n by the angles u_n, k_n. For example, for fixed J_1 similar to (3.11) and (3.12) transformations are

$$\begin{aligned}
J_1' &= J_1,\\
J_2' &= J_2 \cosh(k_1 + u_1)/2 + j_3 \sinh(k_1 + u_1)/2,\\
I' &= I \cosh u_1 - j_1 \sinh u_1,\\
j_3' &= j_3 \cosh(k_1 + u_1)/2 + j_2 \sinh(k_1 + u_1)/2 = J_1' J_2',\\
j_1' &= j_1 \cosh u_1 - I \sinh u_1 = J_1' I',\\
j_2' &= j_2 \cosh(k_1 - u_1)/2 + J_3 \sinh(k_1 - u_1)/2 = J_2' I',\\
J_3' &= J_3 \cosh(k_1 - u_1)/2 + J_2 \sinh(k_1 - u_1)/2 = j_3' I'.
\end{aligned} \tag{3.13}$$

Analogously in the case of fixed I we find that

$$\begin{aligned}
I' &= I,\\
j_1' &= j_1 \cosh\sigma_1 + J_1 \sinh\sigma_1,\\
j_2' &= j_2 \cosh\sigma_2 + J_2 \sinh\sigma_2,\\
J_1' &= j_1' I' = J_1 \cosh\sigma_1 + j_1 \sinh\sigma_1,\\
J_2' &= j_2' I' = J_2 \cosh\sigma_2 + j_2 \sinh\sigma_2,\\
j_3' &= j_1' j_2' = j_3 \cosh(\sigma_1 + \sigma_2) - J_3 \sinh(\sigma_1 + \sigma_2),\\
J_3' &= j_3' I' = J_3 \cosh(\sigma_1 + \sigma_2) - j_3 \sinh(\sigma_1 + \sigma_2).
\end{aligned} \tag{3.14}$$

So for each octonionic basis there are seven independent automorphisms each introducing two angles that correspond to $2 \times 7 = 14$ generators of the algebra G_2^{NC}. For our choice of basis the infinitesimal passive transformation of the coordinates, corresponding to G_2^{NC}, has the form

$$
\begin{aligned}
t' &= t, \\
x_i' &= x_i - \frac{1}{2}\epsilon_{ijk}\left(\alpha^j - \beta^j\right)x^k + c\hbar\beta^i\omega + \frac{\hbar}{2}\left(U_{ik} - \epsilon_{ijk}u^j\right)\lambda^k, \\
\omega' &= \omega - \frac{1}{c\hbar}\beta_i x^i - \frac{1}{c}u_i\lambda^i, \\
\lambda_i' &= \lambda_i - \frac{1}{2}\epsilon_{ijk}\left(\alpha^j + \beta^j\right)\lambda^k - cu^i\omega + \frac{1}{2\hbar}\left(U_{ik} + \epsilon_{ijk}u^j\right)x^k,
\end{aligned}
\tag{3.15}
$$

where U_{ik} is the symmetric matrix

$$
U = \begin{pmatrix} 2\sigma_1 & k_3 & k_2 \\ k_3 & 2\sigma_2 & k_1 \\ k_2 & k_1 & -2(\sigma_1 + \sigma_2) \end{pmatrix}. \tag{3.16}
$$

In the limit $(\hbar\lambda, \hbar\omega \to 0)$ the transformations (3.15) reduce to the standard $O(3)$ rotations of Euclidean 3-space by the Euler angles $\phi_n = \alpha_n - \beta_n$.

The formulas (3.15) represent rotations of (3,4)-sphere that is orthogonal to the time coordinate t. To define the boosts note that active and passive forms of mutual transformations of t with x_n, λ_n, and ω are isomorphic and can be described by the seven operators (3.1) and (3.2) (e.g., the first term in (3.9)), which form the group $O(3,4)$. In the case $(\hbar\lambda, \hbar\omega \to 0)$ we recover the standard $O(3)$ Lorentz boost in the Minkowski space-time governing by the operators $e^{J_n m_n}$, where $m_n = \arctan v_n/c$.

4. Conclusion

In this paper the David's star duality plane, which describes the multiplication table of the basis units of split octonions (instead of the Fano triangle of ordinary octonions), was introduced. Different kind of rotations in the split octonionic space was considered. It was shown that in octonionic space active and passive transformations of coordinates are not equivalent. The group of passive coordinate transformations, which leave invariant the pseudonorms of split octonions, is $SO(4,4)$, while active rotations are done by the direct product of the seven $O(3,4)$-boosts and fourteen G_2^{NC}-rotations. In classical limit these transformations give the standard 6-parametrical Lorentz group.

Acknowledgment

The author would like to acknowledge the support of a 2008-2009 Fulbright Fellowship.

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