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Research Article

Bifurcation Phenomena of Nonlinear Waves in a Generalized Zakharov-Kuznetsov Equation

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We study the bifurcation phenomena of nonlinear waves described by a generalized Zakharov-Kuznetsov equation $u_t + (au^2 + bu^4)u_x + \gamma u_{xxx} + \delta u_{xyy} = 0$. We reveal four kinds of interesting bifurcation phenomena. The first kind is that the low-kink waves can be bifurcated from the symmetric solitary waves, the 1-blow-up waves, the tall-kink waves, and the antisymmetric solitary waves. The second kind is that the 1-blow-up waves can be bifurcated from the periodic-blow-up waves, the symmetric solitary waves, and the 2-blow-up waves. The third kind is that the periodic-blow-up waves can be bifurcated from the symmetric periodic waves. The fourth kind is that the tall-kink waves can be bifurcated from the symmetric periodic waves.

1. Introduction and Preliminary

Zakharov-Kuznetsov (Z-K) equation [1],

$$u_t + auu_x + \left(\nabla^2 u\right)_x = 0, \tag{1}$$

was first derived for describing weakly nonlinear ion-acoustic wave in a strongly magnetized lossless plasma in two dimensions. The Z-K equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [2, 3].

There are lots of research for various generalized Z-K equations [4–13]. For the Z-K equation

$$u_t + \left(au + bu^2\right)u_x + \gamma u_{xxx} + \delta u_{xyy} = 0, \tag{2}$$

Yan and Liu [4] gave some polynomial solutions, triangular function solutions and elliptic periodic solutions, of (2) via a direct symmetry method.

When $a = 0$, $b = 1$, and $\gamma = \delta = 1$, equation (2) reduces to

$$u_t + u^2u_x + u_{xxx} + u_{xyy} = 0; \tag{3}$$

Bekir [5] used the (G'/G)-expansion method to obtain three types of traveling wave solutions of (3).

For the generalized Zakharov-Kuznetsov equation

$$u_t + \left(au^2 + bu^4\right)u_x + \gamma u_{xxx} + \delta u_{xyy} = 0, \tag{4}$$

where a, b, γ, and δ are real constants, Song and Cai [6] got some solitary wave and kink wave solutions of (4).

When $\gamma = \delta$, Zhang [7] used the new generalized algebraic method to obtain some soliton solutions, combined soliton solutions, triangular periodic solutions, Jacobi elliptic function solutions, combined Jacobi elliptic function solutions, and rational function solutions of (4). Biswas and Zerrad [8] obtained 1-soliton solution of (4) with dual-power law nonlinearity.

When $\delta = 0$, Liu and Yan [9] obtained some common expressions and two kinds of bifurcation phenomena for nonlinear waves of (4). Meanwhile, they pointed out that there are two sets of kink waves which are called tall-kink waves and low-kink waves, respectively.

In order to investigate the bifurcation phenomena of (4), letting $c > 0$ be wave speed and substituting $u = \varphi(\xi)$ with $\xi = x + y - ct$ into (4), it follows that

$$-c\varphi' + a\varphi^2\varphi' + b\varphi^4\varphi' + \gamma\varphi''' + \delta\varphi''' = 0. \tag{5}$$

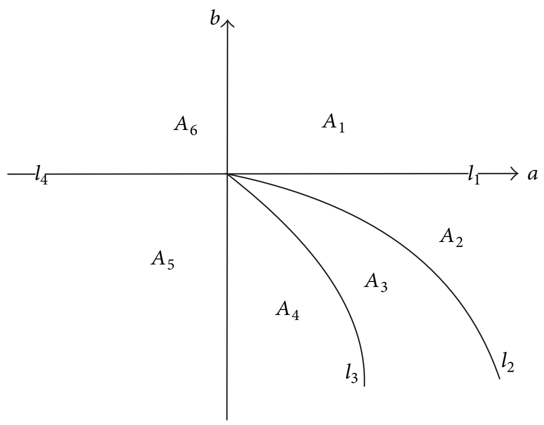

FIGURE 1: The locations of the regions A_i $(i = 1, 2, \ldots, 6)$ and curves l_i $(i = 1, 2, 3, 4)$.

Integrating (5), we get

$$-c\varphi + \frac{a}{3}\varphi^2 + \frac{b}{5}\varphi^5 + \gamma\varphi'' + \delta\varphi'' = 0. \tag{6}$$

Setting $\varphi' = \psi$; yields the following planar system:

$$\varphi' = \psi, \qquad \psi' = \frac{c\varphi - (a/3)\varphi^3 - (b/5)\varphi^5}{\gamma + \delta}. \tag{7}$$

Obviously, system (7) is a Hamiltonian system with Hamiltonian function

$$H(\varphi, \psi) = (\gamma + \delta)\psi^2 - c\varphi^2 + \frac{a}{6}\varphi^4 + \frac{b}{15}\varphi^6 = h, \tag{8}$$

where h is the integral constant.

Let

$$\varphi_1 = \sqrt{-\frac{1}{6b}(5a + \Delta)}, \tag{9}$$

$$\varphi_2 = \sqrt{-\frac{1}{6b}(5a - \Delta)}, \tag{10}$$

$$\Delta = \sqrt{25a^2 + 180bc}. \tag{11}$$

On $a - b$ parametric plane, let l_i $(i = 1, 2, 3, 4)$ represent the following four curves:

$$\begin{gathered} l_1 : b = 0\,(a > 0), \\ l_2 : b = -\frac{16a^2}{75c}, \\ l_3 : b = -\frac{2a^2}{9c}, \\ l_4 : b = 0\,(a < 0). \end{gathered} \tag{12}$$

Let A_i $(i = 1, 2, \ldots, 6)$ represent the regions surrounded by l_i $(i = 1, 2, 3, 4)$ and the coordinate axes (see Figure 1).

In this paper, we employ bifurcation method of dynamical systems [14–23] to investigate the bifurcation phenomena of nonlinear waves described by (4).

We obtain three types of explicit expressions of nonlinear wave solutions. Under different parameters conditions, these expressions represent symmetric and antisymmetric solitary waves, kink and anti-kink waves, symmetric periodic and periodic-blow-up waves, and 1-blow-up and 2-blow-up waves. Furthermore, we reveal four kinds of interesting bifurcation phenomena which are introduced in the abstract above.

This paper is organized as follows. The four kinds of interesting bifurcation phenomena are shown in Sections 2–5. A brief conclusion is given in Section 6.

2. Bifurcation of the Low-Kink Waves

In this section, we show that the low-kink waves can be bifurcated from the symmetric solitary waves, the 1-blow-up waves, the tall-kink waves, and the antisymmetric solitary waves.

2.1. Bifurcation from Symmetric Solitary Waves and 1-Blow-Up Waves

Proposition 1. *For $ab \neq 0$, $\gamma + \delta > 0$, and $H(\varphi, \psi) = H(0, 0)$, (4) has four nonlinear wave solutions as follows:*

$$\begin{gathered} u_a^{\pm} = \pm\sqrt{\frac{4\alpha_1\lambda}{\lambda^2 e^{-\tau_1\xi} - 2\lambda\beta_1 + (\beta_1^2 - 4\alpha_1)e^{\tau_1\xi}}}, \\ u_b^{\pm} = \pm\sqrt{\frac{4\alpha_1\lambda}{\lambda^2 e^{\tau_1\xi} - 2\lambda\beta_1 + (\beta_1^2 - 4\alpha_1)e^{-\tau_1\xi}}}, \end{gathered} \tag{13}$$

where

$$\alpha_1 = -\frac{15c}{b}, \tag{14}$$

$$\beta_1 = \frac{5a}{2b}, \tag{15}$$

$$\tau_1 = 2\sqrt{\frac{c}{\gamma + \delta}}, \tag{16}$$

and $\lambda \neq 0$ is an arbitrary real constant. For $\lambda > 0$, one has the following results and bifurcation phenomena.

