

## Research Article

# The Meaning of Time and Covariant Superderivatives in Supermechanics

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We present a review of the basics of supermanifold theory (in the sense of Berezin-Kostant-Leites-Manin) from a physicist's point of view. By considering a detailed example of what does it mean the expression "to integrate an ordinary superdifferential equation" we show how the appearance of anticommuting parameters playing the role of time is very natural in this context. We conclude that in dynamical theories formulated within the category of supermanifolds, the space that classically parametrizes time (the real line  $\mathbb{R}$ ) must be replaced by the simplest linear supermanifold  $\mathbb{R}^{1|1}$ . This supermanifold admits several different Lie supergroup structures, and we analyze from a group-theoretic point of view what is the meaning of the usual covariant superderivatives, relating them to a change in the underlying group law. This result is extended to the case of  $N$ -supersymmetry.

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## 1. Introduction

The usual interpretation of time in physics (at least in classical mechanics) is as follows. Consider a dynamical system described by a Hamiltonian  $H$ , which is a differentiable function defined on some symplectic manifold  $(M, \omega)$ . Physical trajectories are then identified with integral curves of a vector field  $X_H \in \mathcal{X}(M)$  such that  $i_{X_H}\omega = dH$ . These integral curves define a flow, that is, a differentiable mapping  $\Phi : I \times M \rightarrow M$  given by  $\Phi(t, p) = c_p(t)$ , where  $c_p : I \rightarrow M$  is the maximal integral curve passing through  $p \in M$  at  $t = 0$ , and  $I \subset \mathbb{R}$  is the maximal interval of definition of the set  $\{c_p\}_{p \in M}$  of integral curves. Then, "time" is the space of values of the parameter  $t$ , that is, the subset  $I \subset \mathbb{R}$  with a manifold structure. In some cases (e.g., dynamics on compact manifolds)  $I = \mathbb{R}$  and then becomes a one-dimensional Lie group with the operation of addition although, in general, this operation is only locally defined.

Our goal in this paper is to describe in analog terms what should be considered as “time” when we deal with a dynamical system implementing some kind of supersymmetry or, in other words, when we deal with a dynamical system represented by a supervector field on a supermanifold.

Let us say, in advance, that the answer is not new. The so-called “supertime” has appeared in physics some time ago, but its introduction almost always has been a matter of esthetics (to preserve the supersymmetric duality one has to give a “super” partner of the bosonic time  $t$ ) or an *ad hoc* convenience. For example, in [1] E. Witten observed that the index formula could be understood in terms of a supersymmetric quantum mechanical system in which the parameter space is given by pairs  $(t, \theta)$ , with  $t$  bosonic and  $\theta$  fermionic. In [2], L. Álvarez-Gaumé gave a complete proof of this result using this “supertime.” The paper [3] introduces “supertime” to supersymmetrize the setup of classical mechanics viewed as a field theory in one time and zero space dimensions, an idea found later in P. G. O. Freund’s book [16, see Chapter 10]. Also, a supertime  $(t, \theta)$  was used by D. Freed in [4] to construct a quantization model for the superparticle (a detailed account can be viewed in [5]). In all these works, supertime is a convenient construct, but not a necessary geometric object whose existence is demanded by the mathematical structure of the theory. Indeed, it is not unusual to find in physics literature some “super” generalizations of classical geometric constructions such as geodesics, normal coordinates, and integral curves of Hamiltonian fields, in which the parameter is just the common bosonic time  $t$ . Also, there are applications in physics in which supertime has the form of a single odd parameter (as in [6]) and others with a supertime involving two odd parameters  $\theta, \bar{\theta}$  and an even one  $t$ , these models being referred to as  $N = 2$  supersymmetric models (see [7–9]). Of course, there are also  $N$ -supersymmetric models with  $N \geq 2$  (see, e.g., [10]). As an alternative reference for the study of supermechanics in supermanifolds (in the sense we will consider in these notes, following Leites, Kostant, and Manin) but from the Lagrangian and Hamiltonian viewpoints, we refer the reader to [11, 12].

Thus, in view of this variety of constructions, it seems interesting to establish some criterion to determine what the analog of “time” (as understood in classical mechanics) should be. We offer one based on the interpretation of time exposed above, and we will try to show why it is necessary to introduce  $(t, \theta)$  as the parameter space for dynamical theories on supermanifolds by means of some examples (which, however, are general enough as will be explained later). This is a consequence of a profound result due to J. Monterde and A. Sánchez-Valenzuela (see [13]) that seems to be not very well known by physicists working with supermanifolds, maybe because of its formal statement and proof. The result is a completely general theorem of existence and uniqueness of solutions to differential equations on supermanifolds. It gives the most general answer in the sense that a procedure is provided to prove the existence of solutions for an arbitrary supervector field.

In the second part of the paper, we analyze the construction of “covariant superderivatives” and explore their relationship with supertime. Also, we consider their meaning within the context of the theory of Lie supergroups and superalgebras, showing how their introduction amounts to a redefinition of the underlying Lie supergroup structure of supertime.

The theory of superdifferential equations was initially studied by V. Shander in [14], a paper virtually unknown outside the circle of soviet mathematicians that deserves much more attention. Shander studied some particular classes of supervector fields that can be integrated and classified them into “normal forms,” a classification that, however, was incomplete and superseded by the treatment in [13], but nevertheless contained the basic ideas about how to deal with the problem of integrating supervector fields. Both Shander

and Monterde and Sánchez-Valenzuela also offered some particular examples considering the supermanifold  $(M, \Omega(M))$ , which is well known and understood, as its superfunctions are just differential forms. We will also work with this supermanifold as a concrete example. In this way, we hope that we can explain the mathematical theory behind supertime in the spirit of the physics approach.

## 2. The Algebraic Language in Differential Geometry and Physics

Throughout this paper we assume that the reader has some familiarity with the elementary facts about differential geometry and manifold theory as presented in any of the numerous textbooks on mathematical methods for physicists (see e.g.[15–18].) However, we do not assume any previous exposition to supermanifold theory.

In physics, it is usual to describe a system in terms of its observables. After all, we obtain information about the system by making measurements on it, that is, by evaluating the action of some observable on the state of the system. Now, how do we describe observables (in the classical -non-quantumsetting)?

Think of a system composed by one single particle. Classically, to describe a state we need its position  $\mathbf{x} = (x^1, x^2, x^3)$  and momentum  $\mathbf{p} = (p^1, p^2, p^3)$  (we are assuming that there are some submanifolds of  $\mathbb{R}^3$  in which  $\mathbf{x}$  and  $\mathbf{p}$  take values). Thus, we need a  $2 \cdot 3 = 6$ -dimensional manifold  $M$  whose elements are pairs  $(\mathbf{x}, \mathbf{p})$ , the classical states.  $M$  itself is called the phase space of the system, and it usually has a cotangent bundle structure  $M = T^*Q$  for some submanifold  $Q \subset \mathbb{R}^3$ . Then, an observable is a function on the phase space. For example, the kinetic energy  $T : M \rightarrow \mathbb{R}$  is given by  $T(\mathbf{x}, \mathbf{p}) = (1/2m)\|\mathbf{p}\|^2$ , where  $\|\cdot\|$  denotes the norm on the cotangent space to the submanifold  $Q \subset \mathbb{R}^3$  associated to the induced metric, and  $m$  is the mass of the particle.

