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Research Article

On the Cauchy Problem for the Two-Component Novikov Equation

Yongsheng Mi,[1,2] Chunlai Mu,[1] and Weian Tao[2]

[1] *College of Mathematics and Statistics, Chongqing University, Chongqing 400044, China*
[2] *College of Mathematics and Computer Sciences, Yangtze Normal University, Chongqing, Fuling 408100, China*

Correspondence should be addressed to Yongsheng Mi; miyongshen@163.com

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We are concerned with the Cauchy problem of two-component Novikov equation, which was proposed by Geng and Xue (2009). We establish the local well-posedness in a range of the Besov spaces by using Littlewood-Paley decomposition and transport equation theory which is motivated by that in Danchin's cerebrated paper (2001). Moreover, with analytic initial data, we show that its solutions are analytic in both variables, globally in space and locally in time, which extend some results of Himonas (2003) to more general equations.

1. Introduction

In this paper, we consider the following Cauchy problem of the two-component Novikov equation

$$\begin{aligned} &m_t + uvm_x + 3vu_xm = 0, \quad t > 0, \ x \in \mathbb{R}, \\ &n_t + uvn_x + 3uv_xn = 0, \quad t > 0, \ x \in \mathbb{R}, \\ &m = u - u_{xx}, \quad n = v - v_{xx}, \quad t > 0, \ x \in \mathbb{R}, \\ &u(0, x) = u_0(x), \quad v(0, t) = v_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

The two-component system in (1) was found by Geng and Xue [1]. It was shown in [1] that the system (1) is exactly a negative flow in the hierarchy and admits exact solutions with N-peakons and an infinite sequence of conserved quantities. Moreover, a reduction of this hierarchy and its Hamiltonian structures are discussed.

For $v = 1$, (1) becomes the Degasperis-Procesi equation

$$\begin{aligned} &m_t + um_x + 3u_xm = 0, \quad t > 0, \ x \in \mathbb{R}, \\ &m = u - u_{xx}, \quad t > 0, \ x \in \mathbb{R}, \\ &u(0, x) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{2}$$

Degasperis et al. [2] proved the formal integrability of (2) by constructing a Lax pair. They also showed that it has bi-Hamiltonian structure and an infinite sequence of conserved quantities and admits exact peakon solutions. The direct and inverse scattering approach to pursue it can be seen in [3]. Moreover, in [4], they also presented that the Degasperis-Procesi equation has a bi-Hamiltonian structure and an infinite number of conservation laws and admits exact peakon solutions which are analogous to the Camassa-Holm peakons. It is worth pointing out that solutions of this type are not mere abstractizations: the peakons replicate a feature that is characteristic for the waves of great height-waves of the largest amplitude that are exact solutions of the governing equations for irrotational water waves (cf. the papers [5–7]). The Degasperis-Procesi equation is a model for nonlinear shallow water dynamics (cf. the discussion in [8]). The numerical stability of solitons and peakons, the multisoliton solutions, and their peakon limits, together with an inverse scattering method to compute N-peakon solutions to Degasperis-Procesi equation, have been investigated, respectively, in [9–11]. Furthermore, the traveling wave solutions and the classification of all weak traveling wave solutions to Degasperis-Procesi equation were presented in [12, 13].

For $u = v$, (1) becomes the Novikov equation

$$\begin{aligned} &m_t + u^2 m_x + 3uu_x m = 0, \quad t > 0,\ x \in \mathbb{R}, \\ &m = u - u_{xx}, \quad t > 0,\ x \in \mathbb{R}, \\ &u(0, x) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3}$$

which has been recently discovered by Vladimir Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity [14]. The perturbative symmetry approach yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate (3) and find its first few symmetries, and he subsequently found a scalar Lax pair for it, then proved that the equation is integrable, which can be thought as a generalization of the Camassa-Holm equation. In [15], it is shown that the Novikov equation admits peakon solutions like the Camassa-Holm. Also, it has a Lax pair in matrix form and a bi-Hamiltonian structure. Furthermore, it has infinitely many conserved quantities, like Camassa-Holm. The most important quantity conserved by a solution u to Novikov equation is its H^1-norm $\|u\|_{H^1}^2 = \int_R (u^2 + u_x^2)$, which plays an important role in the study of (1). In [16–19], the authors study well-posedness and dependence on initial data for the Cauchy problem for Novikov equation. Recently, in [20], a global existence result and conditions on the initial data were considered. Existence and uniqueness of global weak solution to Novikov equation with initial data under some conditions was proved in [21]. The Novikov equation with dissipative term was considered in [22]. Multipeakon solutions were studied in [15, 23]. The Cauchy problem of the Novikov equation on the circle was investigated in [24]. An alternative modified Camassa-Holm equation was introduced in [25].

Motivated by the references cited above, the goal of the present paper is to establish the local well-posedness for the strong solutions to the Cauchy problem (1). The proof of the local well-posedness is inspired by the argument of approximate solutions by Danchin [26] in the study of the local wellposedness to the Camassa-Holm equation. However, one problematic issue is that we here deal with two-component system with a higher order nonlinearity in the Besov spaces, making the proof of several required nonlinear estimates somewhat delicate. These difficulties are nevertheless overcome by careful estimates for each iterative approximation of solutions to (1). Moreover, we also prove the analyticity of its solutions $u = u(t, x)$ in both variables, with x in $\mathbb{R}$ and t in an interval around zero, provided that the initial profile u_0 is an analytic function on the real line. Hence, this analytic result can be viewed as a Cauchy-Kowalevski theorem for (1).

Now we are in the position to state the local existence result and analyticity result, where the definition of Besov spaces $B^s_{p,r}$, $E^s_{p,r}(T)$, and and E_{s_0} will be given in Sections 2 and 3.

Theorem 1. *Let $p, r \in [1, \infty]$ and $s > \max\{5/2, 2 + (1/p)\}$. Assume that $(u_0, v_0) \in B^s_{p,r} \times B^s_{p,r}$. There exists a time $T > 0$ such that the initial-value problem* (1) *has a unique solution $(u, v) \in E^s_{p,r}(T) \times E^s_{p,r}(T)$ and the map $(u_0, v_0) \mapsto (u, v)$ is continuous from a neighborhood of (u_0, v_0) in $B^s_{p,r} \times B^s_{p,r}$ into*

$$\begin{aligned} &C\left([0,T]; B^{s'}_{p,r}\right) \cap C^1\left([0,T]; B^{s'-1}_{p,r}\right) \\ &\quad \times C\left([0,T]; B^{s'}_{p,r}\right) \cap C^1\left([0,T]; B^{s'-1}_{p,r}\right) \end{aligned} \tag{4}$$

for every $s' < s$ when $r = \infty$ and $s' = s$ whereas $r < \infty$.

Theorem 2. *If the initial data $\binom{u_0}{v_0}$ is real analytic on the line $\mathbb{R}$ and belongs to a space E_{s_0}, for some $0 < s_0 \le 1$, then there exist an $\varepsilon > 0$ and a unique solution $\binom{u}{v}$ to the Cauchy problem* (1) *that is analytic on $(-\varepsilon, \varepsilon) \times \mathbb{R}$.*

The rest of this paper is organized as follows. In Section 2, we prove the local well-posedness of the initial value problem (1) in the Besov space. Section 3 is devoted to the study of the analyticity of the Cauchy problem (1) based on a contraction type argument in a suitably chosen scale of the Banach spaces.

2. Local Well-Posedness in the Besov Spaces

In this section, we will establish local well-posedness of the initial value problem (1) in the Besov spaces.

First, for the convenience of the readers, we recall some facts on the Littlewood-Paley decomposition and some useful lemmas.

Notation. $\mathcal{S}$ stands for the Schwartz space of smooth functions over $\mathbb{R}^d$ whose derivatives of all order decay at infinity. The set $\mathcal{S}'$ of temperate distributions is the dual set of $\mathcal{S}$ for the usual pairing. We denote the norm of the Lebesgue space $L^p(\mathbb{R})$ by $\|\cdot\|_{L^p}$ with $1 \le p \le \infty$, and the norm in the Sobolev space $H^s(\mathbb{R})$ with $s \in \mathbb{R}$ by $\|\cdot\|_{H^s}$.

