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Research Article

Adaptive Sliding Mode Control of a Novel Class of Fractional Chaotic Systems

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Recently, control and synchronization of fractional chaotic systems have increasingly attracted much attention in the fractional control community. In this paper we introduce a novel class of fractional chaotic systems in the pseudo state space and propose an adaptive sliding mode control scheme to stabilize the chaotic systems in the presence of uncertainties and external disturbances whose bounds are unknown. To verify the effectiveness of the proposed adaptive sliding mode control technique, numerical simulations of control design of fractional Lorenz's system and Chen's system are presented.

1. Introduction

Fractional calculus is an old and yet novel topic whose infancy dates back to the end of the 17th century, the time when Newton and Leibniz established the foundations of classical calculus. For three centuries, fractional calculus developed mainly as a pure theoretical mathematical field without applications. However, in the last two decades it has attracted the interest of researchers in several areas including mathematics, physics, chemistry, material, engineering, finance, and even social science. The stability of fractional differential equations (FDEs) and fractional control have both gained rapid development very recently [1–3].

One of the most important areas of application is the chaos theory. In recent years, fractional chaotic systems have intensively attracted a great deal of attention due to the ease of their electronics implementations and the rapid development of the stability of FDEs. More and more fractional dynamics described in the pseudo state space exhibiting chaos have been found, such as the fractional Chua circuit [4], the fractional Van der Pol oscillator [5–7], the fractional Lorenz system [8, 9], the fractional Chen system [10–12], the fractional Lü system [13], the fractional Liu system [14], the fractional Rössler system [15, 16], the fractional Arneodo system [17], the fractional Newton-Leipnik system [18–20], the fractional Lotka-Volterra system [21, 22], the fractional finance system [20, 23], and the fractional Rucklidge system [24]. Most of the above papers have used numerical methods to present chaotic behaviors.

In particular, control and synchronization of fractional chaotic systems have increasingly attracted much attention in the fractional control community. Moreover, several control and synchronization methods have been proposed based on the stability of fractional differential equations in the pseudo state space [22]. The linear state feedback control algorithm based upon the stability criterion of linear FDEs has been used in [10, 25–33], the nonlinear feedback control scheme in [34–38], the fractional PID control method in [39, 40], and active control technique in [41]. However, it is important to note that the above four control methods have ignored modeling inaccuracies and external noises which can be hardly avoided in real world application. Robust control and adaptive control are two major and complementary approaches to deal with model uncertainty: sliding mode control design provides a systematic approach to the problem of maintaining stability and consistent performance in the face of modeling imprecision, while adaptive control is a suitable approach to maintain consistent performance of a system in the presence of uncertainty or unknown variation in plant parameters [42]. The fractional sliding mode control methodology has been designed in [43–47], and the motivation of choosing the particular structures for sliding

surfaces has been described in [48]. The adaptive sliding mode control, which is good at maintaining robustness and handling inevitable parameter variation and parameter uncertainty, has been used in [49, 50].

Nevertheless, as stated recently in [51, 52], there arise several questionable or wrong ideas in the field of fractional systems analysis and control using the representation *state space description*, particularly if Caputo's definition is used. To correct these wrong ideas, the authors [53–55] have proposed alternative solutions and equivalent frequency distributed model of FDEs. Particularly, Lorenzo and Hartley [56–58] have derived a simple process of initialization of FDEs. Motivated by the continuous frequency distributed model proposed by Trigeassou et al. and the initialization method addressed by Lorenzo and Hartley, this paper utilizes a new Lyapunov approach [59] to analyze the stability of FDEs and impose physically coherent initial conditions to FDEs using the initialization function [60].

The rest of the paper is organized as follows. Section 2 presents some basic definitions about fractional calculus. Section 3 introduces a novel class of fractional chaotic systems. Section 4 proposes the sliding mode control design and adaptive sliding mode control design of fractional chaotic systems. Numerical simulations are presented to show the effectiveness of the proposed method in Section 5. Finally, the paper is concluded in Section 6.

2. Basic Definitions and Preliminaries

Definition 1. The most important function used in fractional calculus is Euler's Gamma function, which is defined as

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt. \tag{1}$$

Definition 2. Another important function is a two-parameter function of the Mittag-Leffler type defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0. \tag{2}$$

Fractional calculus is a generalization of integration and differentiation to noninteger-order fundamental operator ${}_aD_t^\alpha$, where a and t are the bounds of the operation and $a \in R$. The three most frequently used definitions for the general fractional calculus are the Grünwald-Letnikov definition, the Riemann-Liouville definition, and the Caputo definition [23, 61–63].

Definition 3. The Grünwald-Letnikov derivative definition of order α is described as

$${}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t - jh). \tag{3}$$

Definition 4. The Riemann-Liouville derivative definition of order α is described as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}}, \quad n-1 < \alpha < n. \tag{4}$$

Definition 5. The Caputo definition of fractional derivative can be written as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}}, \quad n-1 < \alpha < n. \tag{5}$$

In the rest of the paper, we will use the Caputo approach to describe the fractional chaotic systems and use the Grünwald-Letnikov approach to propose numerical simulations. To simplify the notation, we denote the fractional derivative of order α as D^α instead of ${}_0D_t^\alpha$ in this paper.

3. System Description

In [64], the authors have introduced a class of integer-order chaotic systems covering about half of the recently published integer-order chaotic models. In [43], Yin et al. have designed a novel class of fractional chaotic model covering several fractional chaotic systems. Inspired by the above two contributions, we introduce and control another novel class of fractional chaotic systems (6) which are described in the pseudo state space and cover the fractional Lorenz system [8, 9], the fractional Chen system [10–12], the fractional Lü system [13], the fractional Liu system [14], the fractional Lotka-Volterra system [22], and the fractional Rucklidge system [24], presented in Table 1:

$$\begin{aligned} D^{q_1} x &= f(x, y, z) - \alpha x, \\ D^{q_2} y &= x g(x, y, z) - \beta y, \\ D^{q_3} z &= x h(x, y, z) - \gamma z, \end{aligned} \tag{6}$$

where x, y, and z are the pseudo state variables, α, β, and γ are nonnegative known constants, and q_1, q_2, and $q_3 \in (0, 1]$. Each of the three functions $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ is assumed to be continuous and satisfies the Lipschitz condition to guarantee the existence and uniqueness of solutions of initial value problems.

4. Control Design

4.1. Sliding Mode Control Design. Sliding mode control methodology is a simple approach to robust control and good at dealing with dynamic uncertainty. The control design procedure consists of two steps: first constructing a sliding surface which presents the desired dynamics and second selecting a switching control law so as to verify sliding condition. The control input $u(t)$ is added to the first state equation in order to control chaos. Then, the controlled plant can be described as

$$\begin{aligned} D^{q_1} x &= f(x, y, z) - \alpha x + u, \\ D^{q_2} y &= x g(x, y, z) - \beta y, \\ D^{q_3} z &= x h(x, y, z) - \gamma z. \end{aligned} \tag{7}$$

The main purpose of this paper is to design a suitable controller $u(t)$ to guarantee the stability of the chaotic system (6).

Table 1: List of published fractional chaotic systems covered by (7).

