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Research Article

Conservative Linear Difference Scheme for Rosenau-KdV Equation

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A conservative three-level linear finite difference scheme for the numerical solution of the initial-boundary value problem of Rosenau-KdV equation is proposed. The difference scheme simulates two conservative quantities of the problem well. The existence and uniqueness of the difference solution are proved. It is shown that the finite difference scheme is of second-order convergence and unconditionally stable. Numerical experiments verify the theoretical results.

1. Introduction

KdV equation has been used in very wide applications and undergone research which can be used to describe wave propagation and spread interaction as follows [1–4]:

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using the well-known KdV equation. To overcome this shortcoming of the KdV equation, Rosenau [5, 6] proposed the so-called Rosenau equation:

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \tag{2}$$

The existence and the uniqueness of the solution for (2) were proved by Park [7], but it is difficult to find the analytical solution for (2). Since then, much work has been done on the numerical method for (2) ([8–13] and also the references therein). On the other hand, for the further consideration of the nonlinear wave, the viscous term $+u_{xxx}$ needs to be included [14]

$$u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0. \tag{3}$$

This equation is usually called the Rosenau-KdV equation. Zuo [14] discussed the solitary wave solutions and periodic solutions for (2). Recently, [15–17] discussed the solitary solutions for the generalized Rosenau-KdV equation with usual power law nonlinearity. In [15, 16], the authors also gave the two invariants for the generalized Rosenau-KdV equation. In particular, [16] not only derived the singular 1-solition solution by the ansatz method but also used perturbation theory to obtain the adiabatic parameter dynamics of the water waves. In [17], The ansatz method is applied to obtain the topological soliton solution of the generalized Rosenau-KdV equation. The G'/G method as well as the exp-function method are also applied to extract a few more solutions to this equation. But the numerical method to the initial-boundary value problem of Rosenau-KdV equation has not been studied till now. In this paper, we propose a conservative three-level finite difference scheme for the Rosenau-KdV equation (3) with the boundary conditions

$$u(x_L, t) = u(x_R, t) = 0, \qquad u_x(x_L, t) = u_x(x_R, t) = 0,$$
$$u_{xx}(x_L, t) = u_{xx}(x_R, t) = 0, \quad t \in [0, T], \tag{4}$$

and an initial condition

$$u(x, 0) = u_0(x), \quad x \in [x_L, x_R]. \tag{5}$$

The initial boundary value problem (3)–(5) possesses the following conservative properties [15]:

$$Q(t) = \int_{x_L}^{x_R} u(x,t)\,dx = \int_{x_L}^{x_R} u_0(x)\,dx = Q(0), \quad (6)$$

$$E(t) = \|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0). \quad (7)$$

The solitary wave solution for (3) is [14, 15]

$$\begin{aligned} u(x,t) &= \left(-\frac{35}{24} + \frac{35}{312}\sqrt{313}\right) \\ &\quad \times \operatorname{sech}^4\left[\frac{1}{24}\sqrt{-26+2\sqrt{313}}\right. \\ &\quad \left. \times \left(x - \left(\frac{1}{2} + \frac{1}{26}\sqrt{313}\right)t\right)\right]. \end{aligned} \quad (8)$$

When $-x_L \gg 0$, $x_R \gg 0$, the initial-boundary value problem (3)–(5) and the Cauchy problem (3) are consistent, so the boundary condition (4) is reasonable.

It is known the conservative scheme is better than the nonconservative ones. The nonconservative scheme may easily show nonlinear blow up. A lot of numerical experiments show that the conservative scheme can possesses some invariant properties of the original differential equation [18–29]. The conservative scheme is more suitable for long-time calculations. In [19], Li and Vu-Quoc said "… in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation." In this paper, we propose a three-level linear finite difference scheme for the Rosenau-KdV equation (3)–(5). The difference scheme is conservative which simulates conservative properties (6) and (7) at the same time.

The rest of this paper is organized as follows. In Section 2, we propose a three-level linear finite difference scheme for the Rosenau-KdV equation and discuss the discrete conservative properties. In Section 3, we show that the scheme is uniquely solvable. Then, in Section 4, we prove that the finite difference scheme is of second-order convergence, unconditionally stable. Finally, some numerical tests are given in Section 5 to verify our theoretical analysis.