Proposition 3 (Littlewood-Paley decomposition [27]). *Let $\mathcal{B} \doteq \{\xi \in \mathbb{R}^d, |\xi| \le 4/3\}$ and $\mathcal{C} \doteq \{\xi \in \mathbb{R}^d, 4/3 \le |\xi| \le 8/3\}$. There exist two radial functions $\chi \in C_c^\infty(\mathcal{B})$ and $\varphi \in C_c^\infty(\mathcal{C})$ such that*

$$\begin{aligned} &\chi(\xi) + \sum_{q \ge 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d, \\ &\left|q - q'\right| \ge 2 \Longrightarrow \operatorname{Supp} \varphi(2^{-q}\cdot) \cap \operatorname{Supp} \varphi\left(2^{-q'}\cdot\right) = \emptyset, \\ &q \ge 1 \Longrightarrow \operatorname{Supp} \chi(\cdot) \cap \operatorname{Supp} \varphi\left(2^{-q'}\cdot\right) = \emptyset, \\ &\frac{1}{3} \le \chi(\xi)^2 + \sum_{q \ge 0} \varphi(2^{-q}\xi)^2 \le 1, \quad \forall \xi \in \mathbb{R}^d. \end{aligned} \tag{5}$$

Furthermore, let $h \doteq \mathcal{F}^{-1}\varphi$ and $\widetilde{h} \doteq \mathcal{F}^{-1}\chi$. Then for all $f \in \mathcal{S}'(\mathbb{R}^d)$, the dyadic operators Δ_q and S_q can be defined as follows:

$$\begin{aligned}
\Delta_q f &\doteq \varphi(2^{-q}D) f \\
&= 2^{qd}\int_{\mathbb{R}^d} h(2^q y) f(x-y)\,dy \quad \text{for } q \geq 0, \\
S_q f &\doteq \chi(2^{-q}D) f \\
&= \sum_{-1\leq k\leq q-1} \Delta_k = 2^{qd}\int_{\mathbb{R}^d} \tilde{h}(2^q y) f(x-y)\,dy, \\
\Delta_{-1} f &\doteq S_0 f, \qquad \Delta_q f \doteq 0 \quad \text{for } q \leq -2.
\end{aligned} \tag{6}$$

Hence,

$$f = \sum_{q\geq 0} \Delta_q f \quad \text{in } \mathcal{S}'(\mathbb{R}^d), \tag{7}$$

where the right-hand side is called the nonhomogeneous Littlewood-Paley decomposition of f.

Lemma 4 (Bernstein's inequality [28]). *Let $\mathscr{B}$ be a ball with center 0 in $\mathbb{R}^d$ and $\mathscr{C}$ a ring with center 0 in $\mathbb{R}^d$. A constant C exists so that, for any positive real number λ, any nonnegative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (a,b) with $b \geq a \geq 1$, there hold*

$$\begin{aligned}
&\operatorname{Supp}\hat{u} \subset \lambda\mathscr{B} \\
&\quad\Longrightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1}\lambda^{k+d((1/a)-(1/b))}\|u\|_{L^a}, \\
&\operatorname{Supp}\hat{u} \subset \lambda\mathscr{C} \\
&\quad\Longrightarrow C^{-k-1}\lambda^k\|u\|_{L^a} \leq \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a} \leq C^{k+1}\lambda^k\|u\|_{L^a}, \\
&\operatorname{Supp}\hat{u} \subset \lambda\mathscr{C} \\
&\quad\Longrightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m}\lambda^{m+d((1/a)-(1/b))}\|u\|_{L^a},
\end{aligned} \tag{8}$$

for any function $u \in L^a$.

Definition 5 (Besov space). Let $s \in \mathbb{R}, 1 \leq p, r \leq \infty$. The inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r} \doteq \left\{ f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{B^s_{p,r}} < \infty \right\}, \tag{9}$$

where

$$\|f\|_{B^s_{p,r}} \doteq \begin{cases} \left(\sum_{q\in\mathbb{Z}} 2^{qsr}\|\Delta_q f\|^r_{L_p} \right)^{1/r}, & \text{for } r < \infty, \\ \sup_{q\in\mathbb{Z}} 2^{qs}\|\Delta_q f\|_{L_p}, & \text{for } r = \infty. \end{cases} \tag{10}$$

If $s = \infty$, $B^\infty_{p,r} \doteq \cap_{s\in\mathbb{R}} B^s_{p,r}$.

Proposition 6 (see [28]). *Suppose that $s \in \mathbb{R}$, $1 \leq p, r, p_i, r_i \leq \infty$ $(i = 1, 2)$. One has the following.*

(1) *Topological properties: $B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'$.*

(2) *Density: C^∞_c is dense in $B^s_{p,r} \Leftrightarrow 1 \leq p, r \leq \infty$.*

(3) *Embedding: $B^s_{p_1,r_1} \hookrightarrow B^{s-n(1/p_1)-(1/p_2)}_{p_2,r_2}$, if $p_1 \leq p_2$ and $r_1 \leq r_2$.*
$B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ locally compact, if $s_1 < s_2$.

(4) *Algebraic properties: for all $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. Moreover, $B^s_{p,r}$ is an algebra, provided that $s > n/p$ or $s \geq n/p$ and $r = 1$.*

(5) *Complex interpolation:*

$$\begin{aligned}
&\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \\
&\quad \leq C\|u\|^\theta_{B^{s_1}_{p,r}}\|u\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall u \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}, \forall \theta \in [0,1].
\end{aligned} \tag{11}$$

(6) *Fatou lemma: If $(u_n)_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and $u_n \to u$ in $\mathcal{S}'$, then $u \in B^s_{p,r}$ and*

$$\|u\|_{B^s_{p,r}} \leq \liminf_{n\to\infty}\|u_n\|_{B^s_{p,r}}. \tag{12}$$

(7) *Let $m \in \mathbb{R}$ and f be an S^m-multiplier (i.e., $f : \mathbb{R}^d \to \mathbb{R}$ is smooth and satisfies that for all $\alpha \in \mathbb{N}^d$, there exists a constant C_α, s.t. $|\partial^\alpha f(\xi)| \leq C_\alpha(1 + |\xi|^{m-|\alpha|})$ for all $\xi \in \mathbb{R}^d$). Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems later.

Lemma 7 (see [26, 28]). *Suppose that $(p,r) \in [1,+\infty]^2$ and $s > -(d/p)$. Let v be a vector field such that ∇v belongs to $L^1([0,T]; B^{s-1}_{p,r})$ if $s > 1 + (d/p)$ or to $L^1([0,T]; B^{d/p}_{p,r} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B^s_{p,r}$, $F \in L^1([0,T]; B^s_{p,r})$ and that $f \in L^\infty(L^1([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ solves the d-dimensional linear transport equations*

$$\begin{aligned}
\partial_t f + v \cdot \nabla f &= F, \\
f|_{t=0} &= f_0.
\end{aligned} \tag{T}$$

Then there exists a constant C depending only on s, p, and d such that the following statements hold.

