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Research Article

Relative-Velocity Distributions for Two Effusive Atomic Beams in Counterpropagating and Crossed-Beam Geometries

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Formulas are presented for calculating the relative velocity distributions in effusive, orthogonal crossed beams and in effusive, counterpropagating beams experiments, which are two important geometries for the study of collision processes between atoms. In addition formulas for the distributions of collision rates and collision energies are also given.

1. Introduction

Studies of atomic collisions—and in particular collisions between laser-excited atoms—have produced a large amount of information on atomic interactions. Numerous investigations on excited-atom collisions have been performed at thermal temperatures in vapor cells [1–6] and in atomic beams with different geometries [2, 7–13]. Typically, the cell experiments give results averaged over all collision directions and over the thermal velocity distribution, whereas the beam experiments allow more information to be extracted from the collision process, for example, the dependence of the collision dynamics on the atomic polarization.

The collision velocity is an important parameter in a collision process, and a knowledge of the relative collision velocities in the experimental setups is therefore essential for a careful analysis of a crossed beams experiment. This is often done using numerical simulations, but more insight can be obtained from analytical solutions using kinetic gas theory.

Two important cases are experiments where the two effusive beams are either crossed at right angles or counterpropagating, and for these two cases a number of useful formulas are presented for the distribution of relative velocities, collision velocities, and collision energies.

2. Velocity Distributions

In the following analysis of two crossed or counter-propagating beams, intrabeam collisions will be neglected, that is, the collisions between the atoms within a single beam ("head-tail" collisions). Intrabeam collisions in a single beam have been analyzed by Baylis [14], where formulas for the relative-velocity distribution and moments of the distribution are given. For crossed beams, numerical results and some related integrals have been given by Berkling et al. [15] and by Meijer [16]. A single molecular beam from an oven has been treated by Leiby and Besse [17], and supersonic nozzle beams have been described by Haberland et al. [18]. Recently Battaglia et al. [19] have described the velocity distribution of thermionic electrons.

2.1. Relative-Velocity Distributions

The determination of the distribution of relative velocities requires a knowledge of the velocity distribution of each beam. We will assume that the beams are produced from an effusive source with a thin-walled orifice and the velocity distribution can then be calculated from kinetic gas theory as a Maxwell-Boltzmann distribution, which is confined to one direction through an aperture. The distribution becomes [14]

$$f(v,m,T) = \left(\frac{2m^3}{\pi k^3 T^3}\right)^{1/2} v^2 e^{-mv^2/2kT}, \tag{2.1}$$

where $v \geq 0$ because it is a beam in one direction. Measurements on atomic beams have shown that (2.1) is a good description of the velocity distribution [20].

For the general case of two beams, one with a distribution $f(v_1, m_1, T_1)$ and one with a distribution $f(v_2, m_2, T_2)$, the distribution function F of the relative velocity is found by convoluting the two distributions. Using v_{rel} for the relative velocity, the convolution is written as follows:

$$F(v_{\text{rel}}) = \int_0^\infty dv_1 \int_0^\infty dv_2 f(v_1, m_1, T_1) f(v_2, m_2, T_2) \delta(v_{\text{rel}} - |\vec{v}_1 - \vec{v}_2|). \tag{2.2}$$

In an experiment with well-collimated counter-propagating (cp) beams, the length of the velocity vector $|\vec{v}_1 - \vec{v}_2|$ is given by

$$|\vec{v}_1 - \vec{v}_2|^{\text{cp}} = v_1 + v_2, \tag{2.3}$$

whereas with orthogonal crossed beams (cb) the length is

$$|\vec{v}_1 - \vec{v}_2|^{\text{cb}} = \sqrt{v_1^2 + v_2^2}. \tag{2.4}$$

In a single beam (sb), the length is

$$|\vec{v}_1 - \vec{v}_2|^{\text{sb}} = |v_1 - v_2|, \tag{2.5}$$

and as mentioned in the introduction an analytical expression of $F^{\text{sb}}(v_{\text{rel}})$ has been given by Baylis [14] for the case when $T_1 = T_2$.

With counter-propagating beams, after inserting (2.3) in (2.2) and integrating over v_2, we get (note that the limit of integration has been changed as $v_1 \leq v_{\text{rel}}$, because $0 \leq v_2 = v_{\text{rel}} - v_1$)

$$F^{\text{cp}}(v_{\text{rel}}) = \int_0^{v_{\text{rel}}} dv_1 f(v_1, m_1, T_1) f(v_{\text{rel}} - v_1, m_2, T_2). \tag{2.6}$$

The integration (Several of the integrations were performed using Mathematica.) yields the result:

$$\begin{aligned}
F^{\text{cp}}(v_{\text{rel}}) = {} & \frac{2}{\pi k^3}\sqrt{m_1^3 m_2^3} \\
& \times \Bigg\{ \frac{k\sqrt{T_1T_2}v_{\text{rel}}\left(-4km_2^2T_1^2 - 3km_1m_2T_1T_2 + km_1^2T_2^2 + m_1m_2^2T_1v_{\text{rel}}^2\right)}{(m_1T_2 + m_2T_1)^4 \exp\left(m_1v_{\text{rel}}^2/2kT_1\right)} \\
& \quad + \frac{k\sqrt{T_1T_2}v_{\text{rel}}\left(-4km_1^2T_2^2 - 3km_1m_2T_1T_2 + km_2^2T_1^2 + m_1^2m_2T_2v_{\text{rel}}^2\right)}{(m_1T_2 + m_2T_1)^4 \exp\left(m_2v_{\text{rel}}^2/2kT_2\right)} \\
& \quad + \frac{\sqrt{\pi k/2}\left(3k^2m_2^2T_1^3T_2 + 6k^2m_1m_2T_1^2T_2^2 + 3k^2m_1^2T_1T_2^3 + km_2^3T_1^3v_{\text{rel}}^2\right)}{(m_1T_2 + m_2T_1)^{9/2} \exp\left(m_1m_2v_{\text{rel}}^2/2k(m_1T_2 + m_2T_1)\right)} \\
& \quad + \frac{\sqrt{\pi k/2}\left(-3km_1m_2^2T_1^2T_2v_{\text{rel}}^2 - 3km_1^2m_2T_1T_2^2v_{\text{rel}}^2 + km_1^3T_2^3v_{\text{rel}}^2 + m_1^2m_2^2T_1T_2v_{\text{rel}}^4\right)}{(m_1T_2 + m_2T_1)^{9/2} \exp\left(m_1m_2v_{\text{rel}}^2/2k(m_1T_2 + m_2T_1)\right)} \\
& \quad \times \left[\operatorname{erf}\left(\frac{m_2\sqrt{T_1}v_{\text{rel}}}{\sqrt{2kT_2(m_1T_2 + m_2T_1)}}\right) + \operatorname{erf}\left(\frac{m_1\sqrt{T_2}v_{\text{rel}}}{\sqrt{2kT_1(m_1T_2 + m_2T_1)}}\right)\right]\Bigg\}.
\end{aligned} \tag{2.7}$$