Name	Model	$f(\cdot)$	$g(\cdot)$	$h(\cdot)$
Lorenz system	$D^{q_1}x = -a(x-y)$ $D^{q_2}y = rx - y - xz$ $D^{q_3}z = -bz + xy$	ay	$r-z$	y
Chen system[a]	$D^{q_1}x = (c-a)y - yz + cx$ $D^{q_2}y = a(x-y)$ $D^{q_3}z = xy - bz$	$(c-a)y - yz + cx$	a	y
Liu system	$D^{q_1}x = -a(x-y)$ $D^{q_2}y = bx - kxz$ $D^{q_3}z = -cz + dx^2$	ay	$b-kz$	dx
Lü system[a]	$D^{q_1}x = cx - yz$ $D^{q_2}y = a(x-y)$ $D^{q_3}z = xy - bz$	$cx - yz$	a	y
Lotka Volterra system	$D^{q_1}x = ax - bxy + ex^2 - sx^2z$ $D^{q_2}y = -cy + dxy$ $D^{q_3}z = -pz + sx^2z$	$ax - bxyz + ex^2 - sx^2$	dx	sxz
Rucklidge system[a]	$D^{q_1}x = y$ $D^{q_2}y = -kx + \lambda x - xz$ $D^{q_3}z = -z + x^2$	y	$\lambda - z$	x

[a]We have replaced x with y just to adopt the general class (6). To propose control scheme for systems in this case, we only need to add the control input to the second state equation of the dynamical systems of the original form, see the second example in Section 5.

4.1.1. Sliding Surface Design. In order to achieve the stability of system (6), a sliding surface $S(t)$ is constructed as

$$s(t) = D^{q_1-1}x(t) + D^{-1}\phi(t), \tag{8}$$

where $\phi(t) = yg(x,y,z) + zh(x,y,z) + \alpha x$.

By differentiating (8), one derives

$$\dot{s}(t) = D^{q_1}x(t) + \phi(t). \tag{9}$$

4.1.2. Sliding Mode Dynamics Analysis. In this part we will show that given any initial conditions, the problem of stabilization of system (6) is equivalent to that of remaining on the surface $S(t)$ for all $t > 0$.

When the system operates in the sliding surface, it satisfies $s = 0$ and $\dot{s} = 0$.

This yields the following sliding mode dynamics:

$$\begin{aligned} D^{q_1}x &= -yg(x,y,z) - zh(x,y,z) - \alpha x, \\ D^{q_2}y &= xg(x,y,z) - \beta y, \\ D^{q_3}z &= xh(x,y,z) - \gamma z. \end{aligned} \tag{10}$$

In the following we will prove that the above sliding mode dynamics (10) is globally asymptotically stable, so that it is able to be considered as our desired dynamics. To this end, the FDEs are firstly converted into an exactly equivalent infinite dimensional ODEs, namely, continuous frequency distributed model of the fractional integrator [53–55]. Then, the indirect Lyapunov approach is applied to derive the stability [59] of the fractional dynamics (10).

Theorem 6. *The fractional sliding mode dynamics as in* (10) *is globally asymptotically stable.*

Proof. In terms of the continuous frequency distributed model of the fractional integrator, the fractional system (10) is exactly equivalent to the following infinite dimensional ODEs:

$$\begin{aligned} \frac{\partial z_1(\omega,t)}{\partial t} &= -\omega z_1(\omega,t) - yg(x,y,z) - zh(x,y,z) - \alpha x, \\ x(t) &= \int_0^\infty \mu_1(\omega) z_1(\omega,t)\, d\omega, \\ \frac{\partial z_2(\omega,t)}{\partial t} &= -\omega z_2(\omega,t) + xg(x,y,z) - \beta y, \\ y(t) &= \int_0^\infty \mu_2(\omega) z_2(\omega,t)\, d\omega, \\ \frac{\partial z_3(\omega,t)}{\partial t} &= -\omega z_3(\omega,t) + xh(x,y,z) - \gamma z, \\ z(t) &= \int_0^\infty \mu_3(\omega) z_3(\omega,t)\, d\omega, \end{aligned} \tag{11}$$

with $\mu_i(\omega) = ((\sin(q_i\pi))/\pi)\omega^{-q_i}$, $i = 1,2,3$.

In the above continuous frequency distributed model, $z_i(\omega,t)$ are the true state variables, while $x(t)$, $y(t)$, and $z(t)$ are the pseudo state variables.

Let us define two types of Lyapunov functions:

(i) $v_i(w,t)$, that is, the monochromatic Lyapunov functions corresponding to the elementary frequency;

(ii) $V_i(t)$, that is, the Lyapunov functions summing all the monochromatic $v_i(w,t)$ with the weighting functions $\mu_i(\omega)$.

Exactly,

$$v_i(w,t) = \frac{1}{2} z_i^2,$$

$$V_i(t) = \int_0^\infty \mu_i(\omega) v_i(\omega,t)\, d\omega = \frac{1}{2} \int_0^\infty \mu_i(\omega) z_i^2(\omega,t)\, d\omega. \tag{12}$$

Then

$$\frac{\partial v_i(\omega,t)}{\partial t} = \frac{\partial v_i(\omega,t)}{\partial z_i} \frac{\partial z_i}{\partial t} = z_i \frac{\partial z_i}{\partial t}, \tag{13}$$

$$\frac{dV_1}{dt} = \int_0^\infty \mu_1(\omega) \frac{\partial v_1(\omega,t)}{\partial t} d\omega. \tag{14}$$

Substituting the first equation of (11) into (14) yields

$$\begin{aligned} \frac{dV_1}{dt} &= \int_0^\infty \mu_1(\omega) z_1 (-\omega z_1 - yg - zh - \alpha x)\, d\omega \\ &= -\int_0^\infty \mu_1(\omega) \omega z_1^2 d\omega \\ &\quad + \int_0^\infty \mu_1(\omega) z_1 (-yg - zh - \alpha x)\, d\omega \\ &= -\int_0^\infty \mu_1(\omega) \omega z_1^2 d\omega \\ &\quad + (-yg - zh - \alpha x) \int_0^\infty \mu_1(\omega) z_1 d\omega. \end{aligned} \tag{15}$$

Substituting the second equation of (11) into the integral term of (15) yields

$$\frac{dV_1}{dt} = -\int_0^\infty \mu_1(\omega) \omega z_1^2 d\omega + x(-yg - zh - \alpha x). \tag{16}$$

Similarly,

$$\begin{aligned} \frac{dV_2}{dt} &= -\int_0^\infty \mu_2(\omega) \omega z_2^2 d\omega + y(xg - \beta y), \\ \frac{dV_3}{dt} &= -\int_0^\infty \mu_3(\omega) \omega z_3^2 d\omega + z(xh - \gamma z). \end{aligned} \tag{17}$$

Finally, let us define

$$V(t) = V_1(t) + V_2(t) + V_3(t). \tag{18}$$

Then,

$$\frac{dV}{dt} = -\sum_{i=1}^{3} \int_0^\infty \mu_i(\omega) \omega z_i^2 d\omega - \left(\alpha x^2 + \beta y^2 + \gamma z^2\right). \tag{19}$$

Owing to the Lemma of the appendix, $dV/dt < 0$, which implies the stability of the sliding mode dynamics (10). □

4.1.3. Control Design via Sliding Mode Methodology. A Lyapunov candidate is selected as

$$V(x,y,z) = \frac{1}{2} s^2. \tag{20}$$

Taking time derivative of (29) yields

$$\begin{aligned} \frac{dV}{dt} &= s\dot{s} \\ &= s(D^q x + \phi) \\ &= s\left[f(x,y,z) - \alpha x + u(t)\right. \\ &\qquad \left. + yg(x,y,z) + zh(x,y,z) + \alpha x\right]. \end{aligned} \tag{21}$$

Then the control law is constructed as

$$\begin{aligned} u(t) = &- f(x,y,z) - yg(x,y,z) \\ &- zh(x,y,z) - k_1 \operatorname{sgn}(s) - k_2 s, \end{aligned} \tag{22}$$

where

$$\operatorname{sgn}(s) = \begin{cases} 1, & s > 0 \\ 0, & s = 0 \\ -1, & s < 0 \end{cases} \tag{23}$$

and k is a known strictly positive constant to be chosen later.