As the equations of classical mechanics are differential equations involving observables, we can assume some degree of regularity for these functions; indeed, it is usual to take infinitely differentiable functions, so the space of classical observables is in turn of the type  $C^\infty(M)$ . What is the structure of these spaces  $C^\infty(M)$  with  $M$  an  $n$ -dimensional manifold? In short (see [19, 20] as advanced references), they have the property that each time we take a subset  $U \subset M$  which is open, we have a subset  $C^\infty(U) \subset C^\infty(M)$  in such a way that

- (i)  $C^\infty(U)$  is an algebra (where the sum and product are given by  $(f + g)(p) = f(p) + g(p)$  and  $(f \cdot g)(p) = f(p)g(p)$  for any  $p \in M$  and  $f, g \in C^\infty(U)$ );
- (ii) for each pair of open sets  $V \subset U$  of  $M$ , there is defined a restriction map  $\rho_V^U : C^\infty(U) \rightarrow C^\infty(V)$  such that
  - (1)  $\rho_U^U = id_U$  for all  $U \subset M$  open,
  - (2) whenever we have open sets  $W \subset V \subset U$ ,  $\rho_W^U = \rho_W^V \circ \rho_V^U$ ,
  - (3) if  $U \subset M$  is open and  $\{U_i\}_{i \in I}$  is an open covering of  $U$ , given two  $f, g \in C^\infty(U)$  such that for all  $i \in I$   $\rho_{U_i}^U(f) = \rho_{U_i}^U(g)$  (in other words, the restrictions  $f|_{U_i}, g|_{U_i}$  to an element of the covering coincide), then,  $f = g$ .

These properties are embodied in the mathematical notion of a sheaf. We say that the assignment  $M \supset U \mapsto C^\infty(U) \subset C^\infty(M)$  turns  $C^\infty(M)$  into a sheaf of (commutative) algebras of differentiable functions on  $M$ , and when we want to stress this fact, we write  $C_M^\infty$  instead of  $C^\infty(M)$ . Indeed, if we are given the topological structure of  $M$  (i.e., we are given a way to distinguish the sets  $U \subset M$  which are open) and we specify an algebra  $\mathcal{A}$

(playing the role of  $C^\infty(M)$ ) with some mild properties (see [19]), the manifold structure of  $M$  is completely determined; this is why  $C^\infty(M)$  is called the structural sheaf of the ordinary manifold  $M$ . What is even more: all the differential geometry on  $M$  (and hence “almost all” the physics on  $M$ ) can be described in terms of  $C^\infty(M)$  and some other algebraic structures derived from it.

As an example, take a differentiable vector field  $X$  on  $M$  (this is denoted by  $X \in \mathcal{X}(M)$ ). Traditionally one sees its value at a point  $p$ ,  $X_p$  as an equivalence class of curves  $c : \mathbb{R} \rightarrow M$  on the manifold, where two curves  $c_1, c_2$  are viewed as equivalent at a point  $p \in M$  if and only if  $c_1(0) = p = c_2(0)$  and  $(dc_1/dt)(0) = (dc_2/dt)(0)$ . But it is easy to see that this is the same as to give a mapping  $X_p : C^\infty(M) \rightarrow \mathbb{R}$  such that if  $f, g \in C^\infty(M)$ , then  $X_p$  is  $\mathbb{R}$ -linear and

$$X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g), \quad (2.1)$$

(see [21]). The idea is to define, for a given  $f \in C^\infty(M)$ , the action of  $X_p$  on  $f$  by  $X_p(f) = (d/dt)|_{t=0}(t \mapsto f \circ c)$ , where  $c$  is a representant of the class  $X_p$ , and then to apply the chain rule to get the operator representation of  $X$  as  $X = X^i(\partial/\partial x^i)$ , where  $X^i \in C^\infty(M)$  (for  $1 \leq i \leq n$ ) and  $\partial/\partial x^i$  acts like the usual partial derivative in  $\mathbb{R}^n$ . In particular,

$$\frac{\partial}{\partial x^i}(f \cdot g) = \frac{\partial f}{\partial x^i} \cdot g + f \cdot \frac{\partial g}{\partial x^i}. \quad (2.2)$$

Thus, we can characterize  $\mathcal{X}(M)$  as the set of mappings  $X : C^\infty(M) \rightarrow C^\infty(M)$  which are linear with respect to the sum of functions and products by scalars of  $\mathbb{R}$  and, in addition, satisfy (2.1) for all  $p \in M$ . (It is essential, in order to make this identification, that we deal with the  $C^\infty$  category. For  $C^r$  vector fields ( $0 < r < \infty$ ) this is no longer true.) For obvious reasons, this set is called the set of derivations of  $C^\infty(M)$ , and it is denoted  $\text{Der } C^\infty(M)$ .

Actually,  $\text{Der } C^\infty(M)$  has a lot more of structure. It is a real vector space and a  $C^\infty(M)$ -module, which is much the same as a  $\mathbb{K}$ -vector space but now  $C^\infty(M)$  is a ring, not a field  $\mathbb{K}$ . Nevertheless, this is enough to define the dual  $C^\infty(M)$ -module (akin to the dual  $V^*$  of a  $\mathbb{K}$ -vector space  $V$ )  $\text{Der}^* C^\infty(M)$ , and this turns to be precisely the space of differential 1-forms  $\text{Der}^* C^\infty(M) = \Omega^1(M)$ . From  $\mathcal{X}(M)$  and  $\Omega^1(M)$ , by taking tensor and exterior products, we can construct any tensor field on  $M$ , and develop in this way all the usual concepts of differential geometry (see [21]).

To summarize: what is really important to get a physical description of a dynamical system, is to know what its observables are. Mathematically, this is reflected in the fact that in order to characterize the phase space  $M$ , it is enough to say what are the differentiable functions on  $M$ , that is, to give the structural sheaf  $C^\infty(M)$ .

### 3. Review of the Classical (Nongraded) Case

Consider first the problem of determining the integral curves of a vector field on a classical manifold  $M$ : if in a coordinate system  $\{x^i\}_{i=1}^n$  the vector field  $X \in \mathcal{X}(M)$  has the local expression (we use Einstein's summation convention)

$$X = X^i \frac{\partial}{\partial x^i}, \quad (3.1)$$

what we want is to find the curves  $c : I_c \subset \mathbb{R} \rightarrow M$  satisfying

$$\frac{dc}{dt}(t) = X_{c(t)}, \quad \forall t \in I_c, \quad (3.2)$$

subject to an initial condition  $c(t_0) = p$  and  $(dc/dt)(t_0) = X_p$ . Let us suppose, for the sake of simplicity, that all the integral curves are defined for all the values of the time parameter, that is,  $I_c = \mathbb{R}$  for every  $c$  satisfying (3.3) (this is the case, e.g., of compact manifolds) and take  $t_0 = 0$ . Under this assumption, it can be proved that we have a family of diffeomorphisms  $\{\varphi_t\}_{t \in \mathbb{R}}$ , where

$$\begin{aligned} \varphi_t : M &\longrightarrow M, \\ p &\longmapsto \varphi_t(p) = c_p(t), \end{aligned} \quad (3.3)$$

$c_p$  being the integral curve of  $X$  such that  $c_p(0) = p$ . Moreover, the following basic properties hold true:

$$\begin{aligned} \varphi_{t+s} &= \varphi_t \circ \varphi_s, \\ \varphi_{-t} &= (\varphi_t)^{-1}. \end{aligned} \quad \forall t, s \in \mathbb{R}. \quad (3.4)$$

It is very important to note the converse: each time we have a family  $\{\varphi_t\}_{t \in \mathbb{R}}$  of diffeomorphisms of  $M$  satisfying (3.4), we can construct a vector field  $X \in \mathcal{X}(M)$  associated to it as

$$X(p) = \left. \frac{d}{dt} \right|_{t=0} (t \mapsto \varphi_t(p)). \quad (3.5)$$

*Remark 3.1.* Diffeomorphisms on  $M$  extend themselves to automorphisms on any algebraic structure defined over  $M$  (vector bundles, exterior algebras, etc.). For instance, a diffeomorphism  $\varphi$  on  $M$  can be viewed as an automorphism of the algebra  $C^\infty(M)$  by means of the action  $\varphi(f) = f \circ \varphi^{-1}$  for any  $f \in C^\infty(M)$ .

We insist on the fact that if we have computed in some way a family of automorphisms, we get the associated vector field (the infinitesimal generator) *by taking derivatives with respect to the parameter* of the family. This fact will be essential to later introduce the notion of “supertime,” in Section 5.