The unit of the distribution function is $\text{s}\cdot\text{m}^{-1}$, and the function has been integrated analytically from 0 to ∞ to verify that the result is 1 as expected from the normalization. Unfortunately with typical values for $m_{1,2}$ and $T_{1,2}$, the argument of the error-function is not close to 0 or ∞, so the factor cannot be simplified further.

For orthogonal crossed beams, the result of inserting (2.4) in (2.2) and performing the integration gives

$$\begin{aligned}
& F^{\text{cb}}(v_{\text{rel}}) \\
& = \left(\frac{m_1m_2}{4k^2T_1T_2}\right)^{3/2} v_{\text{rel}}^5 \exp\left(-\frac{(m_1T_2 + m_2T_1)v_{\text{rel}}^2}{4kT_1T_2}\right) {}_0F_1^{\text{reg}}\left(;2; \frac{(m_1T_2 - m_2T_1)^2 v_{\text{rel}}^4}{64k^2T_1^2T_2^2}\right),
\end{aligned} \tag{2.8}$$

where ${}_0F_1^{\mathrm{reg}}$ is the regularized hypergeometric function given by

$$ {}_0F_1^{\mathrm{reg}}(;a;z) = \frac{1}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{z^n}{(a)_n n!}. \tag{2.9} $$

The expressions (2.7) and (2.8) can be integrated to find the mean value of the relative velocity, which is given as

$$ \langle v_{\mathrm{rel}} \rangle = \frac{\int_0^{\infty} dv_{\mathrm{rel}} v_{\mathrm{rel}} F(v_{\mathrm{rel}})}{\int_0^{\infty} dv_{\mathrm{rel}} F(v_{\mathrm{rel}})}. \tag{2.10} $$

Note that because of the normalization of F, the denominator is just 1. The result of inserting (2.7) in (2.10) and performing the integration yields the surprisingly simple result:

$$ \langle v_{\mathrm{rel}} \rangle^{\mathrm{cp}} = \sqrt{\frac{8kT_1}{\pi m_1}} + \sqrt{\frac{8kT_2}{\pi m_2}}, \tag{2.11} $$

which, however, can be understood as stating that the mean relative velocity is just the sum of the two mean velocities, $\langle v_{\mathrm{MB}} \rangle_{1,2}$, of the Maxwell-Boltzmann distribution from the two beam sources. This value is found from (2.1) and yields

$$ \langle v_{\mathrm{MB}} \rangle = \sqrt{\frac{8kT}{\pi m}}. \tag{2.12} $$

For orthogonal crossed beams, the expression for the mean relative velocity, using (2.8) and (2.10), becomes

$$ \langle v_{\mathrm{rel}} \rangle^{\mathrm{cb}} = 15\sqrt{\pi} \left(\frac{m_1 m_2 T_1 T_2}{(m_1 T_2 + m_2 T_1)^2} \right)^{3/2} \sqrt{\frac{kT_1 T_2}{m_1 T_2 + m_2 T_1}}\, {}_2F_1\left(\frac{7}{4}; \frac{9}{4}; 2; \frac{(m_2 T_1 - m_1 T_2)^2}{(m_1 T_2 + m_2 T_1)^2} \right), \tag{2.13} $$

where ${}_2F_1$ is the hypergeometric function given by

$$ {}_2F_1(a;b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{2.14} $$

In the particular case of two identical crossed beams with $m = m_1 = m_2$ and $T = T_1 = T_2$, the expressions above can be simplified considerably. For counter-propagating beams, the relative-velocity distribution (2.7) becomes

$$F^{\mathrm{cp}}(v_{\mathrm{rel}}) = \frac{mv_{\mathrm{rel}}\left(-6kT + mv_{\mathrm{rel}}^2\right)}{4\pi k^2T^2 \exp\left(mv_{\mathrm{rel}}^2/2kT\right)} + \frac{\sqrt{m}\left(12k^2T^2 - 4kmTv_{\mathrm{rel}}^2 + m^2v_{\mathrm{rel}}^4\right)}{8\sqrt{\pi}k^{5/2}T^{5/2}\exp\left(mv_{\mathrm{rel}}^2/4kT\right)} \operatorname{erf}\left(\sqrt{\frac{mv_{\mathrm{rel}}^2}{4kT}}\right), \tag{2.15}$$

and the mean relative velocity is given as

$$\langle v_{\mathrm{rel}}\rangle^{\mathrm{cp}} = \sqrt{\frac{32kT}{\pi m}} = 2\langle v_{\mathrm{MB}}\rangle. \tag{2.16}$$