Substituting the control law (22) into (21) yields

$$\begin{aligned} \frac{dV}{dt} &= s\left[-k_1 \operatorname{sgn}(s) - k_2 s\right] \\ &= -k_1 |s| - k_2 s \\ &= -k_1 \sqrt{V} - k_2 V \end{aligned} \tag{24}$$

which implies that the Lyapunov function V tends to zero in a finite time, and the same holds for the sliding surface (8). Furthermore, the finite time vanishing of the sliding surface guarantees that solutions $x(t)$, $y(t)$, and $z(t)$ of the fractional system (10) will tend globally and asymptotically to zero [65]. This proves that the fractional system (6) can be stabilized via the proposed sliding mode control law (22).

4.2. Adaptive Sliding Mode Control Design. In this subsection, the system uncertainty $\Delta f(x,y,z)$ and external disturbance $d(t)$ will be considered and added to (6). It is assumed that $\Delta f(x,y,z)$ and $d(t)$ are both bounded; that is, $|\Delta f(x,y,z)| < \theta_1$, $|d(t)| < \theta_2$, where both θ_1 and θ_2 are unknown nonnegative constants. $\widehat{\theta}_1$ and $\widehat{\theta}_2$ represent, respectively, the estimates of θ_1 and θ_2. To estimate the two unknown parameters, the adaptive control technique will be employed in the following.

The controlled plant is

$$\begin{aligned} D^{q_1} x &= f(x,y,z) - \alpha x + \Delta f + d + u, \\ D^{q_2} y &= xg(x,y,z) - \beta y, \\ D^{q_3} z &= xh(x,y,z) - \gamma z. \end{aligned} \tag{25}$$

In order to propose the control design, a Lyapunov candidate is chosen as

$$V(x,y,z) = \frac{1}{2}\left[s^2 + \frac{1}{\mu_1}\left(\widehat{\theta}_1 - \theta_1\right)^2 + \frac{1}{\mu_2}\left(\widehat{\theta}_2 - \theta_2\right)^2\right]. \tag{26}$$

By taking its derivative with respect to time, one has

$$\begin{aligned}\frac{dV}{dt} &= s\dot{s} + \frac{1}{\mu_1}\left(\widehat{\theta}_1 - \theta_1\right)\dot{\widehat{\theta}}_1 + \frac{1}{\mu_2}\left(\widehat{\theta}_2 - \theta_2\right)\dot{\widehat{\theta}}_2 \\ &= s\left(D^q x + \phi\right) + \frac{1}{\mu_1}\left(\widehat{\theta}_1 - \theta_1\right)\dot{\widehat{\theta}}_1 + \frac{1}{\mu_2}\left(\widehat{\theta}_2 - \theta_2\right)\dot{\widehat{\theta}}_2 \\ &= s\left[f - \alpha x + \Delta f + d + u + yg + z + \alpha x\right] \\ &\quad + \frac{1}{\mu_1}\left(\widehat{\theta}_1 - \theta_1\right)\dot{\widehat{\theta}}_1 + \frac{1}{\mu_2}\left(\widehat{\theta}_2 - \theta_2\right)\dot{\widehat{\theta}}_2.\end{aligned} \tag{27}$$

If we chose the control law and adaptive law as

$$u(t) = -f - yg - zh - \left(\widehat{\theta}_1 + \widehat{\theta}_2 + -k_1\right)\operatorname{sgn}(s) - k_2 s, \tag{28}$$

$$\begin{aligned}\dot{\widehat{\theta}}_1 &= \mu_1 |s|, \\ \dot{\widehat{\theta}}_2 &= \mu_2 |s|.\end{aligned} \tag{29}$$

Then substituting the control law (28) and adaptive law (29) into (27) follows that

$$\begin{aligned}\frac{dV}{dt} &= s\left[\Delta f + d - \left(\widehat{\theta}_1 + \widehat{\theta}_2 + -k_1\right)\operatorname{sgn}(s) \right. \\ &\quad \left. + \left(\widehat{\theta}_1 - \theta_1\right)|s| + \left(\widehat{\theta}_2 - \theta_2\right)|s|\right] \\ &= \left(\Delta f + d\right)s - \left(\widehat{\theta}_1 + \widehat{\theta}_2 + k_1\right)|s| \\ &\quad + \left(\widehat{\theta}_1 - \theta_1\right)|s| + \left(\widehat{\theta}_2 - \theta_2\right)|s| - k_2 s \\ &\le \left(|\Delta f| + |d|\right)|s| - \left(\widehat{\theta}_1 + \widehat{\theta}_2 + k_1\right)|s| \\ &\quad + \left(\widehat{\theta}_1 - \theta_1\right)|s| + \left(\widehat{\theta}_2 - \theta_2\right)|s| - k_2 s \\ &\le \left(\theta_1 + \theta_2\right)|s| - \left(\widehat{\theta}_1 + \widehat{\theta}_2 + k_1\right)|s| \\ &\quad + \left(\widehat{\theta}_1 - \theta_1\right)|s| + \left(\widehat{\theta}_2 - \theta_2\right)|s| \\ &= -k|s| - k_2 s \\ &= -k_1\sqrt{V} - k_2 V.\end{aligned} \tag{30}$$

By now it is proved that the fractional system (25) with uncertainty and external disturbance can be stabilized via the proposed sliding mode control law (28) and adaptive law (29).

5. Numerical Simulations

In this section, we present two examples, namely, fractional Lorenz's system and fractional Chen's system, to evaluate the performance of the sliding mode control and adaptive sliding mode control technique. Numerical simulations are implemented using the MATLAB software.

5.1. Control of the Fractional Lorenz's System. The fractional Lorenz's system is described as

$$\begin{aligned}D^{q_1} x &= -a\left(x - y\right), \\ D^{q_2} y &= rx - y - xz, \\ D^{q_3} z &= -bz + xy,\end{aligned} \tag{31}$$

where $a = 10$, $r = 28$, and $b = 8/3$.

The equilibrium points of the system with the above parameters are $E_1 = (0, 0, 0)$, $E_2 = (6\sqrt{2}, 6\sqrt{2}, 5/3)$, and $E_3 = (-6\sqrt{2}, -6\sqrt{2}, 5/3)$.