2. Finite Difference Scheme and Conservation Properties

Let $h = (x_R - x_L)/J$ and τ be the uniform step size in the spatial and temporal direction, respectively. Denote $x_j = x_L + jh$ ($j = -1, 0, 1, 2, \ldots, J, J+1$), $t_n = n\tau$ ($n = 0, 1, 2, \ldots, N$, $N = [T/\tau]$), $u_j^n \approx u(x_j, t_n)$ and $Z_h^0 = \{u = (u_j) \mid u_{-1} = u_0 = u_J = u_{J+1} = 0,\ j = -1, 0, 1, 2, \ldots, J, J+1\}$. Throughout this paper, we denote C as a generic positive constant independent of h and τ, which may have different values in different occurrences. We introduce the following notations:

$$\begin{aligned} &\left(u_j^n\right)_x = \frac{u_{j+1}^n - u_j^n}{h}, \qquad \left(u_j^n\right)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h}, \\ &\left(u_j^n\right)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \qquad \left(u_j^n\right)_{\hat{t}} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau}, \\ &\bar{u}_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2}, \qquad \langle u^n, v^n\rangle = h\sum_{j=1}^{J-1} u_j^n v_j^n, \\ &\|u^n\|^2 = \langle u^n, u^n\rangle, \qquad \|u^n\|_\infty = \max_{1\le j\le J-1}\left|u_j^n\right|. \end{aligned} \quad (9)$$

We propose a three-level linear finite difference scheme for the solution of (3)–(5) as follows:

$$\begin{aligned} &\left(u_j^n\right)_{\hat{t}} + \left(u_j^n\right)_{xx\bar{x}\bar{x}\hat{t}} + \left(\bar{u}_j^n\right)_{\hat{x}} + \left(\bar{u}_j^n\right)_{x\bar{x}\hat{x}} \\ &\quad + \frac{1}{3}\left[u_j^n\left(\bar{u}_j^n\right)_{\hat{x}} + \left(u_j^n \bar{u}_j^n\right)_{\hat{x}}\right] = 0, \end{aligned} \quad (10)$$

$$j = 1, 2, 3, \ldots, J-1, \qquad n = 1, 2, 3, \ldots, N-1, \quad (11)$$

$$u_j^0 = u_0\left(x_j\right), \quad j = 0, 1, 2, 3, \ldots, J, \quad (12)$$

$$\begin{aligned} &u^n \in Z_h^0, \qquad \left(u_0^n\right)_{\hat{x}} = \left(u_J^n\right)_{\hat{x}} = 0, \\ &\left(u_0^n\right)_{x\bar{x}} = \left(u_J^n\right)_{x\bar{x}} = 0, \quad n = 1, 2, 3, \ldots, N. \end{aligned} \quad (13)$$

From the boundary conditions (4), we know that (13) is reasonable.

Lemma 1. *It follows from summation by parts that for any two mesh functions $u, v \in Z_h^0$,*

$$\langle u_x, v\rangle = -\langle u, v_x\rangle, \qquad \langle u_{x\bar{x}}, v\rangle = -\langle u_x, v_x\rangle. \quad (14)$$

Then one has

$$\langle u_x, u\rangle = -\langle u, u_{\bar{x}}\rangle, \qquad \langle u_{x\bar{x}}, u\rangle = -\langle u_x, u_x\rangle = -\|u_x\|^2. \quad (15)$$

Furthermore, if $(u_0)_{x\bar{x}} = (u_J)_{x\bar{x}} = 0$, then

$$\langle u_{xx\bar{x}\bar{x}}, u\rangle = \|u_{xx}\|^2. \quad (16)$$

The difference scheme (10)–(13) simulates two conservative properties of the problems (6) and (7) as follows.

Theorem 2. *Suppose that $u_0 \in H_0^2[x_L, x_R]$, $u(x,t) \in C^{5,3}$, then the difference scheme (10)–(13) is conservative:*

$$Q^n = \frac{h}{2}\sum_{j=1}^{J-1}\left(u_j^{n+1} + u_j^n\right) + \frac{h}{6}\tau\sum_{j=1}^{J-1} u_j^n\left(u_j^{n+1}\right)_{\hat{x}} = Q^{n-1} = \cdots = Q^0, \quad (17)$$

$$\begin{aligned} E^n &= \frac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \|u^n\|^2\right) + \frac{1}{2}\left(\left\|u_{xx}^{n+1}\right\|^2 + \left\|u_{xx}^n\right\|^2\right) \\ &= E^{n-1} = \cdots = E^0. \end{aligned} \quad (18)$$

Proof. Multiplying (10) with h, summing up for j from 1 to $J-1$, and considering the boundary condition (13) and Lemma 1, we get

$$h\sum_{j=1}^{J-1}\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}+\frac{h}{6}\sum_{j=1}^{J-1}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}}-u_j^{n-1}\left(u_j^n\right)_{\hat{x}}\right]=0. \tag{19}$$

Then, (17) is gotten from (19).