For identical orthogonal crossed beams, (2.8) reduces to

$$F^{\mathrm{cb}}(v_{\mathrm{rel}}) = \left(\frac{m}{2kT}\right)^3 v_{\mathrm{rel}}^5 \exp\left(-\frac{mv_{\mathrm{rel}}^2}{2kT}\right), \tag{2.17}$$

and (2.13) reduces to

$$\langle v_{\mathrm{rel}}\rangle^{\mathrm{cb}} = \frac{15\pi}{32}\langle v_{\mathrm{MB}}\rangle. \tag{2.18}$$

Note that $15\pi/32 = 1.47 \simeq \sqrt{2}$ as would have been expected for orthogonal crossed beams with the singular velocity $\langle v_{\mathrm{MB}}\rangle$. The expression (2.17) for identical crossed beams has also been obtained by Bezuglov et al. [21], Weiner et al. [22], and Huynh et al. [23]. Note, however, that equation (B6) in Huynh et al. [23] should be replaced by (2.2)–(2.5) of this paper. The expression (2.15) for identical counter-propagating beams has not been presented previously, but numerical integrations were performed in Meijer [16] and Huynh et al. [23]. Numerical results for $\langle v_{\mathrm{rel}}\rangle$ in identical beams can also be found in Meijer [16]. In Figure 1 examples of relative velocity distributions are shown for the case of identical beams (sodium with $T = 575\,\mathrm{K}$) using (2.15) and (2.17) and different beams (sodium with $T = 575\,\mathrm{K}$ and potassium with $T = 475\,\mathrm{K}$) using (2.7) and (2.8). The sodium temperature has been chosen to allow for a comparison with the numerical results of Meijer [16], and the potassium temperature has been selected to yield approximately the same density as a sodium vapor at 575 K.

2.2. Collision-Rate Distributions

The distribution functions $F(v_{\mathrm{rel}})$ describe the distribution of relative velocities of the atom pairs. However, since atom pairs with low relative velocity will have a lower collision frequency compared to atom pairs with high relative velocity and since the atomic beams are

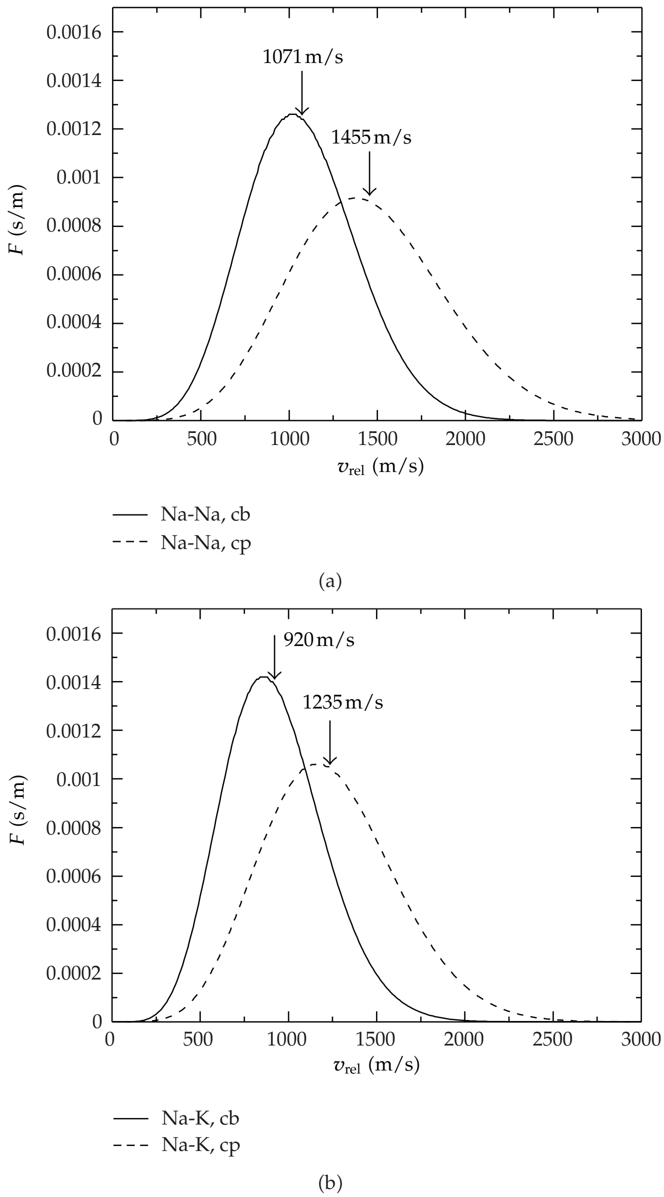

Figure 1: (a) shows relative velocity distributions for two identical beams (sodium with $T = 575\,\mathrm{K}$), for crossed beams (cb) and for counter-propagating (cp) beams, respectively. The arrows indicate the mean relative velocity. (b) shows relative velocity distributions for the two configurations where a sodium beam with $T = 575\,\mathrm{K}$ collides with a potassium beam with $T = 475\,\mathrm{K}$.

used to study collision processes, a more important function is the collision-rate distribution $n_{\rm col}(v_{\rm rel})$, that is, the collision rate as a function of relative velocities, which is given by

$$n_{\rm col}(v_{\rm rel}) = \sigma(v_{\rm rel}) v_{\rm rel} n_0, \tag{2.19}$$

where $\sigma(v_{\rm rel})$ is the collision cross-section and n_0 the number density.

If we assume that $\sigma(v_{\mathrm{rel}})$ is velocity independent and that n_0 is a constant, we obtain (as also pointed out by Meijer [16]) that the distribution function for the collision rate simplifies to

$$F_{\mathrm{col}}(v_{\mathrm{rel}}) = v_{\mathrm{rel}} F(v_{\mathrm{rel}}). \tag{2.20}$$

We note that under these assumptions, the distribution function for the collision rate is identical to the distribution function for the collision velocities and that F_{col} with this choice becomes dimensionless since the units of F are s·m^{-1}.