Owning to the initialization method described in [60], the initial conditions for fractional differential equations with order between 0 and 1 is the *constant* function of time. So the initial conditions for the fractional Lorenz's system can be chosen as

$$\begin{aligned}x(t) &= x\left(0^+\right) = 1, \\ y(t) &= y\left(0^+\right) = 1, \\ z(t) &= z\left(0^+\right) = 1,\end{aligned} \tag{32}$$

for $-\infty \le t \le 0$.

The fractional Lorenz's system exhibits chaos with fractional orders $q_1 = q_2 = q_3 = 0.993$ and the above initial conditions, as depicted in Figure 1.

The numerical algorithm is based on Grünwald-Letnikov's definition:

$$\begin{aligned}x\left(t_k\right) &= -a\left(x\left(t_{k-1}\right) - y\left(t_{k-1}\right)\right)h^{q_1} - \sum_{j=2}^{k} c_j^{(q_1)} x\left(t_{k-j}\right), \\ y\left(t_k\right) &= \left[rx\left(t_k\right) - y\left(t_{k-1}\right) - x\left(t_k\right)z\left(t_{k-1}\right)\right]h^{q_2} \\ &\quad - \sum_{j=2}^{k} c_j^{(q_2)} y\left(t_{k-j}\right), \\ z\left(t_k\right) &= \left[-bz\left(t_{k-1}\right) + x\left(t_k\right)y\left(t_k\right)\right]h^{q_3} \\ &\quad - \sum_{j=2}^{k} c_j^{(q_3)} z\left(t_{k-j}\right),\end{aligned} \tag{33}$$

where T_{sim} is the simulation time, $k = 1, 2, \ldots, N$, for $N = [T_{\text{sim}}/h]$.

5.1.1. Sliding Mode Control Design of Lorenz's System. In this part, we will firstly consider a simple case: the system uncertainty and external disturbance will be ignored. By introducing the control input to the first state equation of fractional Lorenz's system, the controlled system is derived as

$$\begin{aligned}D^{q_1} x &= -a\left(x - y\right) + u(t), \\ D^{q_2} y &= rx - y - xz, \\ D^{q_3} z &= -bz + xy.\end{aligned} \tag{34}$$

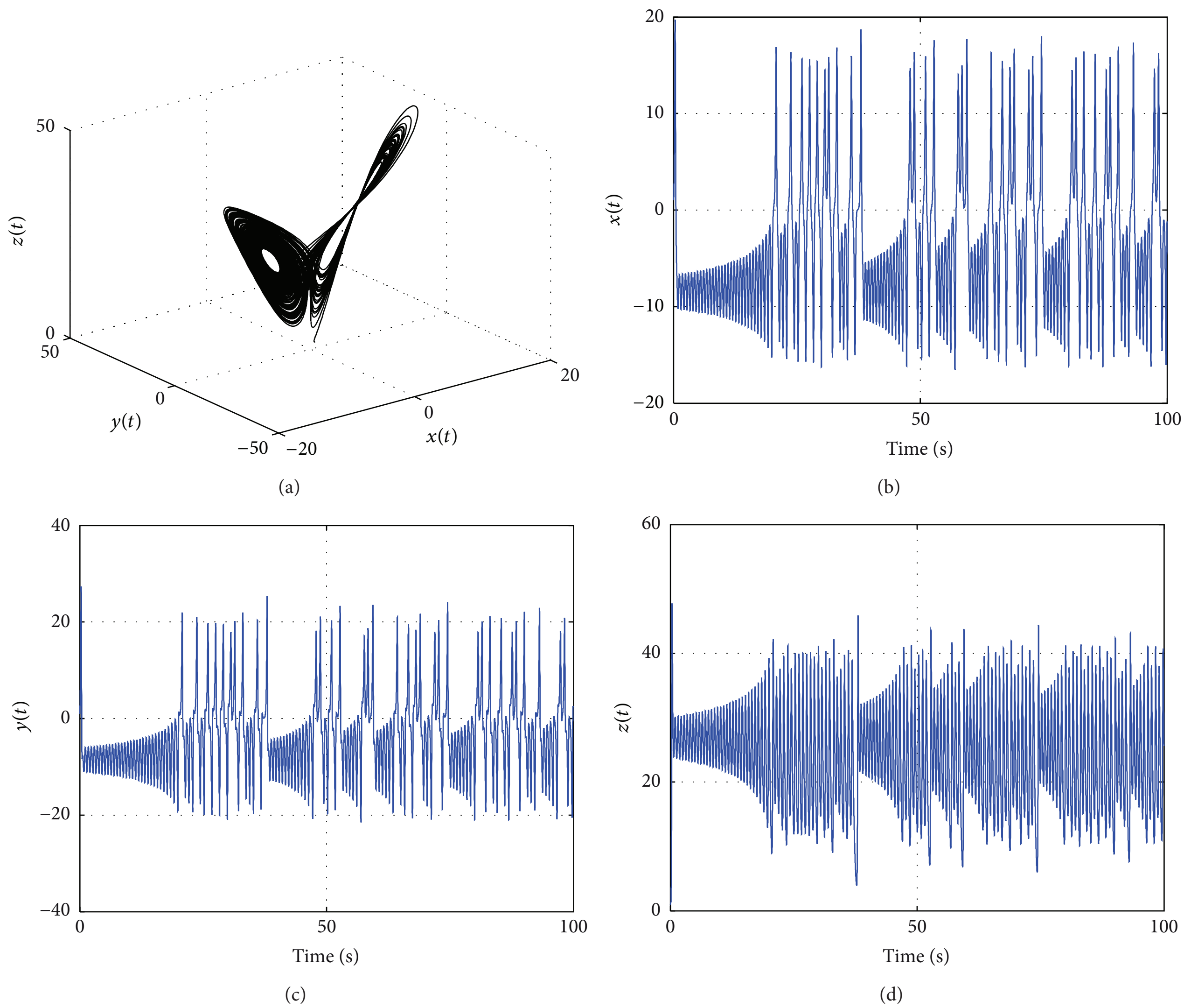

FIGURE 1: The chaotic trajectories of the fractional Lorenz's system with simulation time $T_{\text{sim}} = 100$ s and time step $h = 0.005$: (a) the $x - y - z$ space; (b) the $x - t$ space; (c) the $y - t$ space; (d) the $z - t$ space.

Owning to (8), the sliding surface is

$$s(t) = D^{q_1} x(t) + D^{-1} \phi(t), \tag{35}$$

where

$$\phi(t) = r y(t) + a x(t). \tag{36}$$

In terms of (22), the control law is

$$u(t) = -(a + r) y - k_1 \operatorname{sgn}(s) - k_2 s. \tag{37}$$

The numerical simulations with $k_1 = 0.05$ and $k_2 = 0.1$ are illustrated in Figure 2. It is clear that the control law (37) is efficient for controlling the fractional Lorenz's system.