(1) *If $r = 1$ or $s \neq 1 + (d/p)$, then*

$$\begin{aligned}
\|f\|_{B^s_{p,r}} \leq{} & \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}}\,d\tau \\
& + C\int_0^t V'(\tau)\|F(\tau)\|_{B^s_{p,r}}\,d\tau,
\end{aligned} \tag{13}$$

or

$$\|f\|_{B^s_{p,r}} \leq e^{CV(t)}C\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}}\,d\tau \right) \tag{14}$$

hold, where $V(t)=\int_0^t\|\nabla v(\tau)\|_{B^{d/p}_{p,r}\cap L^\infty}d\tau$ *if* $s<1+(d/p)$ *and* $V(t)=\int_0^t\|\nabla v(\tau)\|_{B^{s-1}_{p,r}}d\tau$ *else.*

(2) *If* $s\le 1+(d/p)$ *and* $\nabla f_0\in L^\infty$, $\nabla f\in L^\infty([0,T]\times\mathbb{R}^d)$ *and* $\nabla F\in L^1([0,T];L^\infty)$, *then*

$$\|f\|_{B^s_{p,r}}+\|\nabla f\|_{L^\infty}\le e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}}+\|\nabla f_0\|_{L^\infty}+\int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}}+\|\nabla F(\tau)\|_{L^\infty}d\tau\right)\tag{15}$$

with $V(t)=\int_0^t\|\nabla v(\tau)\|_{B^{d/p}_{p,r}\cap L^\infty}d\tau$.

(3) *If* $f=v$, *then for all* $s>0$, *the estimate* (14) *holds with* $V(t)=\int_0^t\|\nabla v(\tau)\|_{B^{s-1}_{p,r}}d\tau$.

(4) *If* $r<+\infty$, *then* $f\in C([0,T];B^s_{p,r})$. *If* $r=+\infty$, *then* $f\in C([0,T];B^{s'}_{p,r})$ *for all* $s'<s$.

Lemma 8 (existence and uniqueness see [26, 28]). *Let* $(p,p_1,r)\in[1,+\infty]^3$ *and* $s>-d\min\{1/p_1,1/p'\}$ *with* $p'\doteq(1-(1/p))^{-1}$. *Assume that* $f_0\in B^s_{p,r}$, $F\in L^1([0,T];B^s_{p,r})$. *Let* v *be a time-dependent vector field such that* $v\in L^\rho([0,T];B^{-M}_{\infty,\infty})$ *for some* $\rho>1$, $M>0$ *and* $\nabla v\in L^1([0,T];B^{d/p}_{p,r}\cap L^\infty)$ *if* $s<1+(d/p_1)$ *and* $\nabla v\in L^1([0,T];B^{s-1}_{p_1,r})$ *if* $s>1+(d/p)$ *or* $s=1+(d/p_1)$ *and* $r=1$. *Then the transport equations* (T) *have a unique solution* $f\in L^\infty([0,T];B^s_{p,r})\cap(\cap_{s'<s}C[0,T];B^{s'}_{p,1})$ *and the inequalities in Lemma 7 hold true. Moreover,* $r<\infty$, *then one has* $f\in C[0,T];B^s_{p,1})$.

Lemma 9 (1-D Morse-type estimates [26, 28]). *Assume that* $1\le p,r\le+\infty$, *the following estimates hold.*

(i) *For* $s>0$,

$$\|fg\|_{B^s_{p,r}}\le C\left(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty}+\|g\|_{B^s_{p,r}}\|f\|_{L^\infty}\right).\tag{16}$$

(ii) *For all* $s_1\le 1/p<s_2$ ($s_2\ge 1/p$ *if* $r=1$) *and* $s_1+s_2>0$, *one has*

$$\|fg\|_{B^{s_1}_{p,r}}\le C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.\tag{17}$$

(iii) *In Sobolev spaces* $H^s=B^s_{2,2}$, *one has for* $s>0$,

$$\|f\partial_x g\|_{H^s}\le C\left(\|f\|_{H^{s+1}}\|g\|_{L^\infty}+\|\partial_x g\|_{H^s}\|f\|_{L^\infty}\right),\tag{18}$$

where C *is a positive constant independent of* f *and* g.

Definition 10. For $T>0$, $s\in\mathbb{R}$ and $1\le p\le+\infty$, we set

$$\begin{gathered}E^s_{p,r}(T)\triangleq C\left([0,T];B^s_{p,r}\right)\cap C^1\left([0,T];B^{s-1}_{p,r}\right)\quad\text{if } r<+\infty,\\ E^s_{p,\infty}(T)\triangleq L^\infty\left([0,T];B^s_{p,\infty}\right)\cap \operatorname{lip}^1\left([0,T];B^{s-1}_{p,\infty}\right),\\ E^s_{p,r}\triangleq\cap_{T>0}E^s_{p,r}(T).\end{gathered}\tag{19}$$

In the following, we denote $C>0$ a generic constant only depending on p, r, s. Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 11. *Let* $1\le p,r\le+\infty$ *and* $s>\max\{5/2,2+(1/p)\}$. *Suppose that* $(u^{(i)};v^{(i)})\in\{L^\infty([0,T];B^s_{p,r})\cap C([0,T];\mathcal{S}')\}^2$ $(i=1,2)$ *be two given solutions of the initial-value problem* (1) *with the initial data* $(u_0^{(i)};v_0^{(i)})\in B^s_{p,r}\times B^s_{p,r}$ $(i=1;2)$. *Then for every* $t\in[0;T]$, *one has*

$$\begin{aligned}&\left\|u^{(1)}(t)-u^{(2)}(t)\right\|_{B^{s-1}_{p,r}}+\left\|v^{(1)}(t)-v^{(2)}(t)\right\|_{B^{s-1}_{p,r}}\\&\le\left(\left\|u_0^{(1)}-u_0^{(2)}\right\|_{B^{s-1}_{p,r}}+\left\|v_0^{(1)}-v_0^{(2)}\right\|_{B^{s-1}_{p,r}}\right)\\&\quad\times\exp\left\{C\int_0^t\left(\left\|u^{(1)}(\tau)\right\|^2_{B^s_{p,r}}+\left\|u^{(2)}(\tau)\right\|^2_{B^s_{p,r}}+\left\|v^{(1)}(\tau)\right\|^2_{B^s_{p,r}}+\left\|v^{(2)}(\tau)\right\|^2_{B^s_{p,r}}\right)d\tau\right\}.\end{aligned}\tag{20}$$

Proof. Denote $u^{(12)}=u^{(2)}-u^{(1)}$, $v^{(12)}=v^{(2)}-v^{(1)}$, $m^{(12)}=m^{(2)}-m^{(1)}$ and $n^{(12)}=n^{(2)}-n^{(1)}$. It is obvious that

$$u^{(12)},v^{(12)}\in L^\infty\left([0,T];B^s_{p,r}\right)\cap C\left([0,T];\mathcal{S}'\right),\tag{21}$$

which implies that $u^{(12)},v^{(12)}\in C([0,T];B^{s-1}_{p,r})$ and $(u^{(12)},v^{(12)},m^{(12)},m^{(12)})$ solves the transport equations

$$\begin{gathered}m_t^{(12)}+u^{(1)}v^{(1)}m_x^{(12)}=F,\\ n_t^{(12)}+u^{(1)}v^{(1)}n_x^{(12)}=G,\end{gathered}\tag{22}$$

with

$$\begin{aligned}F&=-\left(u^{(2)}v^{(12)}+u^{(12)}v^{(1)}\right)m_x^{(2)}\\&\quad-3\left(u_x^{(2)}v^{(12)}m^{(2)}+u_x^{(2)}v^{(1)}m^{(12)}+u_x^{(12)}v^{(1)}m^{(1)}\right),\\ G&=-\left(v^{(2)}u^{(12)}+v^{(12)}u^{(1)}\right)n_x^{(2)}\\&\quad-3\left(v_x^{(2)}u^{(12)}n^{(2)}+v_x^{(2)}u^{(1)}n^{(12)}+v_x^{(12)}u^{(1)}n^{(1)}\right).\end{aligned}\tag{23}$$