Now the distribution of the collision rates (or collision velocities) for the two beam geometries can be found using (2.7) and (2.11) or (2.8) and (2.13). The expressions are not reproduced here, but in Figure 2 the distributions for the collision velocities for the same systems as in Figure 1 are shown. By comparing the set of figures it can be seen that the mean relative velocity is not identical to the mean collision velocity but that the distribution of collision rates as expected is shifted to higher velocities.

A useful quantity is the mean value of the collision velocity, which is given by

$$\langle v_{\mathrm{col}} \rangle = \frac{\int_0^\infty dv_{\mathrm{rel}} v_{\mathrm{rel}} F_{\mathrm{col}}(v_{\mathrm{rel}})}{\int_0^\infty dv_{\mathrm{rel}} F_{\mathrm{col}}(v_{\mathrm{rel}})} = \frac{\int_0^\infty dv_{\mathrm{rel}} v_{\mathrm{rel}}^2 F(v_{\mathrm{rel}})}{\int_0^\infty dv_{\mathrm{rel}} v_{\mathrm{rel}} F(v_{\mathrm{rel}})} = \frac{\langle v_{\mathrm{rel}}^2 \rangle}{\langle v_{\mathrm{rel}} \rangle}. \tag{2.21}$$

To find $\langle v_{\mathrm{col}} \rangle$, the second moment of v_{rel}, has to be calculated, which for counter-propagating beams is given by (using that the normalization is 1)

$$\left\langle v_{\mathrm{rel}}^2 \right\rangle^{\mathrm{cp}} = \int_0^\infty dv_{\mathrm{rel}} v_{\mathrm{rel}}^2 F^{\mathrm{cp}}(v_{\mathrm{rel}}) = \frac{3kT_1}{m_1} + \frac{3kT_2}{m_2} + 2\sqrt{\frac{8kT_1}{\pi m_1}}\sqrt{\frac{8kT_2}{\pi m_2}}, \tag{2.22}$$

and the mean collision velocity is then found from (2.11), (2.21), and (2.22) as

$$\langle v_{\mathrm{col}} \rangle^{\mathrm{cp}} = \frac{8 + (3/2)\pi\left(\sqrt{m_1 T_2 / m_2 T_1} + \sqrt{m_2 T_1 / m_1 T_2}\right)}{4\left(\sqrt{\pi m_1 / 8kT_1} + \sqrt{\pi m_2 / 8kT_2}\right)}. \tag{2.23}$$

It is also possible to find the width of the collision velocity distribution from its definition:

$$\Delta v_{\mathrm{col}} = \sqrt{\langle v_{\mathrm{rel}}^2 \rangle_{F_{v\mathrm{col}}} - \langle v_{\mathrm{rel}} \rangle_{F_{v\mathrm{col}}}^2} = \sqrt{\frac{\langle v_{\mathrm{rel}}^3 \rangle}{\langle v_{\mathrm{rel}} \rangle} - \left(\frac{\langle v_{\mathrm{rel}}^2 \rangle}{\langle v_{\mathrm{rel}} \rangle}\right)^2}. \tag{2.24}$$

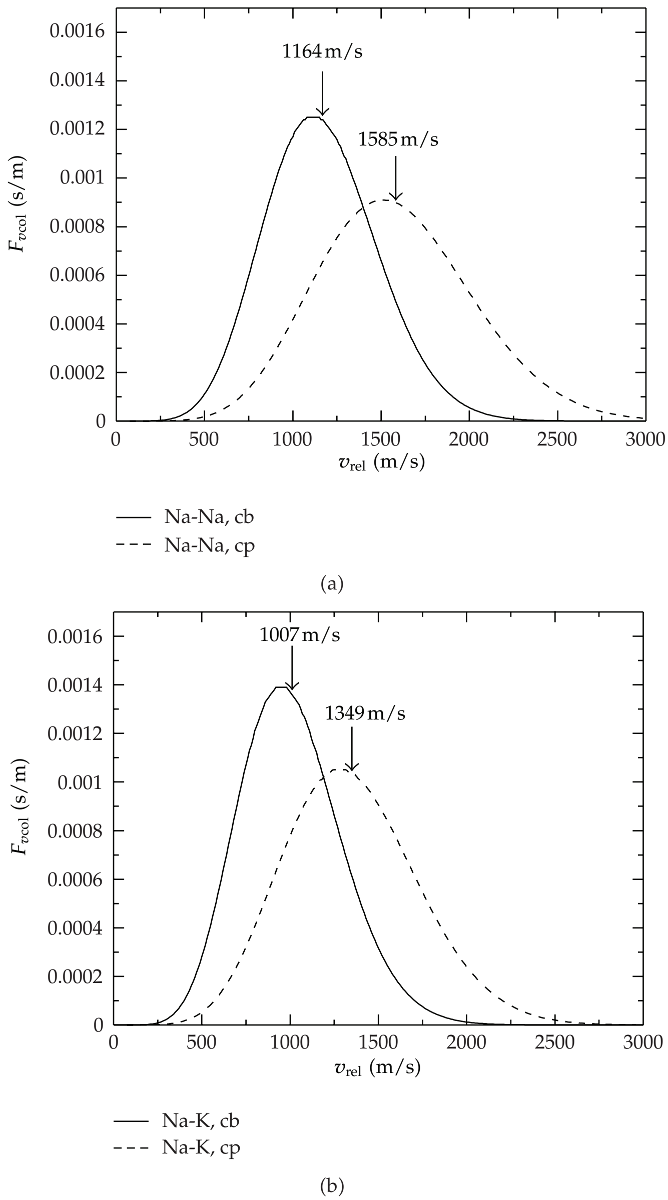

Figure 2: (a) shows collision velocity distributions for identical crossed sodium beams (cb) and counter-propagating (cp) sodium beams with $T = 575\,\mathrm{K}$. The arrows indicate the mean relative collision velocity. (b) shows relative collision velocity distributions for a sodium beam with $T = 575\,\mathrm{K}$ and a potassium beam with $T = 475\,\mathrm{K}$.