At this point, we want to stress yet another feature of this integration procedure. It is well known that every time a Lie group appears, so does its associated Lie algebra. In the case we have just considered (with the specific assumption that  $I_c = \mathbb{R}$ ), the parameter family forms a Lie group: it is just the additive group  $\mathbb{R}$ , whose Lie algebra is trivial, being abelian unidimensional. We will see later, in Section 6, that when working with supermanifolds, several possible Lie supergroup structures for the family of integrating parameters appear, and this has definite consequences regarding the class of vector fields that can be integrated.

#### 4. Differential Equations on Supermanifolds

Now consider the case of supermanifolds. F. Berezin was the first researcher to systematically study the interchange between bosons and fermions in a quantum mechanical system (see [22]. The earliest attempt known to the authors is due to J. Martin [23]), and to treat this operation as if it were a symmetry one. He discovered that a unified formalism, encompassing both fermions and bosons, was possible if one introduces a degree for each characteristic object in the theory, those associated to fermions carrying degree one and those associated to bosons, degree zero. The rules for operating with this degree were easily obtained from the results of applying the fermion-boson exchange symmetry to the system, and they were found to correspond to the rules of the ring  $\mathbb{Z}_2 = \{0, 1\}$  under the sum modulo 2. Of course, observables are examples of what we understand by “characteristic objects in the theory,” so if we want to allow from the start the possibility of having a symmetry in our system with the same properties as the fermion-boson interchange, we must assign a degree to each observable, and we must do that in a way compatible with the rest of algebraic structures that we have defined on  $C^\infty(M)$ . There are different (and some nonequivalent) ways to carry on that “ $\mathbb{Z}_2$  extension,” and we will consider here the one proposed by F. Berezin [22], B. Kostant [24], D. Leites [25], and Y. I. Manin [26]. Alternative approaches can be found in [27–29], and a good review of the general theory and the relations between these approaches in [30].

In this context, a supermanifold can be thought as an ordinary manifold where the structural sheaf of differentiable functions  $C_M^\infty$  on  $M$  (which is a sheaf of commutative algebras) has been replaced by a sheaf of ( $\mathbb{Z}_2$ -commutative) superalgebras  $\mathcal{A}_M$ , so now to each open set  $U \subset M$  we will associate a superalgebra  $\mathcal{A}_U$ . The supermanifold is then written  $\mathcal{M} = (M, \mathcal{A}_M)$ .

*Remark 4.1.* A superalgebra is simply a vector space  $\mathcal{A}$  which has a  $\mathbb{Z}_2$ -grading, that is, it admits a direct sum decomposition  $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$  (the factors are indexed by the elements of  $\mathbb{Z}_2$ ) and has a product adapted to that decomposition, that is, a binary operation  $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  such that  $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{(i+j) \bmod 2}$  (for  $i = 0, 1$ ). The elements of  $\mathcal{A}_i$  ( $i = 0, 1$ ) are called homogeneous of degree  $i$ , and it is clear that any element of the algebra is a linear combination of homogeneous ones. If  $v \in \mathcal{A}_i$  we write  $|v| = i$  to express the degree of  $v$ .

A superalgebra  $\mathfrak{g}$  is a Lie superalgebra if its product  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  satisfy the graded skew-symmetry condition

$$[x, y] = -(-1)^{|x||y|} [y, x], \quad (4.1)$$

and the graded Jacobi identity

$$(-1)^{|x||z|} [[x, y], z] + (-1)^{|y||x|} [[y, z], x] + (-1)^{|z||y|} [[z, x], y] = 0. \quad (4.2)$$

These conditions are just generalizations of the usual ones characterizing a Lie algebra, but taking into account the “golden rule of signs:”  $a \cdot b = (-1)^{|a||b|} b \cdot a$  (i.e., each time two elements of the superalgebra are interchanged, a  $-1$  factor powered to the product of the degrees appear).

A more general setting consists in a sheaf of  $\mathbb{Z}$ -graded algebras. In this case we have a decomposition  $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k$  and a product such that  $\mathcal{A}_k \cdot \mathcal{A}_l \subset \mathcal{A}_{k+l}$ .

Every  $\mathbb{Z}$ -graded algebra can be turned into a superalgebra simply by collecting all the subspaces with an even index in  $\mathcal{A}_0$  and those with an odd index in  $\mathcal{A}_1$ . That is, we write  $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ , where  $\mathcal{A}_0 = \oplus_{r \text{ even}} \mathcal{A}_r$  and  $\mathcal{A}_1 = \oplus_{s \text{ odd}} \mathcal{A}_s$ .

The definition of supermanifold implies some features that are absent in the classical setting. Given an open set  $U$ , we cannot think of  $\mathcal{A}_U$  as an algebra of real functions (as these are commutative); indeed, we cannot evaluate the elements of  $\mathcal{A}_U$  on points of  $M$  as is done for usual functions. This would require specifying first a set  $U$  containing the point, but even if this is the case, it should be specified how to relate the result of the evaluation with the result of any other evaluation taking all the possible sets  $U$  containing the point. Furthermore, if we insist on considering a supermanifold as a set of points, we run into trouble when we deal with observables on it (which should be real-valued functions): all of them must give zero when evaluated on “odd points” (to preserve parity, note that  $\mathbb{R}$  only has even part). Thus, it is useless to speak of “points” of a supermanifold. It is in this sense that sometimes it is said that “a supermanifold does not have points.” (However, it is possible to use the so-called functor of points in such a way that local expressions of supervector fields, superforms, etc., coincide with these usually used in Physics (see [31]).)

We can develop a differential geometry in a supermanifold following the guidelines exposed in Section 2. The crucial point is to keep in mind that all the constructions must be made from the sheaf  $\mathcal{A}_M$ . As a concrete example, take the supermanifold  $\mathcal{M} = (M, \Omega_M)$ , where  $\Omega_M = \oplus_{k \in \mathbb{Z}} \Omega^k(M)$  are the differential forms on  $M$ . Here,  $\Omega^k(M) = \{0\}$  if  $k < 0$  and  $\Omega^0(M) = C^\infty(M)$ , and the product is given by the wedge product of forms  $\wedge$ . Of course, for each  $U \subset M$  open, we can consider the differential forms on  $U$ ,  $\oplus_{k \in \mathbb{Z}} \Omega^k(U)$ , so we have a sheaf of  $\mathbb{Z}$ -graded algebras and, as stated above, also a sheaf of superalgebras (which is sometimes known as the Cartan algebra). A superfunction has to be understood now as an element  $\omega = \sum \omega_{(i)}$ , where  $\omega_{(i)} \in \Omega^i(M)$  are the homogeneous components of  $\omega$  and the sum is taken for  $i \geq 0$ .

The simplest case is the supermanifold  $(\mathbb{R}, \Omega(\mathbb{R}))$ , also denoted by  $\mathbb{R}^{1|1}$ . Although simple, this is a very interesting example because it also illustrates the notion of a Lie supergroup. In the case of this supermanifold, we will denote  $dt$  by  $\theta$ . Thus, note that superfunctions are now differential forms on  $\mathbb{R}$  and these can be written  $f(t) + g(t)\theta$  with  $f, g$  differentiable functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ . A classical theorem of Frölicher-Nijenhuis states that any derivation of a Cartan algebra (such as  $\Omega(\mathbb{R})$ ) can be expressed in the form  $\mathcal{L}_K + i_J$  for some pair of tensors  $J, K$  over the base manifold. In our case, the base is  $\mathbb{R}$  and  $J = \partial/\partial t = K$ , where  $t$  is the canonical global coordinate of  $\mathbb{R}$ . Note that the effect of the basis derivations on an element  $f(t) + g(t)\theta$  is given by

$$\begin{aligned} \mathcal{L}_{\partial/\partial t}(f(t) + g(t)\theta) &= \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t}\theta, \\ i_{\partial/\partial t}(f(t) + g(t)\theta) &= g(t). \end{aligned} \tag{4.3}$$

Due to these formulae, we will write

$$\mathcal{L}_{\partial/\partial t} \equiv \frac{\partial}{\partial t}, \quad i_{\partial/\partial t} \equiv \frac{\partial}{\partial \theta}. \tag{4.4}$$

(Note that the degrees as  $\mathbb{Z}_2$ -graded endomorphisms are  $|\partial/\partial t| = 0$ ,  $|\partial/\partial \theta| = 1$ ).