5.1.2. Adaptive Sliding Mode Control Design of Uncertain Lorenz's System. In this part, we will consider a little intricacy case: the system uncertainty and external disturbance in the fractional Loren's system will be considered. By introducing the control input to the first state equation, the controlled system is derived as

$$\begin{aligned} D^{q_1} x &= -a(x - y) + \Delta f + d + u(t), \\ D^{q_2} y &= rx - y - xz, \\ D^{q_3} z &= -bz + xy. \end{aligned} \tag{38}$$

By (28) and (29), the control law and adaptive law are selected as

$$u(t) = -(a + r) y - \left(\hat{\theta}_1 + \hat{\theta}_2 + k_1\right) \operatorname{sgn}(s) - k_2 s, \tag{39}$$

$$\begin{aligned} \dot{\hat{\theta}}_1 &= \mu_1 |s|, \\ \dot{\hat{\theta}}_2 &= \mu_2 |s|. \end{aligned} \tag{40}$$

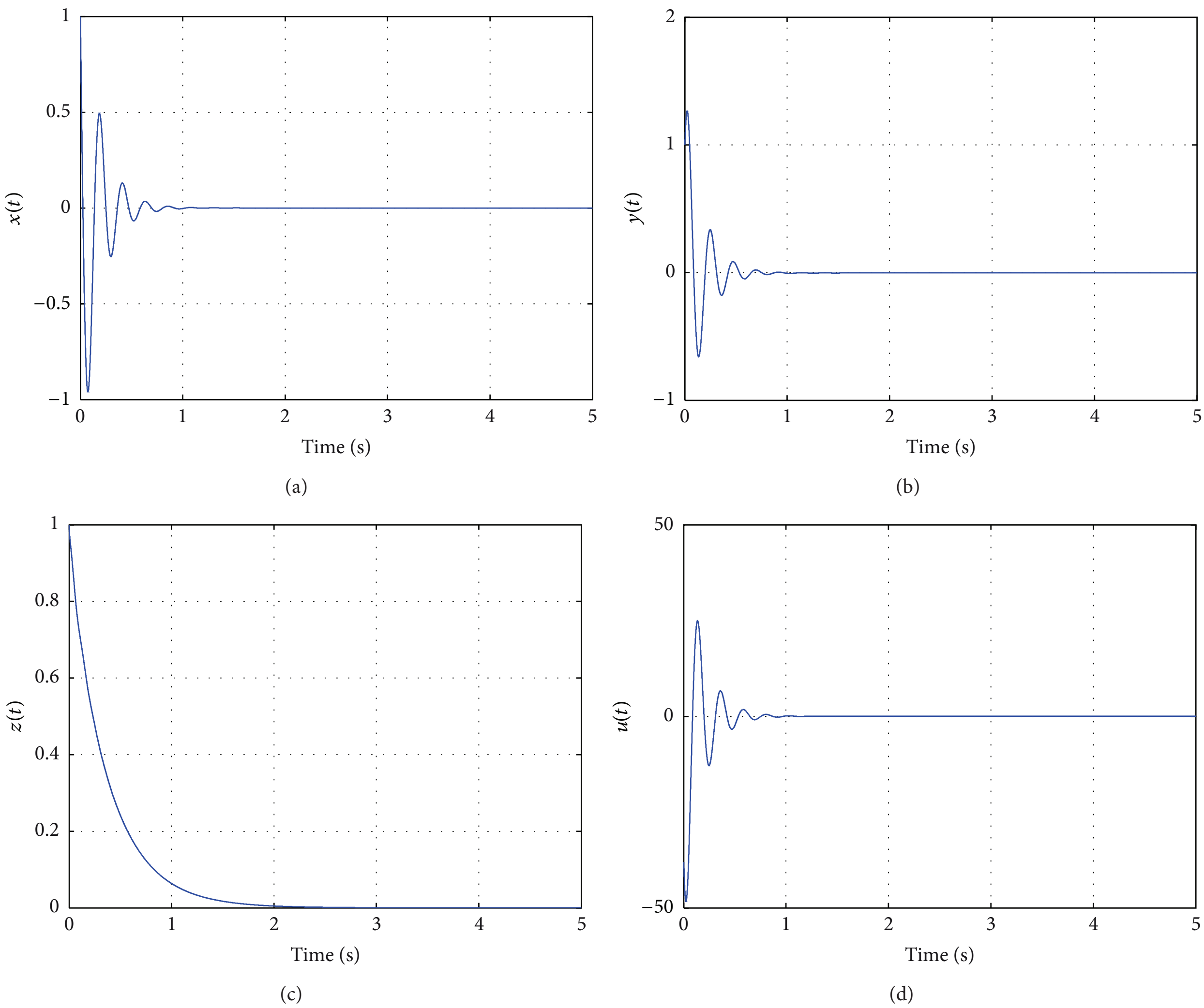

FIGURE 2: Sliding mode control of the fractional Loren's system with simulation time $T_{\text{sim}} = 5$ s and time step $h = 0.0005$: (a) the $x - t$ space; (b) the $y - t$ space; (c) the $z - t$ space; (d) the $u - t$ space.

The numerical simulations with the above control law (39) and adaptive law (40) are illustrated in Figure 3, with the gain of control law $k_1 = 0.02$ and $k_2 = 0.02$, the coefficients of adaptive law $\mu_1 = 0.03$ and $\mu_2 = 0.02$, and the initial conditions of the adaptive parameters $\theta_1(0) = 0.03$, $\theta_2(0) = 0.02$, $\Delta f = 0.1 - 0.1\sin(\pi x)$, and $d(t) = 0.1\cos(t)$. We can see from Figure 3 that the control law (39) and adaptive laws (40) are capable at controlling the fractional Loren's system in the presence of uncertainty and external disturbance.

5.2. Control of the Fractional Chen's System. The fractional Chen's system is described as

$$D^{q_1} x = a(y - x),$$
$$D^{q_2} y = (c - a)x - xz + cy, \quad (41)$$
$$D^{q_3} z = xy - bz,$$

where $a = 35$, $b = 3$, and $c = 28$.

The equilibrium points of the system with the above parameters are $E_1 = (0, 0, 0)$, $E_2 = (3\sqrt{7}, 3\sqrt{7}, 21)$, and $E_3 = (-3\sqrt{7}, -3\sqrt{7}, 21)$.

The initial condition for the fractional Chen's system is chosen as

$$x(t) = x(0^+) = 1,$$
$$y(t) = y(0^+) = 1, \quad (42)$$
$$z(t) = z(0^+) = 1,$$

for $-\infty \le t \le 0$.

The fractional Chen's system exhibits chaos with fractional orders $q_1 = q_2 = q_3 = 0.9$ and the above initial conditions, as depicted in Figure 4. The numerical algorithm is based on Grünwald-Letnikov's definition:

$$x(t_k) = a(y(t_{k-1}) - x(t_{k-1}))h^{q_1} - \sum_{j=2}^{k} c_j^{(q_1)} x(t_{k-j}),$$