Taking an inner product of (10) with $2\overline{u}^n$ (i.e., $u^{n+1}+u^{n-1}$), considering the boundary condition (13) and Lemma 1, we obtain

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2-\left\|u^{n-1}\right\|^2\right)+\frac{1}{2\tau}\left(\left\|u_{xx}^{n+1}\right\|^2-\left\|u_{xx}^{n-1}\right\|^2\right)\\&\quad+2\left\langle\overline{u}_{\hat{x}}^n,\overline{u}^n\right\rangle+2\left\langle\overline{u}_{x\overline{x}\hat{x}}^n,\overline{u}^n\right\rangle\\&\quad+2\left\langle P,\overline{u}^n\right\rangle=0,\end{aligned} \tag{20}$$

where $P_j=(1/3)[u_j^n(\overline{u}_j^n)_{\hat{x}}+(u_j^n\overline{u}_j^n)_{\hat{x}}]$. According to

$$\begin{aligned}\left\langle\overline{u}_{\hat{x}}^n,\overline{u}^n\right\rangle&=0,\\\left\langle\overline{u}_{x\overline{x}\hat{x}}^n,\overline{u}^n\right\rangle&=0,\end{aligned} \tag{21}$$

$$\begin{aligned}\left\langle P,\overline{u}^n\right\rangle&=\frac{1}{3}h\sum_{j=1}^{J-1}\left[u_j^n\left(\overline{u}_j^n\right)_{\hat{x}}+\left(u_j^n\overline{u}_j^n\right)_{\hat{x}}\right]\overline{u}_j^n\\&=\frac{1}{12}\sum_{j=1}^{J-1}\Big[u_j^n\left(u_{j+1}^{n+1}+u_{j+1}^{n-1}-u_{j-1}^{n+1}-u_{j-1}^{n-1}\right)\\&\qquad+u_{j+1}^n\left(u_{j+1}^{n+1}+u_{j+1}^{n-1}\right)-u_{j-1}^n\left(u_{j-1}^{n+1}+u_{j-1}^{n-1}\right)\Big]\\&\qquad\times\left(u_j^{n+1}+u_j^{n-1}\right)\\&=\frac{1}{12}\sum_{j=1}^{J-1}\left(u_j^n+u_{j+1}^n\right)\left(u_{j+1}^{n+1}+u_{j+1}^{n-1}\right)\left(u_j^{n+1}+u_j^{n-1}\right)\\&\quad-\frac{1}{12}\sum_{j=1}^{J-1}\left(u_j^n+u_{j-1}^n\right)\left(u_j^{n+1}+u_j^{n-1}\right)\left(u_{j-1}^{n+1}+u_{j-1}^{n-1}\right)\\&=0,\end{aligned} \tag{22}$$

we have

$$\left(\left\|u^{n+1}\right\|^2-\left\|u^{n-1}\right\|^2\right)+\left(\left\|u_{xx}^{n+1}\right\|^2-\left\|u_{xx}^{n-1}\right\|^2\right)=0. \tag{23}$$

Then, (18) is gotten from (23). □

3. Solvability

Theorem 3. *There exists $u^n\in Z_h^0$ which satisfies the difference scheme* (10)–(13), $(1\le n\le N)$.

Proof. Use mathematical induction to prove it. It is obvious that u^0 is uniquely determined by the initial condition (12). We also can get u^1 in order $O(h^2+\tau^2)$ by two-level *C-N* scheme (i.e., u^0 and u^1 are uniquely determined). Now suppose $u^0,u^1,\dots,u^n$ $(1\le n\le N-1)$ is solved uniquely. Consider the equation of (10) for u^{n+1} :

$$\begin{aligned}&\frac{1}{2\tau}u_j^{n+1}+\frac{1}{2\tau}\left(u_j^{n+1}\right)_{xx\overline{x}\overline{x}}+\frac{1}{2}\left(u_j^{n+1}\right)_{\hat{x}}\\&\quad+\frac{1}{2}\left(u_j^{n+1}\right)_{x\overline{x}\hat{x}}+\frac{1}{6}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}}+\left(u_j^nu_j^{n+1}\right)_{\hat{x}}\right]=0.\end{aligned} \tag{24}$$

Taking an inner product of (24) with u^{n+1}, we get

$$\begin{aligned}&\frac{1}{2\tau}\left\|u^{n+1}\right\|^2+\frac{1}{2\tau}\left\|u_{xx}^{n+1}\right\|^2+\frac{1}{2}\left\langle u_{\hat{x}}^{n+1},u^{n+1}\right\rangle+\frac{1}{2}\left\langle u_{x\overline{x}\hat{x}}^{n+1},u^{n+1}\right\rangle\\&\quad+\frac{h}{6}\sum_{j=1}^{J-1}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}}+\left(u_j^nu_j^{n+1}\right)_{\hat{x}}\right]u_j^{n+1}=0.\end{aligned} \tag{25}$$

Similar to the proof of (21), we have

$$\begin{aligned}\left\langle u_{\hat{x}}^{n+1},u^{n+1}\right\rangle&=0,\\\left\langle u_{x\overline{x}\hat{x}}^{n+1},u^{n+1}\right\rangle&=0.\end{aligned} \tag{26}$$