(1) *If $\lambda \neq \sqrt{\beta_1^2 - 4\alpha_1}$ and $(a, b) \in A_2$, then $u_a^{\pm} \neq u_b^{\pm}$, and they represent four symmetric solitary waves (see Figures 2(a)–2(c)). In particular, when $b \to -(5a^2/48c) + 0$, the four symmetric solitary waves become four low-kink waves (see Figure 2(d))*

$$u_{a0}^{\pm} = \pm\sqrt{\frac{12c}{(-b\lambda/5)\,e^{-2\sqrt{(c/(\gamma+\delta))}\xi} + a}}, \tag{17}$$

$$u_{b0}^{\pm} = \pm\sqrt{\frac{12c}{-(b\lambda/5)\,e^{2\sqrt{(c/(\gamma+\delta))}\xi} + a}}, \tag{18}$$

which were given by Song and Cai [6]. This implies that one extends the previous results. For the varying process, see Figure 2.

(2) *If $\lambda \neq \sqrt{\beta_1^2 - 4\alpha_1}$ and (a, b) belongs to any one of the regions A_3, l_3, A_4, and A_5, then $u_a^\pm \neq u_b^\pm$ and they represent four 1-blow-up waves (see Figures 3(a)–3(c)). In particular, when $(a, b) \in A_3$ and $b \to -(5a^2/48c) - 0$, the four 1-blow-up waves become four low-kink waves with the expressions $u_{a0}^\pm$ and $u_{b0}^\pm$. For the varying process, see Figure 3.*

(3) *If $(a, b) \in A_2$ and $\lambda = \sqrt{\beta_1^2 - 4\alpha_1}$, then $u_a^\pm = u_b^\pm$ equal to the hyperbolic solitary wave solutions*

$$u_{ab}^\pm = \pm\sqrt{-\frac{12c}{\left(\sqrt{25a^2 + 240bc}/5\right)\cosh\left[2\sqrt{c/(\gamma+\delta)}\xi\right] - a}}, \tag{19}$$

which were given by Song and Cai [6]. This implies that one extends the previous results. When $b \to -(5a^2/48c) + 0$, $u_{ab}^\pm$ tend to two trivial solutions $u = \pm\sqrt{12c/a}$.

Proof. In (8), letting $h = H(0, 0)$, it follows that

$$\psi = \pm\sqrt{\frac{1}{\gamma+\delta}\left(c\varphi^2 - \frac{a}{6}\varphi^4 - \frac{b}{15}\varphi^6\right)}. \tag{20}$$

Substituting (20) into $\mathrm{d}\varphi/\mathrm{d}\xi = \psi$ and integrating it, we have

$$\int_\nu^\varphi \frac{\mathrm{d}s}{\sqrt{\left(s^2/(\gamma+\delta)\right)\left(c - (a/6)s^2 - (b/15)s^4\right)}} = \xi, \tag{21}$$

where ν is an arbitrary constant.

Completing the integral above and solving the equation for φ, it follows that

$$\varphi = \pm\sqrt{\frac{4\alpha_1\lambda e^{\tau_1\xi}}{\lambda^2 e^{2\tau_1\xi} - 2\lambda\beta_1 e^{\tau_1\xi} + \left(\beta_1^2 - 4\alpha_1\right)}}, \tag{22}$$

where $\lambda = \lambda(\nu)$ is an arbitrary real number.

Note that if $u = \varphi(\xi)$ is a solution of (4), so is $u = \varphi(-\xi)$. Therefore, from (22) we obtain the solutions $u_a^\pm$ and $u_b^\pm$ as (13).

In (13) letting $b \to -5a^2/48c$, then $\beta_1^2 - 4\alpha_1 \to 0$, and we get (17) and (18). From (13), (17) and (18), we get results (1) and (2) of Proposition 1.

When $\lambda = \sqrt{\beta_1^2 - 4\alpha_1}$, via (13) it follows that

$$\begin{aligned} u_a^\pm &= u_b^\pm \\ &= \pm\sqrt{\frac{4\alpha_1}{\sqrt{\beta_1^2 - 4\alpha_1}\left(e^{-\tau_1\xi} + e^{\tau_1\xi}\right) - 2\beta_1}} \\ &= \pm\sqrt{\frac{2\alpha_1}{\sqrt{\beta_1^2 - 4\alpha_1}\cosh\left(\tau_1\xi\right) - \beta_1}} \\ &= u_{ab}^\pm \quad (\text{see } (19)), \end{aligned} \tag{23}$$

which is result (3) of Proposition 1. □

2.2. Bifurcation from Tall-Kink Waves and Antisymmetric Solitary Waves.

Proposition 2. *If $\gamma + \delta > 0$, $H(\varphi, \psi) = H(\varphi_1, 0)$, and (a, b) belongs to one of the regions A_2, A_3, l_2, and l_3, then (4) has four real nonlinear wave solutions as follows:*

$$\begin{aligned} u_c^\pm &= \pm\frac{\sqrt{-(5a+\Delta)/6b}\left(2\Delta - 5a + 6b\eta e^{\tau_2\xi}\right)}{\sqrt{(5a - 2\Delta)^2 + 36b^2\eta^2 e^{2\tau_2\xi} - 12b(5a + 4\Delta)\eta e^{\tau_2\xi}}}, \\ u_d^\pm &= \pm\frac{\sqrt{-(5a+\Delta)/6b}\left(2\Delta - 5a + 6b\eta e^{-\tau_2\xi}\right)}{\sqrt{(5a - 2\Delta)^2 + 36b^2\eta^2 e^{-2\tau_2\xi} - 12b(5a + 4\Delta)\eta e^{-\tau_2\xi}}}, \end{aligned} \tag{24}$$

where $\eta \neq 0$ is an arbitrary real constant, Δ is given in (11), and

$$\tau_2 = \sqrt{-\frac{\Delta(\Delta + 5a)}{45b(\gamma+\delta)}}. \tag{25}$$

Letting

$$\mu_0 = \frac{5a - 2\Delta}{6b}, \tag{26}$$

corresponding to $\eta > 0$, one has the following results and bifurcation phenomena.

(1) *If $(a, b) \in A_2$ and $\eta \neq |\mu_0|$, then $u_c^\pm \neq u_d^\pm$, and they represent four tall-kink waves (see Figures 4(a)–4(c)). When $b \to -(5a^2/48c) + 0$, the four tall-kink waves become*

$$u_{c0}^\mp = \mp\sqrt{\frac{12c\eta}{a\eta + 48ce^{-2\sqrt{(c/(\gamma+\delta))}\xi}}}, \tag{27}$$

$$u_{d0}^\mp = \mp\sqrt{\frac{12c\eta}{a\eta + 48ce^{2\sqrt{(c/(\gamma+\delta))}\xi}}}, \tag{28}$$

which represent four low-kink waves (see Figure 4(d)). For the varying process, see Figure 4.

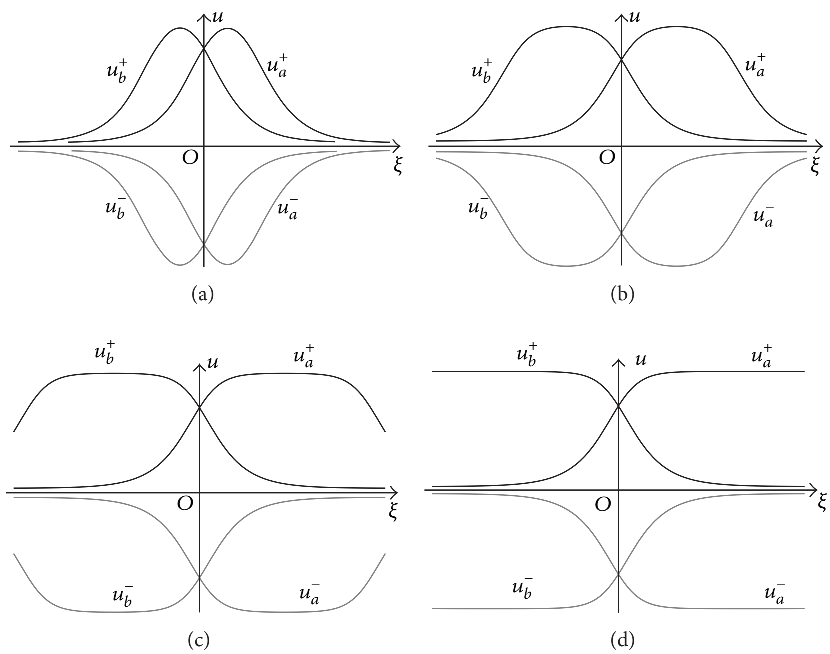

FIGURE 2: Four low-kink waves are bifurcated from four symmetric solitary waves. The varying process for the figures of $u_a^{\pm}$ and $u_b^{\pm}$ when $\lambda > 0$, $\lambda \neq \sqrt{\beta_1^2 - 4\alpha_1}$, $(a,b) \in A_2$, and $b \to -(5a^2/48c)+0$, where $\lambda = 50$, $\gamma = \delta = c = a = 1$, and (a) $b = -(5a^2/48c)+10^{-2}$, (b) $b = -(5a^2/48c)+10^{-4}$, (c) $b = -(5a^2/48c) + 10^{-6}$, and (d) $b = -(5a^2/48c) + 10^{-9}$.