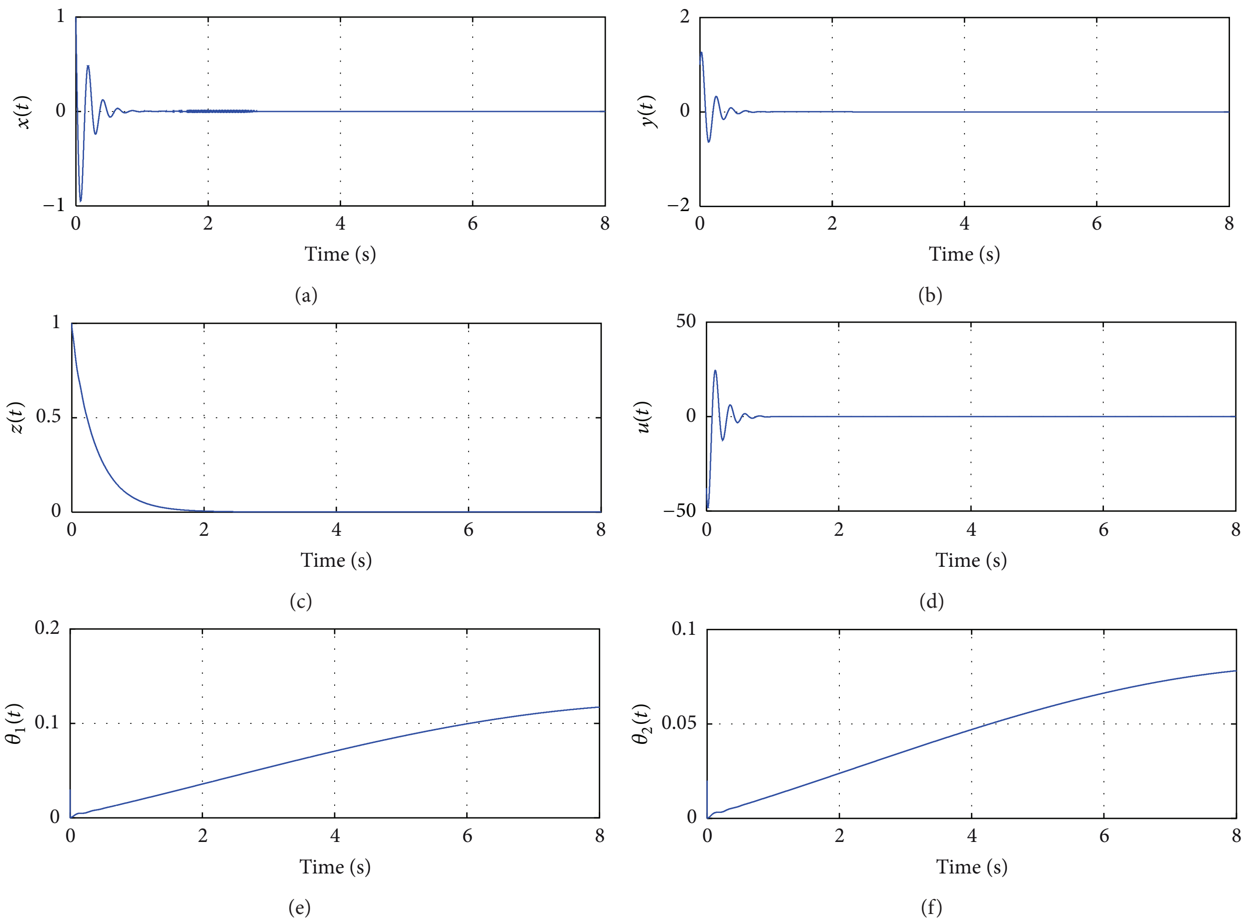

FIGURE 3: Adaptive sliding mode control of the fractional Loren's system with dynamics uncertainty and external disturbance with simulation time $T_{\text{sim}} = 5$ s and time step $h = 0.0005$: (a) the $x - t$ space; (b) the $y - t$ space; (c) the $z - t$ space; (d) the $u - t$ space; (e) online estimate of θ_1; (f) online estimate of θ_2.

$$\begin{aligned} y(t_k) &= \left[(c-a)x(t_k) - x(t_k)z(t_{k-1}) + cy(t_{k-1})\right]h^{q_2} \\ &\quad - \sum_{j=2}^{k} c_j^{(q_2)} y(t_{k-j}), \\ z(t_k) &= \left[x(t_k)y(t_k) - bz(t_{k-1})\right]h^{q_3} \\ &\quad - \sum_{j=2}^{k} c_j^{(q_3)} z(t_{k-j}). \end{aligned} \tag{43}$$

5.2.1. Sliding Mode Control Design of Chen's System. The system uncertainties and external disturbances will not be considered, and a control input is introduced to the second equation. The dynamics of the controlled system is described as

$$\begin{aligned} D^{q_1}x &= a(y-x), \\ D^{q_2}y &= (c-a)x - xz + cy + u, \\ D^{q_3}z &= xy - bz. \end{aligned} \tag{44}$$

Using (8), the sliding surface is constructed as

$$s(t) = D^{q}y(t) + D^{-1}\phi(t), \tag{45}$$

where $\phi(t) = ax + cy + xz$.

By (22), the control law is determined as

$$u(t) = -cx - 2cy - k_1\,\text{sgn}(s) - k_2 s. \tag{46}$$

The numerical simulations with $k_1 = 0.1$ and $k_2 = 0.1$ are illustrated in Figure 5. It is clear that the proposed control law (46) is feasible and efficient for controlling the fractional Chen's system.

5.2.2. Adaptive Sliding Mode Control Design of Uncertain Chen's System. The system uncertainties and external disturbances are added to the second equation and the control input is introduced to the second equation. Then the controlled system can be described as

$$\begin{aligned} D^{q_1}x &= a(y-x), \\ D^{q_2}y &= (c-a)x - xz + cy + \Delta f + d + u, \\ D^{q_3}z &= xy - bz. \end{aligned} \tag{47}$$