By

$$\begin{aligned}&\frac{h}{6}\sum_{j=1}^{J-1}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}}+\left(u_j^nu_j^{n+1}\right)_{\hat{x}}\right]u_j^{n+1}\\&\quad=\frac{1}{12}\sum_{j=1}^{J-1}\left[u_j^n\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right)+\left(u_{j+1}^nu_{j+1}^{n+1}-u_{j-1}^nu_{j-1}^{n+1}\right)\right]u_j^{n+1}\\&\quad=\frac{1}{12}\sum_{j=1}^{J-1}\left[u_j^nu_j^{n+1}u_{j+1}^{n+1}+u_{j+1}^nu_j^{n+1}u_{j+1}^{n+1}\right]\\&\qquad-\frac{1}{12}\sum_{j=1}^{J-1}\left[u_{j-1}^nu_{j-1}^{n+1}u_j^{n+1}+u_j^nu_{j-1}^{n+1}u_j^{n+1}\right]=0,\end{aligned} \tag{27}$$

and from (25)–(27), we have

$$\left\|u^{n+1}\right\|^2+\left\|u_{xx}^{n+1}\right\|^2=0. \tag{28}$$

That is, (24) has only a trivial solution. Therefore, (10) determines u_j^{n+1} uniquely. This completes the proof. □

4. Convergence and Stability

Let $v(x,t)$ be the solution of problem (3)–(5), $v_j^n=v(x_j,t_n)$, then the truncation error of the difference scheme (10)–(13) is as follows:

$$\begin{aligned}r_j^n&=\left(v_j^n\right)_{\hat{t}}+\left(v_j^n\right)_{xx\overline{x}\overline{x}\hat{t}}+\left(\overline{v}_j^n\right)_{\hat{x}}\\&\quad+\left(\overline{v}_j^n\right)_{x\overline{x}\hat{x}}+\frac{1}{3}\left[v_j^n\left(\overline{v}_j^n\right)_{\hat{x}}+\left(v_j^n\overline{v}_j^n\right)_{\hat{x}}\right].\end{aligned} \tag{29}$$

Making use of Taylor expansion, we know that $r_j^n = O(\tau^2 + h^2)$ holds if $h, \tau \to 0$.

Lemma 4. *Suppose that $u_0 \in H_0^2[x_L, x_R]$, then the solution u^n of (3)–(5) satisfies*

$$\|u\|_{L_2} \le C, \qquad \|u_x\|_{L_2} \le C, \\ \|u\|_{L_\infty} \le C, \qquad \|u_x\|_{L_\infty} \le C. \tag{30}$$

Proof. It is follows from (7) that

$$\|u\|_{L_2} \le C, \qquad \|u_{xx}\|_{L_2} \le C. \tag{31}$$

By Holder inequality and Schwarz inequality, we get

$$\begin{aligned} \|u_x\|_{L_2}^2 &= \int_{x_L}^{x_R} u_x u_x \, dx = u u_x \Big|_{x_L}^{x_R} - \int_{x_L}^{x_R} u u_{xx} \, dx \\ &= -\int_{x_L}^{x_R} u u_{xx} \, dx \\ &\le \|u\|_{L_2} \cdot \|u_{xx}\|_{L_2} \le \frac{1}{2}\left(\|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2\right), \end{aligned} \tag{32}$$

which implies that

$$\|u_x\|_{L_2} \le C. \tag{33}$$

Using Sobolev inequality, we get that $\|u\|_{L_\infty} \le C$, $\|u_x\|_{L_\infty} \le C$. □

Lemma 5 (discrete Sobolev's inequality [27]). *There exist two constants C_1 and C_2 such that*

$$\|u^n\|_\infty \le C_1 \|u^n\| + C_2 \|u_x^n\|. \tag{34}$$

Lemma 6 (discrete Gronwall inequality [27]). *Suppose that $w(k)$ and $\rho(k)$ are nonnegative function and $\rho(k)$ is nondecreasing. If $C > 0$, and*

$$w(k) \le \rho(k) + C\tau \sum_{l=0}^{k-1} w(l), \quad \forall k, \tag{35}$$

then

$$w(k) \le \rho(k) e^{C\tau k}, \quad \forall k. \tag{36}$$

Theorem 7. *Suppose $u_0 \in H_0^2[x_L, x_R]$, then the solution of (10)–(13) satisfies: $\|u^n\| \le C$, $\|u_x^n\| \le C$, $\|u_{xx}^n\| \le C$, which yield $\|u^n\|_\infty \le C$, $\|u_x^n\|_\infty \le C$ $(n = 1, 2, \ldots, N)$.*

Proof. It is follows from (18) that

$$\|u^n\| \le C, \qquad \|u_{xx}^n\| \le C. \tag{37}$$

According to (15) and Schwarz inequality, we get

$$\|u_x^n\|^2 \le \|u^n\| \cdot \|u_{xx}^n\| \le \frac{1}{2}\left(\|u^n\|^2 + \|u_{xx}^n\|^2\right) \le C. \tag{38}$$