According to Lemma 7, we have

$$\begin{aligned}&e^{-C\int_0^t\|\partial_x(u^{(1)}v^{(1)})(\tau')\|_{B^{s-2}_{p,r}}d\tau'}\left\|m^{(12)}(t)\right\|_{B^{s-3}_{p,r}}\\&\le\left\|m_0^{(12)}\right\|_{B^{s-3}_{p,r}}+C\int_0^t e^{-C\int_0^\tau\|\partial_x(u^{(1)}v^{(1)})(\tau')\|_{B^{s-2}_{p,r}}d\tau'}\left(\|F\|_{B^{s-3}_{p,r}}\right)d\tau,\\&e^{-C\int_0^t\|\partial_x(u^{(1)}v^{(1)})(\tau')\|_{B^{s-2}_{p,r}}d\tau'}\left\|n^{(12)}(t)\right\|_{B^{s-3}_{p,r}}\\&\le\left\|n_0^{(12)}\right\|_{B^{s-3}_{p,r}}+C\int_0^t e^{-C\int_0^\tau\|\partial_x(u^{(1)}v^{(1)})(\tau')\|_{B^{s-2}_{p,r}}d\tau'}\left(\|G\|_{B^{s-3}_{p,r}}\right)d\tau,\end{aligned}\tag{24}$$

For $s > \max\{5/2, 2 + (1/p)\}$, by Lemma 9, we have

$$
\begin{aligned}
&\|F\|_{B^{s-3}_{p,r}}\\
&\quad = \Big\| -\left(u^{(2)}v^{(12)} + u^{(12)}v^{(1)}\right) m_x^{(2)}\\
&\qquad +3\left(u_x^{(2)}v^{(12)}m^{(2)} + u_x^{(2)}v^{(1)}m^{(12)} + u_x^{(12)}v^{(1)}m^{(1)}\right)\Big\|_{B^{s-3}_{p,r}}\\
&\quad \le C\left\|u^{(2)}v^{(12)} + u^{(12)}v^{(1)}\right\|_{B^{s-3}_{p,r}}\left\|m^{(2)}\right\|_{B^{s-2}_{p,r}}\\
&\qquad + C\left\|u^{(2)}\right\|_{B^{s-2}_{p,r}}\left\|v^{(12)}m^{(2)}\right\|_{B^{s-3}_{p,r}}\\
&\qquad + C\left\|u^{(2)}\right\|_{B^{s-2}_{p,r}}\left\|v^{(1)}m^{(12)}\right\|_{B^{s-3}_{p,r}}\\
&\qquad + C\left\|u^{(12)}\right\|_{B^{s-2}_{p,r}}\left\|v^{(1)}m^{(1)}\right\|_{B^{s-3}_{p,r}}\\
&\quad \le C\left(\left\|u^{(12)}\right\|_{B^{s-1}_{p,r}} + \left\|v^{(12)}\right\|_{B^{s-1}_{p,r}}\right)\\
&\qquad \times\left(\left\|u^{(1)}\right\|^2_{B^{s}_{p,r}} + \left\|v^{(1)}\right\|^2_{B^{s}_{p,r}} + \left\|u^{(2)}\right\|^2_{B^{s}_{p,r}} + \left\|v^{(2)}\right\|^2_{B^{s}_{p,r}}\right),\\
&\|G\|_{B^{s-3}_{p,r}}\\
&\quad = \Big\| -\left(v^{(2)}u^{(12)} + v^{(12)}u^{(1)}\right) n_x^{(2)}\\
&\qquad +3\left(v_x^{(2)}u^{(12)}n^{(2)} + v_x^{(2)}u^{(1)}n^{(12)} + v_x^{(12)}u^{(1)}n^{(1)}\right)\Big\|_{B^{s-3}_{p,r}}\\
&\quad \le C\left\|v^{(2)}u^{(12)} + v^{(12)}u^{(1)}\right\|_{B^{s-3}_{p,r}}\left\|n^{(2)}\right\|_{B^{s-2}_{p,r}}\\
&\qquad + C\left\|v^{(2)}\right\|_{B^{s-2}_{p,r}}\left\|u^{(12)}m^{(2)}\right\|_{B^{s-3}_{p,r}}\\
&\qquad + C\left\|v^{(2)}\right\|_{B^{s-2}_{p,r}}\left\|u^{(1)}n^{(12)}\right\|_{B^{s-3}_{p,r}}\\
&\qquad + C\left\|v^{(12)}\right\|_{B^{s-2}_{p,r}}\left\|u^{(1)}n^{(1)}\right\|_{B^{s-3}_{p,r}}\\
&\quad \le C\left(\left\|v^{(12)}\right\|_{B^{s-1}_{p,r}} + \left\|u^{(12)}\right\|_{B^{s-1}_{p,r}}\right)\\
&\qquad \times\left(\left\|v^{(1)}\right\|^2_{B^{s}_{p,r}} + \left\|u^{(1)}\right\|^2_{B^{s}_{p,r}} + \left\|v^{(2)}\right\|^2_{B^{s}_{p,r}} + \left\|u^{(2)}\right\|^2_{B^{s}_{p,r}}\right).
\end{aligned}
\tag{25}
$$

Therefore, inserting the above estimates to (24), we obtain

$$
\begin{aligned}
&e^{-C\int_0^t \|\partial_x (u^{(1)}v^{(1)})(\tau')\|_{B^{s-2}_{p,r}} d\tau'}\\
&\quad \times\left(\left\|u^{(12)}(t)\right\|_{B^{s-1}_{p,r}} + \left\|v^{(12)}(t)\right\|_{B^{s-1}_{p,r}}\right)\\
&\le \left\|u_0^{(12)}\right\|_{B^{s-1}_{p,r}} + \left\|v_0^{(12)}\right\|_{B^{s-1}_{p,r}}\\
&\quad + C\int_0^t e^{-C\int_0^\tau \|\partial_x (u^{(1)}v^{(1)})(\tau')\|_{B^{s-2}_{p,r}} d\tau'}\\
&\quad \times\left(\left\|v^{(12)}\right\|_{B^{s-1}_{p,r}} + \left\|u^{(12)}\right\|_{B^{s-1}_{p,r}}\right)\\
&\quad \times\left(\left\|v^{(1)}\right\|^2_{B^{s}_{p,r}} + \left\|u^{(1)}\right\|^2_{B^{s}_{p,r}} + \left\|v^{(2)}\right\|^2_{B^{s}_{p,r}} + \left\|u^{(2)}\right\|^2_{B^{s}_{p,r}}\right) d\tau.
\end{aligned}
\tag{26}
$$

Hence, thanks to

$$
\left\|\partial_x\left(u^{(1)}v^{(1)}\right)\right\|_{B^{s-2}_{p,r}} \le C\left(\left\|u^{(1)}\right\|^2_{B^{s}_{p,r}} + \left\|v^{(1)}\right\|^2_{B^{s}_{p,r}}\right), \tag{27}
$$

and then applying the Gronwall's inequality, we reach (20). □

Now let us start the proof of Theorem 1, which is motivated by the proof of local existence theorem about the Camassa-Holm equation in [26]. Firstly, we will use the classical Friedrichs' regularization method to construct the approximate solutions to the Cauchy problem (14).

Lemma 12. *Assume that $u^{(0)} = v^{(0)} = 0$. Let $1 \le p, r \le +\infty$, $s > \max\{5/2, 2 + (1/p)\}$ and $u_0, v_0 \in B^s_{p,r}$. Then there exists a sequence of smooth functions $(u^{(l)}, v^{(l)})_{l\in\mathbb{N}} \in C(R^+; B^\infty_{p,r})^2$ solving the following linear transport equation by induction:*

$$
\begin{aligned}
&\left(\partial_t + \left(u^{(l)}v^{(l)}\right)\partial_x\right) m^{(l+1)}\\
&\qquad = -3v^{(l)}u_x^{(l)}m^{(l)}, \quad t > 0,\ x \in \mathbb{R}\\
&\left(\partial_t + \left(u^{(l)}v^{(l)}\right)\partial_x\right) n^{(l+1)}\\
&\qquad = -3u^{(l)}v_x^{(l)}n^{(l)}, \quad t > 0,\ x \in \mathbb{R}\\
&u^{(l+1)}(x,0) = u_0^{(l+1)}(x) = S_{l+1}u_0, \quad x \in \mathbb{R},\\
&v^{(l+1)}(x,0) = v_0^{(l+1)}(x) = S_{l+1}v_0, \quad x \in \mathbb{R}.
\end{aligned}
\tag{28}
$$