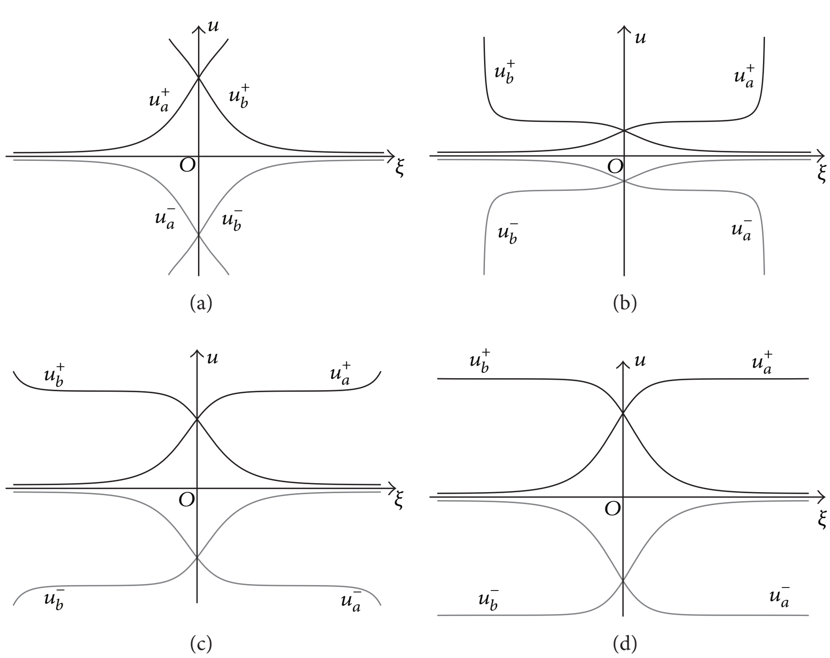

FIGURE 3: Four low-kink waves are bifurcated from four 1-blow-up waves. The varying process for the figures of $u_a^{\pm}$ and $u_b^{\pm}$ when $\lambda > 0$, $\lambda \neq \sqrt{\beta_1^2 - 4\alpha_1}$, $(a,b) \in A_3$, and $b \to -(5a^2/48c)-0$, where $\lambda = 50$, $\gamma = \delta = c = a = 1$, and (a) $b = -(5a^2/48c)-10^{-2}$, (b) $b = -(5a^2/48c)-10^{-5}$, (c) $b = -(5a^2/48c) - 10^{-7}$, and (d) $b = -(5a^2/48c) - 10^{-9}$.

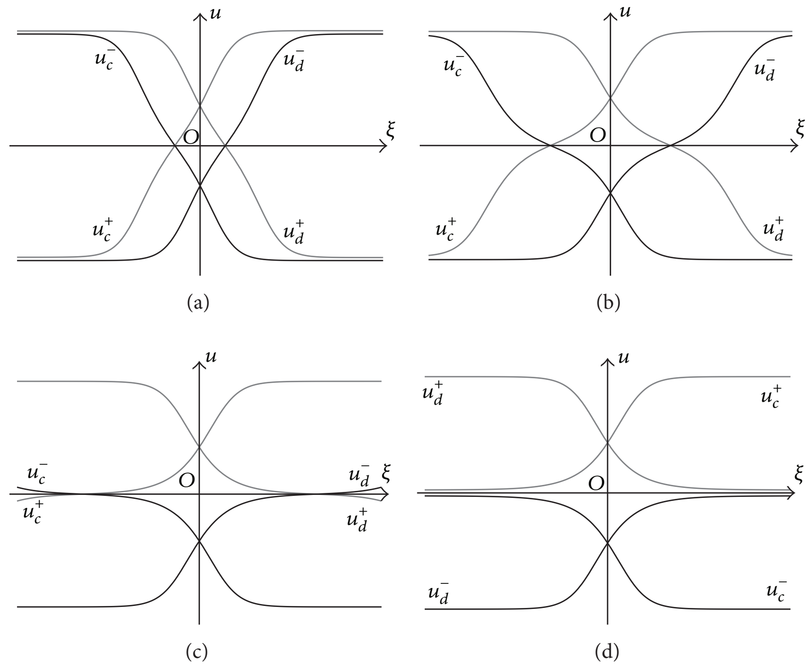

FIGURE 4: Four low-kink waves are bifurcated from four tall-kink waves. The varying process for the figures of $u_c^{\pm}$ and $u_d^{\pm}$ when $\eta > 0, \eta \neq |\mu_0|$, $(a, b) \in A_2$, and $b \to -(5a^2/48c) + 0$, where $\eta = 10$, $\delta = \gamma = c = a = 1$, and (a) $b = -(5a^2/48c) + 10^{-2}$, (b) $b = -(5a^2/48c) + 10^{-3}$, (c) $b = -(5a^2/48c) + 10^{-5}$, and (d) $b = -(5a^2/48c) + 10^{-7}$.

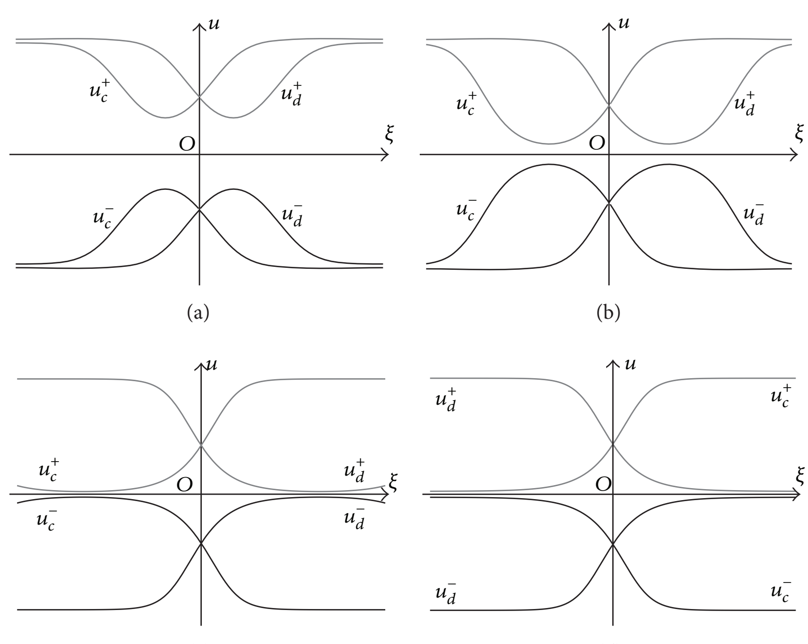

FIGURE 5: Four low-kink waves are bifurcated from four antisymmetric solitary waves. The varying process for the figures of $u_c^{\pm}$ and $u_d^{\pm}$ when $\eta > 0, \eta \neq |\mu_0|$, $(a, b) \in A_3$, and $b \to -(5a^2/48c) - 0$, where $\eta = 10$, $\delta = \gamma = c = a = 1$, and (a) $b = -(5a^2/48c) - 10^{-2}$, (b) $b = -(5a^2/48c) - 10^{-3}$, (c) $b = -(5a^2/48c) - 10^{-5}$, and (d) $b = -(5a^2/48c) - 10^{-7}$.