Using Lemma 5, we have $\|u^n\|_\infty \le C$, $\|u_x^n\|_\infty \le C$. □

Theorem 8. *Suppose $u_0 \in H_0^2[x_L, x_R]$, $u(x,t) \in C^{5,3}$, then the solution u^n of the difference scheme (10)–(13) converges to the solution $v(x,t)$ of the problem (3)–(5) with order $O(\tau^2 + h^2)$ in norm $\|\cdot\|_\infty$.*

Proof. Subtracting (10) from (29) and letting $e_j^n = v_j^n - u_j^n$, we have

$$r_j^n = \left(e_j^n\right)_{\hat{t}} + \left(\bar{e}_j^n\right)_{xx\bar{x}\bar{x}\hat{t}} + \left(\bar{e}_j^n\right)_{\hat{x}} + \left(\bar{e}_j^n\right)_{x\bar{x}\hat{x}} + R_{1,j} + R_{2,j}, \tag{39}$$

where $R_{1,j} = (1/3)[v_j^n(\bar{v}_j^n)_{\hat{x}} - u_j^n(\bar{u}_j^n)_{\hat{x}}]$, $R_{2,j} = (1/3)[(v_j^n\bar{v}_j^n)_{\hat{x}} - (u_j^n\bar{u}_j^n)_{\hat{x}}]$. Computing the inner product of (39) with $2\bar{e}^n$, we obtain

$$\begin{aligned} \left\langle r^n, 2\bar{e}^n \right\rangle &= \frac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \frac{1}{2\tau}\left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) \\ &\quad + \left\langle \bar{e}_{\hat{x}}^n, 2\bar{e}^n \right\rangle + \left\langle \bar{e}_{x\bar{x}\hat{x}}^n, 2\bar{e}^n \right\rangle + \left\langle R_1, 2\bar{e}^n \right\rangle + \left\langle R_2, 2\bar{e}^n \right\rangle. \end{aligned} \tag{40}$$

Similar to the proof of (21), we have

$$\left\langle \bar{e}_{\hat{x}}^n, 2\bar{e}^n \right\rangle = 0, \\ \left\langle \bar{e}_{x\bar{x}\hat{x}}^n, 2\bar{e}^n \right\rangle = 0. \tag{41}$$

Then, (40) can be rewritten as follows:

$$\begin{aligned} &\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n+1}\right\|^2\right) + \left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) \\ &\quad = 2\tau \left\langle r^n, 2\bar{e}^n \right\rangle + 2\tau \left\langle -R_1, 2\bar{e}^n \right\rangle + 2\tau \left\langle -R_2, 2\bar{e}^n \right\rangle. \end{aligned} \tag{42}$$

Using Lemma 4 and Theorem 7, we get

$$\left|v_j^n\right| \le C, \qquad \left|u_j^n\right| \le C, \\ \left|\left(u_j^n\right)_{\hat{x}}\right| \le C, \quad (j = 0, 1, 2, \ldots, J;\ n = 1, 2, \ldots, N). \tag{43}$$

According to the Schwarz inequality, we obtain

$$\begin{aligned} &\left\langle -R_1, 2\bar{e}^n \right\rangle \\ &\quad = -\frac{2}{3} h \sum_{j=1}^{J-1} \left[v_j^n \left(\bar{v}_j^n\right)_{\hat{x}} - u_j^n \left(\bar{u}_j^n\right)_{\hat{x}}\right] \bar{e}_j^n \\ &\quad = -\frac{2}{3} h \sum_{j=1}^{J-1} \left[v_j^n \left(\bar{e}_j^n\right)_{\hat{x}} + e_j^n \left(\bar{u}_j^n\right)_{\hat{x}}\right] \bar{e}_j^n \\ &\quad \le \frac{2}{3} Ch \sum_{j=1}^{J-1} \left(\left|\left(\bar{e}_j^n\right)_{\hat{x}}\right| + \left|e_j^n\right|\right) \left|\bar{e}_j^n\right| \le C\left[\left\|\bar{e}_x^n\right\|^2 + \left\|e^n\right\|^2 + \left\|\bar{e}^n\right\|^2\right] \\ &\quad \le C\left[\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2\right], \end{aligned}$$

$$\begin{aligned}
&\left\langle -R_2, 2\bar{e}^n\right\rangle \\
&= -\frac{2}{3}h\sum_{j=1}^{J-1}\left[\left(v_j^n\bar{v}_j^n\right)_{\hat{x}} - \left(u_j^n\bar{u}_j^n\right)_{\hat{x}}\right]\bar{e}_j^n \\
&= \frac{2}{3}h\sum_{j=1}^{J-1}\left[v_j^n\bar{v}_j^n - u_j^n\bar{u}_j^n\right]\left(\bar{e}_j^n\right)_{\hat{x}} \\
&= \frac{2}{3}h\sum_{j=1}^{J-1}\left[v_j^n\bar{e}_j^n + e_j^n\bar{u}_j^n\right]\left(\bar{e}_j^n\right)_{\hat{x}} \\
&\le \frac{2}{3}Ch\sum_{j=1}^{J-1}\left(\left|\left(\bar{e}_j^n\right)\right| + \left|e_j^n\right|\right)\left|\left(\bar{e}_j^n\right)_{\hat{x}}\right| \\
&\le C\left[\left\|\bar{e}_x^n\right\|^2 + \left\|e^n\right\|^2 + \left\|\bar{e}^n\right\|^2\right] \\
&\le C\left[\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2\right].
\end{aligned} \tag{44}$$