**Table 1:** Lie supergroup structures on  $\mathbb{R}^{1|1}$ .

Type	Product
1	$f_1(t) + f_2(t) + (g_1(t) + g_2(t))\theta$
2	$f_1(t) + f_2(t) + g_1(t)g_2(t) + (g_1(t) + g_2(t))\theta$
3	$f_1(t) + f_2(t) + (e^{f_2(t)}g_1(t) + g_1(t))\theta$

**Table 2:** Lie supergroup structures on  $\mathbb{R}^{1|1}$ .

Type	Product
1	$(t_1 + t_2, \theta_1 + \theta_2)$
2	$(t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)$
3	$(t_1 + t_2, e^{t_2}\theta_1 + \theta_1)$

As we have mentioned,  $\mathbb{R}^{1|1}$  admits several Lie supergroup structures. For reference, we list them here (see [32] for details), giving the result of the composition  $(f_1(t) + g_1(t)\theta) * (f_2(t) + g_2(t)\theta)$  as listed in Table 1.

This is not the standard way of presenting these Lie supergroup structures. Rather, use is made of the global supercoordinates  $t$  and  $\theta$  on  $\mathbb{R}^{1|1}$ , which acts as  $t(f_1(t) + g_1(t)\theta) = f_1(t)$ ,  $\theta(f_1(t) + g_1(t)\theta) = g_1(t)$ , so an  $\alpha \in \mathbb{R}^{1|1}$  can be written  $\alpha = (t(\alpha), \theta(\alpha))$  and we get now Table 2.

What is the analog of a classical vector field in this setting? To characterize it we isolate its main property: it must act as derivations on the algebra  $C^\infty(M)$  (recall (2.1)). Thus, a vector field on the supermanifold  $\mathcal{M} = (M, \Omega_M)$  will be a derivation  $V : \Omega_M \rightarrow \Omega_M$ , and this means that  $V$  is a graded morphism of  $\Omega(M)$  (i.e.,  $V$  is  $\mathbb{R}$ -linear and has a certain degree  $|V|$ :  $V(\Omega^k(M)) \subset \Omega^{k+|V|}(M)$ ), and it verifies Leibniz's rule:  $V(\alpha \wedge \beta) = V(\alpha) \wedge \beta + (-1)^{|\alpha||V|} \alpha \wedge V(\beta)$ . Well-known examples of derivations on  $\Omega_M$  are the exterior differential,  $d$ , the Lie derivative with respect to a vector field  $X \in \mathcal{X}(M)$ ,  $\mathcal{L}_X$ , and the insertion  $i_X$ . Indeed, these operators considered as a subset of the graded endomorphisms of  $\Omega(M)$  (with degrees  $|\mathcal{L}_X| = 0$ ,  $|i_X| = -1$ ,  $|d| = 1$ , for any  $X \in \mathcal{X}(M)$ ) generate a Lie superalgebra  $\mathcal{A} = \langle \{\mathcal{L}_X, d, i_X\} \rangle$ , where the product is the composition of endomorphisms and the Lie superbracket  $[-, -]$  on  $\mathcal{A}$  is given by the graded commutator of graded endomorphisms:  $[E, F] = E \circ F - (-1)^{|E||F|} F \circ E$ , for all  $E, F \in \Omega(M)$ . These Lie superbrackets are very easily computed by using Cartan calculus; for example

$$[\mathcal{L}_X, i_Y] = \mathcal{L}_X \circ i_Y - (-1)^{0 \cdot (-1)} i_Y \circ \mathcal{L}_X = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X, Y]}. \quad (4.5)$$

Now, we would like to know what does it mean to integrate such a vector field. To be precise, consider the problem of integrating the derivation given by the exterior differential  $d : \Omega_M \rightarrow \Omega_M$ . This is a degree 1 derivation, in the sense that  $d(\Omega^i(M)) \subset \Omega^{i+1}(M)$  and

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{1 \cdot |\alpha|} \alpha \wedge d\beta, \quad \forall \alpha, \beta \in \Omega_M. \quad (4.6)$$

By analogy with (3.3) we seek for a curve  $\omega : \mathbb{R} \rightarrow \Omega_M$  (also denoted  $\omega(t)$ ) such that

$$\frac{d\omega}{dt}(t) = d(\omega(t)), \quad \forall t \in \mathbb{R}. \quad (4.7)$$

To find a solution we may proceed formally and construct the exponential of  $d$ , so we would have

$$\omega(t) = e^{td}\omega_0, \quad (4.8)$$

$\omega_0 \in \Omega_M$  being the initial condition  $\omega(0) = \omega_0$ . Here we understand that  $e^{td} = \sum_{k=0}^{\infty} ((t \cdot d)^k / k!)$ , although we do not have any *a priori* notion of convergence for such a series. Of course, for an arbitrary derivation (4.8) would not make sense, but here an important property of the particular derivation  $d$  comes into play: Its nilpotency. As  $d^2 = 0$ , the series in (4.8) reduces to a finite sum and, indeed,

$$\omega(t) = (I + t \cdot d)\omega(0), \quad (4.9)$$

where  $I \in \text{End}\Omega_M$  is the identity morphism.

## 5. The Need for Supertime

Thus, it seems that we have solved the problem of integrating a supervector field such as  $d$ , but still there are some technical questions pending. Recall that in the classical case we had a family of induced diffeomorphisms  $\{\varphi_t\}_{t \in \mathbb{R}}$  (which in turn extend their action as automorphisms to all of  $\Omega(M)$ ); this property is crucial in order to get a well-defined vector field on  $M$  from them. We should ask for a similar property to the morphisms obtained through (4.8). Indeed, if we consider  $C^\infty(M)$  instead of  $\Omega(M)$  (or the sheaf  $\mathcal{A}$  in general), we must recover the usual results, and there is a theorem implying that we must demand that the morphisms  $e^{td} = I + t \cdot d$  be exterior algebra automorphisms (see [33]): let  $E$  be a vector bundle over a manifold  $M$ , then, each automorphism of  $\Gamma(\wedge E)$  induces an isomorphism of  $E$  which, in turn, induces a diffeomorphism on  $M$ .

Now, let us check whether  $\{e^{td}\}_{t \in \mathbb{R}}$  is a family of automorphisms or not. Take  $\alpha, \beta \in \Omega(M)$ , with  $\alpha \in \Omega^{|\alpha|}(M)$ . Then, on the one hand,

$$\begin{aligned} e^{td}(\alpha \wedge \beta) &= (I + t \cdot d)(\alpha \wedge \beta) \\ &= \alpha \wedge \beta + t \cdot d(\alpha \wedge \beta) \\ &= \alpha \wedge \beta + t \cdot d\alpha \wedge \beta + (-1)^{|\alpha|} t \cdot \alpha \wedge d\beta, \end{aligned} \quad (5.1)$$

and on the other,

$$\begin{aligned} e^{td}\alpha \wedge e^{td}\beta &= (\alpha + t \cdot d\alpha) \wedge (\beta + t \cdot d\beta) \\ &= \alpha \wedge \beta + t \cdot d\alpha \wedge \beta + t \cdot \alpha \wedge d\beta + t^2 \cdot d\alpha \wedge d\beta. \end{aligned} \quad (5.2)$$

It is then clear that

$$e^{td}(\alpha \wedge \beta) \neq e^{td}\alpha \wedge e^{td}\beta, \quad (5.3)$$

so  $\{e^{t \cdot d}\}_{t \in \mathbb{R}}$  is *not* a family of automorphisms and we cannot claim to have solved the problem.