The third moment of the relative velocity is found just as the second:

$$\left\langle v_{\mathrm{rel}}^3 \right\rangle = \int_0^\infty dv_{\mathrm{rel}} v_{\mathrm{rel}}^3 F(v_{\mathrm{rel}}). \tag{2.25}$$

However, the expression for the third moment is quite large, so the substitutions $x = m_1T_2$ and $y = m_2T_1$ are used. With this notation one obtains

$$\begin{aligned}\left\langle v_{\mathrm{rel}}^{3}\right\rangle^{\mathrm{cp}} = {} & \left(\frac{k}{m_1m_2}\right)^{3/2}\frac{\sqrt{2xy}}{\sqrt{\pi}(x+y)^{9/2}}\\ & \times\left[3\sqrt{x+y}\left(x^{9/2}-3x^{7/2}y+x^{5/2}y^2+x^2y^{5/2}-3xy^{7/2}+y^{9/2}\right)\right.\\ & \quad+\sqrt{1+\frac{x}{y}}\left(8x^5+50x^4y+120x^3y^2+137x^2y^3+89xy^4+15y^5\right)\\ & \quad\left.+\sqrt{1+\frac{y}{x}}\left(15x^5+89x^4y+137x^3y^2+120x^2y^3+50xy^4+8y^5\right)\right].\end{aligned}\tag{2.26}$$

The width of the collision-velocity distribution then becomes

$$\begin{aligned}\Delta v_{\mathrm{col}}^{\mathrm{cp}} & \\ = {} & \sqrt{\frac{k}{2m_1m_2(\sqrt{x}+\sqrt{y})}}\\ & \times\left\{\frac{\sqrt{xy}(\sqrt{x}+\sqrt{y})}{(x+y)^{9/2}}\times\left[3\sqrt{x+y}\left(x^{9/2}-3x^{7/2}y+x^{5/2}y^2+x^2y^{5/2}-3xy^{7/2}+y^{9/2}\right)\right.\right.\\ & \quad+\sqrt{1+\frac{x}{y}}\left(8x^5+50x^4y+120x^3y^2+137x^2y^3+89xy^4+15y^5\right)\\ & \quad\left.+\sqrt{1+\frac{y}{x}}\left(15x^5+89x^4y+137x^3y^2+120x^2y^3+50xy^4+8y^5\right)\right]\\ & \left.-\frac{3\pi(x+y)+16\sqrt{xy}}{4\pi}\right\}^{1/2}.\end{aligned}\tag{2.27}$$

For orthogonal crossed beams, the second moment of $F^{\mathrm{cb}}(v_{\mathrm{rel}})$ is found as

$$\left\langle v_{\mathrm{rel}}^{2}\right\rangle^{\mathrm{cb}} = \frac{3k(m_1T_2+m_2T_1)}{m_1m_2},\tag{2.28}$$

and the mean collision velocity is found from (2.13), (2.21), and (2.28) as

$$\langle v_{\mathrm{col}}\rangle^{\mathrm{cb}} = \frac{\sqrt{k}(m_1T_2+m_2T_1)^{9/2}}{5\sqrt{\pi}(m_1m_2)^{5/2}(T_1T_2)^2}\frac{1}{{}_2F_1\left(7/4;9/4;2;(m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right)}.\tag{2.29}$$

The third moment of the relative velocity is given as

$$\left\langle v_{\mathrm{rel}}^3 \right\rangle^{\mathrm{cb}} = \frac{210\sqrt{\pi}\left(km_1m_2T_1^2T_2^2\right)^{3/2}}{\left(m_1T_2+m_2T_1\right)^{9/2}}\,{}_2F_1\left(9/4;11/4;2;(m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right), \tag{2.30}$$

and the width of the collision-velocity distribution becomes

$$\begin{aligned}\Delta v_{\mathrm{col}}^{\mathrm{cb}} = \Bigg(& \frac{14kT_1T_2}{m_1T_2+m_2T_1}\frac{{}_2F_1\left(9/4;11/4;2;(m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right)}{{}_2F_1\left(7/4;9/4;2;(m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right)} \\ & -\frac{k(m_1T_2-m_2T_1)^9}{25\pi(m_1m_2)^5(T_1T_2)^4}\frac{1}{{}_2F_1\left(7/4;9/4;2;(m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right)^2}\Bigg)^{1/2}.\end{aligned} \tag{2.31}$$

The results are simplified by setting $m = m_1 = m_2$ and $T = T_1 = T_2$. The second moments of the relative velocity now become

$$\begin{aligned}\left\langle v_{\mathrm{rel}}^2 \right\rangle^{\mathrm{cp}} &= \frac{8+3\pi}{4}\langle v_{\mathrm{MB}}\rangle^2, \\ \left\langle v_{\mathrm{rel}}^2 \right\rangle^{\mathrm{cb}} &= \frac{3\pi}{4}\langle v_{\mathrm{MB}}\rangle^2.\end{aligned} \tag{2.32}$$

The mean collision velocities reduce to

$$\langle v_{\mathrm{col}}\rangle^{\mathrm{cp}} = \frac{8+3\pi}{8}\langle v_{\mathrm{MB}}\rangle, \tag{2.33}$$

$$\langle v_{\mathrm{col}}\rangle^{\mathrm{cb}} = \frac{8}{5}\langle v_{\mathrm{MB}}\rangle. \tag{2.34}$$

By comparison with (2.16) and (2.18), it can be noted that $\langle v_{\mathrm{col}}\rangle^{\mathrm{cp}}$ is 8.9% higher than $\langle v_{\mathrm{rel}}\rangle^{\mathrm{cp}}$ and that $\langle v_{\mathrm{col}}\rangle^{\mathrm{cb}}$ is 8.6% higher than $\langle v_{\mathrm{rel}}\rangle^{\mathrm{cb}}$.