Moreover, there is a positive T such that the solutions satisfying the following properties:

(i) *$(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)\times E^s_{p,r}(T)$,*

(ii) *$(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^{s-1}_{p,r})$.*

Proof. Since all the data $S_{n+1}u_0$ and $S_{n+1}v_0$ belong to $B^\infty_{p,r}$, Lemma 8 enables us to show by induction that for all $l \in \mathbb{N}$, (28) has a global solution which belongs to $C(R^+; B^\infty_{p,r})^2$. Thanks to Lemma 7 and the proof of Proposition 11, we have the following inequality for all $l \in \mathbb{N}$:

$$
\begin{aligned}
&e^{-C\int_0^t \|\partial_x (u^{(l)}v^{(l)})(\tau')\|_{B^{s-1}_{p,r}} d\tau'}\left\|m^{(l+1)}(t)\right\|_{B^{s-2}_{p,r}}\\
&\quad \le \|m_0\|_{B^{s-2}_{p,r}} + C\int_0^t e^{-C\int_0^\tau \|\partial_x (u^{(1)}v^{(1)})(\tau')\|_{B^{s-1}_{p,r}} d\tau'}\\
&\qquad \times\left\|3v^{(l)}u_x^{(l)}m^{(l)}\right\|_{B^{s-2}_{p,r}} d\tau,
\end{aligned}
$$

$$
\begin{aligned}
& e^{-C\int_0^t \|\partial_x(u^{(l)}v^{(l)})(\tau')\|_{B^{s-1}_{p,r}}d\tau'} \left\|n^{(l+1)}(t)\right\|_{B^{s-2}_{p,r}} \\
&\quad \le \left\|m_0\right\|_{B^{s-2}_{p,r}} + C\int_0^t e^{-C\int_0^\tau \|\partial_x(u^{(1)}v^{(1)})(\tau')\|_{B^{s-1}_{p,r}}d\tau'} \\
&\qquad \times \left\|3u^{(l)}v_x^{(l)}n^{(l)}\right\|_{B^{s-2}_{p,r}} d\tau.
\end{aligned} \tag{29}
$$

Thanks to $s > \max\{5/2, 2 + (1/p)\}$, we find $B^{s-2}_{p,r}$ is an algebra. From this, one obtains

$$
\begin{aligned}
\left\|v^{(l)}u_x^{(l)}m^{(l)}\right\|_{B^{s-2}_{p,r}} &\le C\left\|v^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|m^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|u_x^{(l)}\right\|_{B^{s-2}_{p,r}} \\
&\le C\left(\left\|u^{(l)}\right\|_{B^s_{p,r}} + \left\|v^{(l)}\right\|_{B^s_{p,r}}\right)^3, \\
\left\|u^{(l)}v_x^{(l)}n^{(l)}\right\|_{B^{s-2}_{p,r}} &\le C\left\|u^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|n^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|v_x^{(l)}\right\|_{B^{s-2}_{p,r}} \\
&\le \left(\left\|v^{(l)}\right\|_{B^s_{p,r}} + \left\|u^{(l)}\right\|_{B^s_{p,r}}\right)^3,
\end{aligned} \tag{30}
$$

which along with the above inequality leads to

$$
\begin{aligned}
& e^{-C\int_0^t \|\partial_x(u^{(l)}v^{(l)})(\tau')\|_{B^{s-1}_{p,r}}d\tau'} \\
&\quad \times \left(\left\|u^{(l+1)}(t)\right\|_{B^s_{p,r}} + \left\|v^{(l+1)}(t)\right\|_{B^s_{p,r}}\right) \\
&\quad \le \left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}} \\
&\qquad + C\int_0^t e^{-C\int_0^\tau \|\partial_x(u^{(1)}v^{(1)})(\tau')\|_{B^{s-1}_{p,r}}d\tau'} \\
&\qquad \times \left(\left\|u^{(l)}\right\|_{B^s_{p,r}} + \left\|v^{(l)}\right\|_{B^s_{p,r}}\right)^3 d\tau.
\end{aligned} \tag{31}
$$

Let us choose a $T > 0$ such that $4C(\| u_0\|_{B^s_{p,r}} + \| v_0\|_{B^s_{p,r}})^2 T < 1$, and suppose by induction that for all $t \in [0, T]$

$$
\begin{aligned}
& \left\|u^{(l)}(t)\right\|_{B^s_{p,r}} + \left\|v^{(l)}(t)\right\|_{B^s_{p,r}} \\
&\quad \le \frac{\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}}{\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right)^{1/2}}.
\end{aligned} \tag{32}
$$

Indeed, since $B^{s-1}_{p,r}$ is an algebra, one obtains from (32) that for any $0 < \tau < t$

$$
\begin{aligned}
& C\int_\tau^t \left\|\partial_x\left(u^{(l)}v^{(l)}\right)(\tau')\right\|_{B^{s-1}_{p,r}} d\tau' \\
&\quad \le C\int_\tau^t \left(\left\|u^{(l)}(t)\right\|_{B^s_{p,r}} + \left\|v^{(l)}(t)\right\|_{B^s_{p,r}}\right)^2 d\tau \\
&\quad \le C\int_\tau^t \frac{\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2}{1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t} d\tau \\
&\quad = \frac{1}{4}\ln\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 \tau\right) \\
&\qquad - \frac{1}{4}\ln\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right).
\end{aligned} \tag{33}
$$

And then inserting (33) and (32) into (31) leads to

$$
\begin{aligned}
& \left\|u^{(l+1)}(t)\right\|_{B^s_{p,r}} + \left\|v^{(l+1)}(t)\right\|_{B^s_{p,r}} \\
&\quad \le \frac{\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}}{\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right)^{1/4}} \\
&\qquad + \frac{C}{\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right)^{1/4}} \\
&\qquad \times \int_0^t \left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 \tau\right)^{1/4} \\
&\qquad \times \frac{\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2}{\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 \tau\right)^{3/2}} d\tau \\
&\quad \le \frac{\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}}{\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right)^{1/4}} \\
&\qquad \times \left(1 + C\int_0^t \frac{\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2}{\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right)^{5/4}} d\tau\right) \\
&\quad = \frac{\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}}{\left(1 - 4C\left(\left\|u_0\right\|_{B^s_{p,r}} + \left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right)^{1/2}}.
\end{aligned} \tag{34}
$$

Hence, one can see that

$$\left\|u^{(l+1)}(t)\right\|_{B^s_{p,r}}+\left\|v^{(l+1)}(t)\right\|_{B^s_{p,r}} \leq \frac{\left\|u_0\right\|_{B^s_{p,r}}+\left\|v_0\right\|_{B^s_{p,r}}}{\left(1-4C\left(\left\|u_0\right\|_{B^s_{p,r}}+\left\|v_0\right\|_{B^s_{p,r}}\right)^2 t\right)^{1/2}}, \quad (35)$$

which implies that $(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ is uniformly bounded in $C([0;T];B^s_{p,r}) \times C([0;T];B^s_{p,r})$. Using the Moser-type estimates (see Lemma 9), one finds that

$$\begin{aligned}
\left\|u^{(l)}v^{(l)}\partial_x m^{(l+1)}\right\|_{B^{s-3}_{p,r}} &\leq C\left\|u^{(l+1)}\right\|_{B^s_{p,r}}\left(\left\|u^{(l)}\right\|^2_{B^s_{p,r}}+\left\|v^{(l)}\right\|^2_{B^s_{p,r}}\right),\\
\left\|v^{(l)}u^{(l)}\partial_x n^{(l+1)}\right\|_{B^{s-3}_{p,r}} &\leq C\left\|v^{(l+1)}\right\|_{B^s_{p,r}}\left(\left\|u^{(l)}\right\|^2_{B^s_{p,r}}+\left\|v^{(l)}\right\|^2_{B^s_{p,r}}\right),\\
\left\|v^{(l)}u^{(l)}_x m^{(l)}\right\|_{B^{s-2}_{p,r}} &\leq C\left\|v^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|m^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|u^{(l)}_x\right\|_{B^{s-2}_{p,r}}\\
&\leq C\left(\left\|u^{(l)}\right\|_{B^s_{p,r}}+\left\|v^{(l)}\right\|_{B^s_{p,r}}\right)^3,\\
\left\|u^{(l)}v^{(l)}_x n^{(l)}\right\|_{B^{s-2}_{p,r}} &\leq C\left\|u^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|n^{(l)}\right\|_{B^{s-2}_{p,r}}\left\|v^{(l)}_x\right\|_{B^{s-2}_{p,r}}\\
&\leq C\left(\left\|v^{(l)}\right\|_{B^s_{p,r}}+\left\|u^{(l)}\right\|_{B^s_{p,r}}\right)^3.
\end{aligned} \quad (36)$$