A comparison of (5.1) and (5.2) shows us the way out to this impasse. Let us write again these expressions, but now treating  $t$  as a formal parameter, not necessarily in  $\mathbb{R}$ , so we make no assumption about its commutativity with elements of  $\Omega(M)$ . Repeating the computations we get

$$\begin{aligned} e^{t \cdot d}(\alpha \wedge \beta) &= \alpha \wedge \beta + t \cdot d\alpha \wedge \beta + (-1)^{|\alpha|} t \cdot \alpha \wedge d\beta, \\ e^{t \cdot d}\alpha \wedge e^{t \cdot d}\beta &= \alpha \wedge \beta + t \cdot d\alpha \wedge \beta + \alpha \wedge t \cdot d\beta + t \cdot d\alpha \wedge t \cdot d\beta. \end{aligned} \quad (5.4)$$

The problem here is that the term containing two instances of the parameter (i.e.,  $t \cdot d\alpha \wedge t \cdot d\beta$ ) should vanish, and a sign is required to pass from  $\alpha \wedge t \cdot d\beta$  to  $(-1)^{|\alpha|} t \cdot \alpha \wedge d\beta$ . These inconveniences can be solved altogether if we take not  $t \in \mathbb{R}$ , but  $t = \theta$  an anticommuting parameter (so  $\theta^2 = 0$ ) of  $\mathbb{Z}$ -degree 1 (so  $\theta \cdot \alpha = (-1)^{|\alpha|} \alpha \cdot \theta$ ). With these assumptions, it is immediate that

$$e^{\theta \cdot d}(\alpha \wedge \beta) = e^{\theta \cdot d}\alpha \wedge e^{\theta \cdot d}\beta, \quad (5.5)$$

and the problem of integrating the vector field  $d$  on the supermanifold  $\mathcal{M} = (M, \Omega(M))$  will have a solution in the same sense that we integrate vector fields on a manifold: it admits a one-parameter group of integral automorphisms.

In Section 3 we saw that the differential equation is recovered by taking derivatives with respect to the parameter of the family of integrating automorphisms, so we are led to consider  $\partial/\partial\theta$  along with the usual operator  $\partial/\partial t$ . Indeed, note that if we write  $\omega(t, \theta) = (I + \theta \cdot d)\omega_0$  for the solution that we have found, then

$$\frac{d\omega}{dt}(t, \theta) = d(\omega(t, \theta)), \quad \forall t \in \mathbb{R}, \quad (5.6)$$

would not be true, as the left-hand side is trivially zero. Consider now  $D = D_0 + D_1 = (\partial/\partial t) + (\partial/\partial\theta)$  and put  $V = V_0 + V_1 = 0 + d$ , with this new notation we get the *super* differential equation:

$$D\omega(t, \theta) = V(\omega(t, \theta)), \quad (5.7)$$

which can immediately be splitted into two equations:

$$D_0\omega(t, \theta) = V_0(\omega(t, \theta)), \quad D_1\omega(t, \theta) = V_1(\omega(t, \theta)), \quad (5.8)$$

and is easy to see that in this context,  $\omega(t, \theta)$  is a solution to (5.7). Thus, if we accept the usual interpretation of time as the parameter of the family of integrating automorphisms of a dynamical vector field, to be consistent, we must also accept the idea that, when supermanifolds are used, a supertime  $(t, \theta)$  (where  $\theta$  is an anticommuting parameter) should be used instead of  $t$ .

Of course, for certain classes of supervector fields a single commuting parameter  $t$  can suffice (this is the case, e.g., of  $\mathcal{L}_X$  on the supermanifold  $\mathcal{M} = (M, \Omega(M))$ ). In other cases, as we have shown that it is enough with a single anticommuting parameter. By the way, let us remark that a single “odd time derivative”  $D_1 = \partial/\partial\theta$  has been used in physics, notably by O. F. Däyi in his study of quantum field theories (see [6]). But, in general, when considering some dynamical system on a supermanifold as described by a supervector field, we must take integrating parameters which are a mixture of both cases, that is, of the form  $(t, \theta)$ .

## 6. The Interpretation of Initial Conditions and the Question of Uniqueness

A glance at equations (5.8) tells us that when we have a superdifferential equation  $D\omega(t, \theta) = V(\omega(t, \theta))$ , the homogeneous components of  $D$  and  $V$  are  $\omega$ -related, so it follows that the following conditions must be satisfied:

$$[D_0, D_1]\omega(t, \theta) = [V_0, V_1]\omega(t, \theta), \quad [D_1, D_1]\omega(t, \theta) = [V_1, V_1]\omega(t, \theta). \quad (6.1)$$

In the classical case, we have only one operator,  $D_0 = d/dt$ , and only one equation,  $D_0 f = V_0 f$  (for  $f \in C^\infty(M)$ ), which is trivially satisfied as  $[D_0, D_0] = 0 = [V_0, V_0]$  for any  $D_0, V_0 \in \mathcal{X}(M)$ . Moreover, this unique integrating vector field  $D_0 = d/dt$  generates the unique one-dimensional (abelian) Lie algebra, that is, in this case the space of integrating parameters has the structure of a group.

In the setting of supermanifolds, it is natural to expect that relations (6.1) impose some conditions onto the superfields to be integrated because they say that  $\{D_0, D_1\}$  must generate a Lie superalgebra with dimension  $(1, 1)$  (this is because we have two generators, one of which ( $D_0$ ) is even and the other ( $D_1$ ) is odd), and it is well known that there are 3 non-isomorphic Lie superalgebras with this dimension (see [34]). Thus, assuming that  $\{D_0, D_1\}$  close a Lie superalgebra, there must be real constants  $a$  and  $b$  (with  $ab = 0$ ), such that

$$[D_0, D_1] = aD_1, \quad [D_1, D_1] = bD_0, \quad (6.2)$$

and a superdifferential equation  $D\omega(t, \theta) = V(\omega(t, \theta))$  will make sense only for those superfields  $V = V_0 + V_1$  satisfying:

$$[V_0, V_1] = aV_1, \quad [V_1, V_1] = bV_0. \quad (6.3)$$

In contrast with the classical case, these equations are *not* always satisfied for a vector field on a supermanifold. For example, let  $W$  be the supervector field defined by  $W_0 = 0$  and  $W_1 = d + i_X$  ( $X \in \mathcal{X}(M)$ ) on the supermanifold  $\mathcal{M} = (M, \Omega(M))$ . Proceeding as we did in Section 5, we get that

$$\omega(t, \theta) = I + \theta \cdot (d + i_X) \quad (6.4)$$

will define a family of automorphisms. However,

$$D_1\omega(t, \theta) = \frac{\partial}{\partial\theta}(I + \theta \cdot (d + i_X)) = d + i_X, \quad (6.5)$$

and consequently,

$$W_1(\omega(t, \theta)) = (d + i_X)(I + \theta \cdot (d + i_X)) = d + i_X - \theta \cdot \mathcal{L}_X = D_1\omega(t, \theta) - \theta \cdot \mathcal{L}_X, \quad (6.6)$$

so  $D_1\omega(t, \theta) \neq W_1\omega(t, \theta)$ .

What is wrong with this example? In the example of Section 5 the supervector field was  $V = V_0 + V_1 = 0 + d$  and the graded bracket relations between its homogeneous components were

$$[0, 0] = 0, \quad [0, d] = 0, \quad [d, d] = 0, \quad (6.7)$$

that is, the homogeneous components of the vector field  $V = d$  generate a Lie superalgebra structure on the supervector space  $\langle \{V_0, V_1\} \rangle$ . However, for the case at hand  $W = W_1 = d + i_X$ , the brackets between homogeneous components are

$$[0, 0] = 0, \quad [0, d + i_X] = 0, \quad [d + i_X, d + i_X] = 2\mathcal{L}_X \neq 0, \quad (6.8)$$

and these do not define a Lie superalgebra structure.