(2) *If* $(a,b) \in A_3$ *and* $\eta \neq |\mu_0|$, *then* $u_c^{\pm} \neq u_d^{\pm}$, *and they represent four antisymmetry solitary waves (see Figures 5(a)–5(c)). When* $b \to -(5a^2/36c) + 0$, *the four antisymmetry solitary waves become two trivial waves* $u = \pm\sqrt{6c/a}$. *In particular, when* $b \to -(5a^2/48c) - 0$, *the four antisymmetry solitary waves become four low-kink waves with the expressions* $u_{c0}^{\pm}$ *and* $u_{d0}^{\pm}$ *(see Figure 5(d)). For the varying process, see Figure 5.*

(3) *If* $(a,b) \in A_2$ *and* $\eta = |\mu_0|$, *then* $u_c^{\pm} = u_d^{\mp} = u_{cd}^{\mp}$ *of forms*

$$u_{cd}^{\mp} = \mp\sqrt{\frac{(5a+\Delta)(5a-2\Delta)}{3b\left[(2\Delta-5a)\cosh(\tau_2\xi)+5a+4\Delta\right]}}\sinh\left(\frac{\tau_2}{2}\xi\right), \tag{29}$$

which represent two tall-kink waves and tend to a trivial wave $u = 0$ *when* $b \to -(5a^2/48c) + 0$.

(4) *If* $(a,b) \in A_3$ *and* $\eta = |\mu_0|$, *then* $u_c^{\pm} = u_d^{\pm} = u_{cd^*}^{\mp}$ *of forms*

$$u_{cd^*}^{\mp} = \mp\sqrt{\frac{(5a+\Delta)(2\Delta-5a)}{3b\left[(5a-2\Delta)\cosh(\tau_2\xi)+5a+4\Delta\right]}}\cosh\left(\frac{\tau_2}{2}\xi\right), \tag{30}$$

which represent two antisymmetric solitary waves and tend to the trivial wave $u = 0$ *when* $b \to -(5a^2/48c) - 0$ *and tend to* $u = \pm\sqrt{12c/a}$ *when* $b \to -(5a^2/36c) + 0$.

Proof. In (8), letting $h = H(\varphi_1, 0)$, it follows that

$$\psi = \pm\sqrt{-\frac{b}{15(\gamma+\delta)}\left(\varphi_1^2-\varphi^2\right)^2\left(\varphi^2+\mu_0\right)}, \tag{31}$$

where φ_1 and μ_0 are given in (9) and (26), respectively. Substituting (31) into $d\varphi/d\xi = \psi$ and integrating it, we have

$$\int_p^{\varphi} \frac{ds}{\sqrt{-\left(b/15(\gamma+\delta)\right)\left(\varphi_1^2-\varphi^2\right)^2\left(\varphi^2+\mu_0\right)}} = \xi, \tag{32}$$

where p is an arbitrary constant.

Completing the integral above and solving the equation for φ, it follows that

$$\varphi = \pm\sqrt{\varphi_1^2 - \frac{4\alpha_2\eta e^{\tau_2\xi}}{\eta^2 e^{2\tau_2\xi} - 2\eta\beta_2 e^{\tau_2\xi} + \left(\beta_2^2 - 4\alpha_2\right)}}, \tag{33}$$

where τ_2 is given in (25), $\eta = \eta(p)$ is an arbitrary real number, and

$$\alpha_2 = \frac{\Delta(\Delta+5a)}{12b^2}, \qquad \beta_2 = \frac{4\Delta+5a}{6b}. \tag{34}$$

Similarly, if $u = \varphi(\xi)$ is a solution of (4), so is $u = \varphi(-\xi)$. Substituting (34) into (33), we get $u_c^{\pm}$ and $u_d^{\pm}$ (see (24)).

When $b \to -5a^2/48c$, it follows that

$$\tau_2 \longrightarrow 2\sqrt{\frac{c}{\gamma+\delta}}, \qquad 5a - 2\Delta \longrightarrow 0. \tag{35}$$

From (24), it is easy to check that $u_c^{\pm}$ and $u_d^{\pm}$ become $u_{c0}^{\pm}$ and $u_{d0}^{\pm}$ (see (27) and (28)).

If $(a,b) \in A_2$, then $2\Delta - 5a > 0$ and $\eta = |\mu_0| = (5a - 2\Delta)/6b$, and we have

$$\begin{aligned} u_c^{\pm} &= \pm\frac{\sqrt{-(5a+\Delta)/6b}\left(2\Delta-5a+6b\eta e^{\tau_2\xi}\right)}{\sqrt{(5a-2\Delta)^2+36b^2\eta^2e^{2\tau_2\xi}-12b(5a+4\Delta)\eta e^{\tau_2\xi}}} \\ &= \pm\frac{\sqrt{-(5a+\Delta)/6b}\,(2\Delta-5a)\left(1-e^{\tau_2\xi}\right)}{\sqrt{(5a-2\Delta)^2+(5a-2\Delta)^2e^{2\tau_2\xi}-2(5a+4\Delta)(5a-2\Delta)e^{\tau_2\xi}}} \\ &= \mp 2\sqrt{-\frac{5a+\Delta}{6b}} \\ &\quad \cdot\sqrt{\frac{2\Delta-5a}{2(2\Delta-5a)\cosh(\tau_2\xi)+2(4\Delta+5a)}}\sinh\left(\frac{\tau_2}{2}\xi\right) \\ &= \mp\sqrt{\frac{(5a+\Delta)(5a-2\Delta)}{3b\left[(2\Delta-5a)\cosh(\tau_2\xi)+5a+4\Delta\right]}}\sinh\left(\frac{\tau_2}{2}\xi\right) \\ &= u_{cd}^{\mp} \quad (\text{see } (29)). \end{aligned} \tag{36}$$

Similarly, we have $u_d^{\pm} = u_{cd}^{\pm}$.

If $(a,b) \in A_3$, then $5a - 2\Delta > 0$ and $\eta = |\mu_0| = (2\Delta - 5a)/6b$, and we have

$$\begin{aligned} u_c^{\pm} &= \pm\frac{\sqrt{-(5a+\Delta)/6b}\left(2\Delta-5a+6b\eta e^{\tau_2\xi}\right)}{\sqrt{(5a-2\Delta)^2+36b^2\eta^2e^{2\tau_2\xi}-12b(5a+4\Delta)\eta e^{\tau_2\xi}}} \\ &= \pm\frac{\sqrt{-(5a+\Delta)/6b}\,(2\Delta-5a)\left(e^{\tau_2\xi}+1\right)}{\sqrt{(5a-2\Delta)^2+(5a-2\Delta)^2e^{2\tau_2\xi}+2(5a+4\Delta)(5a-2\Delta)e^{\tau_2\xi}}} \\ &= \mp\sqrt{-\frac{5a+\Delta}{6b}} \\ &\quad \cdot\frac{\sqrt{5a-2\Delta}\left(e^{\tau_2\xi/2}+e^{-\tau_2\xi/2}\right)}{\sqrt{(5a-2\Delta)\left(e^{\tau_2\xi/2}+e^{-\tau_2\xi/2}\right)+2(4\Delta+5a)}} \\ &= \mp 2\sqrt{-\frac{5a+\Delta}{6b}} \\ &\quad \cdot\sqrt{\frac{5a-2\Delta}{2(5a-2\Delta)\cosh(\tau_2\xi)+2(4\Delta+5a)}}\cosh\left(\frac{\tau_2}{2}\xi\right) \\ &= \mp\sqrt{\frac{(5a+\Delta)(2\Delta-5a)}{3b\left[(5a-2\Delta)\cosh(\tau_2\xi)+5a+4\Delta\right]}}\cosh\left(\frac{\tau_2}{2}\xi\right) \\ &= u_{cd^*}^{\mp} \quad (\text{see } (30)). \end{aligned} \tag{37}$$

Similarly, we have $u_d^{\pm} = u_{cd^*}^{\mp}$.