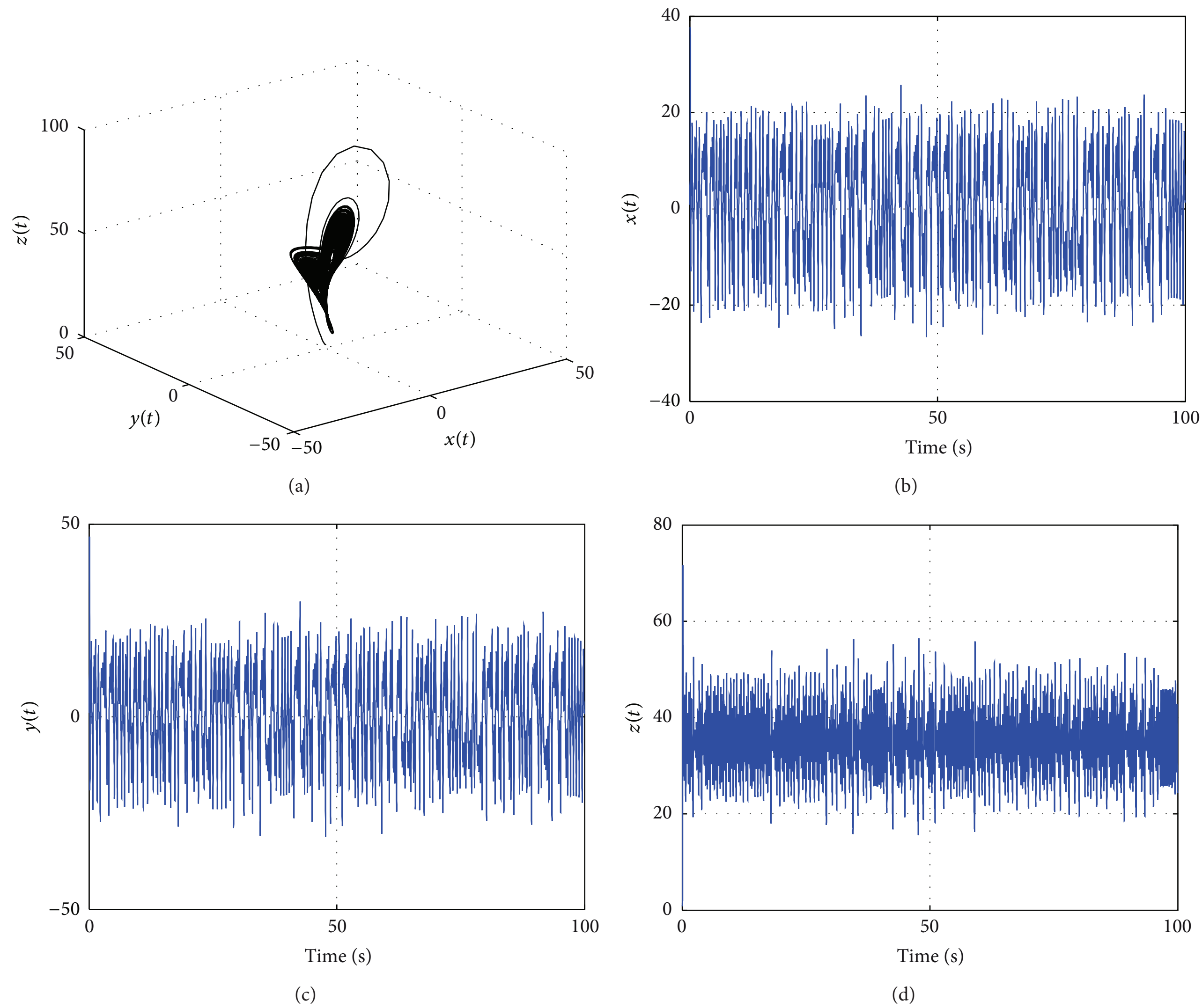

Figure 4: The chaotic trajectories of the fractional Chen's system with simulation time $T_{\text{sim}} = 100$ s and time step $h = 0.005$: (a) the $x - y - z$ space; (b) the $x - t$ space; (c) the $y - t$ space; (d) the $z - t$ space.

By (28) and (29), the control law and adaptive law are selected as

$$u(t) = -cx - 2cy - \left(\widehat{\theta}_1 + \widehat{\theta}_2 + k_1\right)\operatorname{sgn}(s) - k_2 s, \tag{48}$$

$$\begin{aligned} \dot{\widehat{\theta}}_1 &= \mu_1 |s|, \\ \dot{\widehat{\theta}}_2 &= \mu_2 |s|. \end{aligned} \tag{49}$$

The numerical simulations with the above control law (48) and adaptive law (49) are illustrated in Figure 6, with the coefficients of control law $k_1 = 0.175$ and $k_2 = 0.1$, the coefficients of adaptive law $\mu_1 = 0.2$ and $\mu_2 = 0.2$, and the initial conditions of the adaptive parameters $\theta_1(0) = 0.2$, $\theta_2(0) = 0.2$, $\Delta f = 0.1 - 0.1\sin(\pi x)$, and $d(t) = 0.1 - 0.1\sin(\pi x)$. It is clear that the proposed control law (48) and adaptive laws (49) are good at controlling the chaotic system in the presence of uncertainty and external disturbance.

6. Conclusion

In this paper, a novel class of fractional chaotic system in the pseudo state space is introduced and stabilized. To guarantee the stability of sliding mode dynamics, a novel fractional integral type sliding surface is constructed, with which an adaptive sliding mode control law is designed to stabilize the proposed fractional chaotic system with uncertainties and external disturbance whose bounds are unknown. Numerical simulations of fractional Lorenz's system and Chen's system have been presented to demonstrate the effectiveness of the proposed control technique.

Appendix

Lemma A.1. *Consider*

$$W = W_1 + W_2, \tag{A.1}$$

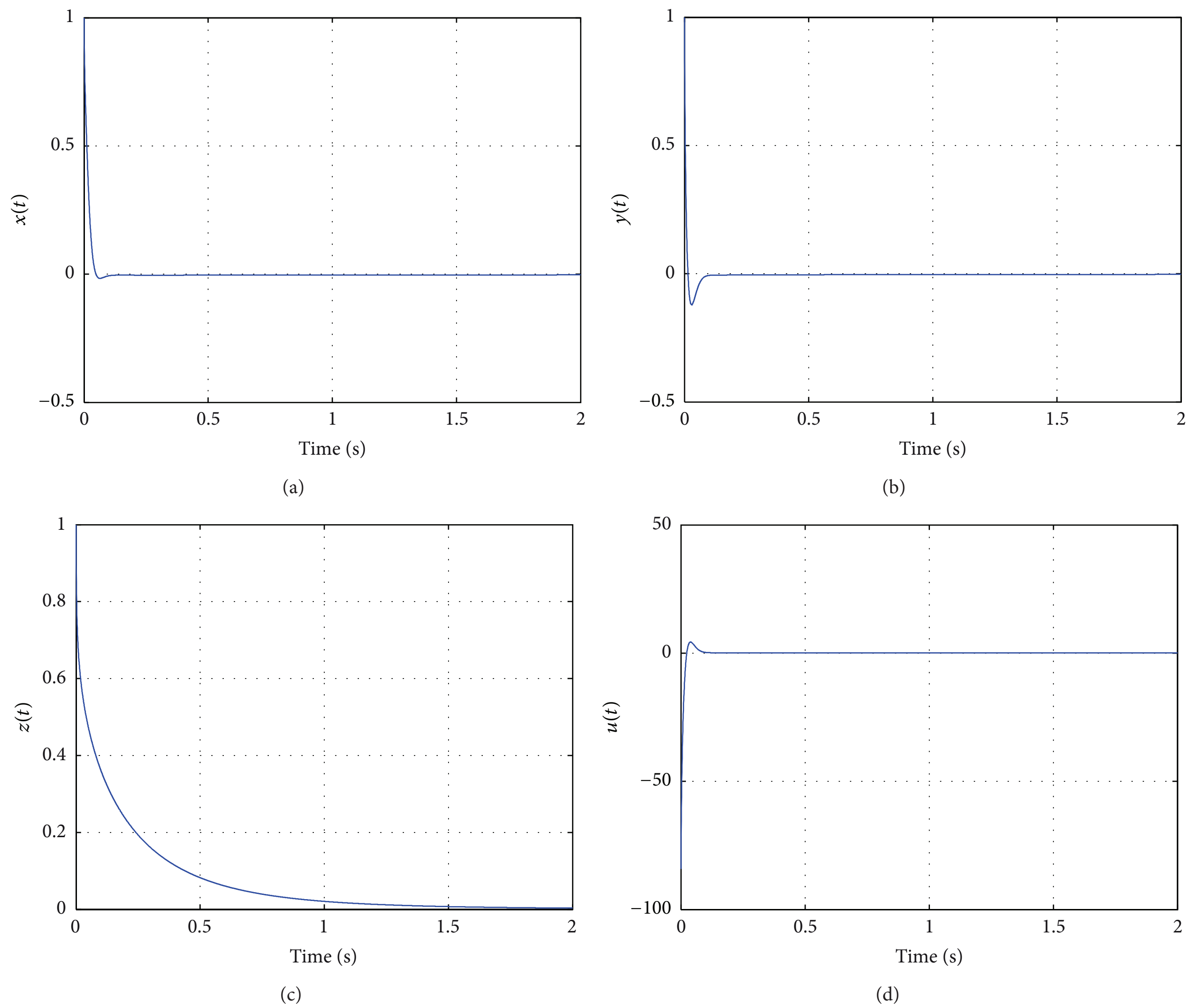

Figure 5: Sliding mode control of the fractional Chen's system with simulation time $T_{\text{sim}} = 5$ s and time step $h = 0.0005$: (a) the $x - t$ space; (b) the $y - t$ space; (c) the $z - t$ space; (d) the $u - t$ space.

where

$$W_1 = \sum_{i=1}^{m} W_{1i},$$
$$W_2 = \sum_{i=1}^{m} a_i W_{2i} \tag{A.2}$$

with $W_{1i} = \int_0^\infty \mu_i(\omega)\omega z_i^2(\omega, t)\, d\omega$ *and* $W_{2i} = x_i^2$, $i = 1, 2, \ldots, m$.