Hence, using (28), we have

$$\left(\partial_x u^{(l+1)}, \partial_x v^{(l+1)}\right)_{l\in\mathbb{N}} \in C\left([0;T];B^{s-1}_{p,r}\right)\times C\left([0;T];B^{s-1}_{p,r}\right) \quad (37)$$

uniformly bounded, which yields that the sequence $(u^{(l)}, v^{(l)})_{\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)\times E^s_{p,r}(T)$.

Now, it suffices to show that $(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ is a Cauchy sequence in $C([0;T];B^{s-1}_{p,r}) \times C([0;T];B^{s-1}_{p,r})$. In fact, for all $l,k \in \mathbb{N}$, from (28), we have

$$\begin{aligned}
\left(\partial_t+\left(u^{(l+k)}v^{(l+k)}\right)\partial_x\right)\left(m^{(l+k+1)}-m^{(l+1)}\right)&=F',\\
\left(\partial_t+\left(u^{(l+k)}v^{(l+k)}\right)\partial_x\right)\left(n^{(l+k+1)}-n^{(l+1)}\right)&=G',
\end{aligned} \quad (38)$$

with

$$\begin{aligned}
F' &= -\left(u^{(l+k)}\left(v^{(l+k)}-v^{(l)}\right)+\left(u^{(l+k)}-u^{(l)}\right)v^{(l)}\right)m^{(l+1)}_x\\
&\quad -3\left(u^{(k+l)}_x\left(v^{(k+l)}-v^{(l)}\right)m^{(k+l)}\right.\\
&\quad +u^{(k+l)}_x v^{(l)}\left(m^{(k+l)}-m^{(l)}\right)\\
&\quad \left.+\left(u^{(k+l)}_x-u^{(l)}_x\right)v^{(l)}m^{(l)}\right),\\
G' &= -\left(v^{(l+k)}\left(u^{(l+k)}-u^{(l)}\right)+\left(v^{(l+k)}-v^{(l)}\right)u^{(l)}\right)n^{(l+1)}_x\\
&\quad -3\left(v^{(k+l)}_x\left(u^{(k+l)}-u^{(l)}\right)n^{(k+l)}\right.\\
&\quad +v^{(k+l)}_x u^{(l)}\left(n^{(k+l)}-n^{(l)}\right)\\
&\quad \left.+\left(v^{(k+l)}_x-v^{(l)}_x\right)u^{(l)}n^{(l)}\right).
\end{aligned} \quad (39)$$

Similar to the proof of Proposition 11, then for every $t \in [0,T]$, we obtain

$$\begin{aligned}
&e^{-C\int_0^t\|\partial_x(u^{(k+l)}v^{(k+l)})(\tau')\|_{B^{s-2}_{p,r}}d\tau'}\\
&\quad\times\left(\left\|\left(u^{(k+l+1)}-u^{(l+1)}\right)(t)\right\|_{B^{s-1}_{p,r}}+\left\|\left(v^{(k+l+1)}-v^{l+1}\right)(t)\right\|_{B^{s-1}_{p,r}}\right)\\
&\leq \left\|u^{(k+l+1)}_0-u^{(l+1)}_0\right\|_{B^{s-1}_{p,r}}+\left\|u^{(k+l+1)}_0-u^{(l+1)}_0\right\|_{B^{s-1}_{p,r}}\\
&\quad + C\int_0^t e^{-C\int_0^\tau\|\partial_x(u^{(l)}v^{(l)})(\tau')\|_{B^{s-2}_{p,r}}d\tau'}\\
&\quad\times\left(\left\|v^{(k+l)}-v^{(l)}\right\|_{B^{s-1}_{p,r}}+\left\|v^{(k+l)}-v^{(l)}\right\|_{B^{s-1}_{p,r}}\right)\\
&\quad\times\left(\left\|v^{(l)}\right\|^2_{B^s_{p,r}}+\left\|u^{(l)}\right\|^2_{B^s_{p,r}}+\left\|v^{(l+k)}\right\|^2_{B^s_{p,r}}\right.\\
&\quad\left.+\left\|u^{(k+l)}\right\|^2_{B^s_{p,r}}+\left\|u^{(l+1)}\right\|^2_{B^s_{p,r}}+\left\|v^{(l+1)}\right\|^2_{B^s_{p,r}}\right)d\tau.
\end{aligned} \quad (40)$$

Since $(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)\times E^s_{p,r}(T)$ and

$$\begin{aligned}
u^{(l+k+1)}_0-u^{(l+1)}_0 &= S_{k+l+1}u_0-S_{l+1}u_0=\sum_{q=l+1}^{l+k}\Delta_q u_0,\\
v^{(l+k+1)}_0-v^{(l+1)}_0 &= S_{k+l+1}v_0-S_{l+1}v_0=\sum_{q=l+1}^{l+k}\Delta_q v_0,
\end{aligned} \quad (41)$$

we get a constant C_T independent of l,k such that for all $t \in [0,T]$,

$$\begin{aligned}
&\left\|\left(u^{(k+l+1)}-u^{(l+1)}\right)(t)\right\|_{B^{s-1}_{p,r}}+\left\|\left(v^{(k+l+1)}-v^{(l+1)}\right)(t)\right\|_{B^{s-1}_{p,r}}\\
&\leq C_T\left(2^{-n}+\int_0^t\left(\left\|\left(u^{(k+l)}-u^{(l)}\right)(\tau)\right\|_{B^{s-1}_{p,r}}\right.\right.\\
&\quad\left.\left.+\left\|\left(v^{(k+l)}-u^{(l)}\right)(\tau)\right\|_{B^{s-1}_{p,r}}\right)d\tau\right).
\end{aligned} \quad (42)$$

Arguing by induction with respect to the index l, one can easily prove that

$$\begin{aligned}
&\left\|\left(u^{(k+l+1)}-u^{(l+1)}\right)(t)\right\|_{L^\infty_T(B^{s-1}_{p,r})}+\left\|\left(v^{(k+l+1)}-v^{(l+1)}\right)(t)\right\|_{L^\infty_T(B^{s-1}_{p,r})}\\
&\leq \frac{(TC_T)^{l+1}}{(l+1)!}\left(\left\|u^k\right\|_{L^\infty_T(B^{s-1}_{p,r})}+\left\|v^k\right\|_{L^\infty_T(B^{s-1}_{p,r})}\right)\\
&\quad + C_T\sum_{q=0}^{l}2^{q-l}\frac{(TC_T)^q}{q!}.
\end{aligned} \quad (43)$$

As $\| u^{(k)}\|_{L^\infty_T(B^{s-1}_{p,r})}$, $\| v^{(k)}\|_{L^\infty_T(B^{s-1}_{p,r})}$, and C are bounded independently of k, there exists constant C'_T independent of l, k such that

$$\begin{aligned}&\left\|\left(u^{(k+l+1)}-u^{(l+1)}\right)(t)\right\|_{L^\infty_T(B^{s-1}_{p,r})}\\&\quad+\left\|\left(v^{(k+l+1)}-v^{(l+1)}\right)(t)\right\|_{L^\infty_T(B^{s-1}_{p,r})}\leq C'_T 2^{-n}.\end{aligned}\tag{44}$$