As mentioned, this example seems to imply that not every differential equation on a supermanifold will make sense and indeed, for a long time, it was thought that this is the case. However, J. Monterde and A. Sánchez-Valenzuela in [13] were able to construct a procedure to integrate *any* superdifferential equation, regardless of whether the homogeneous components of the supervector field  $V = V_0 + V_1$  close a Lie superalgebra or not. Their idea was to get rid of the conditions on the homogeneous components by introducing the so called “evaluation morphism” on points,  $ev|_{t=t_0}$  (this morphism had already appeared in [32]); to do this, we need first to pose the superdifferential equations as

$$ev|_{t=t_0} D\omega = ev|_{t=t_0} V(\omega). \quad (6.9)$$

The meaning of this expression, aside of technicalities, is “first, take a congruence modulo  $\theta$  and then evaluate in  $t = t_0$ .” (It is not easy to give, in a few sentences, a motivation for the introduction of  $ev|_{t=t_0}$  which can be considered as “intuitive”. However, we can offer a quick categorical argument: in the category of supermanifolds the terminal object is  $(\{*\}, \mathbb{R})$ , a point with the algebra of constants on it, with terminal morphism  $C : (M, \mathcal{A}_M) \rightarrow (\{*\}, \mathbb{R})$ . On the other hand, every supermanifold has a preferred embedding  $\delta : (M, C^\infty(M)) \rightarrow (M, \mathcal{A}_M)$  which induces a graded algebra morphism  $f \in \mathcal{A} \mapsto \tilde{f} \in C^\infty(M)$ , and each point  $p \in M$  defines a morphism  $\delta_p : (\{*\}, \mathbb{R}) \rightarrow (M, \mathcal{A}_M)$  simply by declaring that its associated graded algebra morphism is  $f \in \mathcal{A} \mapsto \tilde{f}(x) \in \mathbb{R}$ . Then, the evaluation morphism is determined by these natural morphisms:  $ev|_p = (\delta_p \circ C)^*$ .) We can see how it works by reconsidering the

example of  $W = W_1 = d + i_X$ , for which we have found (see (6.6)) that

$$D_1\omega(t, \theta) = W_1(\omega(t, \theta)) + \theta \cdot \mathcal{L}_X. \quad (6.10)$$

Applying the evaluation morphism, the term  $\theta \cdot \mathcal{L}_X$  vanishes, and we get

$$ev|_{t=t_0} D_1\omega(t, \theta) = ev|_{t=t_0} W_1(\omega(t, \theta)). \quad (6.11)$$

Thus, the introduction of the evaluation morphism allows us to give an interpretation to the “initial conditions” for a superdifferential equation: the imposition of initial conditions is a procedure to project the equation onto homogeneous components in such a way that these verify the Lie superalgebra conditions (6.3), so the equation can be effectively integrated. In this way, the theory in the graded setting is complete and its results are analog to those of the classical case. We refer the reader to [13] for the details and complete proofs.

## 7. Covariant Superderivatives and the Lie Supergroup $\mathbb{R}^{1|1}$

In physics, it is common to find expressions involving the so-called covariant superderivatives. In (1|1)-supermechanics, the covariant superderivative is the superfield  $\theta(\partial/\partial t) + \partial/\partial\theta$ , and in this section we would like to explore the connection between this superfield and the integrating model  $\partial/\partial t + \partial/\partial\theta$ . To be precise, we will see that no matter which supergroup addition law one uses in the parameter superspace  $\mathbb{R}^{1|1}$ , one is always able to provide a unique (and always the same) local solution to a given differential equation. In particular, the pair  $\partial/\partial t, \theta(\partial/\partial t) + \partial/\partial\theta$  plays exactly the same role as the pair  $\partial/\partial t + \partial/\partial\theta$  as far as the integration process is concerned. The only difference is that the pair  $\partial/\partial t, \theta(\partial/\partial t) + \partial/\partial\theta$  is adapted to a different addition law on  $\mathbb{R}^{1|1}$  than  $\partial/\partial t + \partial/\partial\theta$  is. In this way, we will see that the introduction of a covariant superderivative amounts to a change in the Lie supergroup structure of the space of integrating parameters without changing the local expression of the solutions of differential equations.

Suppose that we want to build a supersymmetric action functional on our supermanifold  $(M, \Omega(M))$  (or any other supermanifold, for this purpose). Then, we look for an expression of the form

$$S[\omega(t, \theta)] = \int_{\mathbb{R}^{1|1}} dt d\theta \mathcal{L}(\omega(t, \theta)), \quad (7.1)$$

where the integration over  $t$  is the usual integration over  $\mathbb{R}$ , and the integration over  $\theta$  is the Berezin integration (defined in such a way that  $\int d\theta \omega(t, \theta) = \partial/\partial\theta(\omega(t, \theta))$ ). The superlagrangian  $\mathcal{L}$  must be a function of  $\omega(t, \theta)$  and its “covariant superderivatives”  $D\omega(t, \theta)$ , where  $D$  is an operator to be determined. For example, a simple nontrivial choice (called the “free superlagrangian”) is given by

$$\mathcal{L}(\omega(t, \theta)) = D\omega(t, \theta) \cdot D(D\omega(t, \theta)). \quad (7.2)$$

From this Lagrangian  $\mathcal{L}$  and the action  $S[\omega(t, \theta)]$  we would like to extract the Euler-Lagrange superequations. But, in order to do so, we need to know how to compute the variations induced in  $\omega$ ,  $\delta\omega(t, \theta)$ , and its derivatives,  $\delta D\omega(t, \theta)$ . Now, any transformation of the form

$$\begin{aligned} t &\mapsto t - f_\varepsilon(t, \theta), \\ \theta &\mapsto \theta - g_\varepsilon(t, \theta), \end{aligned} \tag{7.3}$$

where  $\varepsilon$  is an anticommuting parameter, has associated a generator (or supercharge)  $Q$  such that the induced variation on any  $\alpha(t, \theta) \in \mathbb{R}^{1|1}$  is given by

$$\delta\alpha(t, \theta) = \varepsilon \cdot Q\alpha(t, \theta), \tag{7.4}$$

so, in particular,

$$\delta D\omega(t, \theta) = \varepsilon \cdot Q(D\omega(t, \theta)). \tag{7.5}$$

Now, if we choose the covariant superderivative  $D$  as an odd element in the centralizer of the algebra generated by  $Q$  (i.e., if we choose  $D \in Z\langle Q \rangle$ ), we will have

$$0 = \llbracket Q, D \rrbracket = Q \circ D - (-1)^{1 \cdot 1} D \circ Q = QD + DQ, \tag{7.6}$$

so (note that  $\varepsilon$ ,  $Q$  and  $D$  all have odd degree)

$$\delta D\omega(t, \theta) = \varepsilon \cdot Q(D\omega(t, \theta)) = -\varepsilon \cdot D(Q\omega(t, \theta)) = D(\varepsilon \cdot Q\omega(t, \theta)) = D(\delta\omega(t, \theta)), \tag{7.7}$$

that is, the variation of the derivatives,  $\delta D\omega$ , can be computed as the derivatives of the variation  $\delta D\omega = D\delta\omega$ , a condition which directly leads to the usual Euler-Lagrange equations.