Hereto, we have completed the proof for Proposition 2. □

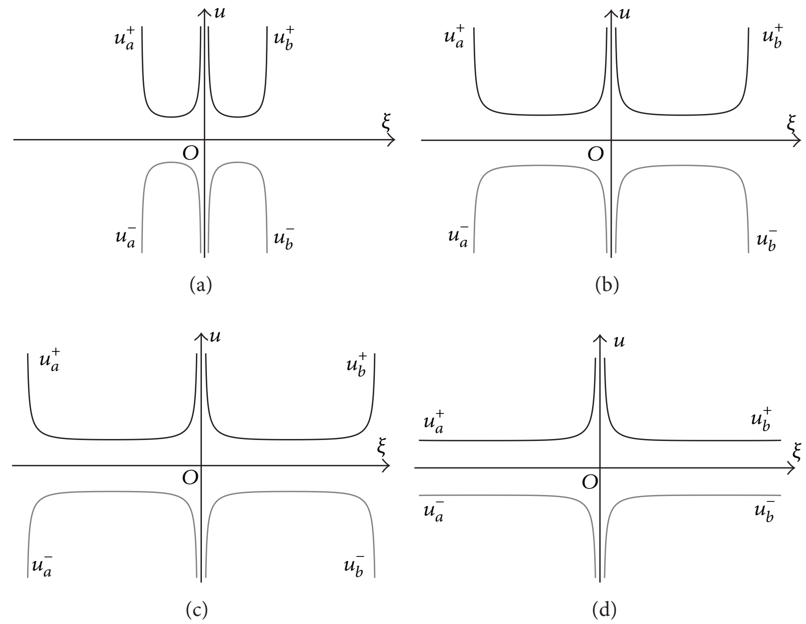

FIGURE 6: Four 2-blow-up waves become four 1-blow-up waves. The varying process for the figures of $u_a^{\pm}$ and $u_b^{\pm}$ when $\lambda < 0, \lambda \neq -\sqrt{\beta_1^2 - 4\alpha_1}$, $(a,b) \in A_2$, and $b \rightarrow -(5a^2/48c) + 0$, where $\lambda = -50$, $\gamma = \delta = c = a = 1$, and (a) $b = -(5a^2/48c) + 10^{-2}$, (b) $b = -(5a^2/48c) + 10^{-4}$, (c) $b = -(5a^2/48c) + 10^{-5}$, and (d) $b = -(5a^2/48c) + 10^{-7}$.

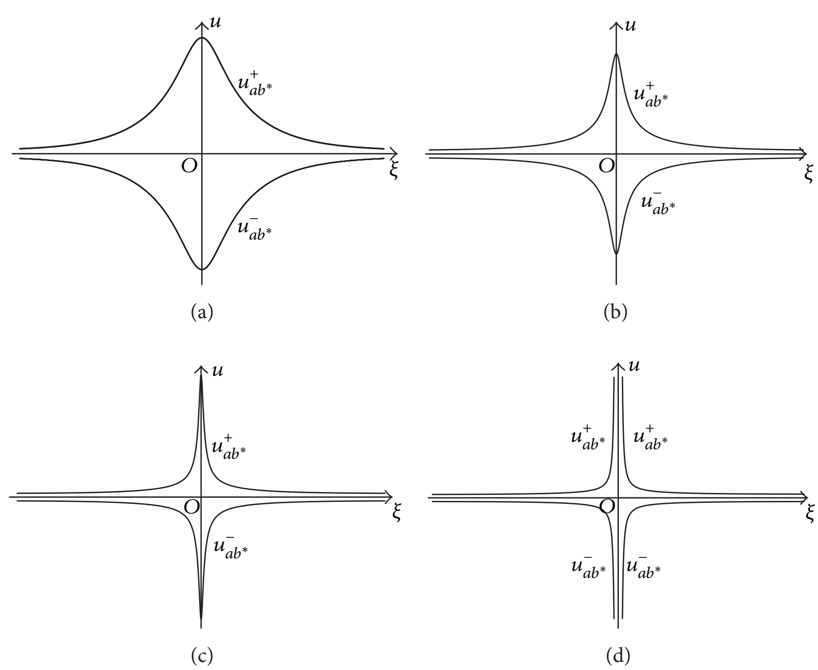

FIGURE 7: Two 1-blow-up waves are bifurcated from two symmetric solitary waves. The varying process for the figures of $u_{ab^*}^{\pm}$ when $\lambda = -\sqrt{\beta_1^2 - 4\alpha_1}$, $(a,b) \in A_6$, and $b \rightarrow 0 + 0$, where $\lambda = a = -1$, $\gamma = \delta = c = 1$, and (a) $b = 0 + 5 \times 10^{-1}$, (b) $b = 0 + 10^{-2}$, (c) $b = 0 + 10^{-4}$, and (d) $b = 0 + 10^{-7}$.

3. Bifurcation of the 1-Blow-Up Waves

In this section, we show that the 1-blow-up waves can be bifurcated from the 2-blow-up waves, the symmetric solitary waves, and the periodic-blow-up waves.

3.1. Bifurcation from 2-Blow-Up Waves and Symmetric Solitary Waves

Proposition 3. *In* (13), *corresponding to* $\lambda < 0$, *one has the following results and bifurcation phenomena.*

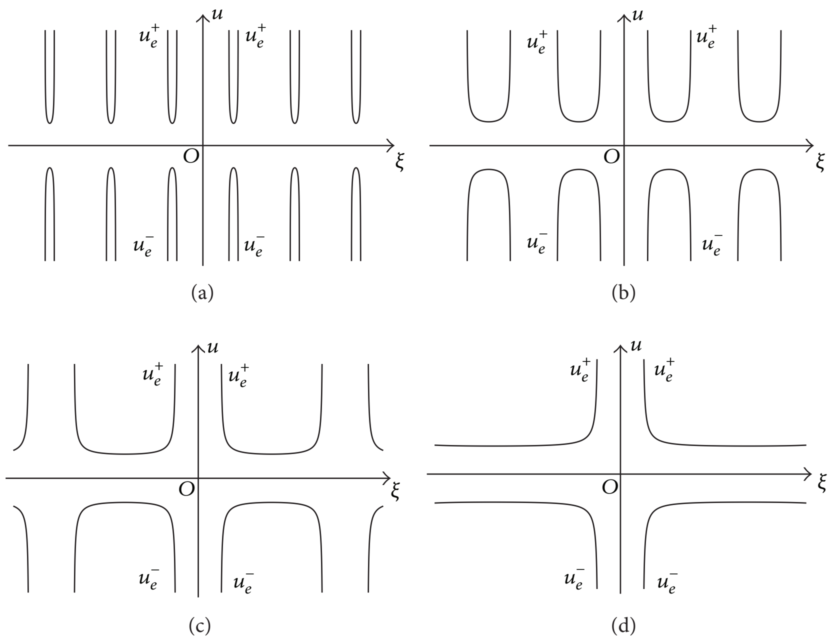

Figure 8: The 1-blow-up waves are bifurcated from the periodic-blow-up waves. The varying process for the figures of $u_e^{\pm}$ when $b \to -(5a^2/36c) + 0$, where $\gamma = \delta = c = a = 1$ and (a) $b = -(5a^2/36c) + 10^{-1}$, (b) $b = -(5a^2/36c) + 10^{-2}$, (c) $b = -(5a^2/36c) + 10^{-3}$, and (d) $b = -(5a^2/36c) + 10^{-4}$.

(1) *If $(a,b) \in A_2$ and $\lambda \neq -\sqrt{\beta_1^2 - 4\alpha_1}$, then $u_a^{\pm} \neq u_b^{\pm}$, and they represent four 2-blow-up waves. When $b \to -(5a^2/48c) + 0$, $u_a^{\pm}$, and $u_b^{\pm}$, respectively, become $u_{a0}^{\pm}$ and $u_{b0}^{\pm}$ (see* (17) *and* (18)*) which represent four 1-blow-up waves (see Figure 6(d)). For the varying process, see Figure 6.*

(2) *If $\lambda = -\sqrt{\beta_1^2 - 4\alpha_1}$, then $u_a^{\pm} = u_b^{\pm}$ and become*

$$u_{ab^*}^{\pm} = \pm\sqrt{\frac{12c}{\left(\sqrt{25a^2 + 240bc}/5\right)\cosh\left[2\sqrt{c/(\gamma+\delta)}\xi\right] + a}}, \tag{38}$$

which were given by Song and Cai [6]. This implies that one extends the previous results.