Noting that

$$\begin{aligned}
\left\langle r^n, 2\bar{e}^n\right\rangle &= \left\langle r^n, e^{n+1} + e^{n-1}\right\rangle \\
&\le \left\|r^n\right\|^2 + \frac{1}{2}\left[\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right],
\end{aligned} \tag{45}$$

and from (42)–(45), we have

$$\begin{aligned}
&\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) \\
&\quad\le C\tau\left[\left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2\right. \\
&\qquad\left. + \left\|e_x^{n+1}\right\|^2 + \left\|e_x^n\right\|^2 + \left\|e_x^{n-1}\right\|^2\right] + 2\tau\left\|r^n\right\|^2.
\end{aligned} \tag{46}$$

Similar to the proof of (38), we have

$$\begin{aligned}
\left\|e_x^{n+1}\right\|^2 &\le \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e_{xx}^{n+1}\right\|^2\right), \\
\left\|e_x^n\right\|^2 &\le \frac{1}{2}\left(\left\|e^n\right\|^2 + \left\|e_{xx}^n\right\|^2\right), \\
\left\|e_x^{n-1}\right\|^2 &\le \frac{1}{2}\left(\left\|e^{n-1}\right\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right).
\end{aligned} \tag{47}$$

Then, (46) can be rewritten as

$$\begin{aligned}
&\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) \\
&\quad\le C\tau\left[\left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2\right. \\
&\qquad\left. + \left\|e_{xx}^{n+1}\right\|^2 + \left\|e_{xx}^n\right\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right] + 2\tau\left\|r^n\right\|^2.
\end{aligned} \tag{48}$$

Let $B^n = \|e^{n+1}\|^2 + \|e_{xx}^{n+1}\|^2 + \|e^n\|^2 + \|e_{xx}^n\|^2$. Then, (48) can be rewritten as follows:

$$B^n - B^{n-1} \le C\tau\left(B^n + B^{n-1}\right) + 2\tau\left\|r^n\right\|^2, \tag{49}$$

which yields

$$(1 - C\tau)\left(B^n - B^{n-1}\right) \le 2C\tau B^{n-1} + 2\tau\left\|r^n\right\|^2. \tag{50}$$

If τ is sufficiently small, which satisfies $1 - C\tau > 0$, then

$$B^n - B^{n-1} \le C\tau B^{n-1} + C\tau\left\|r^n\right\|^2. \tag{51}$$

Summing up (51) from 1 to n, we have

$$B^n \le B^0 + C\tau\sum_{l=1}^{n}\left\|r^l\right\|^2 + C\tau\sum_{l=0}^{n-1}B^l. \tag{52}$$

First, we can get u^1 in order $O(\tau^2 + h^2)$ that satisfies $B^0 = O(\tau^2 + h^2)^2$ by two-level C-N scheme. Since

$$\tau\sum_{l=1}^{n}\left\|r^l\right\|^2 \le n\tau\max_{1\le l\le n}\left\|r^l\right\|^2 \le T\cdot O\left(\tau^2 + h^2\right)^2, \tag{53}$$

then we obtain

$$B^n \le O\left(\tau^2 + h^2\right)^2 + C\tau\sum_{l=0}^{n-1}B^l. \tag{54}$$

From Lemma 6 we get

$$B^n \le O\left(\tau^2 + h^2\right)^2, \tag{55}$$

which implies that

$$\left\|e^n\right\| \le O\left(\tau^2 + h^2\right), \qquad \left\|e_{xx}^n\right\| \le O\left(\tau^2 + h^2\right). \tag{56}$$

From (47) we have

$$\left\|e_x^n\right\| \le O\left(\tau^2 + h^2\right). \tag{57}$$

By Lemma 5 we obtain

$$\left\|e^n\right\|_\infty \le O\left(\tau^2 + h^2\right). \tag{58}$$

□

Finally, we can similarly prove results as follows.

Theorem 9. *Under the conditions of Theorem 8, the solution* u^n *of* (10)–(13) *is stable in norm* $\|\cdot\|_\infty$.