The third moments are

$$\begin{aligned}\left\langle v_{\mathrm{rel}}^3 \right\rangle^{\mathrm{cp}} &= \frac{13\pi}{4}\langle v_{\mathrm{MB}}\rangle^3, \\ \left\langle v_{\mathrm{rel}}^3 \right\rangle^{\mathrm{cb}} &= \frac{105\pi^2}{256}\langle v_{\mathrm{MB}}\rangle^3,\end{aligned} \tag{2.35}$$

and the widths of the distributions reduce to

$$\Delta v_{\mathrm{col}}^{\mathrm{cp}} = \sqrt{\frac{56\pi - 64 - 9\pi^2}{64}} \langle v_{\mathrm{MB}} \rangle, \tag{2.36}$$

$$\Delta v_{\mathrm{col}}^{\mathrm{cb}} = \sqrt{\frac{7\pi}{8} - \frac{64}{25}} \langle v_{\mathrm{MB}} \rangle. \tag{2.37}$$

It is interesting to observe that the ratio $\Delta v_{\mathrm{col}} / \langle v_{\mathrm{col}} \rangle$ is independent on T and m and has a value of about 27% for both configurations. Similar relations for Δv_{rel} can easily be found using the formulas in this section. Again one finds that the ratio $\Delta v_{\mathrm{rel}} / \langle v_{\mathrm{rel}} \rangle$ is independent on T and m and has a value of about 30% for the two configurations considered here.

2.3. Collision-Energy Distributions

A relevant quantity for the physics in the collision processes is collision energy, which is given by $E_{\mathrm{rel}} = (1/2)\mu v_{\mathrm{rel}}^2$, where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. To find the distribution function for the collision energy, we note that the collision rate n_{col} can be written as a function of the relative energy, and using the chain rule and (2.20), one obtains

$$n_{\mathrm{col}}(E_{\mathrm{rel}}) = n_{\mathrm{col}}(v_{\mathrm{rel}}) \frac{dv_{\mathrm{rel}}}{dE_{\mathrm{rel}}} = n_{\mathrm{col}}(v_{\mathrm{rel}}) \frac{1}{\mu v_{\mathrm{rel}}} = \frac{F(v_{\mathrm{rel}})}{\mu}. \tag{2.38}$$

Since μ has dimensions of a mass, $n_{\mathrm{col}}(E_{\mathrm{rel}})$ has the dimensions $\mathrm{s \cdot m^{-1} \cdot kg^{-1}}$.

To find the normalization for the distribution of the collision energy we use (2.38) to calculate

$$\int_0^\infty dE_{\mathrm{rel}} n_{\mathrm{col}}(E_{\mathrm{rel}}) = \int_0^\infty dv_{\mathrm{rel}} v_{\mathrm{rel}} F(v_{\mathrm{rel}}) = \langle v_{\mathrm{rel}} \rangle, \tag{2.39}$$

and the distribution function $F_{E_{\mathrm{col}}}$ for the collision energy is therefore

$$F_{E_{\mathrm{col}}}(v_{\mathrm{rel}}) = \frac{n_{\mathrm{col}}(E_{\mathrm{rel}})}{\langle v_{\mathrm{rel}} \rangle} = \frac{F(v_{\mathrm{rel}})}{\mu \langle v_{\mathrm{rel}} \rangle}. \tag{2.40}$$

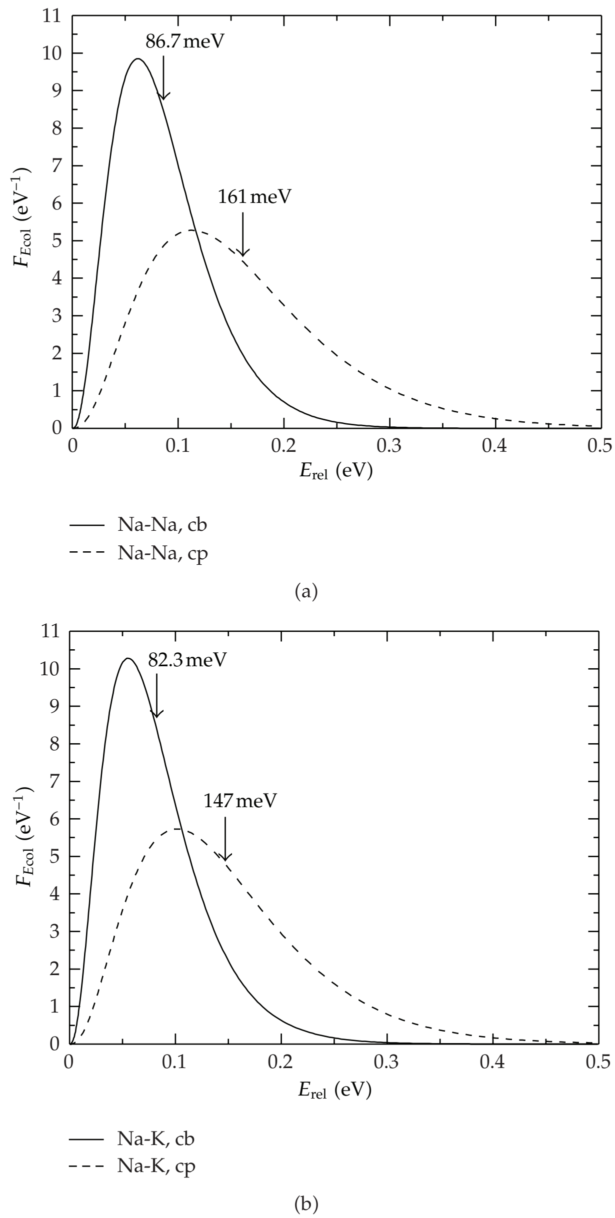

Figure 3: (a) shows distributions of collision energies for identical crossed sodium beams (cb) and counter-propagating (cp) sodium beams with $T = 575\,\mathrm{K}$. The arrows indicate the mean collision energy. (b) shows collision energy distributions for a sodium beam with $T = 575\,\mathrm{K}$ and a potassium beam with $T = 475\,\mathrm{K}$.