Thus, $(u^{(l)}, v^{(l)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r}) \times ([0,T]; B^{s-1}_{p,r})$. □

Proof of Theorem 1. Thanks to Lemma 12, we obtain that $(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^{s-1}_{p,r})$; so, it converges to some function $(u,v) \in C([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^{s-1}_{p,r})$. We now have to check that (u,v) belongs to $E^s_{p,r}(T) \times E^s_{p,r}(T)$ and solves the Cauchy problem (1). Since $(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T]; B^s_{p,r}) \times L^\infty([0,T]; B^s_{p,r})$, according to Lemma 12, the Fatou property for the Besov spaces (Proposition 6) guarantees that (u,v) also belongs to

$$L^\infty\left([0,T]; B^s_{p,r}\right)\times L^\infty\left([0,T]; B^s_{p,r}\right).\tag{45}$$

On the other hand, as $(u^{(l)}, v^{(l)})_{l\in\mathbb{N}}$ converges to (u,v) in $C([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^{s-1}_{p,r})$, an interpolation argument ensures that the convergence holds in $C([0,T]; B^{s'}_{p,r}) \times C([0,T]; B^{s'}_{p,r})$, for any $s' < s$. It is then easy to pass to the limit in (28) and to conclude that (u,v) is indeed a solution to the Cauchy problem (1). Thanks to the fact that u belongs to $L^\infty([0,T]; B^s_{p,r}) \times L^\infty([0,T]; B^s_{p,r})$, the right-hand side of the equation

$$\partial_t m + uv\partial_x m = -3vu_x m\tag{46}$$

belongs to $L^\infty([0,T]; B^s_{p,r})$, and the right-hand side of the equation

$$\partial_t m + uv\partial_x n = -3uv_x n\tag{47}$$

belongs to $L^\infty([0,T]; B^s_{p,r})$. In particular, for the case $r < \infty$, Lemma 8 enables us to conclude that $(u,v) \in C([0,T]; B^{s'}_{p,r}) \times C([0,T]; B^{s'}_{p,r})$ for any $s' < s$. Finally, using the equation again, we see that $(\partial_t u, \partial_t v) \in C([0,T]; B^{s'}_{p,r}) \times C([0,T]; B^{s'}_{p,r})$ if $r < \infty$, and $L^\infty([0,T]; B^{s-1}_{p,r}) \times L^\infty([0,T]; B^{s-1}_{p,r})$ otherwise. Therefore, (u,v) belongs to $E^s_{p,r}(T) \times E^s_{p,r}(T)$. Moreover, a standard use of a sequence of viscosity approximate solutions $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ for the Cauchy problem (1) which converges uniformly in $C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}) \times C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r})$ gives the continuity of the solution (u,v) in $E^s_{p,r} \times E^s_{p,r}$. The proof of Theorem 1 is complete. □

3. Analyticity of Solutions

In this section, we will show the existence and uniqueness of analytic solutions to the system (1) on the line $\mathbb{R}$.

First, we will need a suitable scale of Banach spaces as follows. For any $s > 0$, we set

$$E_s = \left\{u \in C^\infty(\mathbb{R}) : |||u|||_s = \sup_{k\in\mathbb{N}_0}\frac{s^k\left\|\partial^k u\right\|_{H^2}}{k!/(k+1)^2} < \infty\right\},\tag{48}$$

where $H^2(\mathbb{R})$ is the Sobolev space of order two on the real line $\mathbb{R}$ and $\mathbb{N}_0$ is the set of nonnegative integers. One can easily verify that E_s equipped with the norm $|||\cdot|||_s$ is a Banach space and that, for any $0 < s' < s$, E_s is continuously embedded in $E_{s'}$ with

$$|||u|||_{s'} \leq |||u|||_s.\tag{49}$$

Another simple consequence of the definition is that any u in E_s is a real analytic function on $\mathbb{R}$. Crucial for our purposes is the fact that each E_s forms an algebra under pointwise multiplication of functions.

Lemma 13 (see [29]). *Let $0 < s < 1$. There is a constant $C > 0$, independent of s, such that for any u and v in E_s, one has*

$$|||uv|||_s \leq C|||u|||_s|||v|||_s.\tag{50}$$

Lemma 14 (see [29]). *There is a constant $c > 0$ such that for any $0 < s' < s < 1$, one has $|||\partial_x u|||_{s'} \leq (C/(s-s'))|||u|||_s$, $|||(1-\partial_x^2)^{-1}u|||_{s'} \leq |||u|||_s$, and $|||(1-\partial_x^2)^{-1}\partial_x u|||_{s'} \leq |||u|||_s$.*

Theorem 15 (see [30]). *Let $\{X_s\}_{0<s<1}$ be a scale of decreasing Banach spaces; namely, for any $s' < s$, one has $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \leq |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{aligned}\frac{du}{dt} &= F(t, u(t)),\\ u(0) &= 0.\end{aligned}\tag{51}$$

Let T, H, and C be positive constants, and assume that F satisfies the following conditions.

(1) *If for $0 < s' < s < 1$ the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \leq T$ with values in X_s and*

$$\sup_{|t|\leq T}|||u(t)|||_s < H,\tag{52}$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{s'}$.

(2) *For any $0 < s' < s < 1$ and any $u, v \in B(0,H) \subset X_s$, that is, $|||u|||_s < H$, $|||v|||_s < H$, one has*

$$\sup_{|t|\leq T}|||F(t,u) - F(t,v)|||_{s'} \leq \frac{C}{s-s'}|||u-v|||_s.\tag{53}$$

(3) *There exists $M > 0$ such that for any $0 < s < 1$,*

$$\sup_{|t|\leq T}|||F(t,0)|||_s \leq \frac{M}{1-s}.\tag{54}$$

Then, there exists a $T_0 \in (0, T)$ and a unique function $u(t)$, which for every $s \in (0,1)$ is holomorphic in $|t| < (1-s)T_0$ with values in X_s and is a solution to the Cauchy problem (51).

Next, we restate the Cauchy problem (1) in a more convenient form, and we can rewrite the Cauchy problem (1) as follows:

$$\begin{gathered} u_t - u_{xxt} + 4uvu_x - 3u_x u_{xx} v - uu_{xxx} v = 0, \\ v_t - v_{xxt} + 4vuv_x - 3v_x v_{xx} u - vv_{xxx} u = 0, \\ u(0,x) = u_0(x), \qquad v(0,t) = v_0(x). \end{gathered} \tag{55}$$

Applying the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation and second equation in (55), we obtain

$$\begin{aligned} & u_t + uvu_x + \left(1-\partial_x^2\right)^{-1} \\ & \quad \times \left((u_x v + uv_x)_x u_x - u_x^2 v - uv_x u_x \right. \\ & \qquad \left. - 2v(u_x)_x^2 - uv_x u_{xx} + 3u_x vu \right) = 0, \\ & v_t + vuv_x + \left(1-\partial_x^2\right)^{-1} \\ & \quad \times \left((v_x u + vu_x)_x v_x - v_x^2 u - vu_x v_x \right. \\ & \qquad \left. - 2u(v_x)_x^2 - vu_x v_{xx} + 3v_x uv \right) = 0, \\ & u(0,x) = u_0(x), \qquad v(0,t) = v_0(x). \end{aligned} \tag{56}$$