5. Numerical Simulations

Since the three-level implicit finite difference scheme cannot start by itself, we need to select other two-level schemes (such as the C-N Scheme) to get u^1. Then, be reusing initial value u^0, we can work out $u^2, u^3, \ldots$. Iterative numerical calculation is not required, for this scheme is linear, so it saves computing time. Let $x_L = -70$, $x_R = 100$, and $T = 40$,

$$u_0(x) = \left(-\frac{35}{24} + \frac{35}{312}\sqrt{313}\right)\operatorname{sech}^4\left(\frac{1}{24}\sqrt{-26 + 2\sqrt{313}}x\right). \tag{59}$$

TABLE 1: The error at various time steps.

	$\tau = h = 0.1$		$\tau = h = 0.05$		$\tau = h = 0.025$	
	$\|e^n\|$	$\|e^n\|_\infty$	$\|e^n\|$	$\|e^n\|_\infty$	$\|e^n\|$	$\|e^n\|_\infty$
$t = 10$	$1.641934e-3$	$6.314193e-4$	$4.113510e-4$	$1.582641e-4$	$1.028173e-4$	$3.965867e-5$
$t = 20$	$3.045414e-3$	$1.131442e-3$	$7.631169e-4$	$2.835874e-4$	$1.905450e-4$	$7.097948e-5$
$t = 30$	$4.241827e-3$	$1.533771e-3$	$1.062971e-3$	$3.843906e-4$	$2.650990e-4$	$9.610332e-5$
$t = 40$	$5.297873e-3$	$1.878952e-3$	$1.327645e-3$	$4.709118e-4$	$3.306738e-4$	$1.176011e-4$

TABLE 2: The verification of the second convergence.

	$\|e^n(h,\tau)\|/\|e^{2n}(h/2,\tau/2)\|$			$\|e^n(h,\tau)\|_\infty/\|e^{2n}(h/2,\tau/2)\|_\infty$		
	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$
$t = 10$	—	3.991564	4.000797	—	3.989657	3.990655
$t = 20$	—	3.990757	4.004916	—	3.989749	3.995343
$t = 30$	—	3.990539	4.009713	—	3.990136	3.999764
$t = 40$	—	3.990427	4.014970	—	3.990030	4.004314

TABLE 3: Numerical simulations on the two conservation invariants Q^n and E^n.

	$\tau = h = 0.1$		$\tau = h = 0.05$		$\tau = h = 0.025$	
	Q^n	E^n	Q^n	E^n	Q^n	E^n
$t = 0$	5.497722548019	1.984553365290	5.498060684522	1.984390175264	5.498145418391	1.984349335263
$t = 10$	5.497724936513	1.984595075859	5.498060837192	1.984401029470	5.498145479109	1.984352109750
$t = 20$	5.497728744900	1.9846645964099	5.498061080542	1.984414367496	5.498145545374	1.984355520610
$t = 30$	5.497731963790	1.984679827211	5.498061287046	1.984423270337	5.498145609535	1.984357811266
$t = 40$	5.497734235191	1.984701501262	5.498061398506	1.984428974030	5.498145659050	1.984359292230

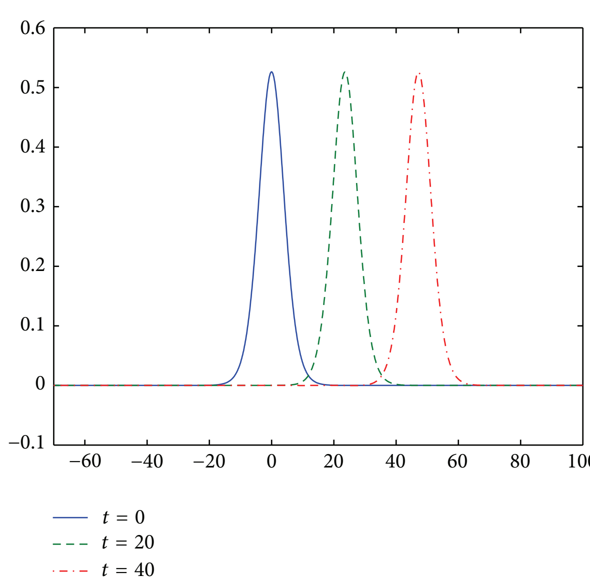

FIGURE 1: When $\tau = h = 0.1$, the wave graph of $u(x,t)$ at various times.

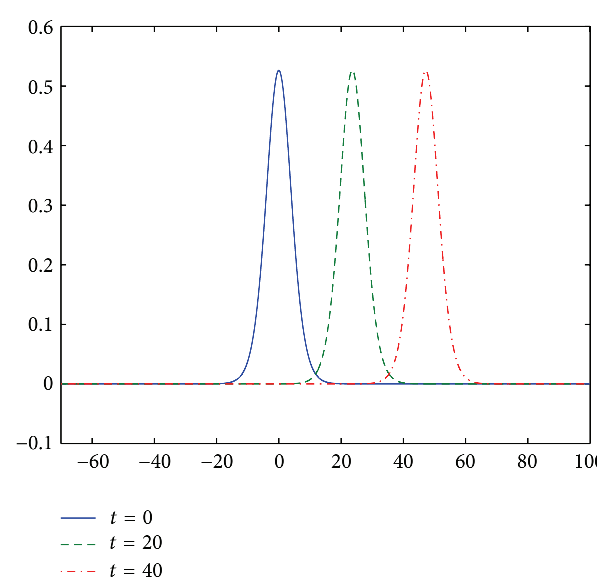

FIGURE 2: When $\tau = h = 0.025$, the wave graph of $u(x,t)$ at various times.