Returning now to (7.3), we take them to be the (left) supertranslations of the form

$$\begin{aligned} t &\mapsto t + \varepsilon \cdot \theta, \\ \theta &\mapsto \theta - \varepsilon, \end{aligned} \tag{7.8}$$

(here  $\theta, \varepsilon \in \Omega^1(\mathbb{R})$  so they are of the form  $\theta = f dt$  and  $\varepsilon = g dt$  for some real functions  $f, g$ . The product  $\varepsilon \cdot \theta$  is then to be understood as  $gf$ ). It is usual in physics to consider the anticommuting parameters  $\varepsilon$  as complex spinors, and the translations in  $t$  of the form  $t + \overline{i\varepsilon} \cdot \theta$ , or even the translations in the whole 4-vector  $x^\mu$  including a gamma matrix as  $x^\mu + \overline{i\varepsilon} \gamma^\mu \theta$ . When one considers (real) Majorana spinors or no spinors at all (as in our setting), the formulae reduce to our expressions (see [10, Chapter 14]). These will induce a variation in  $\omega(t, \theta)$  determined by

$$\omega(t + \varepsilon \cdot \theta, \theta - \varepsilon) = \omega(t, \theta) + \delta\omega(t, \theta), \tag{7.9}$$

and it is immediate that  $\delta\omega(t, \theta) = \varepsilon \cdot Q\omega(t, \theta)$ , where  $Q$  is the generator of these supertranslations, the odd vector field given by

$$Q = \theta \frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}. \quad (7.10)$$

This odd vector field, taken together with the even operator  $H = \partial/\partial t$  (which is the generator of classical time translations) close the simplest Lie superalgebra, as it is easy to check that

$$\begin{aligned} \llbracket Q, H \rrbracket &= 0 = \llbracket H, H \rrbracket, \\ \llbracket Q, Q \rrbracket &= -2H. \end{aligned} \quad (7.11)$$

Now, as stated before, we must look for a supervector field  $D$  with odd degree such that  $D$  lies in the centralizer of  $Q = \theta(\partial/\partial t) - \partial/\partial\theta$ . Indeed, as  $H$  is even,  $D$  must also commute with it, so actually we are looking for an element in the center of the superalgebra  $\langle Q, H \rangle$ . This superalgebra is very simple and well known. In fact, it can be seen that its center is generated by the element

$$D_1 = \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} \quad (7.12)$$

(note the sign difference with  $Q$ ). This is the covariant superderivative used in physics. What is its group theoretical meaning? It results that among the several Lie supergroup structures  $\mathbb{R}^{1|1}$  admits, there is one and just one such that the generators of left invariant supervector fields are precisely

$$D_0 = \frac{\partial}{\partial t}, \quad D_1 = \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}. \quad (7.13)$$

This supergroup structure is different from the structure associated to the integrator  $\partial/\partial t + \partial/\partial\theta$ , with homogeneous components  $D_0 = \partial/\partial t$  and  $D_1 = \partial/\partial\theta$ , that we have considered in previous sections. For  $\partial/\partial t + \partial/\partial\theta$ , an integration gives the multiplication rule  $m$  (for elements  $(t_1, \theta_1)$  and  $(t_2, \theta_2)$  in  $\mathbb{R}^{1|1}$ )

$$m((t_1, \theta_1), (t_2, \theta_2)) = (t_1 + t_2, \theta_1 + \theta_2), \quad (7.14)$$

which correspond to Type 1 product in Table 2, while for the choice of the basis of left invariant supervector fields  $D_0 = \partial/\partial t$  and  $D_1 = \theta(\partial/\partial t) + \partial/\partial\theta$  we get a different (and nonequivalent) rule  $\mu$ , which corresponds to Type 2 in the table:

$$\mu((t_1, \theta_1), (t_2, \theta_2)) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2). \quad (7.15)$$

Thus, we can say that the introduction of the covariant superderivative  $D = \theta(\partial/\partial t) + \partial/\partial\theta$  amounts to a change in the structure of Lie supergroup of  $\mathbb{R}^{1|1}$  from the one given by Type 1 multiplication  $m$ , whose basis of homogeneous left invariant vector fields is  $\{\partial/\partial t, \partial/\partial\theta\}$ ,

to the one determined by  $\mu$ , Type 2, with basis of homogeneous left invariant vector fields  $\{\partial/\partial t, \theta(\partial/\partial t) + \partial/\partial\theta\}$ .

What is the effect of this change with respect to the problem of integrating differential equations? If we shift from the Lie supergroup structure  $(\mathbb{R}^{1|1}, m)$  to  $(\mathbb{R}^{1|1}, \mu)$ , we should be able to integrate any differential equation with  $D_0 = \partial/\partial t$  and  $D_1 = \theta(\partial/\partial t) + \partial/\partial\theta$ , and we know that this requires that these homogeneous vector fields close a Lie superalgebra, which they do:

$$\begin{aligned} \llbracket D_0, D_0 \rrbracket &= 0 = \llbracket D_1, D_0 \rrbracket \\ \llbracket D_1, D_1 \rrbracket &= -2D_0. \end{aligned} \tag{7.16}$$

(Note, by the way, that this is the same Lie superalgebra determined by the supercharge  $Q$  and the even Hamiltonian  $H$ ). But, moreover, regarding the integration of differential equations, this shift has no consequences because of the evaluation morphism: as this morphism implies to take modulo  $\theta$  in the equations, it is equivalent to consider the integrating field  $\partial/\partial t + \partial/\partial\theta$  or any other field of the form  $\partial/\partial t + \partial/\partial\theta + \theta \cdot F$ . And the difference between the integrating fields given by  $D_0 + D_1 = \partial/\partial t + \partial/\partial\theta$  and  $D_0 + D_1 = \partial/\partial t + \theta(\partial/\partial t) + \partial/\partial\theta$  is precisely a term of this form with  $F = \partial/\partial t$ .

## 8. Covariant Superderivatives in $\mathbb{R}^{1|n}$

We want to extend our previous analysis to the case of  $N$ -supersymmetry, that is, the case of several fermionic charges  $Q_i$  (with  $1 \leq n$ ). This leads us to consider the ‘‘supertime space’’  $\mathbb{R}^{1|n}$ .

Proceeding as in the case of  $\mathbb{R}^{1|1}$ , we characterize  $\mathbb{R}^{1|n}$  as the base manifold  $\mathbb{R}$  with a sheaf of superfunctions  $\mathcal{A}$ , which this time can be described as follows. Over each open set  $I \subset \mathbb{R}$ ,  $\mathcal{A}_I$  is the  $C^\infty(I)$ -module with a set of  $n$  generators  $\theta_1, \dots, \theta_n$ , subject to the product rule (which is extended by linearity):

$$\theta_i \theta_j + \theta_j \theta_i = 0 = \llbracket \theta_i, \theta_j \rrbracket. \tag{8.1}$$

So, in particular,  $(\theta_i)^2 = 0$  for any  $1 \leq i \leq n$ . However,

$$\theta_i \cdots \theta_n \neq 0. \tag{8.2}$$

We will consider, for simplicity, the case of globally defined superfunctions, that is,  $I = \mathbb{R}$ . A brief notation for this is then  $\mathcal{A}_{\mathbb{R}} = C^\infty(\mathbb{R})[\theta_1, \dots, \theta_n]$ . We can think about these superfunctions as elements of the form

$$\alpha = \sum_j \sum_{i_j} \alpha_{i_1 \dots i_j} \theta_{i_1} \cdots \theta_{i_j}, \tag{8.3}$$

with the sum over  $j$  running from 0 to  $n$  and the sum over  $i_j$  running from 1 to  $n$  with  $1 \leq i_1 \leq \dots \leq i_j \leq n$ . Each coefficient  $\alpha_{i_1 \dots i_j}$  is an element of  $C^\infty(\mathbb{R})$ .