Differentiating with respect to x on both sides of the previous equation and letting $u_1 = u$, $u_2 = u_x$, $u_3 = v$, and $u_4 = v_x$, then the problem (56) can be written as a system for u_1, u_2, u_2, u_4 as follows:

$$\begin{aligned} \partial_t u_1 &= -u_1 u_3 u_2 - \left(1-\partial_x^2\right)^{-1} \\ & \quad \times \left(u_2 \partial_x (u_2 u_3 + u_1 u_4) - u_2^2 u_3 - u_1 u_4 u_2 \right. \\ & \qquad \left. - 2u_3 \partial_x u_2^2 - u_1 u_4 \partial_x u_2 + 3u_2 u_3 u_1 \right) \\ &= F_1(u_1, u_2, u_3, u_4), \\ \partial_t u_2 &= -(u_2 u_3 u_2 + u_1 \partial_x (u_3 u_2)) - \partial_x \left(1-\partial_x^2\right)^{-1} \\ & \quad \times \left((u_2 u_3 + u_1 u_4)_x u_2 - u_2^2 u_3 - u_1 u_4 u_2 \right. \\ & \qquad \left. - 2u_3 \partial_x u_2^2 - u_1 u_4 \partial_x u_2 + 3u_2 u_3 u_1 \right) \\ &= F_2(u_1, u_2, u_3, u_4), \\ \partial_t u_3 &= -u_3 u_1 u_4 - \left(1-\partial_x^2\right)^{-1} \\ & \quad \times \left(u_4 \partial_x (u_4 u_1 + u_3 u_2) - u_4^2 u_1 - u_3 u_2 u_4 \right. \\ & \qquad \left. - 2u_1 \partial_x u_4^2 - u_3 u_2 \partial_x u_4 + 3u_4 u_1 u_3 \right) \\ &= F_3(u_1, u_2, u_3, u_4), \\ \partial_t u_4 &= -(u_4 u_1 u_4 + u_3 \partial_x (u_1 u_4)) - \left(1-\partial_x^2\right)^{-1} \\ & \quad \times \left(u_4 \partial_x (u_4 u_1 + u_3 u_2) - u_4^2 u_1 - u_3 u_2 u_4 \right. \\ & \qquad \left. - 2u_1 \partial_x u_4^2 - u_3 u_2 \partial_x u_4 + 3u_4 u_1 u_3 \right) \\ &= F_4(u_1, u_2, u_3, u_4), \\ & u_1(0,x) = u_0(x), \\ & u_2(0,x) = u_0'(x), \\ & u_3(0,x) = v_0(x), \\ & u_4(0,x) = v_0'(x). \end{aligned} \tag{57}$$

Define

$$\begin{aligned} U &\equiv (u_1, u_2, u_3, u_4), \\ F(U) &= F(u_1, u_2, u_3, u_4) \\ &\equiv (F_1(u_1, u_2, u_3, u_4), F_2(u_1, u_2, u_3, u_4), \\ & \qquad F_3(u_1, u_2, u_3, u_4), F_4(u_1, u_2, u_3, u_4)). \end{aligned} \tag{58}$$

Then, we have

$$\begin{gathered} \frac{dU}{dt} = F(t, U(t)), \\ U(0) = \left(u_0, u_0', v_0, v_0'\right). \end{gathered} \tag{59}$$

Proof of Theorem 2. Theorem 2 is a straightforward consequence of the abstract Cauchy-Kowalevski Theorem 15. We only need to verify conditions (1)-(2) in the statement of the abstract Cauchy-Kowalevski Theorem 15 for $F_i(u_1, u_2, u_3, u_4)$ $(i = 1,2,3,4)$ in the system (57) since $F_i(u_1, u_2, u_3, u_4)$, $(i = 1,2,3,4)$ do not depend on t explicitly. We observe that, for $0 < s' < s < 1$, by the estimates in Lemmas 13 and 14, condition (1) holds

Next, we verify the second condition. For any u_j and $v_j \in B(0,H) \subset E_s (j = 1,2,3,4)$, we have

$$\begin{aligned} & |||F(u_1, u_2, u_3, u_4) - F(v_1, v_2, v_3, v_4)|||_{s'} \\ & \quad = \sum_{i=1}^{4} |||F_i(u_1, u_2, u_3, u_4) - F_i(v_1, v_2, v_3, v_4)|||_{s'} \\ & \quad \equiv I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{60}$$

We will estimate I_1, I_2, I_3, and I_4, respectively, where

$$\begin{aligned}
I_1 \le{}& |||u_1u_3u_2 - v_1v_3v_2|||_{s'} \\
&+ \left|\left|\left|\left(1-\partial_x^2\right)^{-1}\left(u_2\partial_x\left(u_2u_3+u_1u_4\right)\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left.-v_2\partial_x\left(v_2v_3+v_1v_4\right)\right)\right|\right|\right|_{s'} \\
&+ \left|\left|\left|\left(1-\partial_x^2\right)^{-1}\left(u_1u_4u_2 - v_1v_4v_2\right)\right|\right|\right|_{s'} \\
&+ \left|\left|\left|\left(1-\partial_x^2\right)^{-1}\left(2u_3\partial_x u_2^2 - 2v_3\partial_x v_2^2\right)\right|\right|\right|_{s'} \\
&+ \left|\left|\left|\left(1-\partial_x^2\right)^{-1}\left(u_1u_4\partial_x u_2 - v_1v_4\partial_x v_2\right)\right|\right|\right|_{s'} \\
&+ \left|\left|\left|\left(1-\partial_x^2\right)^{-1}\left(3u_2u_3u_1 - 3v_2v_3v_1\right)\right|\right|\right|_{s'} \\
\le{}& |||u_1u_3u_2 - v_1v_3v_2|||_{s'} \\
&+ |||u_2\partial_x\left(u_2u_3+u_1u_4\right) - v_2\partial_x\left(v_2v_3+v_1v_4\right)|||_{s'} \\
&+ |||u_1u_4u_2 - v_1v_4v_2|||_{s'} \\
&+ |||2u_3\partial_x u_2^2 - 2v_3\partial_x v_2^2|||_{s'} \\
&+ |||u_1u_4\partial_x u_2 - v_1v_4\partial_x v_2|||_{s'} \\
&+ |||3u_2u_3u_1 - 3v_2v_3v_1|||_{s'} \\
\le{}& C\left(|||u_1|||_s|||u_2|||_s|||u_3-v_3|||_s\right. \\
&\quad + |||u_1|||_s|||v_3|||_s|||u_2-v_2|||_s \\
&\quad + |||v_2|||_s|||v_3|||_s|||u_1-v_1|||_s \\
&\quad + |||u_1|||_s|||u_2|||_s|||u_4-v_4|||_s \\
&\quad + |||u_1|||_s|||v_4|||_s|||u_2-v_2|||_s \\
&\quad \left.+|||v_2|||_s|||v_4|||_s|||u_1-v_1|||_s\right) \\
&+ \frac{C}{s-s'}\left(|||u_1|||_s|||u_2|||_s|||u_4-v_4|||_s\right. \\
&\qquad + |||u_1|||_s|||v_4|||_s|||u_2-v_2|||_s \\
&\qquad + |||v_2|||_s|||v_4|||_s|||u_1-v_1|||_s \\
&\qquad + |||u_3|||_s|||u_2+v_2|||_s|||u_2-v_2|||_s \\
&\qquad \left.+|||u_2|||_s^2|||u_3-v_3|||_s\right) \\
\le{}& \frac{C}{s-s'}|||\left(u_1,u_2,u_3,u_4\right) - \left(u_1,u_2,u_3,u_4\right)|||_s.
\end{aligned} \tag{61}$$

In a similar way to what we just did, we can show that the following estimates hold:

$$\begin{aligned}
I_2 &\le \frac{C}{s-s'}|||\left(u_1,u_2,u_3,u_4\right) - \left(u_1,u_2,u_3,u_4\right)|||_s, \\
I_1 &\le \frac{C}{s-s'}|||\left(u_1,u_2,u_3,u_4\right) - \left(u_1,u_2,u_3,u_4\right)|||_s, \\
I_4 &\le \frac{C}{s-s'}|||\left(u_1,u_2,u_3,u_4\right) - \left(u_1,u_2,u_3,u_4\right)|||_s,
\end{aligned} \tag{62}$$

where the constant C depends only on H. This implies that condition (2) also holds. Conditions (1) through (3) are now easily verified once our system (57) is transformed into a new system with zero initial data as in (59). The proof of Theorem 2 is complete. □

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