When $(a,b) \in A_2$, $u_{ab^*}^{\pm}$ represent hyperbolic blow-up waves. Specially, when $b \to -(5a^2/48c) + 0$, $u_{ab^*}^{\pm}$ tend to two trivial solutions $u = \pm\sqrt{12c/a}$.

When (a,b) belongs to any one of the regions A_1, A_6, $u_{ab^*}^{\pm}$ represent two symmetric solitary waves. In particular, when $(a,b) \in A_6$ and $b \to 0+0$, $u_{ab^*}^{\pm}$ become two 1-blow-up waves. For the varying process, see Figure 7.

Similar to the proof of Proposition 1, we get the results of Proposition 3.

3.2. Bifurcation from Periodic-Blow-Up Waves

Proposition 4. *Under $\gamma + \delta > 0$ and $H(\varphi,\psi) = H(\varphi_2, 0)$, one has the following results and bifurcation phenomena.*

(1) *If (a,b) belongs to one of the regions A_2, A_3, and l_2, then* (4) *has two periodic-blow-up wave solutions*

$$u_e^{\pm} = \pm\sqrt{\frac{(5a-\Delta)(5a+2\Delta)}{3b\left[(5a+2\Delta)\cos(\tau_3\xi) + 4\Delta - 5a\right]}}\sin\left(\frac{\tau_3}{2}\xi\right), \tag{39}$$

where

$$\tau_3 = \sqrt{\frac{\Delta(\Delta - 5a)}{45b(\gamma+\delta)}}. \tag{40}$$

(2) *If $(a,b) \in A_3$ and $b \to -(5a^2/36c) + 0$, the periodic-blow-up wave solutions $u_e^{\pm}$ become two fractional wave solutions*

$$u_{e0}^{\pm} = \pm\frac{\sqrt{6}c\xi}{\sqrt{a\left[c\xi^2 - 3(\gamma+\delta)\right]}}, \tag{41}$$

which represent two 1-blow-up waves (see Figure 8(d)). For the varying process, see Figure 8.

Proof. (1) In (8), letting $h = H(\varphi_2, 0)$, it follows that

$$\psi = \pm\sqrt{-\frac{b}{15(\gamma+\delta)}(\varphi^2 - \varphi_2^2)^2(\varphi^2 + \mu_1)}, \tag{42}$$

where φ_2 is given in (10), and

$$\mu_1 = \frac{5a + 2\Delta}{6b}. \tag{43}$$

Substituting (42) into $d\varphi/d\xi = \psi$ and integrating it, we have

$$\int_q^{\varphi} \frac{ds}{\sqrt{-(b/15(\gamma+\delta))(\varphi^2-\varphi_2^2)^2(\varphi^2+\mu_1)}} = \xi, \tag{44}$$

where q is an arbitrary constant.

Completing the integral in (44) and solving the equation for φ, it follows that

$$\varphi = \pm\sqrt{\varphi_2^2 - \frac{2\alpha_3}{\mu_1 \sin(\tau_3\xi+\theta_1)+\beta_3}}, \tag{45}$$

where τ_3 is given in (40), $\theta_1 = \theta_1(q)$ is an arbitrary constant, and

$$\alpha_3 = \frac{\Delta(\Delta-5a)}{12b^2}, \qquad \beta_3 = \frac{4\Delta-5a}{6b}. \tag{46}$$

In (45) letting $\theta_1 = \pi/2$, we obtain the solutions $u_e^{\pm}$ as (39).

(2) Note that

$$\begin{aligned} \cos(\tau_3\xi) &= 1 - \frac{\tau_3^2\xi^2}{2} + \frac{\tau_3^4\xi^4}{4!} + \cdots \\ &= 1 - \frac{\Delta(\Delta+5a)\xi^2}{90b(\gamma+\delta)} + O(\Delta^2), \end{aligned} \tag{47}$$

$$\lim_{b\to-(5a^2/36c)+0}\Delta = \lim_{b\to-(5a^2/36c)+0}\sqrt{25a^2+180bc} = 0. \tag{48}$$

Thus, we have

$$\begin{aligned} u_e^{\pm} &= \pm\sqrt{\frac{(5a+\Delta)(5a-2\Delta)[1-\cos(\tau_3\xi)]}{6b[(5a-2\Delta)\cos(\tau_3\xi)-5a-4\Delta]}} \\ &= \pm\sqrt{\frac{(1/90b(\gamma+\delta))(5a+\Delta)(5a-2\Delta)\Delta(\Delta+5a)\xi^2+O(\Delta^2)}{6b[(5a-2\Delta)(1-(\Delta(5a+\Delta)\xi^2)/90b(\gamma+\delta))-5a-4\Delta+O(\Delta^2)]}} \\ &= \pm\sqrt{\frac{(5a+\Delta)^2(5a-2\Delta)\xi^2+O(\Delta)}{6b[-540b(\gamma+\delta)-(5a-2\Delta)(5a+\Delta)\xi^2+O(\Delta)]}}. \end{aligned} \tag{49}$$

Furthermore, we get

$$\begin{aligned} &\lim_{b\to-(5a^2/36c)+0} u_e^{\pm} \\ &= \lim_{b\to-(5a^2/36c)+0} \pm\sqrt{\frac{(5a+\Delta)^2(5a-2\Delta)\xi^2+O(\Delta)}{6b[-540b(\gamma+\delta)-(5a-2\Delta)(5a+\Delta)\xi^2+O(\Delta)]}} \\ &= \pm\sqrt{\frac{25a^2\times5a\xi^2}{6\times(-5a^2/36c)[-540\times(-5a^2/36c)(\gamma+\delta)-25a^2\xi^2]}} \\ &= \pm\frac{\sqrt{6}c\xi}{\sqrt{a[c\xi^2-3(\gamma+\delta)]}} \\ &= u_{e0}^{\pm} \quad (\text{see } (41)). \end{aligned} \tag{50}$$

Hereto, we have completed the proof for Proposition 4. □

4. Bifurcation of the Periodic-Blow-Up Waves

In this section, we show that the periodic-blow-up waves can be bifurcated from symmetric periodic waves.

4.1. Bifurcation from Periodic Waves

Proposition 5. *If $ab \neq 0$, $\gamma+\delta < 0$, and $H(\varphi,\psi) = H(0,0)$, (4) has two nonlinear wave solutions*

$$u_f^{\pm} = \pm\sqrt{\frac{-2\alpha_1}{\sqrt{\beta_1^2-4\alpha_1}\cos(\tau_4\xi)+\beta_1}}, \tag{51}$$

where

$$\tau_4 = 2\sqrt{-\frac{c}{\gamma+\delta}}. \tag{52}$$

One has the following results and bifurcation phenomena.

(1) *If (a,b) belongs to any one of the regions A_1 and A_6, then $u_f^{\pm}$ represent periodic-blow-up waves.*

(2) *If $(a,b) \in A_2$, then $u_f^{\pm}$ represent periodic waves. In particular, when $b \to 0-0$, the periodic waves become periodic-blow-up waves as follows:*

$$u_{f0}^{\pm} = \pm\sqrt{\frac{12c}{a\left[1-\cos\left(\tau_4\xi\right)\right]}}. \tag{53}$$

For the varying process, see Figure 9.

When $b \to -(5a^2/48c)+0$, the periodic wave tends to two trivial waves $u = \pm\sqrt{12c/a}$. For the varying process, see Figure 10.

Proof. Completing the integral in (21) and solving the equation for φ, it follows that

$$\varphi = \pm\sqrt{\frac{-2\alpha_1}{\sqrt{\beta_1^2-4\alpha_1}\sin\left(\tau_4\xi+\theta_2\right)+\beta_1}}, \tag{54}$$

where τ_4 is given in (52) and $\theta_2 = \theta_2(\nu)$ is an arbitrary constant.