The graphs for $F_{E_{\mathrm{col}}}(v_{\mathrm{rel}})$ can then be obtained by a simple transformation of the ordinate in Figure 2. Another option is to show the distribution using E_{rel} as the argument. The result of transforming the abscissa in Figure 2 from v_{rel} to E_{rel} and using (2.40) for the two beam geometries is shown in Figure 3, again for the two systems, Na-Na and Na-K.

The mean collision energy is then

$$\begin{aligned}\langle E_{\mathrm{col}}\rangle = \langle E_{\mathrm{rel}}\rangle_{F_{E\mathrm{col}}} &= \int_0^\infty dE_{\mathrm{rel}} E_{\mathrm{rel}} F_{E_{\mathrm{col}}}(v_{\mathrm{rel}}) \\ &= \frac{1}{\mu\langle v_{\mathrm{rel}}\rangle}\int_0^\infty dv_{\mathrm{rel}}\mu v_{\mathrm{rel}}\frac{1}{2}\mu v_{\mathrm{rel}}^2 F(v_{\mathrm{rel}}) = \frac{\mu}{2}\frac{\langle v_{\mathrm{rel}}^3\rangle}{\langle v_{\mathrm{rel}}\rangle}.\end{aligned} \tag{2.41}$$

By comparison with (2.21), it can be noted that this is different from $(1/2)\mu\langle v_{\mathrm{col}}\rangle^2$. Finally, after deriving $\langle E_{\mathrm{col}}^2\rangle$ analogous to (2.41), the width of the collision energy distribution can be found as

$$\Delta E_{\mathrm{col}} = \sqrt{\left\langle E_{\mathrm{rel}}^2\right\rangle_{F_{E\mathrm{col}}} - \langle E_{\mathrm{rel}}\rangle_{F_{E\mathrm{col}}}^2} = \sqrt{\frac{\mu^2}{4}\cdot\frac{\langle v_{\mathrm{rel}}^5\rangle}{\langle v_{\mathrm{rel}}\rangle} - \left(\frac{\mu}{2}\cdot\frac{\langle v_{\mathrm{rel}}^3\rangle}{\langle v_{\mathrm{rel}}\rangle}\right)^2}. \tag{2.42}$$

Using (2.11) and (2.26), the mean collision energy for counter-propagating beams is found from (2.41) as follows:

$$\begin{aligned}\langle E_{\mathrm{col}}\rangle^{\mathrm{cp}} = &\frac{k\sqrt{xy}}{4(m_1+m_2)\left(\sqrt{x}+\sqrt{y}\right)(x+y)^{9/2}} \\ &\times\left[3\sqrt{x+y}\left(x^{9/2}-3x^{7/2}y+x^{5/2}y^2+x^2y^{5/2}-3xy^{7/2}+y^{9/2}\right)\right. \\ &\quad+\sqrt{1+\frac{x}{y}}\left(8x^5+50x^4y+120x^3y^2+137x^2y^3+89xy^4+15y^5\right) \\ &\quad\left.+\sqrt{1+\frac{y}{x}}\left(15x^5+89x^4y+137x^3y^2+120x^2y^3+50xy^4+8y^5\right)\right].\end{aligned} \tag{2.43}$$

The fifth moment is given by

$$
\begin{aligned}
\left\langle v_{\mathrm{rel}}^{5}\right\rangle^{\mathrm{cp}}
&= 3\sqrt{\frac{2}{\pi}}\left(\frac{k}{m_1 m_2}\right)^{5/2}\frac{1}{(x+y)^{9/2}} \\
&\quad\times\left[x^{1/2}y^{5/2}\sqrt{1+\frac{x}{y}}\left(466x^4+744x^3y+631x^2y^2+295xy^3+35y^4\right)\right. \\
&\qquad + x^{5/2}y^{1/2}\sqrt{1+\frac{y}{x}}\left(35x^4+295x^3y+631x^2y^2+744xy^3+466y^4\right) \\
&\qquad + \sqrt{x+y}\left(16x^{13/2}+15x^6y^{1/2}+144x^{11/2}y-15x^5y^{3/2}+5x^4y^{5/2}\right) \\
&\qquad \left. + \sqrt{x+y}\left(5x^{5/2}y^4-15x^{3/2}y^5+144xy^{11/2}+15x^{1/2}y^6+16y^{13/2}\right)\right],
\end{aligned}
\tag{2.44}
$$

and using (2.42), the width of the collision-energy distribution becomes

$$
\begin{aligned}
\Delta E_{\mathrm{col}}^{\mathrm{cp}}
&= \frac{k}{4(m_1+m_2)(\sqrt{x}+\sqrt{y})} \\
&\quad\times\left\{\frac{6(\sqrt{x}+\sqrt{y})}{(x+y)^{9/2}}\times\left[x^{1/2}y^{5/2}\sqrt{1+\frac{x}{y}}\left(466x^4+744x^3y+631x^2y^2+295xy^3+35y^4\right)\right.\right. \\
&\qquad + x^{5/2}y^{1/2}\sqrt{1+\frac{y}{x}}\left(35x^4+295x^3y+631x^2y^2+744xy^3+466y^4\right) \\
&\qquad + \sqrt{x+y}\left(16x^{13/2}+15x^6y^{1/2}+144x^{11/2}y-15x^5y^{3/2}+5x^4y^{5/2}\right) \\
&\qquad \left. + \sqrt{x+y}\left(5x^{5/2}y^4-15x^{3/2}y^5+144xy^{11/2}+15x^{1/2}y^6+16y^{13/2}\right)\right] \\
&\qquad - \frac{x^2y^2}{(x+y)^9}\left[3\sqrt{x+y}\left(x^{9/2}-3x^{7/2}y+x^{5/2}y^2+x^2y^{5/2}-3xy^{7/2}+y^{9/2}\right)\right. \\
&\qquad + \sqrt{1+\frac{x}{y}}\left(8x^5+50x^4y+120x^3y^2+137x^2y^3+89xy^4+15y^5\right) \\
&\qquad \left.\left. + \sqrt{1+\frac{y}{x}}\left(15x^5+89x^4y+137x^3y^2+120x^2y^3+50xy^4+8y^5\right)\right]^2\right\}^{1/2}.
\end{aligned}
\tag{2.45}
$$