In Table 1, we give the error at various time steps. Using the method in [30, 31], we verified the second convergence of the difference scheme in Table 2. Numerical simulations on two conservation invariants Q^n and E^n are given in Table 3.

The wave graph comparison of $u(x,t)$ between $\tau = h = 0.1$ and $\tau = h = 0.025$ at various times is given in Figures 1 and 2.

Numerical simulations show that the finite difference scheme is efficient.

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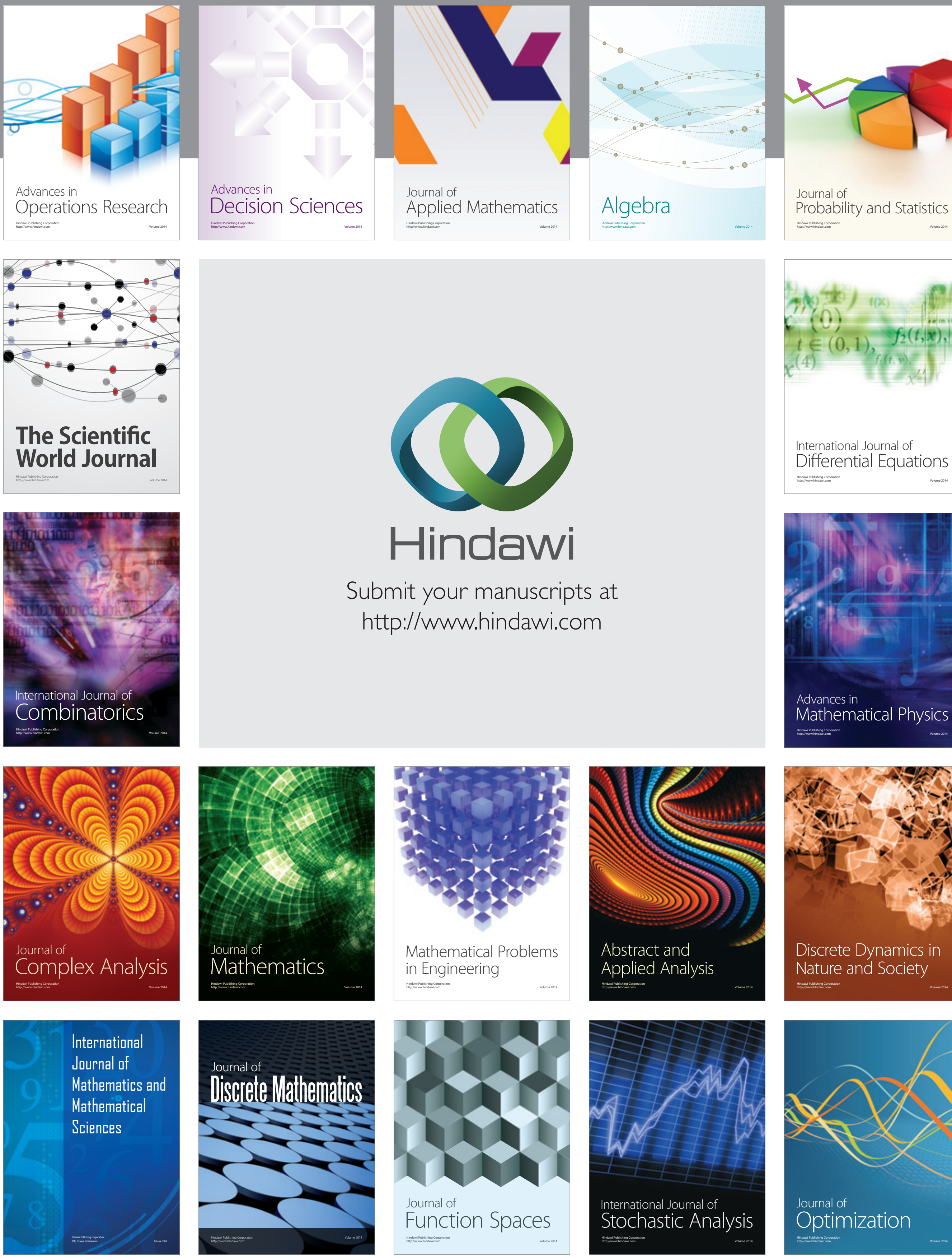

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