In (54) letting $\theta_2 = \pi/2$, we obtain the solutions $u_f^{\pm}$ as (51).

From (14) and (15), we have

$$\begin{aligned}\sqrt{\beta_1^2-4\alpha_1} &= \sqrt{\frac{240bc+25a^2}{4b^2}}\\ &= -\frac{\sqrt{240bc+25a^2}}{2b}.\end{aligned} \tag{55}$$

Letting $b \to 0-0$, then

$$\begin{aligned}\lim_{b\to 0-0} u_f^{\pm} &= \lim_{b\to 0-0} \pm\sqrt{\frac{-2\alpha_1}{\sqrt{\beta_1^2-4\alpha_1}\cos\left(\tau_4\xi\right)+\beta_2}}\\ &= \lim_{b\to 0-0} \pm\sqrt{\frac{60c}{-\sqrt{240bc+25a^2}\cos\left(\tau_4\xi\right)+5a}}\\ &= \pm\sqrt{\frac{12c}{a\left[1-\cos\left(\tau_4\xi\right)\right]}}\\ &= u_{f0}^{\pm} \quad (\text{see } (53)).\end{aligned} \tag{56}$$

Hereto, we have completed the proof for Proposition 5. □

5. Bifurcation of the Tall-Kink Waves

In this section, we show that the tall-kink waves can be bifurcated from the symmetric periodic waves.

5.1. Bifurcation from Symmetric Periodic Waves

Proposition 6. *Under $\gamma+\delta<0$ and $H(\varphi,\psi) = H(\varphi_1, 0)$, one has the following results and bifurcation phenomena.*

(1) *If (a,b) belongs to the region A_3, then (4) has two periodic wave solutions*

$$u_g^{\pm} = \sqrt{\frac{(5a+\Delta)(5a-2\Delta)}{3b\left[(5a-2\Delta)\cos\left(\tau_5\xi\right)-5a-4\Delta\right]}}\sin\left(\frac{\tau_5}{2}\xi\right), \tag{57}$$

where

$$\tau_5 = \sqrt{\frac{\Delta(\Delta+5a)}{45b(\gamma+\delta)}}. \tag{58}$$

(2) *If $(a,b) \in A_3$ and $b \to -(5a^2/36c)+0$, the periodic wave solutions $u_g^{\pm}$ tend to two fractional wave solutions $u_{g0}^{\pm}$ which have the expressions as $u_{e0}^{\pm}$ (see (41)) and represent two tall-kink waves (see Figure 11(d)). For the varying process, see Figure 11.*

Proof. Completing the integral in (31) and solving the equation for φ, it follows that

$$\varphi = \pm\sqrt{\varphi_1^2 - \frac{2\alpha_2}{\mu_0\sin\left(\tau_5\xi+\theta_3\right)-\beta_2}}, \tag{59}$$

where τ_5 is given in (58) and $\theta_3 = \theta_3(p)$ is an arbitrary constant.

In (59) letting $\theta_3 = \pi/2$, we obtain the solutions $u_g^{\pm}$ as (57).

Similar to the proof of Proposition 4, we get the results of Proposition 6.

Besides these bifurcation phenomena above, there is another bifurcation phenomenon as follows. □

Proposition 7. *If $\gamma+\delta<0$ and (a,b) belongs to one of the regions A_2, A_3, and l_2, then (4) has four symmetric solitary wave solutions (see Figures 12(a)–12(c)) as follows:*

$$\begin{aligned}u_h^{\pm} &= \pm\frac{\sqrt{(\Delta-5a)/6b}\left(6b\omega e^{\tau_6\xi}-5a-2\Delta\right)}{\sqrt{36b^2\omega^2e^{2\tau_6\xi}-12b(5a-4\Delta)\omega e^{\tau_6\xi}+(5a+2\Delta)^2}},\\ u_i^{\pm} &= \pm\frac{\sqrt{(\Delta-5a)/6b}\left(6b\omega e^{-\tau_6\xi}-5a-2\Delta\right)}{\sqrt{36b^2\omega^2e^{-2\tau_6\xi}-12b(5a-4\Delta)\omega e^{-\tau_6\xi}+(5a+2\Delta)^2}},\end{aligned} \tag{60}$$

where

$$\tau_6 = \sqrt{\frac{\Delta(5a-\Delta)}{45b(\gamma+\delta)}}. \tag{61}$$

In particular, when $(a,b) \in A_3$ and $b \to -(5a^2/36c)+0$, $u_h^{\pm}$ and $u_i^{\pm}$ tend to two trivial solutions $u = \pm\sqrt{12c/a}$. For the varying process, see Figure 12.

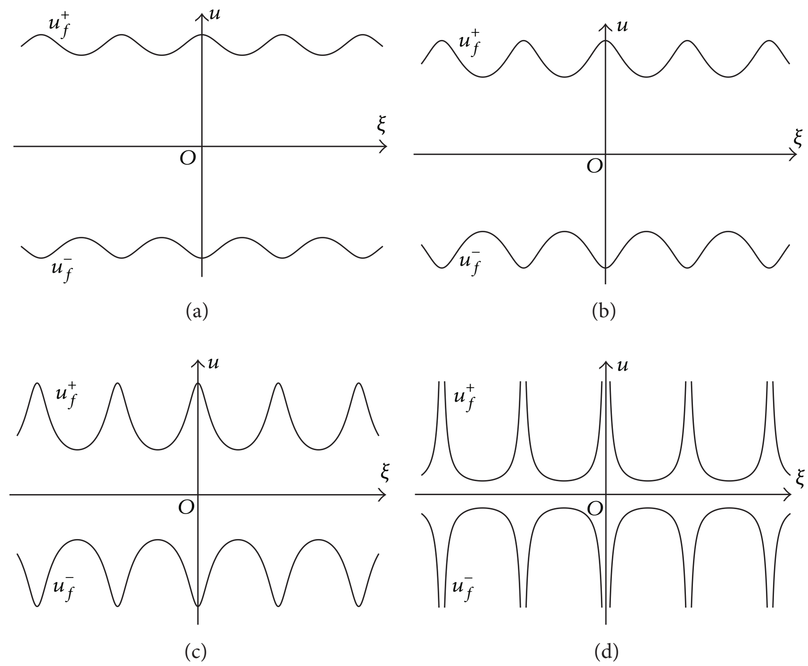

FIGURE 9: The periodic-blow-up waves are bifurcated from the symmetric periodic waves. The varying process for the figures of $u_f^{\pm}$ when $(a,b) \in A_2$ and $b \to 0-0$, where $c = a = 1$, $\gamma = \delta = -1$, and (a) $b = 0 - 10^{-1}$, (b) $b = 0 - 10^{-2}$, (c) $b = 0 - 10^{-4}$, and (d) $b = 0 - 10^{-7}$.

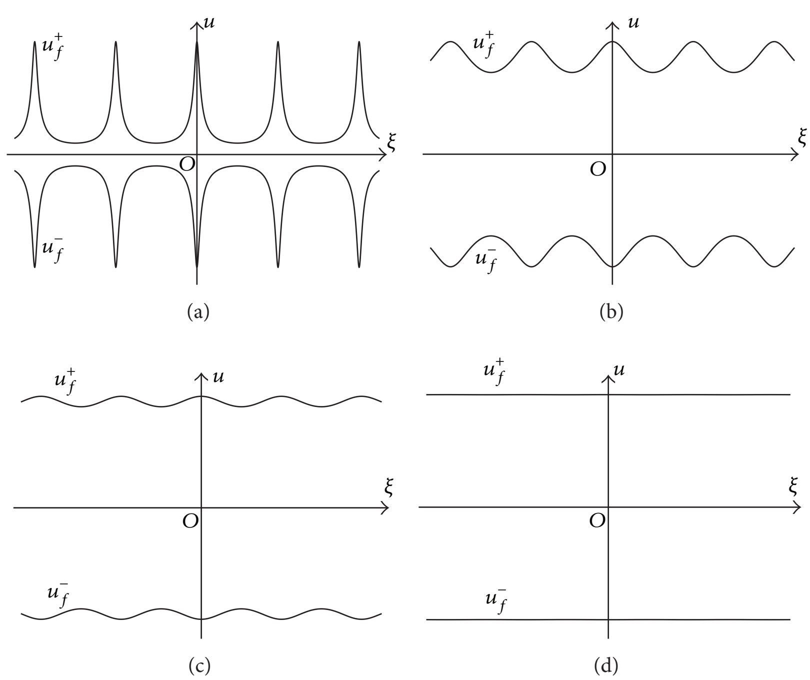

FIGURE 10: The periodic waves become the trivial waves. The varying process for the figures of $u_f^{\pm}$ when $(a,b) \in A_2$ and $b \to -(5a^2/48c)+0$, where $c = a = 1$, $\gamma = \delta = -1$, and (a) $b = -(5a^2/48c)+10^{-1}$, (b) $b = -(5a^2/48c)+10^{-2}$, (c) $b = -(5a^2/48c)+10^{-3}$, and (d) $b = -(5a^2/48c)+10^{-7}$.