The frequency discretizations of W_{1i} *give*

$$W_{1i} = \sum_{i=1}^{J} \omega_{ij}\mu_i\left(\omega_{ij}\right) z_{ij}^2\left(\omega_{ij}, t\right) \times \Delta\omega_{ij} = \sum_{i=1}^{J} \omega_{ij} c_{ij} z_i^2\left(\omega_{ij}, t\right). \tag{A.3}$$

It is clear that W_{1i}, $i = 1, 2, \ldots, m$, *are all positive definite quadratic forms and can be expressed in the matrix form as*

$$W_{1i} = Z_i^T M_i Z_i \tag{A.4}$$

with

$$M_i = \begin{pmatrix} \omega_{i1}c_{i1} & & & \\ & \omega_{i2}c_{i2} & & \\ & & \ddots & \\ & & & \omega_{iJ}c_{iJ} \end{pmatrix}. \tag{A.5}$$

It is clear that W_{2i}, $i = 1, 2, \ldots, m$, *are also positive. Because* $x_i = C_i^T Z_i$, W_{2i} *can be rewritten as*

$$W_{2i} = Z_i^T C_i C_i^T Z_i. \tag{A.6}$$

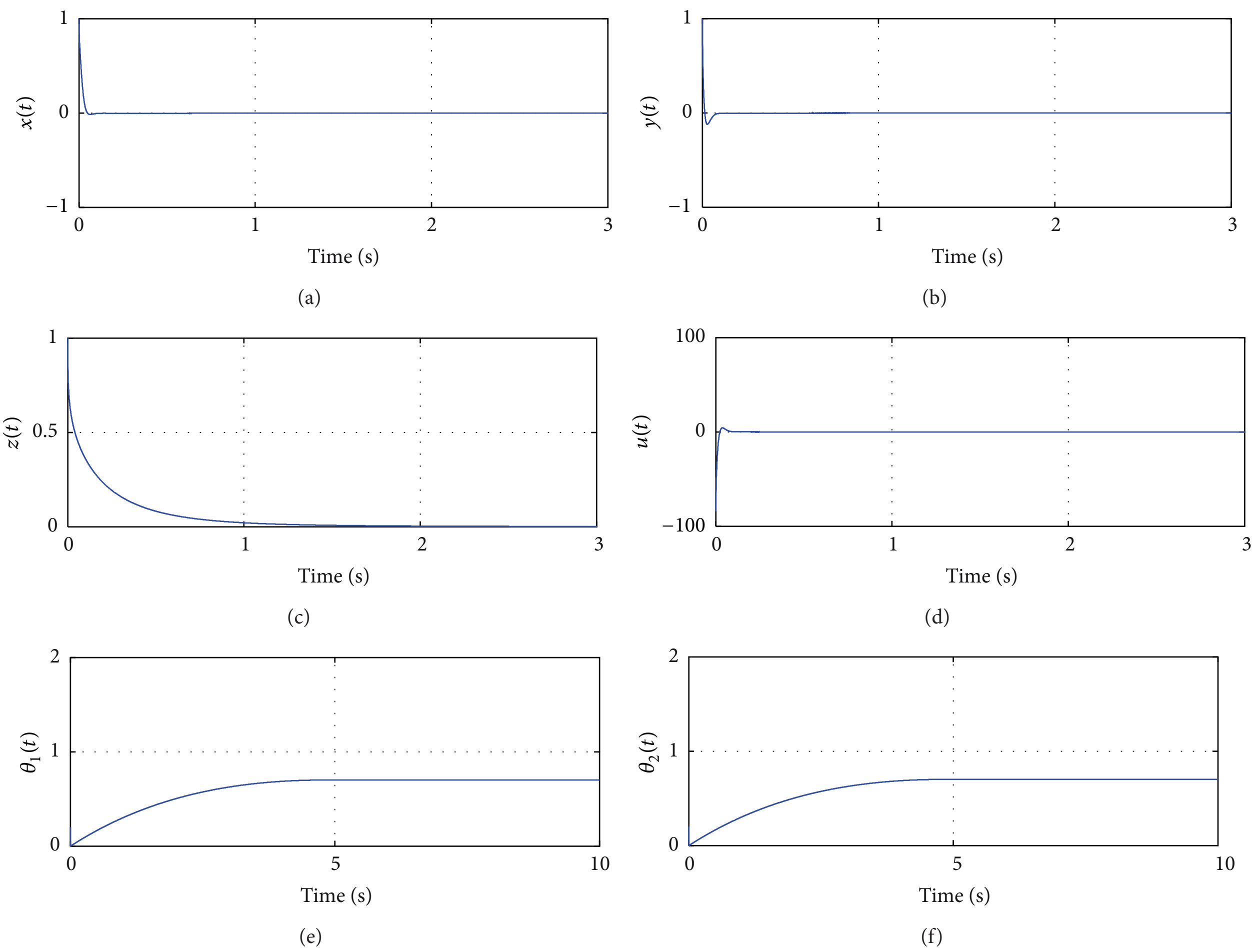

Figure 6: Adaptive sliding mode control of the fractional Chen's system with dynamics uncertainty and external disturbance with simulation time $T_{\text{sim}} = 5$ s and time step $h = 0.0005$: (a) the $x - t$ space; (b) the $y - t$ space; (c) the $z - t$ space; (d) the $u - t$ space; (e) online estimate of θ_1; (f) online estimate of θ_2.

Let us define

$$Z = \left(Z_1 \;\; Z_2 \;\; \cdots \;\; Z_m\right)^T,$$

$$M_C = \begin{pmatrix} M_1 + a_1 C_1 C_1^T & & & \\ & M_2 + a_2 C_2 C_2^T & & \\ & & \ddots & \\ & & & M_m + a_m C_m C_m^T \end{pmatrix}. \tag{A.7}$$

Then $W = Z^T M_C Z$.

Since W_{1i} *and* W_{2i}, $i = 1, 2, \ldots, m$, *are all positive, then nonnegative values of* a_i, $i = 1, 2, \ldots, m$, *implies* $W \geq 0$, *and positive values of* a_i *implies* $W > 0$.

Consequently, we have the following lemma.

The quadratic form W is positive semi-definite if $a_i \geq 0$ and is positive definite if $a_i > 0$, $i = 1, 2, \ldots, m$.

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