For orthogonal crossed beams, the corresponding relations are

$$\langle E_{\text{col}} \rangle^{\text{cb}} = \frac{7km_1m_2T_1T_2}{(m_1+m_2)(m_1T_2+m_2T_1)} \frac{{}_2F_1\left(9/4; 11/4; 2; (m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right)}{{}_2F_1\left(7/4; 9/4; 2; (m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right)}, \tag{2.46}$$

the fifth moment is

$$\left\langle v_{\text{rel}}^5 \right\rangle^{\text{cb}} = \frac{3780\sqrt{\pi}k^{5/2}(m_1m_2)^{3/2}(T_1T_2)^4}{(m_1T_2+m_2T_1)^{11/2}} {}_2F_1\left(11/4; 13/4; 2; \frac{(m_1T_2-m_2T_1)^2}{(m_1T_2+m_2T_1)^2}\right), \tag{2.47}$$

and the width of the collision-energy distribution is

$$\begin{aligned}\Delta E_{\text{col}}^{\text{cp}} = {} & \frac{\sqrt{7}km_1m_2T_1T_2}{(m_1+m_2)(m_1T_2+m_2T_1)} \frac{1}{{}_2F_1\left(7/4; 9/4; 2; (m_1T_2-m_2T_1)^2/(m_1T_2+m_2T_1)^2\right)} \\ & \times \left[9\,{}_2F_1\left(7/4; 9/4; 2; \frac{(m_1T_2-m_2T_1)^2}{(m_1T_2+m_2T_1)^2}\right) {}_2F_1\left(11/4; 13/4; 2; \frac{(m_1T_2-m_2T_1)^2}{(m_1T_2+m_2T_1)^2}\right) \right. \\ & \left. - 7\,{}_2F_1\left(9/4; 11/4; 2; \frac{(m_1T_2-m_2T_1)^2}{(m_1T_2+m_2T_1)^2}\right)^2\right]^{1/2}. \end{aligned} \tag{2.48}$$

Finally, the reduced formulas obtained by setting $m = m_1 = m_2$ and $T = T_1 = T_2$ are found as

$$\langle E_{\text{col}} \rangle^{\text{cp}} = \frac{13}{4}kT, \tag{2.49}$$

$$\langle E_{\text{col}} \rangle^{\text{cb}} = \frac{7}{4}kT, \tag{2.50}$$

$$\left\langle v_{\text{rel}}^5 \right\rangle^{\text{cp}} = \frac{219\pi^2}{32}\langle v_{\text{MB}} \rangle^5, \tag{2.51}$$

$$\left\langle v_{\text{rel}}^5 \right\rangle^{\text{cb}} = \frac{945\pi^3}{2048}\langle v_{\text{MB}} \rangle^5, \tag{2.52}$$

$$\Delta E_{\text{col}}^{\text{cp}} = \frac{5}{\sqrt{8}}kT, \tag{2.53}$$

$$\Delta E_{\text{col}}^{\text{cb}} = \sqrt{\frac{7}{8}}kT. \tag{2.54}$$

Table 1: Values (see text) for counter-propagating identical sodium beams with temperatures $T = 575$, 823 and 1,000 K.

T (K)	$\langle v_{\mathrm{rel}} \rangle$ (m/s)	$\langle v_{\mathrm{col}} \rangle$ (m/s)	Δv_{col} (m/s)	$\langle E_{\mathrm{col}} \rangle$ (meV)	ΔE_{col} (meV)	$v(E_{\mathrm{col}})$ (m/s)
575	1,455	1,585	437	161	88	1,644
823	1,741	1,896	523	230	125	1,967
1,000	1,919	2,090	576	280	152	2,168

Note that $\Delta E_{\mathrm{col}} / \langle E_{\mathrm{col}} \rangle$ is a constant and approximately equal to 0.54 for the two geometries. As also discussed by Meijer [16], the collision energy may be related to a velocity (different from the mean collision velocity) given by

$$v(E_{\mathrm{col}}) = \sqrt{\frac{2\langle E_{\mathrm{col}} \rangle}{\mu}}, \tag{2.55}$$

and for the two beam configurations, one then obtains

$$v(E_{\mathrm{col}})^{\mathrm{cp}} = \sqrt{\frac{13\pi}{8}} \langle v_{\mathrm{MB}} \rangle, \tag{2.56}$$

$$v(E_{\mathrm{col}})^{\mathrm{cb}} = \langle v_{\mathrm{MB}} \rangle. \tag{2.57}$$

By comparison with the results above, these velocities are found to be a factor of 1.04 larger than the mean collision velocities and a factor of 1.13 larger than the mean relative velocities.

2.4. Numerical Values

In Table 1 some numerical results for two identical counter-propagating sodium beams at three different temperatures are given (using (2.16), (2.33), (2.36), (2.49), (2.53), and (2.56)). The lowest temperature of 575 K is chosen to allow a comparison with the numerical results obtained by Meijer [16], and our formulas reproduce his results exactly. The higher temperature of 823 K is a typical working temperature in the laboratory [24] and has been included together with results for 1,000 K to indicate the temperature dependence of the expressions.

3. Conclusion

Atomic (and molecular) beam studies provide valuable information on collision studies, and the collision parameters such as collision velocities and collision energies are important quantities. In this work a number of useful and relatively simple analytic formulas are presented for the cases where two effusive beams are either crossed at right angles or counterpropagating. In particular the difference between the mean relative velocity and mean collision velocity has been underlined.

Often an important number in an experiment is the fraction of collision energies that will be above the threshold for an endothermic reaction. This number can be obtained using the formula given in the paper and by integrating the distribution function in (2.40).

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