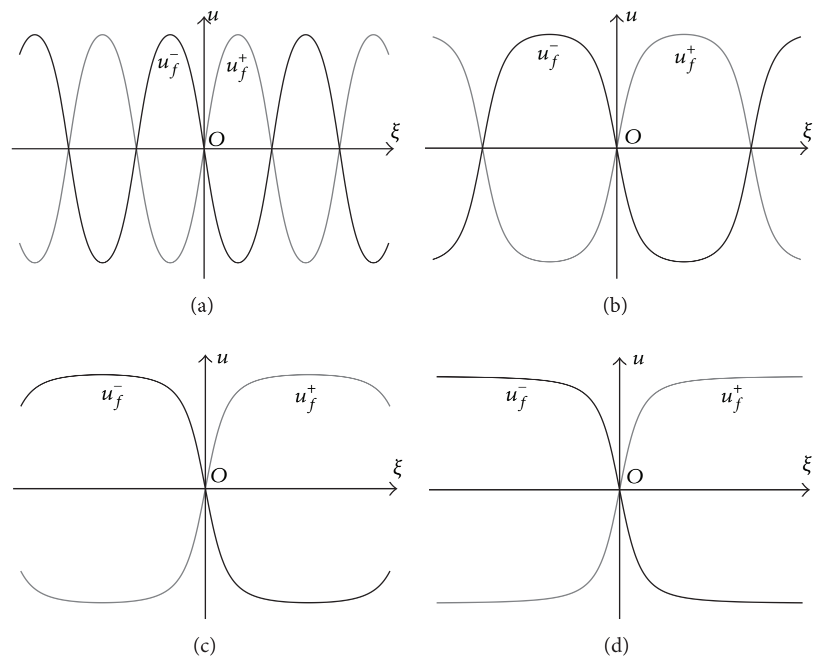

Figure 11: Two tall-kink waves are bifurcated form the symmetric periodic waves. The varying process for the figures of $u_g^{\pm}$ when $(a,b) \in A_3$ and $b \to -(5a^2/36c)+0$, where $c = a = 1$, $\delta = \gamma = -1$, and (a) $b = -(5a^2/36c) + 10^{-2}$, (b) $b = -(5a^2/36c) + 10^{-3}$, (c) $b = -(5a^2/36c) + 10^{-4}$, and (d) $b = -(5a^2/36c) + 10^{-7}$.

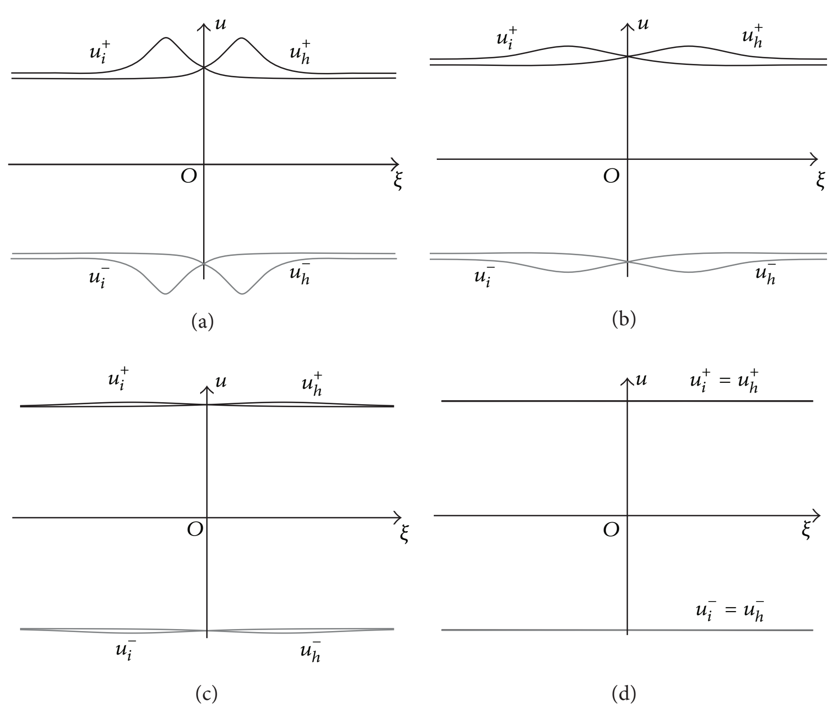

Figure 12: Four symmetric solitary waves become two trivial waves. The varying process for the figures of $u_h^{\pm}$ and $u_i^{\pm}$ when $b \to -(5a^2/36c)+0$, where $c = a = \omega = 1$, $\gamma = \delta = -1$, and (a) $b = -(5a^2/36c) + 10^{-2}$, (b) $b = -(5a^2/36c) + 10^{-3}$, (c) $b = -(5a^2/36c) + 10^{-4}$, and (d) $b = -(5a^2/36c) + 10^{-6}$.

Proof. Completing the integral in (44) and solving the equation for φ, it follows that

$$\varphi = \sqrt{\varphi_2^2 - \frac{4\alpha_3 \omega e^{\tau_6 \xi}}{\omega^2 e^{2\tau_6 \xi} + 2\omega\beta_3 e^{\tau_6 \xi} + \left(\beta_3^2 - 4\alpha_3\right)}}, \tag{62}$$

where α_3, β_3, and τ_6 are given in (46) and (61) and $\omega = \omega(q)$ is an arbitrary constant.

Similar to the derivations for $u_c^{\pm}$ and $u_d^{\pm}$, we get $u_h^{\pm}$ and $u_i^{\pm}$ (see (60)) from (62).

Hereto, we have completed the proofs for all propositions. □

6. Conclusion

In this paper, we have studied the bifurcation behavior of the nonlinear waves in a generalized Z-K equation. Firstly, we obtained three types of explicit nonlinear wave solutions. The first type is the exp-function expressions $u_a^{\pm}$, $u_b^{\pm}$, $u_c^{\pm}$, $u_d^{\pm}$, $u_h^{\pm}$, and $u_i^{\pm}$ (see (13), (24), and (60)). The second type is the trigonometric expressions $u_e^{\pm}$, $u_f^{\pm}$, and $u_g^{\pm}$ (see (39), (51), and (57)). The third type is the fractional expressions $u_{e0}^{\pm}$ (see (41)). Furthermore, four kinds of interesting bifurcation phenomena have been revealed. The first kind is that the low-kink waves can be bifurcated from four types of nonlinear waves, the symmetric solitary waves, the 1-blow-up waves, the tall-kink waves, and the antisymmetric solitary waves (see Propositions 1 and 2). The second kind is that the 1-blow-up waves can be bifurcated from the 2-blow-up waves, the symmetric solitary waves, and the periodic-blow-up waves (see Propositions 3 and 4). The third kind is that the periodic-blow-up waves can be bifurcated from the symmetric periodic waves (see Proposition 5). The fourth kind is that the tall-kink waves can be bifurcated from the symmetric periodic waves (see Proposition 6). Some previous results are our some special cases (see (17), (19), and (38)).

Conflict of Interests

The authors declare that they have no conflict of interests.

Acknowledgments

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