Let  $D$  be a derivation of  $\mathcal{A}_{\mathbb{R}^{1|n}}$ , that is,  $D \in \text{Der}(\mathcal{A}_{\mathbb{R}^{1|n}})$ . We can decompose it into homogeneous even and odd parts,  $D = D_0 + D_1$  with  $D_i \in (\text{Der } \mathcal{A}_{\mathbb{R}^{1|n}})_i$ ,  $i = 0, 1$ , and as derivations are particular cases of endomorphisms of a superalgebra, we can consider the Lie superalgebra structure given by the restriction of the graded commutator of endomorphisms:

$$[[D_1, D_2]] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1. \quad (8.4)$$

It is easy to verify that  $\{\partial/\partial t, \theta_j(\partial/\partial\theta_i)\} \subset (\text{Der } \mathcal{A}_{\mathbb{R}^{1|n}})_0$  whereas  $\{\theta_i(\partial/\partial t), \partial/\partial\theta_i\} \subset (\text{Der } \mathcal{A}_{\mathbb{R}^{1|n}})_1$ . Moreover,

$$\begin{aligned} \left[ \theta_j \frac{\partial}{\partial\theta_i}, \theta_k \frac{\partial}{\partial t} \right] &= \delta_i^k \theta_j \frac{\partial}{\partial t}, & \left[ \theta_j \frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_k} \right] &= -\delta_k^j \frac{\partial}{\partial\theta_i}, \\ \left[ \theta_j \frac{\partial}{\partial t}, \frac{\partial}{\partial\theta_i} \right] &= \delta_i^j \frac{\partial}{\partial t}, \end{aligned} \quad (8.5)$$

are the nontrivial Lie superbrackets. We can define a Lie subsuperalgebra of  $\text{Der } \mathcal{A}_{\mathbb{R}^{1|n}}$  taking the  $\mathbb{R}$ -expand of  $\{\partial/\partial t, \theta_j(\partial/\partial\theta_i), \theta_i(\partial/\partial t), \partial/\partial\theta_i\}$ , and it is straightforward to verify that the result is isomorphic to  $\mathfrak{gl}(\mathbb{R}^{1|n})$  (see [34] for more on  $\mathfrak{gl}(V)$  for  $V$  a vector superspace). That is, we will restrict ourselves to the linear expressions in  $\{\partial/\partial t, \theta_j(\partial/\partial\theta_i), \theta_i(\partial/\partial t), \partial/\partial\theta_i\}$  with real constant coefficients. We will consider in the following this Lie superalgebra  $\mathfrak{gl}(\mathbb{R}^{1|n})$ .

Following the ideas expressed in previous sections, we want to consider now the left supertranslations  $L_i (1 \leq i \leq n)$  of the form

$$\begin{aligned} t &\longmapsto t + \varepsilon \cdot \theta_i, \\ \theta_i &\longmapsto \theta_i - \varepsilon, \\ \theta_j &\longmapsto \theta_j \quad (i \neq j). \end{aligned} \quad (8.6)$$

And, as before, we get that the generators of these supertranslations are given by the odd vector fields:

$$Q_i = \theta_i \frac{\partial}{\partial t} - \frac{\partial}{\partial\theta_i}, \quad 1 \leq i \leq n. \quad (8.7)$$

It is easy to verify that taking  $H = \partial/\partial t$  as the even generator, they close in a Lie superalgebra with dimension  $(1, n)$ . The nontrivial relations are

$$[[Q_i, Q_i]] = -2H, \quad 1 \leq i \leq n. \quad (8.8)$$

The Lie superalgebra these supercharges generate is a subalgebra of  $\mathfrak{gl}(\mathbb{R}^{1|n})$ :

$$\langle H, Q_i \rangle_{1 \leq i \leq n} \subset \mathfrak{gl}(\mathbb{R}^{1|n}) \quad (8.9)$$

Now, let  $D \in (\mathfrak{gl}(\mathbb{R}^{1|n}))_1$ , so,

$$D = \sum_{k=1}^n a_k \theta_k \frac{\partial}{\partial t} + b_k \frac{\partial}{\partial \theta_k}. \quad (8.10)$$

We are looking for  $D$  odd such that

$$[[Q_i, D]] = 0, \quad 1 \leq i \leq n \quad (8.11)$$

but now,

$$\begin{aligned} [[Q_i, D]] &= \left[ \left[ \theta_i \frac{\partial}{\partial t} - \frac{\partial}{\partial \theta_i}, \sum_{k=1}^n a_k \theta_k \frac{\partial}{\partial t} + b_k \frac{\partial}{\partial \theta_k} \right] \right] \\ &= \sum_{k=1}^n \left( a_k \left[ \left[ \theta_i \frac{\partial}{\partial t}, \theta_k \frac{\partial}{\partial t} \right] + b_k \left[ \left[ \theta_i \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta_k} \right] \right] \right. \right. \\ &\quad \left. \left. - a_k \left[ \left[ \frac{\partial}{\partial \theta_i}, \theta_k \frac{\partial}{\partial t} \right] - b_k \left[ \left[ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_k} \right] \right] \right] \right) \\ &= \sum_{k=1}^n \delta_k^i (b_k - a_k) \frac{\partial}{\partial t} \\ &= (b_i - a_i) \frac{\partial}{\partial t}. \end{aligned} \quad (8.12)$$

So we get that  $[[Q_i, D]] = 0$  if and only if  $a_i = b_i$ ; then, the centralizer of  $\langle H, Q_i \rangle$  in  $\mathfrak{gl}(\mathbb{R}^{1|n})$  is what we expected:

$$\left( Z_{\mathfrak{gl}(\mathbb{R}^{1|n})} \langle H, Q_i \rangle \right)_1 = \left\langle \theta_i \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta_i} \right\rangle_{1 \leq i \leq n}. \quad (8.13)$$

In fact, it is easy to prove that

$$\left( Z_{\mathfrak{gl}(\mathbb{R}^{1|n})} \langle H, Q_i \rangle \right)_0 = \langle H \rangle. \quad (8.14)$$

Moreover, if we denote by  $D_i = \theta_i(\partial/\partial t) + \partial/\partial \theta_i$ , then

$$\langle H, Q_i \rangle \simeq \langle H, D_i \rangle, \quad (8.15)$$

as Lie superalgebras.

## 9. Conclusions

The previous exposition can raise an important question: what is the meaning of the examples we have presented? Are they general enough? The answer to that question is yes. First,

let us note that superspaces as used in physics can be seen as images of graded manifolds (in the sense of Berezin-Kostant-Leites) under the functor of points (see [35].) Although this correspondence is not functorial, it is enough to describe these superspaces within our framework (see [31].) Second, there is a theorem (due to M. Batchelor and K. Gawedzki, see [36, 37]) stating that *any* real supermanifold  $\mathcal{M}$  can be considered in the form  $(M, \Gamma(\wedge E))$  for some vector bundle  $\pi : E \rightarrow M$ , and this is equivalent to giving a graded connection on  $\mathcal{M}$  (see [38], note that this correspondence is not canonical). The case we have just analyzed corresponds to the choice  $E = T^*M$  (the cotangent bundle), but the changes needed to deal with the general case are mainly notational in character.

The moral is that, from a mathematical point of view, to integrate an arbitrary vector field on a supermanifold we need to work with a parameter space more general than  $\mathbb{R}$ . Our examples above tell us that in order to integrate a degree-1 vector field, such as  $d$  on  $(M, \Omega(M))$ , we need an anticommuting parameter, and that, in general, a pair  $(t, \theta)$  with  $t$  commuting and  $\theta$  anticommuting will suffice to deal with an arbitrary vector field. The supermanifold of which these pairs are elements is the graded generalization (or supersymmetrization) of  $\mathbb{R}$ , and is denoted by  $\mathbb{R}^{1|1}$ . So, from a physical point of view we can say that in order to solve a dynamical theory on a supermanifold (one given by an arbitrary dynamical vector field, e.g., in the Hamiltonian setting), we must use the notion of supertime understood as an integrating parameter in  $\mathbb{R}^{1|1}$ .

However, the choice of this integrating parameter is flexible. If one is interested just in the solution to some differential equation, then this choice is irrelevant as long as the different generators of supertime differ by a term linear in  $\theta$ . But if we wish to construct some dynamics through a supersymmetric Lagrangian  $\mathcal{L}$ , then we must pay some attention to the underlying group structure on  $\mathbb{R}^{1|1}$ , as this structure is intimately tied to the covariant superderivatives that enter in the definition of  $\mathcal{L}$ .

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